Radical

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ABSTRACT

Approximations of n^{th} roots are discussed and their relationship with the AKS test. The aim of this paper is also to discuss equations leading to the AKS test from a different angle of logic and find new or interesting information.

There are many approximations of n^{th} roots. One common approach is Newton's method for example. It is not hard to create one so this will not be the interest of this paper and I will not include the proof of my own which will be used here. Proofs on such methods can be provided if it becomes of interest, but the interest in this paper will be exploring and finding relationships.

It is a fairly easy process to find a working and very close approximation to square roots, or any root for that matter. Here are a few of my own.

$$\sqrt{n}\sqrt{n+1} \approx \frac{2n(n+1)}{2n+1} + \frac{2n+1}{4(2n+1)^2+1}$$

$$\sqrt{n^2+1} \approx \frac{2n(n^2+1)}{2n^2+1} + \frac{2n^2+1}{n(4(2n^2+1)^2+1)}$$

$$\sqrt{n^2-1} \approx \frac{2n(n^2-1)}{2n^2-1} + \frac{2n^2-1}{n(4(2n^2-1)^2+1)}$$

$$\sqrt{n}\sqrt{2n+1} \approx \frac{1}{\sqrt{2}}\left(\frac{4n(2n+1)}{4n+1} + \frac{4n+1}{4(4n+1)^2+1}\right)$$

$$\sqrt{n}\sqrt{2n-1} \approx \frac{1}{\sqrt{2}}\left(\frac{4n(2n-1)}{4n-1} + \frac{4n-1}{4(4n-1)^2+1}\right)$$

From this information one can derive approximations to other types of functions, as an example:

$$|n| \approx \frac{1}{\sqrt{n^2+1}}\left(\frac{2n(n^2+1)}{2n^2+1}+\frac{2n^2+1}{n(4(2n^2+1)^2+1)}\right)$$

$$Sgn(n) \approx \frac{1}{n\sqrt{n^2+1}}\left(\frac{2n(n^2+1)}{2n^2+1}+\frac{2n^2+1}{n(4(2n^2+1)^2+1)}\right)$$

This is not that interesting unless perhaps it can help to very closely approximate an integral with no known solution. But, in searching for one a relationship previously unknown can be discovered. This is because there are some properties of roots that seem to be fundamental in the understanding of prime numbers. As an example, $(x+1)^n - x^n$ gives the number of n^{th} roots that have x in the integer position. If you expand this for several n, you can find that there are two separate representations for odd and even n.

$$(x+1)^{2n-1} - x^{2n-1} = (2n-1)\sum_{j=1}^{n}\frac{\binom{n-2+j}{2j-2}}{2j-1}(x^2+x)^{n-j}$$

$$(x+1)^{2n} - x^{2n} = (2x+1)\sum_{j=1}^{n}\binom{n-1+j}{2j-1}(x^2+x)^{n-j}$$

If we evaluate this at $x=1$, we will find

$$2^{2n-1} - 1 = (2n-1)\sum_{j=1}^{n}\frac{\binom{n-2+j}{2j-2}}{2j-1}2^{n-j}$$

$$2^{2n} - 1 = 3\sum_{j=1}^{n}\binom{n-1+j}{2j-1}2^{n-j}$$

This shows that the Mersenne numbers $2^n - 1$ may be represented as the number of n^{th} roots that have 1 in the integer position. The forms of the sums are interesting too. Further investigation will reveal,

$$L_{2n-1} = (2n-1)\sum_{j=1}^{n}\frac{\binom{n-2+j}{2j-2}}{2j-1}$$

$$F_{2n} = \sum_{j=1}^{n}\binom{n-1+j}{2j-1}$$

which are bisections of the Lucas and Fibonacci numbers. If one forms a triangle of the coefficients of the expansion of $(x+1)^n - x^n$, it can be quickly discovered that,

$$(x+1)^n - x^n - 1 \equiv mod\ n\ \ iff\ n\ is\ prime$$

This is the basis of the AKS test. Further investigation reveals that this is not the end all to the story. In fact, this should actually say,

$$(x+1)^n - x^n - 1 \equiv mod\ p\ \ iff\ n =\ p^k \qquad k \geq 1$$

The AKS test is capitalizing on $k = 1$. But, there are other interesting relationships here as well. If we restrict n to the even numbers $(x+1)^{2n} - x^{2n} - 1$, than one can find that in this form, the smallest number that will be coprime to each and every coefficient is in fact the greatest prime factor of 2n+1, $GPF(2n+1)$. So,

$GPF(2n+1) = min\{k: \forall \in\ (x+1)^n - x^n - 1, k \nmid (x+1)^n - x^n - 1\ \}$.

If that notation makes any sense. If one replaces x with i, the imaginary number one can find,

$$\frac{(i+1)^{2n} - i^{2n} - 1}{2n-1} = \mathbb{Z}\ iff\ \ 2n-1 = p\ \wedge\ p \in 4m-1$$

If you explore this, you can find that,

$$\frac{2^{\frac{n-1}{2}} - (-1)^{\frac{n^2-1}{24}} - 1}{n}\ is\ and\ integer\ iff\ n = p \quad n \geq 5$$

In conclusion, there is much more to discover within this area. This is not the direction that the authors of Primes in P followed to derive their algorithm. It is a separate way that it might have been discovered. It is interesting that working on something as trivial as square roots or any roots for that matter can lead to a polynomial time test for primes. There may be deeper relationships here. At worst, it may be an interesting area to explore within mathematics.