Pi Revisited

Abstract :Five notes related with the number Pi

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Pi Revisited One

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Resumen

En esta nota mostramos una fórmula que involucra las constantes Pi y Phi:

$$\pi = 2 + 8\int_0^1 \frac{x\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\,dx = 3.141592\ldots$$

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618033\ldots$$

Fórmula

(1) $$\pi^2 = \frac{12}{7}(7-4\varphi)\sum_{n=0}^{\infty} c_n\, I_n = \frac{12}{7}(7-4\varphi)\sum_{n=0}^{\infty} a_n\, 5^{-n} I_n$$

donde

(2) $$I_n = \int_0^1\int_0^1 \left(x\,y - \frac{1}{2}\right)^n (6\varphi - 4 + 6xy)\,dx\,dy\,, n \in \mathbb{N}\cup\{0\}$$

(3) $$c_{n+2} = -\frac{4}{5}(6\varphi-8)c_{n+1} - \frac{4}{5}(7-4\varphi)c_n\,, c_0 = 1, c_1 = -\frac{4}{5}(6\varphi-8)$$

(4) $$a_{n+2} = -4(6\varphi-8)a_{n+1} - 20(7-4\varphi)a_n\,, a_0 = 1, a_1 = -4(6\varphi-8)$$

Poniendo $a_n = A_n + B_n\varphi$, se tiene:

(5) $$\begin{cases} A_{n+2} = 32A_{n+1} - 24B_{n+1} - 140A_n + 80B_n \\ B_{n+2} = -24A_{n+1} - 8B_{n+1} + 80A_n - 60B_n \\ \quad A_0 = 1, B_0 = 0, A_1 = 32, B_1 = -24 \end{cases}$$

Desarrollando (2) , se tiene:

(6) $$I_n = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^{n-k} \left(\frac{6\varphi - 4}{(k+1)^2} + \frac{6}{(k+2)^2}\right), n \in \mathbb{N} \cup \{0\}$$

La fórmula (6) , se puede escribir como:

(7) $$I_n = F_n + G_n \varphi$$

donde

(8) $$\begin{cases} F_n = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^{n-k} \left(-\frac{4}{(k+1)^2} + \frac{6}{(k+2)^2}\right) \\ G_n = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^{n-k} \frac{6}{(k+1)^2} \end{cases}$$

La fórmula (1) , se puede escribir como:

(9) $$\pi^2 = \frac{12}{7}(7 - 4\varphi) \sum_{n=0}^{\infty} 5^{-n} (A_n + B_n \varphi)(F_n + G_n \varphi)$$

La fórmula (9) , se puede simplificar:

(10) $$\pi^2 = \frac{12}{7}(22\varphi - 31) \sum_{n=0}^{\infty} b_n J_n = \frac{12}{7}(22\varphi - 31) \sum_{n=0}^{\infty} 5^{-n} d_n J_n$$

donde

(11) $$J_n = \int_0^1 \int_0^1 \left(x\, y - \frac{1}{2}\right)^n dx\, dy, n \in \mathbb{N} \cup \{0\}$$

(12) $$J_n = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^{n-k} \frac{1}{(k+1)^2}, n \in \mathbb{N} \cup \{0\}$$

(13) $$b_{n+2} = -\frac{4}{5}(6\varphi - 8) b_{n+1} - \frac{4}{5}(7 - 4\varphi) b_n, b_0 = 1, b_1 = \frac{1162 - 804\varphi}{205}$$

(14) $$d_{n+2} = -4(6\varphi - 8) d_{n+1} - 20(7 - 4\varphi) d_n, d_0 = 1, d_1 = \frac{1162 - 804\varphi}{41}$$

Poniendo $d_n = u_n + v_n \varphi$, se tiene:

$$(15)\quad \begin{cases} u_{n+2} = 32u_{n+1} - 24v_{n+1} - 140u_n + 80v_n \\ v_{n+2} = -24u_{n+1} + 8v_{n+1} + 80u_n - 60v_n \\ u_0 = 1, v_0 = 0, u_1 = \dfrac{1162}{41}, v_1 = -\dfrac{804}{41} \end{cases}$$

La fórmula (10) , se puede escribir como:

$$(16)\quad \pi^2 = \frac{12}{7}(22\varphi - 31)\sum_{n=0}^{\infty} 5^{-n}\,(u_n + v_n\varphi)J_n$$

Referencias

[1] Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions. Nueva York: Dover , 1965.

