Optimisation of dynamical systems subject to meta-rules

Chris Goddard

July 1, 2013
Outline

The Basic Problem

Jet bundles

Geometry

Optimisation

Concluding remarks
Outline

The Basic Problem

Jet bundles

Geometry

Optimisation

Concluding remarks
Dynamical systems with metarules

- Suppose we have a simple dynamical system, e.g., a Morse function on a torus.
Dynamical systems with metarules

- Suppose we have a simple dynamical system, e.g., a Morse function on a torus.
- But suppose that it is not so simple. Suppose the shape of the system depends on the location in the system that we are currently at.
Dynamical systems with metarules

- Suppose we have a simple dynamical system, eg a Morse function on a torus.
- But suppose that it is not so simple. Suppose the shape of the system depends on the location in the system that we are currently at.
- So if the current state of the system is at the top of the torus, and we were to draw a trajectory from this point, we would expect suddenly the shape of the torus to change.
Dynamical systems with metarules

Why is this a useful way of modelling a real dynamical system?

- Because in reality the way a system changes depends on the direction a system is pushed from one state to another. The system is not static, but depends on the trajectories that are traced through it.
Dynamical systems with metarules

Why is this a useful way of modelling a real dynamical system?

- Because in reality the way a system changes depends on the direction a system is pushed from one state to another. The system is not static, but depends on the trajectories that are traced through it.

- In practice, this means that if we were to consider a system holistically, and consider a unique choice of initial tangent vector from each point - a vector field - in parameter space (ignoring situations where such is forbidden, since I am assuming Lorentzian geometry), then we would like to measure how a system would evolve / change in structure in a natural way, given that initial choice, or "push" in parameter space.
More primitively, consider the idea of a Markov process. One has a set of states, with transition probabilities between them. One can characterise this with a transition matrix.
More primitively, consider the idea of a Markov process. One has a set of states, with transition probabilities between them. One can characterise this with a transition matrix.

But suppose now that we wish to consider a set of transition matrices, and transition probabilities between these, which depend on the last state and the current state. In other words, a "meta-Markov" process. Then this is closer to the general idea I am trying to aim at.
More primitively, consider the idea of a Markov process. One has a set of states, with transition probabilities between them. One can characterise this with a transition matrix.

But suppose now that we wish to consider a set of transition matrices, and transition probabilities between these, which depend on the last state and the current state. In other words, a "meta-Markov" process. Then this is closer to the general idea I am trying to aim at.

We are now ready to ask the central question.
Central Question

Given a meta-dynamical system as loosely defined above, how can one describe the geometry of the associated object?

- If we can describe the geometry, we can compute geodesics (avoidance of tipping points).
Central Question

Given a meta-dynamical system as loosely defined above, how can one describe the geometry of the associated object?

- If we can describe the geometry, we can compute geodesics (avoidance of tipping points).
- If we can describe the geometry, it suggests ways that the system can be understood.
Central Question

Given a meta-dynamical system as loosely defined above, how can one describe the geometry of the associated object?

▶ If we can describe the geometry, we can compute geodesics (avoidance of tipping points).
▶ If we can describe the geometry, it suggests ways that the system can be understood.
▶ If we can describe the geometry, it suggests ways that the system can be controlled.
Outline

The Basic Problem

Jet bundles

Geometry

Optimisation

Concluding remarks
The first jet bundle

- The tangent bundle to $M$ is given by tuples $(p, v)$, where $v$ is an element of $T_pM$. 

- The tangent space to the tangent space gives $T(TM)$, given by tuples $(p, v, w)$. 

- Iterating this process a countably infinite number of times, we obtain the first jet bundle $J^1M$, given by tuples $(p, V)$, where $V$ is an infinite matrix. 

- In practice, however, $V$ is of rank $\dim(M)$. 
The first jet bundle

- The tangent bundle to $M$ is given by tuples $(p, v)$, where $v$ is an element of $T_p M$.
- The tangent space to the tangent space gives $T_{(2)} M$, given by tuples $(p, v, w)$.
The first jet bundle

- The tangent bundle to $M$ is given by tuples $(p, v)$, where $v$ is an element of $T_pM$.
- The tangent space to the tangent space gives $T_{(2)}M$, given by tuples $(p, v, w)$.
- Iterating this process a countably infinite number of times, we obtain the first jet bundle $J^1M$, given by tuples $(p, V)$, where $V$ is an infinite matrix.
The first jet bundle

- The tangent bundle to $M$ is given by tuples $(p, v)$, where $v$ is an element of $T_p M$.
- The tangent space to the tangent space gives $T_{(2)} M$, given by tuples $(p, v, w)$.
- Iterating this process a countably infinite number of times, we obtain the first jet bundle $\mathcal{J}M$, given by tuples $(p, V)$, where $V$ is an infinite matrix.
- In practice, however, $V$ is of rank $\text{dim}(M)$. 
Elements of the jet bundle associated to trajectories

Suppose now we have two points, \( p \) and \( q \) in our parameter space \( M \).

