The density of primes

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Abstract

The prime numbers has very irregular pattern. The problem of finding pattern in the prime numbers is the long-open problem in mathematics. In this paper, we try to solve the problem axiomatically. And we propose some natural properties of prime numbers.

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1 Introduction

In 1859, Riemann [Rie59] showed a deep connection between non-trivial zeros of the Riemann zeta-function and the prime numbers. Our goal is to axiomatize the structure of primes.

2 The pattern of numbers

These below are some patterns of number.

Let t_n denote the nth triangular number. Then

$$t_n = \binom{n+1}{2} \qquad n \geq 1,$$

where $\binom{n+1}{2}$ is the binomial coefficients [Bur02, p. 15].

Let F_n be the nth Fibonacci number. Then

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}},$$

where n is a positive integer [Wei03, p. 1042].

Let B_n be the nth Bernoulli number. Then

$$B_n = (-1)^{n+1} n\zeta(1-n),$$

where $\zeta(1-n)$ is the Riemann zeta-function [Wei03, p. 189].

If $p(n)$ denotes the total number of partitions of n, then

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}},$$

where n is a positive integer [Har99, p. 116].

Postulate 2.1 (Peano Postulates)**.** Given the number 0, the set $\mathbf{N}$, and the function σ. Then:

1. $0 \in \mathbf{N}$.
2. $\sigma : \mathbf{N} \to \mathbf{N}$ is a function from $\mathbf{N}$ to $\mathbf{N}$.
3. $0 \notin \text{range}(\sigma)$.
4. The function σ is one-to-one.
5. If $I \subset \mathbf{N}$ such that $0 \in I$ and $\sigma(n) \in I$ whenever $n \in I$, then $I = \mathbf{N}$.

We define $1 = \sigma(0)$, $2 = \sigma(1)$, $3 = \sigma(2)$, etc. [Eis96, p. A64].

3 The axiomatic approach

In this section, we try to solve the problem of prime number axiomatically. We propose the basic characteristics of prime numbers. The prime numbers stands on several basic assumptions. Given the prime numbers p, q and the set $\mathbf{N}$. Then

Postulate 3.1. *$2 \leq p$.*

Postulate 3.1 says that 2 is the least prime number. 2 is the only one even prime number. 2 is also assumed as a generator of primes.

Postulate 3.2. *$4 \nmid p$.*

Postulate 3.2 holds generally for any prime number. Postulate 3.2 expresses the indivisibility of primes over the number 4. The number 4 is the least composite number which cannot divide any prime number.

Definition 3.3. For an integer $n > 1$, where $\tau(n)$ denote the number of positive divisors of n. The function $\chi(n)$ is defined by

$$\chi(n) = \begin{cases} 0 & \text{if } \tau(n) = 2 \\ 1 & \text{if } \tau(n) > 2. \end{cases}$$

Postulate 3.4. $(-1)^{\chi(p)} = 1$.

Postulate 3.4 shows the connection between -1 and the prime number p. For a composite number k, then $(-1)^{\chi(k)} = -1$. Hence, $(-1)^{\chi(n)} = \pm 1$ for $n \geq 2$.

$\sigma(n)$ denotes the sum of positive divisors of n. Then

Postulate 3.5. $3 \leq \sigma(p)$.

Postulate 3.5 expresses that 3 is the lower bound of $\sigma(p)$. Postulate 3.1 can deduce Postulate 3.5. Given $2 \leq p$. Then $2+1 \leq p+1$ implies $3 \leq \sigma(p)$. Postulate 3.5 is the one of basic patterns of the function $\sigma(n)$.

Definition 3.6. Given an integer $n > 1$, let $\Delta(n)$ denote the number of positive divisors of n besides 1 and n.

Postulate 3.7. $\Delta(p) = 0$.

Postulate 3.7 shows that the zeros of such a function are any prime number.

Postulate 3.8. If $p, q \in \mathbf{N}$, then $pq \in \mathbf{N}$.

Postulate 3.8 explains the closure property of multiplication over the prime numbers p and q. By using induction, Postulate 3.8 can imply the prime factorization of a positive integer. And Postulate 3.8 shows the existence of semiprimes.

By our observation, we get the estimation. Let p_n be the nth prime, where n is a positive integer. Then

$$\frac{p_{n+1}}{p_n} \leq 1.7.$$

We have tested for several positive integers of n. The result can be seen in the table below.

n	p_{n+1}/p_n
1	1.500
2	1.667
3	1.400
4	1.571
5	1.182
6	1.308
7	1.118
8	1.211
9	1.261
10	1.069

Table 1: The ratio of p_{n+1}/p_n

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