[2] Boros, G., Moll, V.: Irresistible Integrals.Cambridge University Press,2004.

[3] Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals,Series and Products. 5th ed.,ed. Alan Jeffrey. Academic Press, 1994.

[4] Spiegel,M.R.: Mathematical Handbook, McGraw-Hill Book Company , New York , 1968.

[5] Valdebenito, E.:, Pi Handbook , manuscript , unpublished , 1989 , (20000 formulas).

Pi Revisited Two

Edgar Valdebenito

Septiembre 16 , 2009

Resumen

En esta nota mostramos algunas fórmulas que involucran la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

Introducción

En esta nota mostramos algunas series que involucran la constante Pi , las series son obtenidas de la teoría de integrales elípticas.Una fórmula importante es:

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\, dx \, , 0 < k < 1$$

Conocida como integral elíptica completa de primera especie.un desarrollo en serie para $K(k)$,es:

$$K(k) = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right\} , 0 < k < 1$$

Desarrollando en serie la integral para $K(k)$, se pueden obtener una infinidad de fórmulas que involucran la constante Pi.

Fórmulas

(1) $$\pi \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{k}{4}\right)^{2n}$$
$$= \frac{4}{\sqrt{2(1-k^2-a)}} \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s} \binom{2n-2m}{n-m}\binom{2m}{m}\binom{m}{s}\binom{s}{r} \frac{(-1)^{s+r}\, 2^{-3n+m+s-r} a^{m-s} k^{2s}}{(1-k^2+a)^m (2n-2m+2s+2r+1)}$$

$$0 < k < 1 \, , a > k^2 - \frac{1}{2}$$

$$(2)\qquad \pi\sum_{n=0}^{\infty}\binom{2n}{n}^2\left(\frac{k}{4}\right)^{2n}$$
$$=\frac{4}{\sqrt{2-k^2}}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{2n-2m}{n-m}\binom{2m}{m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{s+r}\,2^{-3n+m+2s-r}}{2n-2m+2s+2r+1}\left(\frac{k^2}{2-k^2}\right)^m$$
$$0<k<1$$

$$(3)\qquad \pi\sum_{n=0}^{\infty}\binom{2n}{n}^2\left(\frac{2q^2-p^2}{16q^2}\right)^{n}$$
$$=\frac{4q}{p}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{2n-2m}{n-m}\binom{2m}{m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{s+r}\,2^{-3n+m+2s-r}}{2n-2m+2s+2r+1}\left(\frac{2q^2-p^2}{p^2}\right)^m$$
$$p,q\in\mathbb{N}\,|\ \left|2-\frac{p^2}{q^2}\right|<1\ \vee\ q^2<p^2<2q^2$$

Sean p_n , q_n , las sucesiones definidas como sigue:

$$\begin{cases} p_{n+1}=p_n+2q_n\ ,n\in\mathbb{N} \\ q_{n+1}=p_n+q_n\ ,n\in\mathbb{N} \\ p_1=3\ ,q_1=2 \end{cases}$$

Algunos valores de $\{(p_n,q_n):n\in\mathbb{N}\}$, son:

$$\{(p_n,q_n)\}=\{(3,2),(7,5),(17,12),(41,29),\ldots\}$$

Las sucesiones p_n,q_n , satisfacen las condiciones de la fórmula (3) , por lo tanto tenemos una infinidad de identidades que involucran la constante Pi.