- Consider the set of index preserving diffeomorphisms \( \text{Aut}(M) \) on \( M \). This will have a basis given by \( \{ f_{ij} : x_i \mapsto x_j \} \).
Suppose now we have two points, \( p \) and \( q \) in our parameter space \( M \).

- Consider the set of index preserving diffeomorphisms \( \text{Aut}(M) \) on \( M \). This will have a basis given by \( \{f_{ij} : x_i \mapsto x_j\} \).
- Consider a trajectory \( \gamma \) joining \( p \) and \( q \) in \( M \).
Elements of the jet bundle associated to trajectories

Suppose now we have two points, \( p \) and \( q \) in our parameter space \( M \).

- Consider the set of index preserving diffeomorphisms \( \text{Aut}(M) \) on \( M \). This will have a basis given by \( \{ f_{ij} : x_i \mapsto x_j \} \).
- Consider a trajectory \( \gamma \) joining \( p \) and \( q \) in \( M \).
- Then relative to any point \( \gamma(t) \) we have a vector pointing in the direction of the perturbation of the point relative to the \( ij \)th element of \( \text{Aut}(M) \) at \( \gamma(t) \).
Elements of the jet bundle associated to trajectories

Suppose now we have two points, \( p \) and \( q \) in our parameter space \( M \).

- Consider the set of index preserving diffeomorphisms \( Aut(M) \) on \( M \). This will have a basis given by \( \{ f_{ij} : x_i \mapsto x_j \} \).
- Consider a trajectory \( \gamma \) joining \( p \) and \( q \) in \( M \).
- Then relative to any point \( \gamma(t) \) we have a vector pointing in the direction of the perturbation of the point relative to the \( ij \)th element of \( Aut(M) \) at \( \gamma(t) \).
- This gives us a matrix of tangents (relative to these perturbations of \( \gamma \)), or an element of the first jet bundle, associated to each point of the path \( \gamma \).
Meta-markov processes again

I claim that to specify the structure associated to the first jet bundle, we need a 6-tensor $\kappa$.

- Consider again meta rules for a markov process. Note that $GL(n)$ as a matrix group has tangent group $GL(n)$. 
Meta-markov processes again

I claim that to specify the structure associated to the first jet bundle, we need a 6-tensor $\kappa$.

- Consider again meta rules for a markov process. Note that $GL(n)$ as a matrix group has tangent group $GL(n)$.
- Then if $T_{ij}$ is a unit transition probability, and $U_{kl}$, $V_{mn}$ are unit tangent probabilities sitting in the tangent group $GL(n)$, we have that $\kappa_{ijklmn}$ determines the result of acting on $T_{ij}$ with $U_{kl}$ "on the left" and $V_{mn}$ "on the right". It is the "meta-rule transition to transition probability".
Meta-markov processes again

I claim that to specify the structure associated to the first jet bundle, we need a 6-tensor $\kappa$.

- Consider again meta rules for a markov process. Note that $GL(n)$ as a matrix group has tangent group $GL(n)$.
- Then if $T_{ij}$ is a unit transition probability, and $U_{kl}, V_{mn}$ are unit tangent probabilities sitting in the tangent group $GL(n)$, we have that $\kappa_{ijklmn}$ determines the result of acting on $T_{ij}$ with $U_{kl}$ "on the left" and $V_{mn}$ "on the right". It is the "meta-rule transition to transition probability".
- The analogy for left and right action is that a left action occurs subsequent to the state - it is where the trajectory is moving to, and a right action occurs prior - it is where the trajectory is moving from.
Outline

The Basic Problem

Jet bundles

Geometry

Optimisation

Concluding remarks
As in Riemannian geometry, we have structural coefficients given by

\[ \Gamma^p_{ijklmn} = \langle \partial_p E_{ij}, E_{kl}, \partial_q E_{mn} \rangle_\kappa \]

where \( \{ E_{ij} \} \) forms a basis for the (first) jet bundle of the space.
As in Riemannian geometry, we have structural coefficients given by

\[ \Gamma_{ijklmn}^{pq} = \langle \partial_p E_{ij}, E_{kl}, \partial_q E_{mn} \rangle_{\kappa} \]

where \( \{ E_{ij} \} \) forms a basis for the (first) jet bundle of the space.

