How the imaginary unit is inherent in quantum indeterminacy

Reliance of mixed states on the logic of self-referentially generated unitarity

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Abstract Between 2008 and 2010, Tomasz Paterek et al published ingenious work linking quantum randomness with logical independence. From a foundational point of view, this is evidence that quantum randomness, and therefore indeterminacy, have mathematical origins. The logical independence of Paterek et al is seen in a system of Boolean propositions. Here, I explain the origins of that logical independence in terms of standard quantum theory, showing it has symmetry foundations in a *'unitary switch'* – and whose logic originates in logically circular self-reference. The profound finding is that indeterminacy becomes a visible feature of quantum mathematics when unitarity (or self-adjointness) — *by Postulate* — is given up.

Keywords foundations of quantum theory, quantum mechanics, quantum randomness, quantum indeterminacy, quantum information, prepared state, measured state, pure eigenstates, mixed states, unitary, redundant unitarity, orthogonal, scalar product, inner product, mathematical logic, logical independence, self-reference, logical circularity, mathematical undecidability.

1 Introduction

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. The 'randomness' stems from ignorance of *physical information* in the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [8], the inequalities of John Bell [4], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, Tomasz Paterek et al provide an explanation in *mathematical information.* They demonstrate a link between quantum randomness and *logical independence* in a *formal system* of *Boolean propositions* [9,10]. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. In experiments measuring photon polarisation, Paterek et al demonstrate statistics correlating *predictable* outcomes with logically dependent mathematical propositions, and *random* outcomes with propositions that are logically independent.

Whilst, from the Paterek research, we may reliably infer that the machinery of quantum randomness *does* entail logical independence, the fact that this logical independence is seen in a *Boolean* system, rather obscures any insight. To understand the workings of quantum randomness, theory must be written exhibiting logical independence in context of *standard textbook quantum theory* — specifically, in terms of the Pauli algebra $\mathsf{su}(2)$.

Here, in this paper, I show what the Paterek Boolean information means for the system of Pauli operators. The interesting surprise revealed, is that although every measurement of polarisation is representable by the Pauli algebra $\mathsf{su}(2)$, only the

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measurement of mixed states *requires* this algebra. Measurement of pure eigenstates does not. For pure states, the unitary component of the Pauli algebra is not involved.

In predictable experiments, where measurement is on pure states, unitarity is shown to be 'redundant' — *possible* but *not necessary.* And in experiments whose outcomes are random, where measurement is on mixed states, unitarity is shown unavoidably *necessary.* My conclusion is that there is a *unitary switch-on* in passing from pure states to mixed and a *unitary switch-off* in passing from mixed to pure.

Logically, this regime can be viewed in two ways. It can be viewed as a system that is always unitary, but where unitarity switches between possible and necessary: such a *possible / necessary* system constitutes a *modal logic.* Or otherwise it can be seen as a complete switch between different symmetries, where unitarity is new, *logically independent*, extra information required for the transition. To adequately describe the transition between pure and mixed states, either modal logic is needed, or logical independence. The classical logic of *true* and *false* is not an option.

The question of where the newly formed unitary information comes from is resolved. I show that it has origins in uncaused, unprevented, logically circular *self-reference.* By uncaused and unprevented, I mean that no information already present in the system implies nor denies the logically circular self-reference.

In experiments measuring mixed states, whose outcomes are random, the system symmetry may be represented, in the usual way, by the (unitary) Pauli matrices:

$$\sigma_{\mathsf{x}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_{\mathsf{y}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

This unconventional ordering of the Pauli matrices is chosen to agree with Paterek et al's choice of σ_{x} & σ_{z}, in representing all their experiments, rather than a choice of σ_{y} & σ_{x}, say.

but for measurements on pure states, whose outcomes are predictable, the system symmetry is represented by this set of non-unitary, involutory[1] matrices:

$$\mathsf{s}_{\mathsf{x}}(\eta) = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \qquad \sigma_{\mathsf{y}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

where η is a scalar of any value. It can be seen that σ_{x} is a particular value of $\mathsf{s}_{\mathsf{x}}(\eta)$. The crucial distinction between (1) and (2) is that, whereas the three Pauli matrices (1) are all mutually orthogonal, in the non-unitary matrices (2), there is no orthogonality between $\mathsf{s}_{\mathsf{x}}(\eta)$ and σ_{z}, except in the accidental coincidence of $\eta + \eta^{-1} = 0$. That is: $\eta = \pm i$. See Appendix.

Actually, no orthogonality is needed in (2), at all, but these matrices sufficiently fit the purpose; and indeed, it may not be possible to find 2×2 involutory matrices that are all non-orthogonal.

My overall reasoning, in this paper, is to argue that Paterek logical independence is an intrinsic feature of the unitary (3-way orthogonal) content of the Pauli algebra and this is exactly identical (for one representation at least) to circular logical connectivity, not required by pure states, but inherent in mixed states.

Sections 2 – 5 explain the Paterek thesis and method. The Paterek approach treats the experiment like computer hardware, whose input and output is machine code. The machine code 'zeros' and 'ones' register hard involutory and orthogonal items of information witnessable in an experiment, and relate this to *separated* involutory and orthogonal information, extracted from the Pauli algebra, rather than the unseparated Pauli algebra itself. Ingress of logical independence occurs as hardware interacts with the photon density matrix.

Section 6 shows the Pauli algebra consists of 6 logically independent items of algebraic information – 3 involutory and 3 orthogonal.

Section 7 shows that all polarisation states need involutory information. And that only mixed states need the 3 orthogonal items of algebraic information.

Section 8 takes the non-unitary, involutory algebraic system, and makes a purely logical alteration that assumes circular self-reference. The resultant is the unitary Pauli system.

Whereas the Boolean system, used by Paterek et al, is an axiomatic system that can be completed, and is therefore not subject to Gödel's Incompleteness Theorems; elements of matrices (1) and (2) are scalars belonging to Elementary Algebra, a system that cannot be completed, that is subject to Gödel.

2 Information and logic

In Mathematical Logic, a *formal system* is a system of mathematical formulae, treated as propositions, where focus in on *provability* and *non-provability.*

[1] An involutory operator is one whose square is the identity operator. e.g.. $\mathsf{aa} = \mathbb{1}$.

A formal system comprises: a precise language, rules for writing formulae, and further rules of deduction. Within such a formal system, any two propositions are **either** *logically dependent* — in which case, one proves, or disproves the other — **or otherwise** they are *logically independent*, in which case, neither proves, nor disproves the other. A helpful perspective on this is the viewpoint of Gregory Chaitin's information-theoretic formulation [5]. In that, logical independence is seen in terms of information content. If a proposition contains information, not contained in some given set of axioms, then those axioms can neither prove nor disprove the proposition.

Axioms are propositions presupposed to be 'true' and adopted *a priori.*

Central to the formal system used in the Paterek et al research are these Boolean functions of a binary argument:

$$x \in \{0,1\} \mapsto f(x) \in \{0,1\}$$

Typical propositions, stemming from those functions, are these:

$$\begin{array}{lll} f(0)=0 & f(1)=0 & f(0)=f(1) \\ f(0)=1 & f(1)=1 & f(0)\neq f(1) \end{array} \tag{3}$$

Each one of these propositions is an item of information, taken as being openly true or openly false. Our interest lies, not so much, in their truth or falsity, but in, which statements prove which, which disprove which, and which do neither. In other words, which are logically dependent and which are logically independent.