Ejemplo 1: $(p_1,q_1)=(3,2)$

$$(4)\qquad \pi\sum_{n=0}^{\infty}\binom{2n}{n}^2\left(-\frac{1}{64}\right)^{n}=\frac{8}{3}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{2n-2m}{n-m}\binom{2m}{m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{s+r+m}\,2^{-3n+m+2s-r}\,3^{-2m}}{2n-2m+2s+2r+1}$$

Ejemplo 2: $(p_2,q_2)=(7,5)$

$$(5)\qquad \pi\sum_{n=0}^{\infty}\binom{2n}{n}^2\left(\frac{1}{400}\right)^{n}=\frac{20}{7}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{2n-2m}{n-m}\binom{2m}{m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{s+r}\,2^{-3n+m+2s-r}\,7^{-2m}}{2n-2m+2s+2r+1}$$

Ejemplo 3: $(p_5,q_5)=(99,70)$

(6) $$\pi \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(-\frac{1}{78400}\right)^n$$
$$= \frac{280}{99} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=0}^{m} \sum_{r=0}^{s} \binom{2n-2m}{n-m} \binom{2m}{m} \binom{m}{s} \binom{s}{r} \frac{(-1)^{s+r+m} 2^{-3n+m+2s-r} 99^{-2m}}{2n-2m+2s+2r+1}$$

Ejemplo 4: $(p_6, q_6) = (239,169)$

(7) $$\pi \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{1}{456976}\right)^n$$
$$= \frac{676}{239} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{s=0}^{m} \sum_{r=0}^{s} \binom{2n-2m}{n-m} \binom{2m}{m} \binom{m}{s} \binom{s}{r} \frac{(-1)^{s+r} 2^{-3n+m+2s-r} 239^{-2m}}{2n-2m+2s+2r+1}$$

Observación. Todas las fórmulas se han tomado de la referencia (5).

Referencias

[1] Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions. Nueva York: Dover , 1965.

[2] Boros, G., Moll, V.: Irresistible Integrals.Cambridge University Press,2004.

[3] Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals,Series and Products. 5th ed.,ed. Alan Jeffrey. Academic Press, 1994.

[4] Spiegel,M.R.: Mathematical Handbook, McGraw-Hill Book Company , New York , 1968.

[5] Valdebenito, E.:, Pi Handbook , manuscript , unpublished , 1989 , (20000 formulas).

Pi Revisited Three

Edgar Valdebenito

Septiembre 23 , 2009

Resumen

Se muestran algunas fórmulas que involucran la constante Pi:

$$\pi = 2\int_0^1 \frac{1}{\sqrt{2x-x^2}}dx = 2\int_1^2 \frac{1}{\sqrt{2x-x^2}}dx = 3.141592\ldots$$

Introducción

En esta nota mostramos algunas series que involucran la constante Pi , las series se obtienen de la teoría de integrales elípticas.Una fórmula importante es:

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx \, , 0 < k < 1$$

Conocida como integral elíptica completa de segunda especie.un desarrollo en serie para $E(k)$ es:

$$E(k) = \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right\} , 0 < k < 1$$

Desarrollando en serie la integral para $E(k)$, se pueden obtener una infinidad de fórmulas que involucran la constante Pi.

Notación.

$$\binom{p}{k} = \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!} = \frac{p!}{k!\,(p-k)!} \, , \binom{p}{0} = 1$$

Fórmulas

(1) $$\pi\left(1 - \sum_{n=1}^{\infty}\binom{2n}{n}^2\left(\frac{k^2}{16}\right)^n \frac{1}{2n-1}\right)$$

$$= 2\sqrt{2(1-k^2+a)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{1/2}{m}\binom{-1/2}{n-m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{n+s+r}2^{-n+m+s-r}a^{m-s}k^{2s}}{(1-k^2+a)^m(2n-2m+2s+2r+1)}$$

$$a > k^2 - \frac{1}{2}\ , 0 < k < 1$$

(2) $$\pi\left(1 - \sum_{n=1}^{\infty}\binom{2n}{n}^2\left(\frac{k^2}{16}\right)^n\frac{1}{2n-1}\right)$$
$$= \sqrt{2-k^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{1/2}{m}\binom{-1/2}{n-m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{n+s+r}2^{-n+m+2s-r}}{2n-2m+2s+2r+1}\left(\frac{k^2}{2-k^2}\right)^m$$

