Simple equation for the free-fall time in air

I. INTRODUCTION

One of the most popular topics of physics is the free fall. The first systematic studies was done by Galilei (who allegedly dropped various bodies off the Leaning Tower of Pisa) and Huygens [1]. It should be interesting for students to estimate and compare the free-fall time in vacuum and air. The free vertical fall with quadratic air resistance is well studied analytically [2 - 4, etc.] but the accurate equation for the free-fall time in air is cumbersome for use and interpretation. In this paper, the free-fall time in air is calculated using the new dimensionless number mentioned as the Galilei-Huygens number, a combination of the falling body mass and effective cross-section area, air density, and air drag coefficient. As shown, in most practical cases the free-fall time in air can be closely approximated as the product of the free-fall time in vacuum and a linear function of the Galilei-Huygens number. To illustrate the accuracy of this simple equation, the free-fall time is calculated for various spherical bodies (ping-pong and tennis balls, hailstones, basketball, and track-and-field men's shot) if dropped off the Leaning Tower of Pisa.

II. ACCURATE AND APPROXIMATE SOLUTIONS FOR FREE_FALL IN AIR

The mathematical model of the free fall is described by the equation

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$$M\frac{dV}{dt} = Mg - \frac{1}{2}C_d\,\rho_0\,S\,V^2 \tag{1}$$

with the initial conditions $V(0) = 0$, $X(0) = 0$ at time $t = 0$ when the body starts freely falling down from the altitude H. Here, $V(t) = \frac{dX}{dt}$ and $X(t)$ are the instantaneous velocity and displacement of the falling body at time t; M, S, and C_d are respectively the mass, effective cross-section area, and the drag coefficient (a dimensionless number depending on the geometry and velocity of the falling body); ρ_0 is the air density; g is the acceleration of gravity.

The free-fall time in air is accurately calculated by Eq. (A5) derived in APPENDIX as the product of two factors: the free-fall time in vacuum

$$t_v = \sqrt{\frac{2H}{g}}. \tag{2}$$

and a function of the dimensionless parameter

$$G_H = \frac{C_d \rho_0 S H}{M} \tag{3}$$

defined here as the Galilei-Huygens number ((in favor of Galileo Galilei and Christian Huygens for their important inputs in the physics of free fall [1]). The parameter G_H can be also interpreted as the ratio of the air drag resistance force $C_d\rho_0 S g H$ calculated for the final velocity $V = \sqrt{2gH}$ of the freefall in vacuum, to the gravity force Mg.

Since Eq. (A2) is cumbersome for use and interpretation, its approximation is derived in APPENDIX as

$$t_a \approx t_v \left(1 + \frac{G_H}{12}\right), \tag{4}$$

As follows from Eq. (4), the higher the Galilei-Huygens number, the bigger the ratio of the free-fall times in air and vacuum. It is noteworthy that Eq. (4) can be transformed to a more cumbersome equation derived earlier [3]. But Eq. (4) is more convenient for use and interpretation, its application limits are better defined, and the derivation method is simpler.

The accurate and linear approximation solutions described by Eqs (A5) and (4) are graphically compared in FIG. 1. As seen, both plots are about similar for $G_H \leq 10$ and are close at $G_H \leq 15$.