As illustration, if $f(0)=0$ were considered to be true, the statement $f(0)=1$ would be proved false. More simply, we could say: $f(0)=0$ disproves $f(0)=1$, and accordingly, $f(0)=1$ is *logically dependent* on $f(0)=0$.

On the other hand, again, if $f(0)=0$ were considered to be true, that would not prove, or disprove $f(1)=0$. We could say: $f(0)=0$ neither proves, nor disprove $f(1)=0$, and accordingly, $f(0)=0$ and $f(1)=0$ are *logically independent.*

Over and above the propositions in (3), I introduce *permanent axioms*, that Paterek et al take for granted, but do not state. They are:

$$f(0)=0 \Rightarrow f(1)=1 \qquad f(1)=0 \Rightarrow f(0)=1 \tag{4}$$

These prohibit the combination $f(0)=0$, $f(1)=0$.

3 The Paterek et al experiments

The Paterek et al research involves polarised photons as information carriers through measurement experiments. Experiment hardware consists of a sequence of three segments, that I denote: State preparation, Black box and Measurement. Respectively, these three segments prepare, then transform, then measure states. Data gathered from experiments is read from an X–Y–Z reference system fixed to the hardware. All 27 possible combinations of the following were performed, very many times, and statistical information was gathered.

1. State preparation
   Photons prepared, either as $|z+\rangle$, $|x+\rangle$ or $|y+\rangle$ eigenstates, by filtering, directly after one of these Pauli transformations:
   (a) σ_z, aligned with the Z axis.
   (b) σ_x, aligned with the X axis.
   (c) σ_y, aligned with the Y axis.

2. Black box
   The prepared eigenstates are altered through one of these Pauli transformations:
   (a) σ_z, aligning states with the Z axis,
   (b) σ_x aligning states with the X axis,
   (c) σ_y aligning states with the Y axis.

3. Measurement
   Measurement is performed, by detecting photon capture, directly after one of these Pauli transformations:
   (a) σ_z, aligned with the Z axis.
   (b) σ_x, aligned with the X axis.
   (c) σ_y, aligned with the Y axis.

My analysis compares just 2 of these experiment configurations: the pure state measurement: $\sigma_z \to \sigma_z \to \sigma_z$, and the mixed state $\sigma_z \to \sigma_x \to \sigma_z$.

Within all these experiments, there exist two classes of orientational information. The more obvious is *segment alignment*; this is the orientation of individual hardware segments with respect to the X–Y–Z reference system. Normally, in standard theory, segment alignment would be represented as Pauli information, through the σ_x, σ_y, σ_z operators. In the Paterek et al research, alignment information is fully conveyed in two bits, as *Boolean pairs* — $(0,1)$, $(1,0)$, $(1,1)$.

The less obvious class of information, I refer to as *orthogonality index*. This is the degree of orthogonality between one hardware segment and the next — either orthogonal, or not orthogonal. Photons need to know this, in order to prepare their states. Orthogonality index is conveyed through the experiment, as information propagated in the *density matrix*.

SF: This makes me wonder if the polarisation density matrix is the conveyor of 3-space. It would be interesting to replace the Pauli matrices by the Dirac matrices to see if they convey spacetime geometry.

In Section 5 we see how Boolean pairs, representing X–Y–Z information from State preparation and Black box, feed into the orthogonality index, and then how Measurement attempts to read that Boolean information.

4 Boolean pairs and 4-sequences

The Boolean values, used by Paterek et al, are based on the Pauli operators and products between them. In their treatment of the mathematics, Paterek et al represent all configurations of their experiments, through the use of only σ_x and σ_z — just two of the Pauli operators. This is done by specifying each of the three Pauli operators, using products of the form $\sigma_x^i \sigma_z^j$, where i and j are interpreted as integers, modulo 2. Thus:

$$\sigma_z = \sigma_x^0 \sigma_z^1 \qquad \sigma_x = \sigma_x^1 \sigma_z^0 \qquad -i\sigma_y = \sigma_x^1 \sigma_z^1 \tag{5}$$

Permanent axioms (4) deny the Boolean pair $(0,0)$ and the 'null' formula

$$\mathbb{1} = \sigma_x^0 \sigma_z^0$$

This formula's null action represents absence of state preparation, the contribution of which is randomness, as direct consequence of unprepared, random input.

By way of the indices on these operators, Paterek et al link the three *Boolean pairs* $(0,1)$, $(1,0)$, $(1,1)$, with the three operators: σ_z, σ_x, σ_y.

Stringing together sequences of Pauli operators to form '*quad-products*' invokes corresponding Boolean *'4-sequences'* that represent orientational information linking two consecutive segments of the experiment hardware. Examples are:

$$\sigma_z \sigma_z = \sigma_x^0 \sigma_z^1 \sigma_x^0 \sigma_z^1 \to (0,1)\,(0,1) \tag{6}$$

$$\sigma_x \sigma_z = \sigma_x^1 \sigma_z^0 \sigma_x^0 \sigma_z^1 \to (1,0)\,(0,1) \tag{7}$$

$$-i\sigma_y \sigma_z = \sigma_x^1 \sigma_z^1 \sigma_x^0 \sigma_z^1 \to (1,1)\,(0,1) \tag{8}$$

These can be used to represent the action of the State preparation followed by the action of the Black box; or, the action of the Black box followed by the action of the Measurement.

Now consider a specific experiment where the action of the State preparation is encoded thus: $\sigma_x^m \sigma_z^n \to (m,n)$; where the action of the Black box is encoded thus: $\sigma_x^{f(0)} \sigma_z^{f(1)} \to (f(0), f(1))$; and the action of Measurement is encoded thus: $\sigma_x^p \sigma_z^q \to (p,q)$. In this experiment, the joint action for the State preparation and Black box is encoded in the *quad-product* and *4-sequence*:

Variables p and q are not used by Paterek et al. I introduce them for the sake of clarity.

$$\sigma_x^{f(0)} \sigma_z^{f(1)} \sigma_x^m \sigma_z^n \to (f(0), f(1))\,(m,n)$$

Here, $f(0)$ and $f(1)$ are the Boolean functions that give us the propositions written in (3). The Measurement, $\sigma_x^p \sigma_z^q \to (p,q)$, comes into play subsequently.

5 Logical independence from the Boolean viewpoint

Propagation of information, about whether states are mixed or pure, is conveyed in the density matrix. The *input* density matrix, on entry into the Black box is:

$$\rho = \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} i^{mn} \sigma_x^m \sigma_z^n\right]$$

with $\lambda = \pm 1$. Under the action of the Black box the state evolves to:

$$U \rho U^\dagger = \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} (-1)^{nf(0)+mf(1)}\, i^{mn} \sigma_x^m \sigma_z^n\right]$$

The index on the factor $(-1)^{nf(0)+mf(1)}$, I call the *orthogonality index* and give the label $\mathscr{N}_{\mathsf{B}}$:

$$\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1)$$

The suffix B stands for 'leaving the Black box'. Depending on whether the Black box imparts orthogonal information, the value of $\mathscr{N}_{\mathsf{B}}$ is either 0 or 1.

All sums are taken modulo 2.

