

Solution for Euler Equations – Lagrangian and Eulerian Descriptions

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Abstract – We find an exact solution for the system of Euler equations, supposing that there is some solution, following the Eulerian and Lagrangian descriptions, for spatial dimension $n = 3$. As we had seen in other previous articles, it is possible that there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Euler equations, velocity, pressure, Eulerian description, Lagrangian description, formulation, classical mechanics, Newtonian mechanics, Newton's law, second law of Newton, equivalent systems, exact solutions, Bernoulli's law.

§ 1

Essentially the Euler (and Navier-Stokes) equations relating the velocity u and pressure p suffered by a volume element dV at position (x, y, z) and time t . In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx dy dz$ at time t , and this position varies with the movement of this same element dV .

Basically, the deduction of the Euler equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz^[1] or Prandtl's book^[2] for a more detailed description of the deduction of these equations (including Navier-Stokes equations).

In spatial dimension $n = 3$, the Euler equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1) \quad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = f_3 \end{cases}$$

where $u(x, y, z, t) = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$, $u: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3 , p is the pressure, $p: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}$, and $f(x, y, z, t) = (f_1(x, y, z, t), f_2(x, y, z, t), f_3(x, y, z, t))$, $f: \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$, is the density of external force applied in the fluid in point (x, y, z) and at the instant of time t , for example, gravity force per mass unity, with $x, y, z, t \in \mathbb{R}$, $t \geq 0$. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

The non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$, $1 \leq i \leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$ and these x, y, z also vary in time, $x = x(t)$, $y = y(t)$, $z = z(t)$, then, by the chain rule,

$$(2) \quad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}.$$

Defining $\frac{dx}{dt} = u_1$, $\frac{dy}{dt} = u_2$, $\frac{dz}{dt} = u_3$, comes

$$(3) \quad \frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u_1 + \frac{\partial u}{\partial y} u_2 + \frac{\partial u}{\partial z} u_3,$$

and therefore

$$(4) \quad \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}, \quad 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluation of $\frac{Du}{Dt}$ (or $\frac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x, y, z) variable in time (Lagrangian formulation) and then consider (x, y, z) fixed (Eulerian formulation), seems to be subject to criticism. In our present deduction, starting from Euler equations in Eulerian description, implicitly with a solution (u, p) , next the pressure, and its

corresponding gradient, they travel with the volume element $dV = dx dy dz$, i.e., obeys to the Lagrangian description of motion, as well as the external force f , in order to avoid contradictions. The velocity u also will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = \text{const.}$, $u = u(t)$. Done the solution in Lagrangian description, the solution for pressure in Eulerian description will be given explicitly.

Following this idea, the system (1) above can be transformed into

$$(5) \quad \begin{cases} \frac{1}{u_1} \frac{\partial p}{\partial t} + \frac{Du_1}{Dt} = f_1 \\ \frac{1}{u_2} \frac{\partial p}{\partial t} + \frac{Du_2}{Dt} = f_2 \\ \frac{1}{u_3} \frac{\partial p}{\partial t} + \frac{Du_3}{Dt} = f_3 \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity, since that the partial derivatives of the pressure and velocities were correctly transformed to the variable time, using $\partial x = u_1 \partial t$, $\partial y = u_2 \partial t$, $\partial z = u_3 \partial t$. Likewise for the calculation of $\frac{Du}{Dt}$, according (4), and external force f , using $x = x(t)$, $y = y(t)$, $z = z(t)$. The integration of the system (5) shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then this is a condition to the occurrence of solutions. In the sequence the procedure in more details for obtaining the pressure in Lagrangian formulation, a time dependent function, starting by solution for pressure in Eulerian description.

Given $u = u(x, y, z, t) \in C^1(\mathbb{R}^3 \times [0, \infty))$ obeying the initial conditions and a vector function f (both when in Eulerian description) such that the difference $f - \frac{Du}{Dt}$ is gradient, the system's solution (1) for p , using the condensed notation given by (4), is

$$(6) \quad p = \int_L \left(f - \frac{Du}{Dt} \right) \cdot dl + \theta(t),$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$.

In Eulerian description and in special case when $f - \frac{Du}{Dt}$ is a constant vector or a dependent function only on the time variable, we come to

$$(7) \quad \begin{aligned} p &= p^0 + S_1(t) (x - x_0) + S_2(t) (y - y_0) + S_3(t) (z - z_0), \\ S_i(t) &= f_i - \frac{Du_i}{Dt}, \end{aligned}$$

where $p^0 = p^0(t)$ is the pressure in the point (x_0, y_0, z_0) at time t .