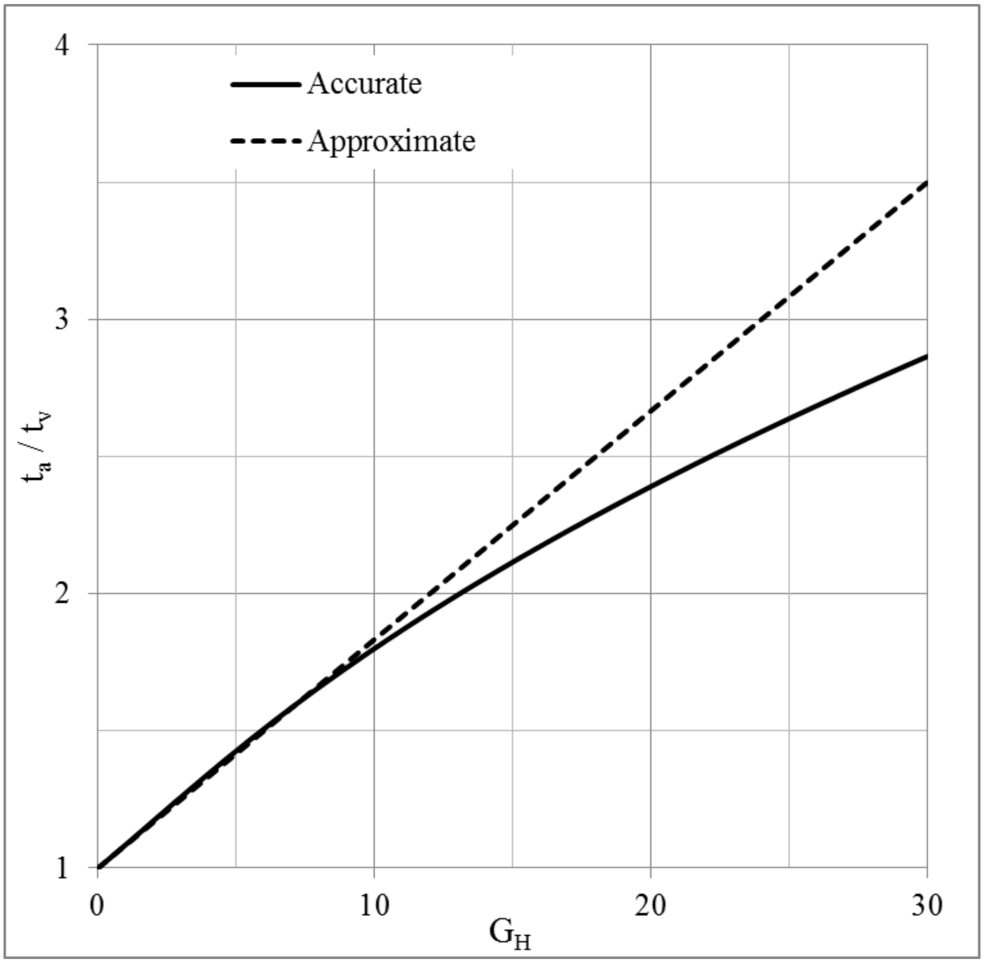

FIG. 1. The dimensionless free-fall time in air calculated using the accurate equation (A5) and approximate equations (4).

III. CALCULATION EXAMPLES

For spherical bodies, the drag coefficient $C_D \approx 0.4$ [1-3], the area $S = \pi R^2$, and the mass $M = 4\pi \rho R^3/3$, so, Eq. (3) can be reduced to the form

$$G_H = 0.4 \frac{\rho_0}{\rho} \frac{H}{R}. \tag{5}$$

where R is the radius and ρ is the average density of the body, kg/m^3. The trends described by Eq. (5) are plotted in FIG. 2 for the uniform balls made of steel, aluminum, ice, and wood (here,

$\rho = 7800$, 2700, 900, and 500 kg/m^3, respectively; $\rho_a = 1.25$ kg/m^3). As seen, in most practical cases parameter $G_H << 10$. But for hollow and/or small bodies, the G_H values can be relatively high and the free-fall time in air may notably exceed that in vacuum.