$\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1) = 0$	zero orthogonality imparted by the Black box
$\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1) = 1$	unit orthogonality imparted by the Black box

Downstream of the Black box and prior to Measurement, $\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1)$ is determined, logically dependent on values set (by the operator) for (m, n) and $(f(0), f(1))$. That determination can be thought of as an information process where (m, n) and $(f(0), f(1))$ are copied from the State preparation and Black box, then given as input to $nf(0) + mf(1)$, to compute $\mathscr{N}_{\mathsf{B}}$ as output.

The value of $\mathscr{N}_{\mathsf{B}}$, leaving the Black box, continues its propagation through the experiment, to be read as input, by the Measurement hardware. Once the Measurement hardware knows that value for $\mathscr{N}_{\mathsf{B}}$, given the Measurement orientation, set by $\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q} \rightarrow (p, q)$, computation of $f(0)$ and $f(1)$ is attempted from $\mathscr{N}_{\mathsf{B}} = qf(0) + pf(1)$. However, $f(0)$ and $f(1)$ are not both determinable from $\mathscr{N}_{\mathsf{B}}$ and (p, q), because, one or the other of $f(0)$ and $f(1)$, will be logically independent unless $\mathscr{N}_{\mathsf{B}}$ is zero.

To demonstrate this, it is sufficient to set the Measurement configuration (p, q) to the same basis (m, n), set for the State preparation. See Table 1.

Information sources		**Flow of orthogonality information through experiment**	
		parallel alignment	orthogonal alignment
State preparation			
configuration input	(m, n)	$(0, 1)$	$(0, 1)$
Black box			
configuration input	$(f(0), f(1))$	$(0, 1)$	$(1, 0)$
After Black box			
compute $\mathscr{N}_{\mathsf{B}}$	$\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1)$	$\mathscr{N}_{\mathsf{B}} = 1 \times 0 + 0 \times 1 = 0$	$\mathscr{N}_{\mathsf{B}} = 1 \times 1 + 0 \times 0 = 1$
Measurement			
configuration input	(m, n)	$(0, 1)$	$(0, 1)$
compute $f(0)$ & $f(1)$	$\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1)$	$0 = 1 \times f(0) + 0 \times f(1)$	$1 = 1 \times f(0) + 0 \times f(1)$
		$f(0) = 0$; $f(1) = 0$ or 1	$f(0) = 1$; $f(1) = 0$ or 1
permanent axiom	$f(0) = 0 \Rightarrow f(1) \neq 0$	$f(0) = 0$; $f(1) = 1$	$f(0) = 1$; $f(1) = 0$ or 1
		logically dependent	logically independent $f(1)$

Table 1 The Paterek research involves polarised photons as information carriers through measurement experiments. Orthogonality index $\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1)$ is a Boolean quantity, conveyed through experiments by the density matrix. For the case of photons conveying orthogonal information, the diagram shows how $\mathscr{N}_{\mathsf{B}}$ does not convey enough information for a measurement to determine the whole of the information imparted by the Black box.

6 Information content of the Pauli algebra

It is instructive to review the information content of the Pauli algebra, or more significantly, the information implied in the formula: $-i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^{1}\sigma_{\mathsf{z}}^{1}$; or more strictly, information content in formulae of that general unitary form:

$$-i\mathsf{b} = \mathsf{ac} \tag{9}$$

That review means going through the process of constructing (9), from scratch, and noting all information needed. The procedure I give is an adaption of a proof given by W E Baylis, J Huschilt and Jiansu Wei [3].

This may originate from D. Hestenes (1971), in which case, reference needs adding.

The Pauli algebra is a Lie algebra; and hence, is a linear vector space. Therefore, I begin with information inherited from the vector space axioms, and then add other information peculiar to the Pauli, Lie algebra.

Closure: For any two vectors u and v, there exists a vector w such that

$$\mathsf{w} = \mathsf{u} + \mathsf{v}$$

Identities: There exist additive and multiplicative identities, $\mathbb{0}$ and $\mathbb{1}$. For any arbitrary vector v:

$$\mathsf{v}\mathbb{1} = \mathbb{1}\mathsf{v} = \mathsf{v} \tag{10}$$
$$\mathsf{v} + \mathbb{0} = \mathbb{0} + \mathsf{v} = \mathsf{v} \tag{11}$$
$$\mathsf{v}\mathbb{0} = \mathbb{0}\mathsf{v} = \mathbb{0} \tag{12}$$

Additive inverse: For any arbitrary vector v, there exists an additive inverse $-\mathsf{v}$ such that

$$(-\mathsf{v}) + \mathsf{v} = \mathbb{0} \tag{13}$$

Scaling: For any arbitrary vector v, and any scalar a, there exists a vector u such that

$$\mathsf{u} = a\mathsf{v} \tag{14}$$

Products: A feature of Lie algebras is that products of vectors are members of the vector space. Between any two arbitrary vectors, u and v, there exist products uv and vu.
Dimension: Assume a 3 dimensional vector space, with independent basis a, b, c.

The six items of information

Involutory information: Assume all three basis vectors are involutory. Thus:

$$\mathsf{aa} = \mathbb{1} \qquad \mathsf{a} \text{ involutory} \tag{15}$$
$$\mathsf{bb} = \mathbb{1} \qquad \mathsf{b} \text{ involutory} \tag{16}$$
$$\mathsf{cc} = \mathbb{1} \qquad \mathsf{c} \text{ involutory} \tag{17}$$

Orthogonal information: Assume products between complimentary basis vectors are orthogonal. Thus:

$$\mathsf{ab} + \mathsf{ba} = \mathbb{0} \qquad \mathsf{ab} \text{ orthogonal} \tag{18}$$
$$\mathsf{bc} + \mathsf{cb} = \mathbb{0} \qquad \mathsf{bc} \text{ orthogonal} \tag{19}$$
$$\mathsf{ca} + \mathsf{ac} = \mathbb{0} \qquad \mathsf{ca} \text{ orthogonal} \tag{20}$$

Bringing items of information together, the Pauli algebra is constructed thus:

$$\begin{aligned} \mathsf{bc} + \mathsf{cb} &= \mathbb{0} && \text{by (19)} \quad , \quad \mathsf{bc} \text{ orthogonal} \\ \mathsf{b} + \mathsf{cbc} &= \mathbb{0} && \text{by (17)} \quad , \quad \mathsf{c} \text{ involutory} \\ \mathsf{ba} + \mathsf{cbca} &= \mathbb{0} && \text{by (12)} \end{aligned} \tag{21}$$

And similarly:

$$\begin{aligned} \mathsf{ca} + \mathsf{ac} &= \mathbb{0} && \text{by (20)} \quad , \quad \mathsf{ca} \text{ orthogonal} \\ \mathsf{cac} + \mathsf{a} &= \mathbb{0} && \text{by (17)} \quad , \quad \mathsf{c} \text{ involutory} \\ \mathsf{cacb} + \mathsf{ab} &= \mathbb{0} && \text{by (12)} \end{aligned} \tag{22}$$