$$0 < k < 1$$

(3) $$\pi\left(1 - \sum_{n=1}^{\infty}\binom{2n}{n}^2\left(\frac{2q^2-p^2}{16q^2}\right)^n\frac{1}{2n-1}\right)$$
$$= \frac{2p}{q}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{1/2}{m}\binom{-1/2}{n-m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{n+s+r}2^{-n+m+2s-r}}{2n-2m+2s+2r+1}\left(\frac{2q^2-p^2}{p^2}\right)^m$$

$$p, q \in \mathbb{N} \,|\, \left|2 - \frac{p^2}{q^2}\right| < 1 \ \vee \ q^2 < p^2 < 2q^2$$

Sean p_n, q_n , las sucesiones definidas como sigue:

$$\begin{cases} p_{n+1} = p_n + 2q_n\ , n \in \mathbb{N} \\ q_{n+1} = p_n + q_n\ , n \in \mathbb{N} \\ p_1 = 3\ , q_1 = 2 \end{cases}$$

Algunos valores de $\{(p_n, q_n) : n \in \mathbb{N}\}$, son:

$$\{(p_n, q_n)\} = \{(3,2), (7,5), (17,12), (41,29), \dots\}$$

Las sucesiones p_n, q_n , satisfacen las condiciones de la fórmula (3) , por lo tanto tenemos una infinidad de identidades que involucran la constante Pi.

Ejemplo 1: $(p_1, q_1) = (3,2)$

(4) $$\pi\left(1 - \sum_{n=1}^{\infty}\binom{2n}{n}^2\left(-\frac{1}{64}\right)^n\frac{1}{2n-1}\right)$$
$$= 3\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{1/2}{m}\binom{-1/2}{n-m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{n+m+s+r}2^{-n+m+2s-r}3^{-2m}}{2n-2m+2s+2r+1}$$

Ejemplo 2: $(p_2, q_2) = (7,5)$

(5) $$\pi\left(1-\sum_{n=1}^{\infty}\binom{2n}{n}^{2}\left(\frac{1}{400}\right)^{n}\frac{1}{2n-1}\right)$$
$$=\frac{14}{5}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{r=0}^{s}\binom{1/2}{m}\binom{-1/2}{n-m}\binom{m}{s}\binom{s}{r}\frac{(-1)^{n+s+r}2^{-n+m+2s-r}7^{-2m}}{2n-2m+2s+2r+1}$$

Observación. Todas las fórmulas se han tomado de la referencia (5).

Referencias

[1] Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions. Nueva York: Dover , 1965.

[2] Boros, G., Moll, V.: Irresistible Integrals.Cambridge University Press,2004.

[3] Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals,Series and Products. 5th ed.,ed. Alan Jeffrey. Academic Press, 1994.

[4] Spiegel,M.R.: Mathematical Handbook, McGraw-Hill Book Company , New York , 1968.

[5] Valdebenito, E.:, Pi Handbook , manuscript , unpublished , 1989 , (20000 formulas).

Pi Revisited Four

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Enero 20 , 2011

Resumen

En esta nota recordamos algunas fórmulas para la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.141592\ldots$$

Fórmulas

(1) $$\pi = 3\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}s_{k+1}c_{k+1}$$

donde

$$s_1 = \frac{\sqrt{3}}{2}, s_2 = s_1, s_3 = 0, s_4 = -s_1, s_5 = -s_1, s_6 = 0, s_{k+6} = s_k, k \in \mathbb{N}$$

$$c_1 = \frac{1}{\sqrt{2}}, c_2 = 0, c_3 = -c_1, c_4 = -1, c_5 = -c_1, c_6 = 0, c_7 = c_1, c_8 = 1$$

$$c_{k+8} = c_k, k \in \mathbb{N}$$

(2) $$\pi = 3\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}s_{k+1}c_{k+1}$$

donde s_k , se define como en la fórmula (1) , y c_k se define como sigue:

$$c_1 = \frac{\sqrt{3}}{2}, c_2 = \frac{1}{2}, c_3 = 0, c_4 = -c_2, c_5 = -c_1, c_6 = -1$$