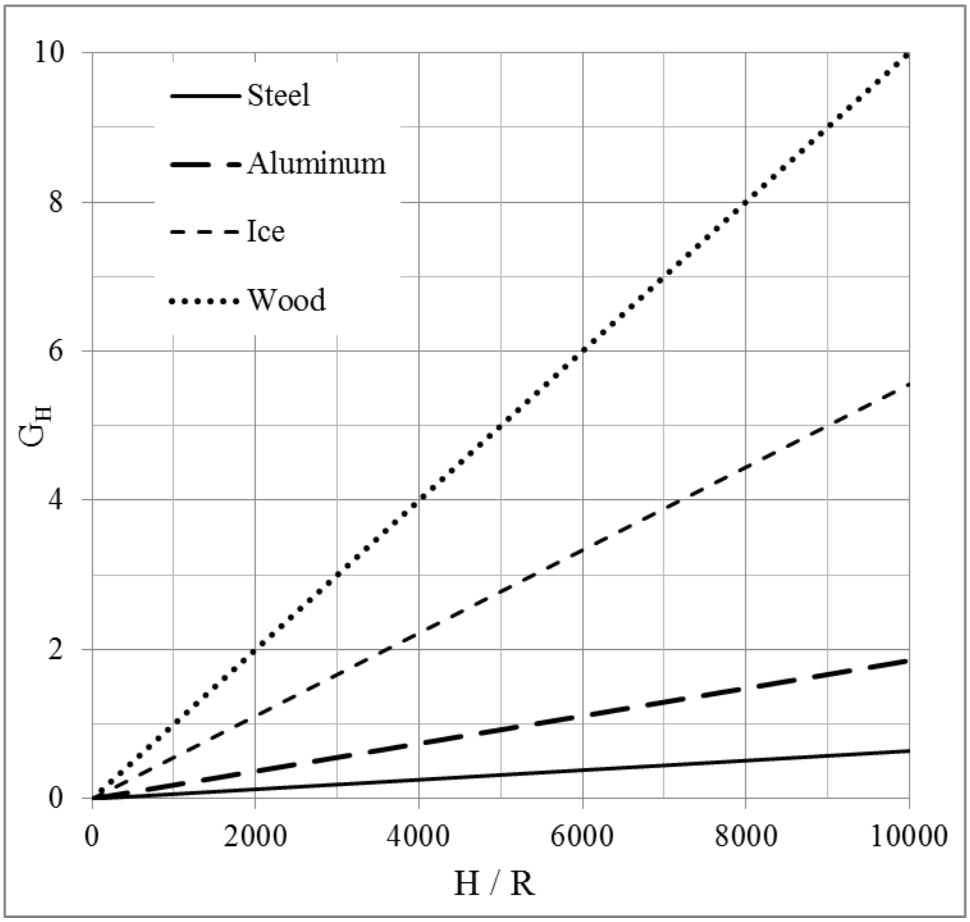

FIG. 2. Dimensionless numbers G_H vs. the dimensionless ratio H/R for uniform spherical bodies made of various materials.

Some calculation examples for the precise and approximate equations are compared in Table 1. As seen, the accuracy of Eq. (5) is sufficient to estimate the effect of the air drag on the free-fall time for various bodies if dropped off the Leaning Tower of Pisa, the alleged place of Galilei's experiments with falling bodies. Generally, the human visual resolution is good enough to distinguish the free-fall times for men's shot (track and field) and basketball since the estimated difference is 0.7 s and the mean human reaction time is ≈0.2 s. For the ping-pong balls or common hailstones, such a difference is even more significant.

Table 1. Free-fall times calculated with the Eqs (A5) and (4) for various bodies dropped off the Leaning Tower of Pisa (≈50 m high).

Falling body	Diam., m	Mass, kg	Average density, kg/m^3	G_H	Free-fall time, s	
					Accurate Eq. (13)	Approximate Eq. (14)
Ping-pong ball	0.038	0.0025	87	15.1	6.8	7.2
Hailstone	0.005	-	900	11.1	6.0	6.1
Hailstone	0.010	-	900	5.6	4.7	4.7
Tennis ball	0.067	0.058	368	2.0	3.7	3.7
Basketball	0.245	0.610	79	2.6	3.9	3.9
Track-and-field men's shot	0.120	7.260	8024	0.1	3.2	3.2
Any free-fall in vacuum	-	-	-	0	3.2	3.2

Notes: The results were rounded to the accuracy of 0.1 s which is considered as the best human reaction time.

IV. CONCLUSIONS

A close-form relationship for the free-fall time in air was derived as a function of two factors: the free-fall time in vacuum and dimensionless parameter G_H (mentioned as the Galilei-Huygens number in favor of Galileo Galilei and Christian Huygens for their important inputs in the physics of free fall). This parameter may be interpreted as the ratio of the air drag resistance force, calculated for the final velocity of the freefall in vacuum, to the gravity force. For most practical cases, the relationship is reduced to a quite simple form: the product of the free-fall time in vacuum and a linear function of the parameter G_H. The accuracy and simplicity of the approximate equation are illustrated for various spherical bodies (ping-pong and tennis balls, hailstones, basketball, and track-and-field men's shot) if they were dropped off the Leaning Tower of Pisa. The results are clear and traceable and can be of educational value and interest for physics teachers and students.