Adding (22) and (21) gives:

$$\begin{aligned}
\mathsf{cacb} + \mathsf{ab} + \mathsf{ba} + \mathsf{cbca} &= \mathbb{0} \\
\mathsf{cacb} + \mathsf{cbca} &= \mathbb{0} && \text{by (18)} \quad , \quad \mathsf{ab} \text{ orthogonal} \\
\mathsf{acb} + \mathsf{bca} &= \mathbb{0} && \text{by (17)} \quad , \quad \mathsf{c} \text{ involutory} \\
\mathsf{acba} + \mathsf{bc} &= \mathbb{0} && \text{by (15)} \quad , \quad \mathsf{a} \text{ involutory} \\
\mathsf{acbac} + \mathsf{b} &= \mathbb{0} && \text{by (17)} \quad , \quad \mathsf{c} \text{ involutory} \\
\mathsf{acbacb} + \mathbb{1} &= \mathbb{0} && \text{by (16)} \quad , \quad \mathsf{b} \text{ involutory} \\
(\mathsf{acb})^2 &= -\mathbb{1} && \text{by (13)} \\
(\mathsf{acb})^2 &= (-1)\,\mathbb{1} \\
\mathsf{acb} &= \pm\, i\mathbb{1} \\
\mathsf{ac} &= \pm\, i\mathsf{b} && \text{by (16)} \quad , \quad \mathsf{b} \text{ involutory} && (23)
\end{aligned}$$

And a couple of extra steps gives the Pauli algebra:

$$\begin{aligned}
\mathsf{ca} &= \mp\, i\mathsf{b} && \text{by (23)} \quad , \quad \mathsf{a}, \mathsf{b}, \mathsf{c} \text{ involutory} && (24) \\
\mathsf{ac} - \mathsf{ca} &= \pm\, 2i\mathsf{b} && \text{by (23) \& (24)} && (25)
\end{aligned}$$

The six formulae (15) – (20) constitute six items of logically independent information. They are logically independent because none can be proved nor disproved from the others. All six are needed in proving $\mathsf{ac} = \pm i\mathsf{b}$.

7 Logical independence from the viewpoint of symmetry

Quantitatively, standard Pauli theory is superbly successful. But, in terms of representing the logic of experiments, it would seem the Paterek Boolean system is an improvement. Accepting that as important, the Boolean system must be traced through for information that standard theory misses.

The Paterek research shows that mathematics encoding the measurement of *mixed* states has logically independent structure; and that the measurement of *pure* states does not. And therefore, logically at least, any mathematical structure *faithfully* representing the measurement of mixed states cannot *faithfully* represent pure eigenstates, also. For the faithful representation of pure, and of mixed states, two structures are needed which are not mutually isomorphic: meaning that no one, single mathematical structure can be isomorphic with every polarisation measurement experiment. This contradicts standard theory, where the Pauli algebra is understood to represent every measurement configuration.

Consequently, the Paterek paper establishes, that measurement of arbitrarily prepared polarised photons, cannot, in general, be isomorphically represented by any single, exclusive, mathematical structure. Specifically, the Pauli algebra cannot be relied upon as a general theory, isomorphically representing every configuration of measurement experiment. Instead, measurement aligned parallel to the prepared state – and – measurement aligned orthogonal against it, are separately represented by distinct mathematical structures, not isomorphic with one another.

Having said all the above, *quantitatively*, the Pauli theory *does* work. Resolution to this *quantitative* versus *logical* dichotomy, as will be seen, is in the fact that one of those distinct mathematical structures agrees with the other, but the other does not agree with the one.

The above is helpful news. Of course, we take for granted the fact that individual experiments are independent of one another; but extra to that, the above tells us, also, experiments are independent, to the extent that, all Pauli experiments do not share one same algebraic environment. Algebra conformed to by one experiment does not extrapolate to all others.

In practice, this means that the formula (8) does not confer existence of σ_{y} upon the formulae (6). Also that (8) does not confer its value of σ_{z} upon (6). Et cetera. We must regard all such formulae, entailing the Pauli quad-products, as individual constructs of information, in isolation from one another, without passing information between them.

The Paterek findings rely on a *logical isomorphism*, linking the Boolean system with Pauli experiments. That isomorphism is a one – one correspondence that connects the logic of experiments with the logic of the Boolean system. The Paterek paper remarks on this logical isomorphism in its conclusion.

In contrast, the Pauli system lacks that one – one logical correspondence with experiment. The position is that the Pauli system faithfully represents experiments *quantitatively* whilst the Boolean system faithfully represents experiments *logically*. In order that the Pauli system should be logical also, it must connect logically, one – one, with Pauli experiments. That means Pauli experiments must connect logically, one – one, with the Boolean system (as they do); and then in turn, the Boolean system must connect logically, one – one, with the Pauli system. Thus:

$$\text{Pauli system} \rightleftarrows \text{Boolean system} \rightleftarrows \text{Pauli experiments}$$

To approach this, we must examine the exact nature of the link relating the Pauli and Boolean systems to see where logical correspondence between them currently fails.

Readers of the Paterek paper might infer that there is one – one correspondence linking the Pauli products with Boolean pairs. The actual picture is one – way. Implication is only directed from the Pauli products, to the Boolean pairs, in the sense of the arrows shown here:

$$\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^0\sigma_{\mathsf{z}}^1 \longrightarrow (0,1) \qquad \sigma_{\mathsf{x}} = \sigma_{\mathsf{x}}^1\sigma_{\mathsf{z}}^0 \longrightarrow (1,0) \qquad -i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^1\sigma_{\mathsf{z}}^1 \longrightarrow (1,1) \tag{26}$$

If the Pauli system were to connect logically, one – one, with the Boolean system, we would witness a backwards implication, also, in the sense of these reverse arrows:

$$\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^0\sigma_{\mathsf{z}}^1 \longleftarrow (0,1) \qquad \sigma_{\mathsf{x}} = \sigma_{\mathsf{x}}^1\sigma_{\mathsf{z}}^0 \longleftarrow (1,0) \qquad -i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^1\sigma_{\mathsf{z}}^1 \longleftarrow (1,1) \tag{27}$$

But, as they stand, the formulae in (27) are invalid. Generally, the Boolean pairs invoke operators that are not necessarily Paulian; they invoke operators belonging to some wider algebra. The Pauli operators are merely the special case that happens to be unitary. And so, we must either abandon the backwards implication or accept the replacement of Pauli operators with operators that maintain backwards validity.

The situation is made clearer when all Pauli notation is dropped and replaced by abstract symbols c, a, b. Formulae can then be seen for the information they *assert*, rather than content we *presume* – that stems from meaning we place on the symbols they contain.

Restating (27):

$$\mathsf{c} = \mathsf{a}^0\mathsf{c}^1 \longleftarrow (0,1) \qquad \mathsf{a} = \mathsf{a}^1\mathsf{c}^0 \longleftarrow (1,0) \qquad -i\mathsf{b} = \mathsf{a}^1\mathsf{c}^1 \longleftarrow (1,1) \tag{28}$$

The first two of these formulae imply *involutory* information only; whereas the last formulae, corresponding to $(1,1)$, implies information that is both *involutory* and *unitary*.

Now consider these Boolean 4-sequences:

$$\mathsf{cc} = \mathsf{a}^0\mathsf{c}^1\mathsf{a}^0\mathsf{c}^1 \longleftarrow (0,1)\,(0,1) \tag{29}$$
$$\mathsf{ac} = \mathsf{a}^1\mathsf{c}^0\mathsf{a}^0\mathsf{c}^1 \longleftarrow (1,0)\,(0,1) \tag{30}$$
$$-i\mathsf{bc} = \mathsf{a}^1\mathsf{c}^1\mathsf{a}^0\mathsf{c}^1 \longleftarrow (1,1)\,(0,1) \tag{31}$$

These express information representing three independent experiments. For the ‘straight-through’ experiment (29), the equality holds true for values of $\mathsf{a} \neq \sigma_{\mathsf{x}}$. This experiment invokes only the formulae $\mathsf{c} = \mathsf{a}^0\mathsf{c}^1$ from (28). The 4-sequence $(0,1)\,(0,1)$ implies only that a be any *involutory* operator, nothing more; and not that it should be a Pauli operator belonging to the Pauli algebra. No unitary information is implied and any unitarity attributed is redundant. I should add that (29) implies nothing whatever about c, except that it complies with rules for forming operator products.

