Solution for Euler Equations –
Lagrangian and Eulerian Descriptions

Valdir Monteiro dos Santos Godoi
valdir.msgodoi@gmail.com

Abstract – We find an exact solution for the system of Euler equations, following the description of the Lagrangian movement of an element of fluid, for spatial dimension n = 3. As we had seen in other previous articles, there are infinite solutions for pressure and velocity, given only the condition of initial velocity.

Keywords – Euler equations, exact solutions, equivalent systems, Lagrangian description, Eulerian description.

§

Essentially the Euler (and Navier-Stokes) equations relate to the velocity u and pressure p suffered for a volume element dV at position (x, y, z) and time t. In the formulation or description Eulerian the position (x, y, z) is fixed in time, running different volume elements of fluid in this same position, while the time varies. In the Lagrangian formulation the position (x, y, z) refers to the instantaneous position of a specific volume element $dV = dx\, dy\, dz$ at time t, and this position varies with the movement of this same element dV.

Basically, the deduction of the Euler equations is a classical mechanics problem, a problem of Newtonian mechanics, which use the 2nd law of Newton $F = ma$, force is equal to mass multiplied by acceleration. We all know that the force described in Newton's law may have different expressions, varying only in time or also with the position, or with the distance to the source, varying with the body's velocity, etc. Each specific problem must to define how the forces involved in the system are applied and what they mean. I suggest consulting the classic Landau & Lifshitz[1] or Prandtl book[2] for a more detailed description of the deduction of these equations (including Navier-Stokes equations).

In spatial dimension $n = 3$, the Euler equations can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(1)\qquad \begin{cases} \dfrac{\partial p}{\partial x} + \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x} + u_2 \dfrac{\partial u_1}{\partial y} + u_3 \dfrac{\partial u_1}{\partial z} = f_1 \\ \dfrac{\partial p}{\partial y} + \dfrac{\partial u_2}{\partial t} + u_1 \dfrac{\partial u_2}{\partial x} + u_2 \dfrac{\partial u_2}{\partial y} + u_3 \dfrac{\partial u_2}{\partial z} = f_2 \\ \dfrac{\partial p}{\partial z} + \dfrac{\partial u_3}{\partial t} + u_1 \dfrac{\partial u_3}{\partial x} + u_2 \dfrac{\partial u_3}{\partial y} + u_3 \dfrac{\partial u_3}{\partial z} = f_3 \end{cases}$$

where $u(x,y,z,t) = \big(u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t)\big)$, $u: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$, and $f(x,y,z,t) = \big(f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t)\big)$, $f: \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x,y,z) and at the instant of time t, for example, gravity force per mass unity, with $x,y,z,t \in \mathbb{R}$, $t \geq 0$. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

The non-linear terms $u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z}$, $1 \leq i \leq 3$, are a natural consequence of the Eulerian formulation of motion, and corresponds to part of the total derivative of velocity with respect to time of a volume element dV in the fluid, i.e., its acceleration. If $u = (u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$ and these x, y, z also vary in time, then, by the chain rule,

(2) $$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Defining $\frac{dx}{dt} = u_1$, $\frac{dy}{dt} = u_2$, $\frac{dz}{dt} = u_3$, comes

(3) $$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u_1 + \frac{\partial u}{\partial y}u_2 + \frac{\partial u}{\partial z}u_3,$$

and therefore

(4) $$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_1\frac{\partial u_i}{\partial x} + u_2\frac{\partial u_i}{\partial y} + u_3\frac{\partial u_i}{\partial z}, 1 \leq i \leq 3,$$

which contain the non-linear terms that appear in (1).