1. S. Ducheyne, Steffen "Galileo and Huygens on free fall: Mathematical and methodological differences". *Dynamis : Acta Hispanica ad Medicinae Scientiarumque. Historiam Illustrandam* **28**. pp. 243–274 (2008).
2. J. Lindemuth, "The effect of air resitance on falling balls", Am.J.Phys. **39**, 757 – 759 (1971).
3. P. Timmerman and J.P. van der Weele, "On the rise and fall of a ball with linear or quadratic drag", Am.J.Phys. **67** (6), 538-546 (1998).
4. C.F. Bohren, "Dimensionless analysis, falling bodies, and the fine art of not solving differential equations", Am. J. Phys. 72 (4), 534-537 (2004).
5. H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, Forth Edition (The Macmillan Company, NY, 1961).
6. E. Kamke, *Differentialgleichungen: Losungsmethoden und Losungen, I, Gewohnliche Differentialgleichungen* (B. G. Teubner, Leipzig, 1977).

APPENDIX

Accurate solution

To reduce the number of the parameters in Eq. (1), introduce the dimensionless variables

$$\xi = \frac{X}{H}, \quad \tau = \frac{t}{t_v} \qquad \text{(A1)}$$

where the free-fall time in vacuum is given by Eq. (2). Substituting Eqs (A1) into Eq. (1), obtain equation

$$\frac{d\upsilon}{dt} = 2 - \frac{1}{2} G_H \upsilon^2 \tag{A2}$$

for the dimensionless velocity $\upsilon(\tau) = \frac{d\xi}{d\tau}$. Here, the dimensionless parameter G_H is defined by Eq. (3). As known [6], the solution of Eq. (4) with the initial condition $\upsilon(0) = 0$ is given by

$$\upsilon(\tau) = \frac{d\xi}{d\tau} = \frac{2\tanh\left(\sqrt{G_H}\,\tau\right)}{\sqrt{G_H}} \tag{A3}$$

for $G_H > 0$. Integrating Eq. (A3) with the initial condition $\xi(0) = 0$, obtain

$$\xi(\tau) = \frac{2}{G_H} \ln\left[\cosh\left(\sqrt{G_H}\,\tau\right)\right] \tag{A4}$$

The dimensionless free-fall time in air is calculated from Eq. (A4) at the condition $\xi = 1$ (that is, for $X = H$):

$$\tau_a = \frac{t_a}{t_v} = \frac{1}{\sqrt{G_H}} \operatorname{acosh}\left[\exp\left(\frac{G_H}{2}\right)\right]. \tag{A5}$$

As can be shown, the dimensionless time $\tau_a \to 1$ and therefore fits the free-fall time in vacuum if $G_H \to 0$.

Approximate solution

Eq. (A5) can be rewritten as

$$\cosh\left(\sqrt{G_H}\,\tau_a\right) = \exp\left(G_H/2\right)$$

and simplified to the approximate polynomial equation using the Taylor-Maclaurin series expansions: $\cosh(x) = 1 + x^2/2 + x^4/24 + ...$ and $\exp(x) = 1 + x + x^2/2 + ...$. Considering

$G_H << 1$ and neglecting the terms of the second and higher order to the parameter G_H, obtain the biquadratic equation with the unknown variable τ_a:

$$\frac{1}{12} G_H \tau_a^4 + \tau_a^2 - 1 - \frac{G_H}{4} = 0 \qquad \text{(A6)}$$

Consider $\tau_a = 1 + \varepsilon$ where $\varepsilon = \alpha G_H << 1$, so, $\tau_a^4 \approx 1 + 4\varepsilon$ and $\tau_a^2 \approx 1 + 2\varepsilon$. Substituting such approximate relationships into Eq. (A6) and neglecting the terms of the second order to the parameter G_H, obtain $\alpha \approx 1/12$ and therefore Eq (4).