Considering (30). The right hand side of the equality invokes $\mathsf{c} = \mathsf{a}^0\mathsf{c}^1$ and $\mathsf{a} = \mathsf{a}^1\mathsf{c}^0$ from (28), implying involutory c and a. The left hand side invokes unitarity through $-i\mathsf{b} = \mathsf{a}^1\mathsf{c}^1$. As for (31); this implies unitarity, directly through the formula $-i\mathsf{b} = \mathsf{a}^1\mathsf{c}^1$. See Table 2 for the other 4-sequences.

The fact these different experiments invoke different sets of information taken from (28) shows the variables a, b and c should not be regarded as fixed across all experiments. For some experiments they are unitary, others, not.

For (29) a is satisfied by any matrix of this form:

$$\mathsf{a} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad a^2 + bc = 1$$

Cases of interest are:

$$\mathsf{a} = \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} \qquad a^2 - b^2 = 1$$

$$\mathsf{a} = \begin{pmatrix} a & b^{-1} \\ b & -a \end{pmatrix} \qquad a^2 + 1 = 1$$

Measurement		**Logio – symmetry properties**		**Algebraic Information**		**Algebra implied by Boolean 4-sequences**				
Random outcomes	state	Unitarity	Circularly Self-referent	Involutory $\mathsf{aa}=\mathbb{1}$ $\mathsf{bb}=\mathbb{1}$ $\mathsf{cc}=\mathbb{1}$	Orthogonal $\mathsf{ab}+\mathsf{ba}=\mathbb{0}$ $\mathsf{bc}+\mathsf{cb}=\mathbb{0}$ $\mathsf{ca}+\mathsf{ac}=\mathbb{0}$	Implied algebra		Implied quad product		Boolean 4-sequence
no	pure	redundant	no	yes	no	$\mathsf{a}^2=\mathbb{1}$	$\leftarrow$	$\mathsf{a}^0\mathsf{c}^1\,\mathsf{a}^0\mathsf{c}^1$	$\leftarrow$	$(0,1)(0,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ac}=-i\mathsf{b}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^0\,\mathsf{a}^0\mathsf{c}^1$	$\leftarrow$	$(1,0)(0,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{bc}=+i\mathsf{a}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^1\,\mathsf{a}^0\mathsf{c}^1$	$\leftarrow$	$(1,1)(0,1)$
no	pure	redundant	no	yes	no	$\mathsf{c}^2=\mathbb{1}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^0\,\mathsf{a}^1\mathsf{c}^0$	$\leftarrow$	$(1,0)(1,0)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ba}=-i\mathsf{c}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^0$	$\leftarrow$	$(1,1)(1,0)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ca}=+i\mathsf{b}$	$\leftarrow$	$\mathsf{a}^0\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^0$	$\leftarrow$	$(0,1)(1,0)$
no	pure	redundant	no	yes	no	$(\mathsf{ac})^2=-\mathbb{1}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^1$	$\leftarrow$	$(1,1)(1,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{cb}=-i\mathsf{a}$	$\leftarrow$	$\mathsf{a}^0\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^1$	$\leftarrow$	$(0,1)(1,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ab}=+i\mathsf{c}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^0\,\mathsf{a}^1\mathsf{c}^1$	$\leftarrow$	$(1,0)(1,1)$

Table 2 Comparison of randomness in experiment outcomes, and logical independence in symmetry information, implied by the Paterek Boolean system.

8 Logical independence from the viewpoint of self-reference

An orthogonal vector space (or tensor space) can be thought of as a composite of information – consisting of – information that comprises a general, arbitrary vector space, plus additional information that renders that space orthogonal. More formally we might think of axioms imposing rules for vector spaces with additional axioms imposing orthogonality. However, the information of orthogonality need not originate in axioms; it can originate through *self-reference* or *logical circularity* [11].

This has profound implications for the logical standing of vector spaces used in representing quantum states — in particular, the logical standing of pure states, in relation to, the logical standing of mixed states. For it is this self-reference, at the interface between pure and mixed states, that is the root of logical independence in quantum systems — and is the root of *information deficiency* that manifests as quantum randomness. The self-reference sets up mathematical structure, but this structure lacks definite quantitative information.

In the case of Pauli systems, before this self-reference may proceed, a triplet of vector spaces forms into a closed system. Information of orthogonality does not enter from outside, but instead arises through the transfer of information, passed cyclicly from each vector space, to the next. This exchange is possible because the process is *logically independent* of axioms, so no information in the system opposes the exchange. Specifically, there is no *syntactic* information in the system axioms (axioms of Linear Algebra or Elementary Algebra) that causes or prevents (implies or contradicts) the information transfer.

Generally in any self-referent orthogonal system, the *fact* of Logical independence is conclusively confirmed by the fact that existence of imaginary unit is always a consequence [7], and that that number's logical independence, in Elementary Algebra, is well-known [6].

In momentum-position wave mechanics, a dual-pair of spaces forms into a closed system. The reason this is *dual* rather than a *triplet* is that the system algebra:

$$[\mathsf{p},\mathsf{x}]=-i\mathbb{1}$$

has $\mathbb{1}$ as its third operator. So the third vector space is trivial.

Examining Table 2, we see that all experiments demand the 3 involutory axioms (15) – (17); in particular, that includes pure states. They are satisfied by this set of involutory matrices:

$$\mathsf{a}(\eta)=\begin{pmatrix}0&\eta^{-1}\\\eta&0\end{pmatrix}\qquad \mathsf{b}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad \mathsf{c}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\tag{32}$$

That claim is confirmed in the Appendix. If (32) is permitted to take the value $\eta=\sqrt{-1}$ then (32) is *unitary-possible.* To progress from pure states to mixed states, the orthogonal axioms (18) – (20) are needed, in addition. That combined involutory plus orthogonal information is the Pauli algebra, which is *unitary-necessary,* and of course, satisfied by:

$$\mathsf{a}=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\qquad \mathsf{b}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad \mathsf{c}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\tag{33}$$

I now show how the unitary-switch in stepping from the *unitary-possible* symmetry (32), into the *unitary-necessary* symmetry (33), is permitted through logical independence.

Starting with the three matrices of (32), I begin by writing the general arbitrary transformation of which each of these matrices is capable.

$$\forall\eta\,\forall a_1\forall a_2\exists\psi_1\exists\psi_2 \;\Bigg|\; \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = \begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} \tag{34}$$

$$\forall b_1\forall b_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} \tag{35}$$

$$\forall c_1\forall c_2\exists\chi_1\exists\chi_2 \;\Bigg|\; \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} \tag{36}$$

Note that these formulae do not assert equality, they assert existence. I now explore the possibility of (34), (35) and (36) accepting information, circularly, from one another, through a cyclic mechanism where:

$$\begin{bmatrix}a_1\\ a_2\end{bmatrix}\text{ feeds off }\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \qquad \begin{bmatrix}b_1\\ b_2\end{bmatrix}\text{ feeds off }\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \qquad \begin{bmatrix}c_1\\ c_2\end{bmatrix}\text{ feeds off }\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \tag{37}$$

This forms a closed flow of information. There is no *cause* implying this self-reference; the idea is that it is *prevented* by no information occupying the system.