When the variables x, y, z in (6) as well as f and u are in Lagrangian description, representing a motion over time of a hypothetical volume element dV or particle of fluid, we need eliminate the dependence of the position using in (6)

$$(8) \quad dl = (dx, dy, dz) = (u_1 dt, u_2 dt, u_3 dt)$$

and integrating over time. The result is

$$(9) \quad p(t) = p^0 + \int_0^t \sum_{i=1}^3 S_i(t) u_i(t) dt,$$

$$p^0 = p(0) = \text{const.}$$

This expression can be more facilitated making $u_i \frac{Du_i}{Dt} dt = u_i du_i$ and $\int_0^t u_i \frac{Du_i}{Dt} dt = \int_{u_i^0}^{u_i} u_i du_i = \frac{1}{2}(u_i^2 - u_i^{0^2})$, so (9) is equal to

$$(10) \quad p(t) = p^0 - \frac{1}{2} \sum_{i=1}^3 (u_i^2 - u_i^{0^2}) + \int_0^t \sum_{i=1}^3 f_i(t) u_i(t) dt,$$

i.e.,

$$(11) \quad p(t) = p^0 - \frac{1}{2}(u^2 - u^{0^2}) + \int_0^t f \cdot u dt,$$

$p, p^0 \in \mathbb{R}, u, u^0, f \in \mathbb{R}^3, u = (u_1, u_2, u_3)(t), u^0 = (u_1^0, u_2^0, u_3^0) = u(0), f = (f_1, f_2, f_3)(t)$, in Lagrangian description. $u^2 = u \cdot u$ and $u^{0^2} = u^0 \cdot u^0$ are the square modules of the respective vectors u and u^0 .

When $f = 0$ the solution (11) is simply

$$(12) \quad p = p^0 - \frac{1}{2}(u^2 - u^{0^2}),$$

which then can be considered an exact solution for Euler equations in Lagrangian description, and similarly to Bernoulli's law without external force (gravity, in special).

Unfortunately, in Eulerian description, neither

$$(13) \quad p(x, y, z, t) = p^0(x, y, z) - \frac{1}{2}(u^2 - u^{0^2}) + \int_L f \cdot dl,$$

$p^0(x, y, z) = p(x, y, z, 0), u^0 = u^0(x, y, z) = u(x, y, z, 0)$, nor

$$(14) \quad p(x, y, z, t) = p^0(t) - \frac{1}{2}(u^2 - u^{0^2}) + \int_L f \cdot dl,$$

$p^0(t) = p(x_0, y_0, z_0, t)$, $u^0 = u^0(t) = u(x_0, y_0, z_0, t)$, solve (1) for all cases of velocities, both formulations supposing f a gradient vector function ($\nabla \times f = 0$, $f = \nabla\phi$, ϕ potential function of f).

For example, for $f = 0$ the solution (14) is valid only when

$$(15) \quad \frac{\partial p}{\partial x_i} = - \sum_{j=1}^3 u_j \frac{\partial u_j}{\partial x_i} = - \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} \right),$$

i.e.,

$$(16) \quad \frac{\partial u_i}{\partial t} = \sum_{j=1}^3 u_j \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right).$$

How to return to the Eulerian formulation if only was obtained a complete solution in the Lagrangian formulation? As well as we can choose any convenient velocity $u(t) = (u_1(t), u_2(t), u_3(t))$ to calculate the pressure (11) that complies with the initial conditions (Lagrangian formulation), we also can choose appropriate $u(x, y, z, t)$ (Eulerian formulation) and $x(t), y(t), z(t)$ to the velocities and positions of the system and taking the corresponding inverse functions in the obtained solution. This choose is not completely free because will be necessary to calculate a system of ordinary differential equations to obtain the correct set of $x(t), y(t), z(t)$, such that

$$(17) \quad \begin{cases} \frac{dx}{dt} = u_1(x, y, z, t) \\ \frac{dy}{dt} = u_2(x, y, z, t) \\ \frac{dz}{dt} = u_3(x, y, z, t) \end{cases}$$

Nevertheless, this yet can save lots calculation time.

