Radical

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ABSTRACT

Approximations of square roots are discussed. A very close approximation to their decimal expansion is derived in the form of a simple fraction. Their relationship with the AKS test is also discussed.

There are many approximations of square roots. One common approach is Newton's method for example. It is a fairly easy process to find a working and very close approximation to square roots, or any root for that matter. I explain here an approach of my own, in which I approximate the decimal expansion itself.

List the values of square roots of numbers from 1 to some number. I will list them up to 9 here.

$$\sqrt{1} = 1$$

$$\sqrt{2} = 1.41421356$$

$$\sqrt{3} = 1.73205080$$

$$\sqrt{4} = 2$$

$$\sqrt{5} = 2.23606797$$

$$\sqrt{6} = 2.44948974$$

$$\sqrt{7} = 2.64575131$$

$$\sqrt{8} = 2.82842712$$

$$\sqrt{9} = 3$$

The whole numbers place in this list can be seen to follow the sequence $2n + 1$, where n is the whole numbers place value of $\sqrt{x}$. Meaning there are 3 ones, 5 twos, 7 threes,...ect. So, if you wanted to know how many square roots of whole numbers begin with the number 3, the equation would yield $2 * 3 + 1 = 7$, which is correct. Further, because each first new whole number is also the square root of the perfect square in the beginning of each interval of numbers, $n = \lfloor\sqrt{x}\rfloor$, which is the perfect square before any number x. So, n^2 is this perfect square.

It is also intuitive to note that in each interval of values, for example (4,8), the values of the decimal places nearly have a uniform distribution with respect to the interval. It will become more uniform for higher intervals, meaning for higher n. Using this assumption, its possible to estimate the decimal place values with a fraction. We can represent the location of any number x in an interval using the perfect square before it as, $n - x^2$. Knowing the location of x and the distance in the interval, this fraction can take the form of location over distance as follows,

$$\frac{x - n^2}{2n + 1}$$

This is a beginning approximation to the decimal expansion of $\sqrt{x}$. Consequently, in knowing that the whole number value will be n, a beginning approximation for $\sqrt{x}$ can be formulated as follows,

$$\sqrt{x} \approx \frac{x - n^2}{2n + 1} + n \quad = \quad \frac{x + n + n^2}{2n + 1} \quad = \quad \frac{x + n(n + 1)}{2n + 1}.$$

The error of this approximation can be seen in a graph of $\sqrt{x} - \frac{x+n(n+1)}{2n+1}$. You can find from this that the maximum error in this function occurs at $x = n(n + 1)$. If you divide the distance $2n + 1$ in each interval by the maximum error on the interval, you will get a sequence that is rapidly approaching integers, because the decimal expansion of the terms are approaching 0. This sequence truncated is, $\{37,101,197,325,...\}$. This sequence can be represented as $4(2n + 1)^2 + 1$. So, if the maximum error divided by $2n + 1$ rapidly approaches this sequence, then this sequence divided by $2n + 1$ must rapidly approach the maximum error. Thus, the error at $n(n + 1)$ is closely approximated by,

$$\frac{2n + 1}{4(2n + 1)^2 + 1}.$$

Now, if we make x equal to $n(n + 1)$, the maximum points of error, and then add the approximation of this error, we get a very accurate approximation of $\sqrt{n(n + 1)}$. And so,

$$\sqrt{n(n+1)} \approx \frac{2n(n+1)}{2n+1} + \frac{2n+1}{4(2n+1)^2+1}.$$

This is very accurate. By simple manipulation of the above equation, these other forms can be derived as well.

$$\sqrt{n^2+1} \approx \frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}$$

$$\sqrt{n^2-1} \approx \frac{2n(n^2-1)}{2n^2-1} + \frac{2n^2-1}{n(4(2n^2-1)^2+1)}$$

$$\sqrt{n}\sqrt{2n+1} \approx \frac{1}{\sqrt{2}}\left(\frac{4n(2n+1)}{4n+1} + \frac{4n+1}{4(4n+1)^2+1}\right)$$

$$\sqrt{n}\sqrt{2n-1} \approx \frac{1}{\sqrt{2}}\left(\frac{4n(2n-1)}{4n-1} + \frac{4n-1}{4(4n-1)^2+1}\right)$$