To proceed, the strategy followed will be to posit a hypothesis that such self-reference does occur, then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

Hypothesised coincidences:

$$\forall\phi_1\forall\phi_2\exists a_1\exists a_2 \;\Bigg|\; \begin{bmatrix}a_1\\ a_2\end{bmatrix} = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{38}$$

$$\forall\chi_1\forall\chi_2\exists b_1\exists b_2 \;\Bigg|\; \begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \tag{39}$$

$$\forall\psi_1\forall\psi_2\exists c_1\exists c_2 \;\Bigg|\; \begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{40}$$

Note: there is no guarantee that any such coincidence should exist. We proceed to investigate.

Manipulations involve all three transformations (34) – (36), plus all three assumed hypothesised coincidences (38) – (40).

Begin by taking transformation (35). Because any involutory matrix is its own inverse, inheriting identical quantifiers, the inverse transformation of (35) is:

$$\forall b_1\forall b_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{41}$$

Substituting hypothesis (39) into (41) gives:

$$\forall\chi_1\forall\chi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{42}$$

Now taking transformation (36) and substituting hypothesis (40)

$$\forall\psi_1\forall\psi_2\exists\chi_1\exists\chi_2 \;\Bigg|\; \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{43}$$

Substituting transformation (34) into (43):

$$\forall\eta\,\forall a_1\forall a_2\exists\chi_1\exists\chi_2 \;\Bigg|\; \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} \tag{44}$$

And substituting hypothesis (40) into (44) gives:

$$\forall\eta\,\forall\phi_1\forall\phi_2\exists\chi_1\exists\chi_2 \;\Bigg|\; \begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & \eta^{-1}\\ \eta & 0\end{pmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{45}$$

Substitution involving quantifiers

$$\begin{aligned} \forall\beta\underline{\forall\gamma}\exists\alpha \;&|\; \alpha = \beta + \gamma \\ \forall\lambda\underline{\exists\gamma} \;&|\; \gamma = 2\lambda \\ \Rightarrow\; \forall\lambda\forall\beta\exists\alpha \;&|\; \alpha = \beta + 2\lambda \end{aligned}$$

An *existential* quantifier of one proposition is matched with a *universal* quantifier of the other. Those matched are underlined.

By direct comparison of (42) and (45), we have:

$$\begin{aligned} \forall\eta \Big| \quad & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \Rightarrow \forall\eta \Big| \quad & \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{46}$$

But (46) is a contradiction because, for one problem alone, it states that $\forall\eta\,(\eta = 1)$; but 1 cannot equal every other number η. To remove the contradiction, the bound variable η must be freed from its quantifier $\forall\eta$, and η be allowed to take a particular value, which I denote $\boldsymbol{\eta}$ (bold). Then we may say:

$$\begin{pmatrix} \boldsymbol{\eta} & 0 \\ 0 & \boldsymbol{\eta}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{47}$$

Up to this point, no imaginary information exists in the system. (47) cannot be solved without new, logically independent information, which must be formally assumed:

Hypothesised existence:

$$\exists\eta \mid \quad \eta^2 = -1$$

and

$$\boldsymbol{\eta}^2 = -1 \tag{48}$$

In context of Elementary Algebra, logical independence of this number is very well-known to Mathematical Logicians [6].

9 Discussion – Redundant unitarity in the free particle

A different quantum system – that of the *free particle* – mirrors this same unitary logic, between pure and mixed states.

It is instructive to understand the difference between *syntactical information* versus *semantical information.* Syntax concerns rules used for constructing or transforming symbols and formulae – the rules of Elementary Algebra, say. Semantics, on the other hand, concerns interpretation. Here, *interpretation* does not refer to *physical* meaning, but to *mathematical* meaning: whether symbols might be understood to mean: complex scalars, real scalars, or rational. Such interpretation has null logical connectivity with the rules of algebra — the syntax. Indeed, typically, the interpretation may be only in the theorist's mind and not asserted by the mathematics, at all.

A most relevant illustration is the comparison of syntax versus semantics in the mathematics representing pure eigenstates, set against mixed states, in the system of the quantum free particle. Consider the *eigenformulae pair:*

$$\frac{d}{dx}\left[\mathit{\Phi}\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right] = +i\mathsf{k}\left[\mathit{\Phi}\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right] \tag{49}$$

$$\frac{d}{dk}\left[\mathit{\Psi}\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right] = -i\mathsf{x}\left[\mathit{\Psi}\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right] \tag{50}$$

This pair of formulae is true, irrespective of any interpretation placed on the variable i. But in contrast, the *superposition pair:*

$$\mathit{\Psi}\left(x\right) = \int\left[\mathit{\Phi}\left(\mathsf{k}\right)\exp\left(+i\mathsf{k}x\right)\right]d\mathsf{k} \tag{51}$$

$$\mathit{\Phi}\left(k\right) = \int\left[\mathit{\Psi}\left(\mathsf{x}\right)\exp\left(-ik\mathsf{x}\right)\right]d\mathsf{x} \tag{52}$$

is true, only if we interpret i as *pure imaginary.* (And if k is restricted to real or rational k; and if x is restricted to real or rational x.) In the case of the eigenvalue pair (49) & (50) the imaginary interpretation is purely in the mind of the theorist, but for the superposition pair (51) & (52), the imaginary interpretation is implied by the mathematics. Whilst for the superposition pair (51) & (52), specific interpretation is *necessary*, for the eigenvalue pair (49) & (50), interpretation is *possible*, but *not necessary.*

In Mathematical Logic, '*necessary information* versus *possible information*' is recognised as constituting what is known as a 'modal logic'. However, in textbook quantum theory, the distinction separating possible from necessary is not noticeable, nor is it recognised; and the logical distinction between pure states and mixed states is lost. The crucial difference in expressing pure states is that their information derives from pure syntax. The transition in forming mixed states from pure states demands the creation of new information[2]. That creation goes unopposed.

The important point is that the logical status of pure states and mixed is distinct, not only in experiments, but also in current Theory.