It will be necessary find solutions of (17) such that it is always possible to make any point (x, y, z) of the velocity domain can be achieved for each time t , introducing for this initial positions (x_0, y_0, z_0) conveniently calculated according to (17). Thus we will have velocities and pressures that, in principle, can be calculated for any position and time, independently of one another, not only for a single position for each time. For different values of (x, y, z) and t we will, in the general case, obtain the velocity and pressure of different volume elements dV , starting from different initial positions (x_0, y_0, z_0) .

We can escape the need to solve (17), but admitting its validity and the corresponding existence of solution, previously choosing differentiable functions $x = x(t), y = y(t), z = z(t)$ and then calculating directly the solution for velocity in the Lagrangian formulation,

$$(18) \quad \begin{cases} u_1(t) = \frac{dx}{dt} \\ u_2(t) = \frac{dy}{dt} \\ u_3(t) = \frac{dz}{dt} \end{cases}$$

Concluding, answering the question, in the result of pressure in Lagrangian formulation given by (9) or (11), conveniently transforming the initial position (x_0, y_0, z_0) as function of a generic position (x, y, z) and time t , we will have a correct value of the pressure in Eulerian formulation, since that keeping the same essential original significance. The same is valid for the velocity in Lagrangian formulation, if the correspondent Eulerian formulation was not previously obtained.

§ 2

It is worth mentioning that the Euler equations in the standard Lagrangian format, traditional one, are different than previously deduced.

Based on [5] the Euler equations without external force and with mass density $\rho = 1$ are

$$(19.1) \quad \frac{\partial^2 X_i}{\partial t^2} = - \sum_{j=1}^3 \frac{\partial A_j}{\partial x_i} \frac{\partial p}{\partial a_j},$$

$$(19.2) \quad \frac{\partial A_j}{\partial x_i} \equiv \frac{\partial}{\partial x_i} X_j(x_n, t)|_{x_n=X_n(a_m, s|t)},$$

where a_m is the label given to the fluid particle at time s . Its position and velocity at time t are, respectively, $X_n(a_m, s|t)$ and $u_n(a_m, s|t)$.

The significant difference between (19) and (5) is that our pressure (5) is varying only with time, as the initial position is a constant for each particle, not variable. In (19) the pressure varies with the initial position (label) and there is a summation on the three coordinates. We did in (5) $\partial x_i = u_i \partial t$.

§ 3

Without passing through the Lagrangian formulation, given a differentiable velocity $u(x, y, z, t)$ and an integrable external force $f(x, y, z, t)$, perhaps a better expression for the solution of the equation (1) is, in fact,

$$(20) \quad p = \sum_{i=1}^3 \int_{P_i^0}^{P_i} \left[- \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} \right) + f_i \right] dx_i + \theta(t),$$

supposing possible the integrations and that the vector $S = - \left[\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right] + f$ is a gradient function. This is the development of the solution (6) for the specific path

L going parallelly (or perpendicularly) to axes X, Y and Z from $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ to $(x_1, x_2, x_3) \equiv (x, y, z)$, since that the solution (6) is valid for any piecewise smooth path L . We choose $P_1^0 = (x_0, y_0, z_0)$, $P_2^0 = (x, y_0, z_0)$, $P_3^0 = (x, y, z_0)$ and $P_1 = (x, y_0, z_0)$, $P_2 = (x, y, z_0)$, $P_3 = (x, y, z)$. $\theta(t)$ is a generic time function, physically and mathematically reasonable, for example with $\theta(0) = 0$ or adjustable for any given condition. Again we have seen that the system of Euler equations has no unique solution, only given initial conditions, supposing that there is some solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

The remark given for system (5), when used here, leads us to the following conclusion: the integration of the system (1), confronting with (5), shows that anyone of its equations can be used for solve it, and the results must be equals each other. Then again this is a condition to the occurrence of solutions, which shows to us the possibility of existence of breakdown solutions, as will become clearer in §5.

§ 4

Another way to solve (1) with $f = 0$ seems to me to be the best of all, for its extreme ease of calculation, also without we need to resort to Lagrangian formulation and its conceptual difficulties. If $u(x, y, z, 0) = u^0(x, y, z)$ is the initial velocity of the system, valid solution in $t = 0$, then $u(x, y, z, t) = u^0(x + t, y + t, z + t)$ is a solution for velocity in $t \geq 0$, a non-unique solution. Similarly, $p(x, y, z, t) = p^0(x + t, y + t, z + t)$ is the correspondent solution for pressure in $t \geq 0$, being $p^0(x, y, z)$ the initial condition for pressure. The velocities $u^0(x + t, y, z)$, $u^0(x, y + t, z)$ and $u^0(x, y, z + t)$ are also solutions, and respectively also the pressures $p^0(x + t, y, z)$, $p^0(x, y + t, z)$ and $p^0(x, y, z + t)$. Other solutions may be searched, for example in the kind $u(x, y, z, t) = u^0(x + T_1(t), y + T_2(t), z + T_3(t))$, $T_i(0) = 0$, and therefore $p(x, y, z, t) = p^0(x + T_1(t), y + T_2(t), z + T_3(t))$. See [6].