These can be computed as

\[ \Gamma_{ijklmn}^{pq} = \kappa_{ijk}^{abc}(\sum_{g \in C_8 \otimes C_7} \{ g \cdot \partial_p \partial_q \kappa_{abclmn} \}) \]

where summation is over the group product \( C_8 \otimes C_7 \) acting on the indices of \( \partial_p \partial_q \kappa_{abclmn} \).
Geodesics

- $\gamma$ is geodesic with respect to $\kappa$ if

$$\nabla_{(X_{ij}, \kappa)} X_{kl} = 0$$

where $X_{ij} : [0, 1] \rightarrow JM$ is the one parameter jet field associated to $\gamma$. 
Geodesics

- $\gamma$ is geodesic with respect to $\kappa$ if
  \[ \nabla_{(X_{ij}, \kappa)} X_{kl} = 0 \]
  where $X_{ij} : [0, 1] \rightarrow J M$ is the one parameter jet field associated to $\gamma$.

- $\nabla_{(X, \kappa)}$ is the affine connection with respect to $\kappa$, uniquely determined by
  \[ \partial_{ij} \langle \ddot{X}, \ddot{Y}, \ddot{Z} \rangle_{\kappa} = \langle \partial_{ij} \ddot{X}, \ddot{Y}, \ddot{Z} \rangle + \langle \ddot{X}, \partial_{ij} \ddot{Y}, \ddot{Z} \rangle + \langle \ddot{X}, \ddot{Y}, \partial_{ij} \ddot{Z} \rangle \]
The cybernetic information functional

We wish to know what choice of $\kappa$ is most natural, ie how a "physical" system will place constraints on allowable behaviour for $\kappa$.

Define $Cyb(M) := \{(\mathcal{J}M)^3 \to \mathcal{J}M\}$ as the space of left and right actions on the first jet bundle of $M$.

- We have an information functional given by

$$I := \int_M \int_{Cyb_m(M)} f(\partial_{ij}\partial_k \log f)^3 dmdV$$

where $f = f(m, V) = \delta(\kappa(m) - V)$, with $m \in M$ a point in parameter space and $V \in Cyb_m(M)$ is a point in the space of meta-rules at $m$. 
The cybernetic information functional

We wish to know what choice of $\kappa$ is most natural, i.e., how a "physical" system will place constraints on allowable behaviour for $\kappa$.

Define $Cyb(M) := \{ (\mathcal{J} M)^3 \to \mathcal{J} M \}$ as the space of left and right actions on the first jet bundle of $M$.

- We have an information functional given by

$$I := \int_M \int_{Cyb_m(M)} f(\partial_{ij} \partial_k \log f)^3 \, dm \, dV$$

where $f = f(m, V) = \delta(\kappa(m) - V)$, with $m \in M$ a point in parameter space and $V \in Cyb_m(M)$ is a point in the space of meta-rules at $m$.

- $\partial_{ij}$ is the derivative on function space. $\partial_k$ is the derivative on normal space.
The key result

- I conjecture that, after some considerable work, it can be demonstrated that this simplifies to

\[ \int_M \text{Inv}(\kappa)\,dm \]

where \( \text{Inv}(\kappa) \) is a geometric invariant defined by

\[ \text{Inv}(\kappa) := \kappa_{ijklmn} \Gamma_{ijabcdef} \Gamma_{klghpabc} \Gamma_{mndefghp} \]
The key result

- I conjecture that, after some considerable work, it can be demonstrated that this simplifies to

\[ \int_M \text{Inv}(\kappa) \, dm \]

where \( \text{Inv}(\kappa) \) in a geometric invariant defined by

\[
\text{Inv}(\kappa) := \kappa_{ijklmn} \Gamma_{ijabcdef} \Gamma_{klghpabc} \Gamma_{mndefghp}
\]

- This allows us to understand the geometric behaviour of a meta-dynamical system as \( \text{Inv}(\kappa) = 0 \), as a physical system will minimise the information associated to its relevant information functional.
Outline

The Basic Problem

Jet bundles

Geometry

Optimisation

Concluding remarks
Final comments

- In this talk I have indicated how one might go about modelling dynamical systems using meta-rule type considerations.
Final comments

- In this talk I have indicated how one might go about modelling dynamical systems using meta-rule type considerations.
- This talk has been intended only as the starting point for a conversation on said matters.
Final comments

- In this talk I have indicated how one might go about modelling dynamical systems using meta-rule type considerations.
- This talk has been intended only as the starting point for a conversation on said matters.
- Naturally a great deal of work remains to be done.
Questions