Numerically, searching a computational result, i.e., in practical terms, there can be no difference between the Eulerian and Lagrangian formulations for the evaluation of $\frac{Du}{Dt}$ (or $\frac{du}{dt}$, it is the same physical and mathematical entity). Only conceptually and formally there is difference in the two approaches. I agree, however, that you first consider (x,y,z) variable in time (Lagrangian formulation) and then consider (x,y,z) fixed (Eulerian formulation), seems to be subject to criticism. As in the Newton equations of motion we can consider that there are forces traveling with a body, and also there are forces that may be fixed in the each space position, in a same configuration or system model, in our present formulation the pressure, and its corresponding gradient, they do not travel with the volume element $dV = dx\,dy\,dz$, i.e., obeys to the Eulerian description of motion, and the external force f can do it or not, depending on the definition of the problem. The velocity u, however, will obey to the Lagrangian description, and it is representing the velocity of a generic volume element dV over time, initially at position (x_0, y_0, z_0) and with initial velocity $u^0 = u(0) = cte., u = u(t)$.

Following this definition, the system (1) above is transformed into

$$(5) \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{Du_1}{Dt} = f_1 \\ \frac{\partial p}{\partial y} + \frac{Du_2}{Dt} = f_2 \\ \frac{\partial p}{\partial z} + \frac{Du_3}{Dt} = f_3 \end{cases}$$

thus (1) and (5) are equivalent systems, according (4) validity.

The system (5) always has a solution if the external force f is a gradient function, for example, dependent only on the time variable, and the components velocity are $C^2([0,\infty))$ class.

Given $u = u(t) \in C^2([0,\infty))$ obeying the initial conditions, the system solution (5) is

$$(6) \qquad p = \int_L \left(f - \frac{Du}{Dt}\right) \cdot dl + \theta(t),$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z) and $\theta(t)$ is a generic time function, for example with $\theta(0) = 0$.

In special case when f is a constant or a function dependent only on the time variable, we come to

$$(7) \qquad p = p^0 + S_1(t)\,(x - x_0) + S_2(t)\,(y - y_0) + S_3(t)\,(z - z_0), \\ S_i(t) = f_i - \frac{Du_i}{Dt},$$

where p^0 is the pressure at point (x_0, y_0, z_0) in initial time $t = 0$. We can add to pressure given in (7) the generic time function $\theta(t), \theta(0) = 0$, assuming that $\theta(t)$ is a function on time physically and mathematically reasonable. Note that the variables x, y, z in (7) are in Eulerian description, i.e. are fixed coordinates in space while time t varies, while $\frac{Du}{Dt}$ it is in Lagrangian description, representing a motion over time.

Again we have seen that the system of Euler equations has no unique solution. We can choose different velocities that have the same initial velocity and also result, in general, in different pressures.

How to return to the Eulerian formulation obtained a solution in the Lagrangian formulation? This is very free. We can choose any convenient velocity $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ to calculate the pressure (7) that complies with the initial conditions, as we also can choose appropriate $x(t), y(t), z(t)$ and $u(x, y, z, t)$ to the positions and velocities of the system, calculating (4) and carrying the result in (7). This can save lots calculation time.

Apply these results to the Navier-Stokes Equations and to the famous 6th Millennium Problem[3] apparently is not so difficult at the same time also it is not absolutely trivial. It takes some time. I hope to do it soon, with dignity. On the other hand, apply these results to the case $n = 2$ is almost immediate.

To Leonard Euler, in memorian,
the greatest mathematician of all time.
Even without being 100% accurate for today's standards,
he was brilliant, great intuitive genius.

References

[1] Landau, Lev D. and Lifshitz, Evgenii M., *Fluid Mechanics*, Course of Theoretical Physical, vol. 6. New York: Elsevier Butterworth Heinemann (2004).
[2] Prandtl, L. and Tietjens, O. G., *Fundamentals of Hydro- and Aeromechanics*. New York: Dover Publications, Inc. (1957).
[3] Fefferman, Charles L., *Existence and Smoothness of the Navier-Stokes Equation*, in http://www.claymath.org/sites/default/files/navierstokes.pdf (2000).