§ 5

This article would not be complete without mentioning the potential flows. When there is a potential function ϕ such that $u = \nabla\phi$ then $\nabla \times u = 0$, i.e., the velocity is an irrotational field. When the incompressibility condition is required, i.e., $\nabla \cdot u = 0$, the velocity is solenoidal, and if the field is also irrotational then $\nabla^2 u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) = 0$, i.e., the Navier-Stokes equations are reduced to Euler's equations and the velocity-potential ϕ must satisfied the Laplace's equation, $\nabla^2 \phi = 0$, as well as the velocity.

According Courant^[7] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w, t)$ be analytic in the

complex variable w for fixed real t . Then, for arbitrary values of t , both the real and imaginary parts of the function

$$(21) \quad u = f(z + ix \cos t + iy \sin t, t)$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

$$(22) \quad u = \int_a^b f(z + ix \cos t + iy \sin t, t) dt.$$

For example, if we set

$$(23) \quad f(w, t) = w^n e^{iht},$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

$$(24) \quad u = \int_{-\pi}^{\pi} (z + ix \cos t + iy \sin t)^n e^{iht} dt$$

in x, y, z , following example given by Courant. Introducing polar coordinates $z = r \cos \theta, x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi$, we obtain

$$(25) \quad \begin{aligned} u &= 2r^n e^{ih\phi} \int_0^{\pi} (\cos \theta + i \sin \theta \cos t)^n \cos ht \, dt \\ &= r^n e^{ih\phi} P_{n,h}(\cos \theta), \end{aligned}$$

where $P_{n,h}(\cos \theta)$ are the associated Legendre functions.

On the other hand, according Tokaty^[8], Lagrange^[9] came to the conclusion that Euler's equations could be solved only for two specific conditions: (1) for potential (irrotational) flows, and (2) for non-potential (rotational) but steady flows. The external force in [9] is considered with potential, $f = \nabla V$, and the fluid is incompressible.

Lagrange also proved, as well as Laplace (*Mécanique Céleste*), Poisson (*Traité de Mécanique*), Cauchy (*Mémoire sur la Théorie des Ondes*) and Stokes (*On the Friction of Fluids in Motion and the Equilibrium and Motion of Elastic Solids*), that if the differential of the fluid's velocity $u_1 dx + u_2 dy + u_3 dz$ is a differential exact in some instant of time (for example, in $t = 0$) then it is also for all time ($t \geq 0$) of this movement on the same conditions. This means that a potential flow is always potential flow, since $t = 0$. Then, from the previous paragraph, if the initial velocity have not an exact differential (i.e., if the initial velocity is not a gradient function, irrotational, with potential) and the external force have potential then the Euler's equations have no solution in this case of incompressible and potential flows, for non-steady flows.

For steady flows, where $\frac{\partial u}{\partial t} \equiv 0$ and $u = u^0$ for all $t \geq 0$, the condition for existence of solution (obtaining the pressure) is that

$$(26) \quad \frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$$

for all pair (i, j) , $1 \leq i, j \leq 3$, defining

$$(27) \quad S_i = f_i - \sum_{j=1}^3 u_j^0 \frac{\partial u_i^0}{\partial x_j},$$

where $f \equiv f^0$ is the stationary external force. This is a common condition for existence of solution for a system $\nabla p = S$, representing the stationary Euler's equations, that is $\nabla \times S = 0$.

§ 6

I think that this is better than nothing... It is no longer true that the Euler equations do not have a general solution (when there is some).

Apply some of these methods to the Navier-Stokes equations and to the famous 6th Millennium Problem^[4] on existence and smoothness of the Navier-Stokes equations apparently is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon. On the other hand, apply these methods to the case $n = 2$ is almost immediate.

*To Leonard Euler, in memorian,
the greatest mathematician of all time.
He was brilliant, great intuitive genius.*

*Euler, and mathematical community,
forgive me for my mistakes...
This subject is very difficult!*

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