$$c_7 = -c_1, c_8 = -c_2 = c_9 = 0, c_{10} = c_2, c_{11} = c_1, c_{12} = 1$$

$$c_{k+12} = c_k \text{ , } k \in \mathbb{N}$$

(3) $$\pi = 4\sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} s_{k+1} c_{k+1}$$

donde

$$s_1 = \frac{1}{\sqrt{2}}, s_2 = 1, s_3 = s_1, s_4 = 0, s_5 = -s_1, s_6 = -1, s_7 = -s_1, s_8 = 0$$

$$s_{k+8} = s_k \text{ , } k \in \mathbb{N}$$

y c_k , se define como en la fórmula (2).

(4) $$\pi = 4\sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} s_{k+1} c_{k+1}$$

donde s_k , se define como en la fórmula (3), y c_k , se define como sigue:

$$c_{k+2} = \frac{4pq}{p^2 + q^2} c_{k+1} - c_k \text{ , } c_0 = 1, c_1 = \frac{2pq}{p^2 + q^2}$$

$$p, q \in \mathbb{N}, q < p, p^2 - q^2 < 2pq$$

Ejemplo: $p = 3, q = 2$

$$c_{k+2} = \frac{24}{13} c_{k+1} - c_k, c_0 = 1, c_1 = \frac{12}{13}$$

Ejemplo: $p = 4, q = 2$

$$c_{k+2} = \frac{8}{5} c_{k+1} - c_k, c_0 = 1, c_1 = \frac{4}{5}$$

(5) $$\pi = 4\sum_{n=0}^{\infty}\left\{\left(\frac{1}{(4m+2)n+1}-\frac{1}{(4m+2)n+4m+1}\right)\sin\left(\frac{\pi}{2m+1}\right)\right.$$
$$+\left(\frac{1}{(4m+2)n+3}-\frac{1}{(4m+2)n+4m-1}\right)\sin\left(\frac{3\pi}{2m+1}\right)$$
$$+\left(\frac{1}{(4m+2)n+5}-\frac{1}{(4m+2)n+4m-3}\right)\sin\left(\frac{5\pi}{2m+1}\right)+\cdots$$
$$\left.+\left(\frac{1}{(4m+2)n+2m-1}-\frac{1}{(4m+2)n+2m+3}\right)\sin\left(\frac{2m-1}{2m+1}\pi\right)\right\}$$

$$m \in \mathbb{N}$$

Ejemplos: $m = 2, m = 3$

$$\pi = 4\sum_{n=0}^{\infty}\left\{\left(\frac{1}{10n+1}-\frac{1}{10n+9}\right)\sin\left(\frac{\pi}{5}\right)+\left(\frac{1}{10n+3}-\frac{1}{10n+7}\right)\sin\left(\frac{3\pi}{5}\right)\right\}$$

$$\pi = 4\sum_{n=0}^{\infty}\left\{\left(\frac{1}{14n+1}-\frac{1}{14n+13}\right)\sin\left(\frac{\pi}{7}\right)+\left(\frac{1}{14n+3}-\frac{1}{14n+11}\right)\sin\left(\frac{3\pi}{7}\right)\right.$$
$$\left.+\left(\frac{1}{14n+5}-\frac{1}{14n+9}\right)\sin\left(\frac{5\pi}{7}\right)\right\}$$

(6) $$\pi = \frac{4}{\sqrt{m^2+1}}\sum_{n=1}^{\infty}\frac{1}{2n-1}c_n(m)$$

donde

$$c_{n+2}(m) = 2\left(\frac{m^2-1}{m^2+1}\right)c_{n+1}(m) - c_n(m)\,, n \in \mathbb{N}$$

$$c_1(m) = 1, c_2(m) = \frac{3m^2-1}{m^2+1}, m \in \mathbb{N}$$

Ejemplos: $m = 1, m = 2, m = 3$

$$\pi = \frac{4}{\sqrt{2}}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\cdots\right)$$

$$\pi = \frac{4}{\sqrt{5}}\left(1 + \frac{1}{3}\left(\frac{6}{5}\right) + \frac{1}{5}\left(\frac{11}{25}\right) - \frac{1}{7}\left(\frac{84}{125}\right) - \cdots\right)$$

$$\pi = \frac{4}{\sqrt{10}}\left(1 + \frac{1}{3}\left(\frac{8}{5}\right) + \frac{1}{5}\left(\frac{39}{25}\right) + \frac{1}{7}\left(\frac{112}{125}\right) - \cdots\right)$$