The fact is that quantum theory for pure states need not be unitary (or self-adjoint); whereas, for mixed states, unitarity is necessary. The jump between pure states and mixed states represents a logical jump between *possible unitarity* and *necessary unitarity.*

Historically, the distinction between necessary and possible unitarity has not been noticed, as any point of significance. No doubt, standard quantum theory ignores the fact for reasons of consistency. But, rewriting (49) – (52) as formulae in *first order logic* shows there is no contradiction, and that the possible / necessary information can be conveyed by a single theory. Thus, for pure states:

$$\forall\eta \mid \; \frac{d}{dx}\left[\Phi\left(\mathsf{k}\right)\exp\left(\eta^{+1}\mathsf{k}x\right)\right] \;=\; \eta^{+1}\mathsf{k}\left[\Phi\left(\mathsf{k}\right)\exp\left(\eta^{+1}\mathsf{k}x\right)\right] \tag{53}$$

$$\forall\eta \mid \; \frac{d}{dk}\left[\Psi\left(\mathsf{x}\right)\exp\left(\eta^{-1}\mathsf{x}k\right)\right] \;=\; \eta^{-1}\mathsf{x}\left[\Psi\left(\mathsf{x}\right)\exp\left(\eta^{-1}\mathsf{x}k\right)\right] \tag{54}$$

And for mixed:

$$\exists\eta \mid \; \Psi\left(x\right) = \int\left[\Phi\left(\mathsf{k}\right)\exp\left(\eta^{+1}\mathsf{k}x\right)\right]d\mathsf{k} \tag{55}$$

$$\exists\eta \mid \; \Phi\left(k\right) = \int\left[\Psi\left(\mathsf{x}\right)\exp\left(\eta^{-1}\mathsf{x}k\right)\right]d\mathsf{x} \tag{56}$$

But having rewritten formulae as (53) – (56), these new formulae are inconsistent with the Postulates of Quantum Mechanics. Specifically, (53) & (54) disagree with unitarity (or self-adjointness) – imposed *by Postulate.* Whilst (53) – (56) represent a mathematical system that is logically self-consistent, conveying the *whole information* of unitarity; that conveyance of whole information is gained at the expense of textbook quantum theory's most treasured fact — the self-adjointness of operators.

Not to worry. The postulated unitarity (or self-adjointness) is not needed. Unitarity is implied where it is needed – in the mathematics of the mixed states. Elsewhere, unitarity (or self-adjointness) is redundant.

10 Discussion – Self-reference in the free particle

As in the Pauli system, the transition (53) – (56) from pure to mixed states, again involves logical self-reference.

Consider the following pair of formulae.

I use the notation $\int_{\mathsf{k}} f\left(\mathsf{k}\right) = \int_{-\infty}^{+\infty} f\left(\mathsf{k}\right)d\mathsf{k}$.

$$\forall\eta\forall x\exists a\exists\Psi \mid \quad \Psi\left(x\right) = \int_{\mathsf{k}}\left[\exp\left(\eta^{+1}x\mathsf{k}\right)a\left(\mathsf{k}\right)\right] \tag{57}$$

$$\forall\eta\forall k\exists b\exists\Phi \mid \quad \Phi\left(k\right) = \int_{\mathsf{x}}\left[\exp\left(\eta^{-1}k\mathsf{x}\right)b\left(\mathsf{x}\right)\right] \tag{58}$$

In writing these, the san-serif notated k and x are the dummy (bound) variables over the integrals. The italicised variables η, k, x, a, b are all bound variables over the existential quantifier $\exists$ and universal quantifier $\forall$. I have laid out the ordering of variables to mirror the convention of repeated dummy indices used in summations of discrete quantities.

Note that these formulae do not assert equality, they assert existence. Also note that the integrals exist, and the pair of propositions are true, only if a and b are functions restricted to the Banach space L^1 (at least).

I now explore the possibility of (57) and (58) accepting information, circularly, from one another, through a mechanism where $a\left(\mathsf{k}\right)$ feeds off $\Phi\left(k\right)$ and $b\left(\mathsf{x}\right)$ feeds off $\Psi\left(x\right)$. There is no *cause* implying this self-reference; the idea is that it is *prevented* by nothing. Indeed, the fact of this self-reference is information, logically independent of all algebraic rules in operation.

[2] In some way, yet to be understood, this information is lost again during measurement.

To proceed, the strategy followed will be to posit a hypothesis that such self-reference does occur, then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

Hypothesised coincidence:

$$\forall \Phi \exists a \mid \quad a = \Phi; \tag{59}$$
$$\forall \Psi \exists b \mid \quad b = \Psi. \tag{60}$$

When these assumptions are substituted into (57) and (58) we get:

$$\forall \eta \forall x \exists \Phi \exists \Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} \left[\exp\left(\eta^{+1} x \mathsf{k}\right) \Phi(\mathsf{k}) \right] \tag{61}$$
$$\forall \eta \forall k \exists \Psi \exists \Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} \left[\exp\left(\eta^{-1} k \mathsf{x}\right) \Psi(\mathsf{x}) \right] \tag{62}$$

and that allows cross-substitution of Φ and Ψ, invoking a simultaneous pair of propositions, which together, will force particular values on η. Before the pair can be considered as simultaneous, in order to preserve validity, the repeated $\forall\eta$ quantifier must be lost, leaving the particularised (bold) $\boldsymbol{\eta}$. Substituting (62) into (61), and (61) into (62), we get:

$$\forall x \exists \Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} \left[\exp\left(\boldsymbol{\eta}^{+1} x \mathsf{k}\right) \int_{\mathsf{x}} \left[\exp\left(\boldsymbol{\eta}^{-1} \mathsf{k}\mathsf{x}\right) \Psi(\mathsf{x}) \right] \right] \tag{63}$$
$$\forall k \exists \Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} \left[\exp\left(\boldsymbol{\eta}^{-1} k \mathsf{x}\right) \int_{\mathsf{k}} \left[\exp\left(\boldsymbol{\eta}^{+1} \mathsf{x}\mathsf{k}\right) \Phi(\mathsf{k}) \right] \right] \tag{64}$$

Taking the integral signs outside and reversing their order, these tidy up to become:

$$\forall x \exists \Psi \mid \quad \Psi(x) = \int_{\mathsf{x}} \int_{\mathsf{k}} \exp\left[\left(\boldsymbol{\eta}^{+1} x + \boldsymbol{\eta}^{-1} \mathsf{x} \right) \mathsf{k} \right] \Psi(\mathsf{x}) \tag{65}$$
$$\forall k \exists \Phi \mid \quad \Phi(k) = \int_{\mathsf{k}} \int_{\mathsf{x}} \exp\left[\left(\boldsymbol{\eta}^{-1} k + \boldsymbol{\eta}^{+1} \mathsf{k} \right) \mathsf{x} \right] \Phi(\mathsf{k}) \tag{66}$$

These integrals, over the exponentials, exist only when $\boldsymbol{\eta} = \pm\,\mathsf{i}$. And therefore this pair of propositions is true — with the **Hypothesised coincidence** guaranteed — only for $\boldsymbol{\eta} = \pm\,\mathsf{i}$.

Up to this point, no imaginary information exists in the system. In order to validate the pair of integrals, new information must be introduced. This information must be assumed. To properly document this assumption, the hypothesis is formally declared, thus:

Hypothesised existence:

$$\exists \boldsymbol{\eta} \mid \quad \boldsymbol{\eta}^2 = -1$$

Setting the *particular* number $\mathsf{i} = \sqrt{\ } - 1$ and also $\boldsymbol{\eta} = \mathsf{i}$:

$$\forall x \exists \Psi \mid \quad \Psi(x) = \int_{\mathsf{x}} \int_{\mathsf{k}} \exp\left[+\mathsf{i}\,(x - \mathsf{x})\,\mathsf{k} \right] \Psi(\mathsf{x}) \tag{67}$$
$$\forall k \exists \Phi \mid \quad \Phi(k) = \int_{\mathsf{k}} \int_{\mathsf{x}} \exp\left[-\mathsf{i}\,(k - \mathsf{k})\,\mathsf{x} \right] \Phi(\mathsf{k}) \tag{68}$$

and in conclusion, claim that this pair of formulae are true, providing they are allowed self-referential information.