As an example of the accuracy of these formulas, for $n = 35$, the calculation of $\sqrt{35^2+1}$ using the second approximation formula above,

$$\sqrt{n^2+1} \approx \frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}$$

is accurate to 19 decimal places. And as discussed, this accuracy increases for higher n. The error reaches a limit of 0 at infinity. From these formulas, one can derive approximations to other types of functions, as an example:

$$|n| \approx \frac{1}{\sqrt{n^2+1}}\left(\frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}\right)$$

$$Sgn(n) \approx \frac{1}{n\sqrt{n^2+1}}\left(\frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}\right)$$

There are some properties of roots that seem to be fundamental in the understanding of prime numbers. As an example, $(x+1)^n - x^n$ gives the number of n^{th} roots that have x in the integer position. If you expand this for several n, you can find that there are two separate representations for odd and even n.

$$(x+1)^{2n-1} - x^{2n-1} = (2n-1)\sum_{j=1}^{n} \frac{\binom{n-2+j}{2j-2}}{2j-1}(x^2+x)^{n-j}$$

$$(x+1)^{2n} - x^{2n} = (2x+1)\sum_{j=1}^{n} \binom{n-1+j}{2j-1}(x^2+x)^{n-j}$$

If we evaluate this at $x = 1$, we will find

$$2^{2n-1} - 1 = (2n-1)\sum_{j=1}^{n} \frac{\binom{n-2+j}{2j-2}}{2j-1}2^{n-j}$$

$$2^{2n} - 1 = 3\sum_{j=1}^{n} \binom{n-1+j}{2j-1}2^{n-j}.$$

This shows that the Mersenne numbers $2^n - 1$ may be represented as the number of n^{th} roots that have 1 in the integer position. The forms of the sums are interesting too. Further investigation will reveal,

$$L_{2n-1} = (2n-1)\sum_{j=1}^{n} \frac{\binom{n-2+j}{2j-2}}{2j-1}$$

$$F_{2n} = \sum_{j=1}^{n} \binom{n-1+j}{2j-1}$$

which are bisections of the Lucas and Fibonacci numbers. If one forms a triangle of the coefficients of the expansion of $(x+1)^n - x^n$, it can be quickly discovered that,

$$(x+1)^n - x^n - 1 \equiv mod\ n\ \ iff\ n\ is\ prime.$$

This is the basis of the AKS test. Further investigation reveals that this is not the end all to the story. In fact, this should actually say,

$$(x+1)^n - x^n - 1 \equiv mod\ p\ \ iff\ n = \ p^k \quad\ k \geq 1.$$

The AKS test is capitalizing on $k = 1$. But, there are other interesting relationships here as well. If we restrict n to the even numbers like so $(x+1)^{2n} - x^{2n} - 1$, then one can find that in this form, the smallest number that will be coprime to each and every coefficient is in fact the greatest prime factor of 2n+1, $GPF(2n+1)$. So,

$$GPF(2n+1) = min\{k: \forall\in\ (x+1)^n - x^n - 1, k \nmid (x+1)^n - x^n - 1\}.$$

If one replaces x with i, the imaginary number, one can find,

$$\frac{(i+1)^{2n} - i^{2n} - 1}{2n-1} = \mathbb{Z}\ \ if\ \ 2n-1 = p\ \wedge\ p \in 4m-1.$$

If you explore this, you can find that,

$$\frac{2^{\frac{n-1}{2}} - (-1)^{\frac{n^2-1}{24}}}{n} \in \mathbb{N}\ if\ n = p\ \ for\ p\ prime\ \wedge\ \ n \geq 5.$$

In conclusion, there is still much more to discover and learn within the study of radicals. There may be deeper relationships hiding in the numbers. At worst, it may be an interesting area to explore within mathematics.