(7) $$\pi = \frac{5}{2}\sum_{n=0}^{\infty}(-1)^n\left(\left(\frac{1}{5n+1} + \frac{1}{5n+4}\right)\sin\left(\frac{\pi}{5}\right) + \left(\frac{1}{5n+2} + \frac{1}{5n+3}\right)\sin\left(\frac{3\pi}{5}\right)\right)$$

(8) $$\pi\sqrt{2}\sqrt{5-\sqrt{5}} = 8 - 16\sum_{n=0}^{\infty}\left(\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{1}{(10n+2)^2-1} + \frac{1}{(10n+8)^2-1}\right) - \left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{1}{(10n+4)^2-1} + \frac{1}{(10n+6)^2-1}\right) + \frac{1}{(10n+10)^2-1}\right)$$

(9) $$\pi\prod_{n=1}^{\infty}\frac{(5n)^2-a^2}{(5n)^2-(5a)^2} = 5\sum_{n=0}^{\infty}\left(\left(\frac{\sqrt{10-2\sqrt{5}}}{2}\right)\left(\frac{5n+1}{(5n+1)^2-a^2} - \frac{5n+4}{(5n+4)^2-a^2}\right) - \left(\frac{\sqrt{10+2\sqrt{5}}}{2}\right)\left(\frac{5n+2}{(5n+2)^2-a^2} - \frac{5n+3}{(5n+3)^2-a^2}\right)\right)$$

$$a \in \mathbb{R} - \mathbb{Z}$$

Para $a = 0$, se tiene:

$$\pi = 5\sum_{n=0}^{\infty}\left(\left(\frac{\sqrt{10-2\sqrt{5}}}{2}\right)\left(\frac{1}{5n+1} - \frac{1}{5n+4}\right) - \left(\frac{\sqrt{10+2\sqrt{5}}}{2}\right)\left(\frac{1}{5n+2} - \frac{1}{5n+3}\right)\right)$$

$$(10)\quad \pi\prod_{n=1}^{\infty}\frac{(5n)^2+a^2}{(5n)^2+(5a)^2}$$

$$=5\sum_{n=0}^{\infty}\left(\left(\frac{\sqrt{10-2\sqrt{5}}}{2}\right)\left(\frac{5n+1}{(5n+1)^2+a^2}-\frac{5n+4}{(5n+4)^2+a^2}\right)\right.$$

$$\left.-\left(\frac{\sqrt{10+2\sqrt{5}}}{2}\right)\left(\frac{5n+2}{(5n+2)^2+a^2}-\frac{5n+3}{(5n+3)^2+a^2}\right)\right)$$

$$a\in\mathbb{R}$$

Referencias

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Pi Revisited Five

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Resumen

En esta nota hacemos algunas observaciones sobre una fórmula para la constante Pi:

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.141592 \ldots$$

Introducción

Recordamos una fórmula para la constante Pi , que aparece en la referencia (5):

(1) $$\pi = 12 \sum_{n=0}^{\infty} c_n \, r^{2n+1} \left(\frac{1}{2n+1} - \frac{3r^2}{2n+3} \right)$$

donde los números c_n , se obtienen de la recurrencia:

(2) $$c_{n+3} = -c_{n+2} + 2c_{n+1} - c_n \; , c_0 = 1, c_1 = -1, c_2 = 3$$

y el número $r = 0.293138 \ldots$, satisface la ecuación:

(3) $$r^3 - r + 2 - \sqrt{3} = 0$$

En esta nota mostramos algunas representaciones para el número r.