Simultaneous propositions
Illustrating. Taking the two propositions:

$$\forall \eta : \ \mathsf{y} = \mathsf{a}\eta + \mathsf{b}$$
$$\forall \eta : \ \mathsf{y} = \mathsf{c}\eta + \mathsf{d}$$

If these are to be solved simultaneously, the *repeated* $\forall\eta$ must lost, with instances of η from each formulae, being particularised first. Their joint solution then:

$$\mathsf{a}\boldsymbol{\eta} + \mathsf{b} = \mathsf{c}\boldsymbol{\eta} + \mathsf{d}$$

where $\boldsymbol{\eta}$ (bold) is the *particular value* variable.

As a final point, it is rather noticeable that these logical phenomena in quantum theory, surround the presence of the imaginary unit. And so it is important to say that, within Elementary Algebra, this number's existence is very well-known, by Mathematical Logicians, to be logically independent [6].

11 Conclusions

Quantum indeterminacy is strictly a phenomenon of *mixed* states. Measurement outcomes from pure eigenstates are never random. That is well-known. In alignment with that, the new research of Tomasz Paterek et al shows that *logical independence,* also, is a strict feature of mixed states – pure states being *logically dependent* [9, 10].

That logical dependence and in-dependence is mathematical information. The transition from pure states to mixed is reflected in corresponding mathematical transition stepping from dependence to in-dependence. The information comprising that mathematical transition represents the information of quantum indeterminacy. This paper examines that transition.

Here I have written *independence* with a hyphen, as *in-dependence*. This is for nothing more than clarity.

Textbook quantum theory demands: Hilbert space, self-adjoint operators and unitary symmetries, as features. From the viewpoint of the transition, none of these are required by pure eigenstates; they are required only by mixed states. A truly faithful, isomorphic theory would need to be *non-unitary* on the pure state side of the transition, and *unitary* on the mixed state side.

Whilst the mathematician might feel free to simply declare a theory *unitary*, by declaring that observable operators should be Hermitian, say — although such declaration might seem to impose purely *quantitative* restriction on variables, that eigenvalues be real, for instance — such declaration includes hidden logical structure, not noticed. This is logic that sits at the interface between Elementary Algebra (school algebra) and *orthogonal* Linear Algebra. The juxtaposition of these two algebras, in a single environment, is inherent in quantum mathematics, placing that logical structure squarely and unavoidably in the domain of quantum theory.

The logical structure is logically circular self-reference, going on within a symmetry. Unlike energy or momentum, that self-reference is perfectly free and not subject to any conservation law. There is no resistance to its onset. Self-reference is a spontaneous logical option, neither caused nor prevented (implied nor denied) by any information in the mathematical environment — it is *logically independent* of all information in that mathematical environment.

The effect of the self-reference is to create the consequent existence of a unitary symmetry, along with structures that follow from it: self-adjoint operators and Hilbert space, et cetera – all logically independent within the mathematics as a whole. The impact of all this is that unitarity or self-adjointness, imposed – *by Postulate* – is redundant.

The conclusion of this research is that a quantum theory that adheres strictly to the *faithful* representation of (non-unitary) pure states – *that switches to* – the strict and faithful representation of (unitary) mixed states, automatically invokes representation of quantum indeterminacy. Those faithful representations require isomorphisms under two distinct symmetries: a non-unitary symmetry representing pure states, and a unitary symmetry representing mixed. Transition between these is logically self-referential. To allow this logical mechanism to operate, unitarity (and self-adjointness) must be free to switch on and off. But in standard theory, unitarity (or self-adjointness) is imposed – *by Postulate* – and this freedom is blocked.

The most profound conclusion, therefore, is that unitarity or self-adjointness, imposed – *by Postulate* – must be given up; the benefit being a quantum theory that expresses theory and logic of quantum indeterminacy.

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A Appendix: Products of Pauli and involutory operators

$$\sigma_{\mathsf{y}}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{1}_2$$

$$\mathsf{s}_{\mathsf{z}}^2 = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} = \mathbb{1}_2$$

$$\sigma_{\mathsf{x}}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbb{1}_2$$

$$\sigma_{\mathsf{z}}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbb{1}_2$$

$$\sigma_{\mathsf{y}}\mathsf{s}_{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \qquad (\sigma_{\mathsf{y}}\mathsf{s}_{\mathsf{z}})^2 = \begin{pmatrix} \eta^2 & 0 \\ 0 & \eta^{-2} \end{pmatrix}$$

$$\mathsf{s}_{\mathsf{z}}\sigma_{\mathsf{y}} = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta \end{pmatrix} \qquad (\mathsf{s}_{\mathsf{z}}\sigma_{\mathsf{y}})^2 = \begin{pmatrix} \eta^{-2} & 0 \\ 0 & \eta^2 \end{pmatrix}$$

$$\sigma_{\mathsf{y}}\mathsf{s}_{\mathsf{z}} + \mathsf{s}_{\mathsf{z}}\sigma_{\mathsf{y}} = \left(\eta + \eta^{-1}\right)\mathbb{1}_2$$

$$\mathsf{s}_{\mathsf{z}}\sigma_{\mathsf{x}} = \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\eta^{-1} \\ \eta & 0 \end{pmatrix} \qquad (\mathsf{s}_{\mathsf{z}}\sigma_{\mathsf{x}})^2 = -\mathbb{1}_2$$

$$\sigma_{\mathsf{x}}\mathsf{s}_{\mathsf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \eta^{-1} \\ -\eta & 0 \end{pmatrix} \qquad (\sigma_{\mathsf{x}}\mathsf{s}_{\mathsf{z}})^2 = -\mathbb{1}_2$$

$$\mathsf{s}_{\mathsf{z}}\sigma_{\mathsf{x}} + \sigma_{\mathsf{x}}\mathsf{s}_{\mathsf{z}} = \mathbb{0}_2$$

$$\sigma_{\mathsf{y}}\sigma_{\mathsf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (\sigma_{\mathsf{y}}\sigma_{\mathsf{x}})^2 = -\mathbb{1}_2$$

$$\sigma_{\mathsf{x}}\sigma_{\mathsf{y}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (\sigma_{\mathsf{x}}\sigma_{\mathsf{y}})^2 = -\mathbb{1}_2$$

$$\sigma_{\mathsf{y}}\sigma_{\mathsf{x}} + \sigma_{\mathsf{x}}\sigma_{\mathsf{y}} = \mathbb{0}_2$$

$$\sigma_{\mathsf{y}}\sigma_{\mathsf{z}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad (\sigma_{\mathsf{z}}\sigma_{\mathsf{x}})^2 = -\mathbb{1}_2$$

$$\sigma_{\mathsf{z}}\sigma_{\mathsf{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \qquad (\sigma_{\mathsf{y}}\sigma_{\mathsf{z}})^2 = -\mathbb{1}_2$$

$$\sigma_{\mathsf{y}}\sigma_{\mathsf{z}} + \sigma_{\mathsf{z}}\sigma_{\mathsf{y}} = \mathbb{0}_2$$

$$\sigma_{\mathsf{x}}\sigma_{\mathsf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\sigma_{\mathsf{z}}\sigma_{\mathsf{x}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\sigma_{\mathsf{x}}\sigma_{\mathsf{z}} + \sigma_{\mathsf{z}}\sigma_{\mathsf{x}} = \mathbb{0}_2$$

Table 3 Algebraic properties of operator products arising the Pauli and involutory algebras.