Fórmulas

(4) $$r = 2 - \sqrt{3} + \left(2 - \sqrt{3} + \left(2 - \sqrt{3} + \cdots \right)^3 \right)^3$$

(5) $$r = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}$$

donde

$$a_{n+6} = 4a_{n+5} - a_{n+4} - 4a_{n+3} + 2a_{n+2} - a_n$$

$$a_0 = 1, a_1 = 4, a_2 = 15, a_3 = 52, a_4 = 179, a_5 = 612$$

(6) $r = \lim_{n\to\infty} \frac{b_n}{b_{n+1}}$

donde

$$b_{n+3} = \left(2+\sqrt{3}\right)(b_{n+2} - b_n)$$

$$b_0 = 1, b_1 = 2+\sqrt{3}, b_2 = 7+4\sqrt{3}$$

(7) $r = \frac{1}{2}\left(s - \sqrt{4-3s^2}\right)$

donde

$$s = \sqrt[3]{2-\sqrt{3}+\sqrt[3]{2-\sqrt{3}+\sqrt[3]{2-\sqrt{3}+\cdots}}}$$

(8) $r = u - v\sqrt{3}$

donde

$$t = \sqrt[6]{2}\,\sqrt[3]{\frac{2-\sqrt{3}}{2}}\left(\frac{\sqrt{6}+\sqrt{2}}{4} + i\,\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

$$z = \frac{1}{2}(1+i)\left(\frac{1}{3}\sqrt{\frac{1+16\sqrt{3}}{3}} - 1\right)$$

$$u + v\,i = t\,F\left(-\frac{1}{3}, 1; 1; -z\right)$$

$$i = \sqrt{-1}$$

$F(a,b;c;x)$, es la función hypergeometrica de Gauss.

(9) $r = \lim_{n\to\infty} y_n$

donde

$$y_{n+1} = \frac{5y_n^6 - 6y_n^4 + 8y_n^3 + y_n^2 - 1}{6y_n^5 - 8y_n^3 + 12y_n^2 + 2y_n - 4}\ , y_0 = \frac{3}{10}$$

(10) $r = \lim_{n\to\infty}\left(a_n - b_n\sqrt{3}\right)$

donde

$$\begin{cases} a_{n+1} = a_n^3 + 9a_n b_n^2 + 2 \\ b_{n+1} = 3a_n^2 b_n + 3b_n^3 + 1 \\ a_1 = 2\ , b_1 = 1 \end{cases}$$

(11) $r = \frac{1}{2\sqrt[3]{c}}\left(\sqrt{4\sqrt[3]{c^2} - 3} - 1\right)$

donde

$$c = \left(1 + \left(2 - \sqrt{3}\right)\left(1 + \left(2 - \sqrt{3}\right)\left(1 + \left(2 - \sqrt{3}\right) \dots\right)^{\frac{3}{2}}\right)^{\frac{3}{2}}\right)^{\frac{3}{2}}$$

(12) $r = \lim_{n\to\infty}\left(x_n - y_n\sqrt{3}\right)$

donde

$$x_{n+1} = \frac{a_n c_n - 3b_n d_n}{c_n^2 - 3d_n^2}$$

$$y_{n+1} = -\frac{a_n d_n - b_n c_n}{c_n^2 - 3d_n^2}$$

$$a_n = 2x_n^3 + 18x_n y_n^2 - 2$$

$$b_n = 6x_n^2 y_n + 6y_n^3 - 1$$

$$c_n = 3x_n^2 + 9y_n^2 - 1$$

$$d_n = 6x_n y_n$$

$$x_0 = 2\ , y_0 = 1$$

Fórmulas alternativas para Pi son:

(13) $\pi = 12\sum_{n=0}^{\infty}\frac{r^{2n+1}}{2n+1}c_n$

donde

$$c_{n+3} = -c_{n+2} + 2c_{n+1} - c_n \ , c_0 = 1, c_1 = -4, c_2 = 6$$

(14) π

$$= 12\sum_{n=0}^{\infty}(-1)^n \, r^{2n+1} \sum_{k=0}^{\left[\frac{2n+1}{3}\right]} \binom{2n-2k+1}{k} \frac{1}{2n-2k+1}$$

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