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Modified PCR Rules of Combination with Degrees of Intersections

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***Abstract*—In this paper, we propose a modification of PCR5 and PCR6 fusion rules with degrees of intersections for taking into account the cardinality of focal elements of each source of evidence to combine. We show in very simple examples the interest of these new fusion rules w.r.t. classical Dempster-Shafer, PCR6, Zhang's and Jaccard's Center rules of combination.**

Keywords: Information fusion, belief functions, DSmT, PCR6, degrees of intersection.

I. Introduction

In this paper, we propose modifications of the Proportional Conflict Redistribution rule no. 6 (PCR6) [2] (Vol. 3) for the combination of basic belief assignments (BBA's) which integrate the degrees of intersections of focal elements of each source of evidence to combine. Because we consider two possible definitions of degrees of intersections (i.e. Zhang's and Jaccard's degrees) and also two normalization methods (simplest and sophisticate), we propose four modified versions of PCR6 rules[1]. After a brief presentation of classical rules of combination and a detailed presentation of our modified PCR6 rules, we evaluate and compare their behaviors in different emblematic examples to guide the choice of the most interesting one.

II. Belief functions and classical fusion rules

Belief functions have been introduced by Shafer in 1976 from Dempster's works [1] in Dempster-Shafer's theory (DST) of evidence. DST is mainly characterized by a frame of discernment (FoD), sources of evidence represented by basic belief assignment (BBA), belief (Bel) and plausibility (Pl) functions, and Dempster's rule of combination, denoted as DS rule in the sequel[2] of combination. DST has been modified and extended into Dezert-Smarandache theory [2] (DSmT) to work with quantitative or qualitative BBA and to combine the sources of evidence in a more efficient way thanks to new proportional conflict redistribution (PCR) fusion rules – see [3]–[6] for discussion and examples.

More precisely, let's consider a finite discrete FoD $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$, with $n > 1$, of the fusion problem under consideration and its fusion space G^Θ which can be chosen either as the power-set 2^Θ, the hyper-power set[3] D^Θ, or the super-power set S^Θ depending on the model that fits with the problem [2]. A BBA associated with a given source of evidence is defined as the mapping $m(.) : G^\Theta \rightarrow [0,1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A\in G^\Theta} m(A) = 1$. The quantity $m(A)$ is called mass of belief of A committed by the source of evidence. Belief and plausibility functions are defined by

$$\text{Bel}(A) = \sum_{\substack{B\subseteq A\\ B\in G^\Theta}} m(B) \quad \text{and} \quad \text{Pl}(A) = \sum_{\substack{B\cap A\neq\emptyset\\ B\in G^\Theta}} m(B) \qquad (1)$$

If for some $A \in G^\Theta$, $m(A) > 0$ then A is called a focal element of the BBA $m(.)$. When all focal elements are singletons and $G^\Theta = 2^\Theta$ then the BBA $m(.)$ is called a Bayesian BBA [1] and its corresponding belief function $Bel(.)$ is homogeneous to a (possibly subjective) probability measure, and one has $Bel(A) = P(A) = Pl(A)$, otherwise in general one has $Bel(A) \leq P(A) \leq Pl(A)$, $\forall A \in G^\Theta$. The vacuous BBA , or VBBA for short, representing a totally ignorant source is defined as $m_v(I_t) = 1$, where the total ignorance defined as $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ if the FoD is $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$. Since in Shafer's book [1], the total ignorance I_t is also denoted Θ, we will adopt this notation in the sequel.

Many rules have been proposed in the literature over the decades (see [2], Vol. 2 for a detailed list of fusion rules) to combine several distinct sources of evidence represented by the BBA's $m_1(.)$, $m_2(.)$, …, $m_s(.)$ $(s \geq 2)$ defined on same fusion space G^Θ. In DST, the combination of $s \geq 2$ BBA's is traditionally accomplished with Dempster-Shafer (DS) rule [1] defined by $m^{DS}_{1,\dots,s}(\emptyset) = 0$ and for all $X \neq \emptyset$ in 2^Θ

$$m^{DS}_{1,\dots,s}(X) \triangleq \frac{1}{1-m_{1,\dots,s}(\emptyset)} \sum_{\substack{X_1,\dots,X_s\in 2^\Theta\\ X_1\cap\ldots\cap X_s=X}} \prod_{i=1}^{s} m_i(X_i) \qquad (2)$$

where the numerator of (2) is the mass of belief on the conjunctive consensus on X. The denominator $1-m_{1,\dots,s}(\emptyset)$ is a

[1]The methodology proposed in this paper is general and can also be applied to modify similarly other PCR rules. Since we consider PCR6 rule the most efficient one [6], we focus our presentation on PCR6 only

[2]DS acronym standing for *Dempster-Shafer* since Dempster's rule has been widely promoted by Shafer in the development of his mathematical theory of evidence.

[3]which corresponds to a Dedekind's lattice, see [2] Vol. 1.

normalization constant. The total degree of conflict $m_{1,\ldots,s}(\emptyset)$ between the s sources of evidences is defined by

$$m_{1,\ldots,s}(\emptyset) \triangleq \sum_{\substack{X_1,\ldots,X_s\in 2^\Theta\\ X_1\cap\ldots\cap X_s=\emptyset}} \prod_{i=1}^{s} m_i(X_i) \tag{3}$$

DS rule is associative and commutative and preserves the neutrality of the VBBA. s sources of evidence are said in total conflict if $m_{1,\ldots,s}(\emptyset)=1$. In this case the combination of the sources by DS rule cannot be done because of the mathematical 0/0 indeterminacy in (2). In DS rule, $m_{1,\ldots,s}(\emptyset)$ is redistributed to all focal elements of the conjunctive operator only proportionally to their mass (i.e. without taking care of their cardinalities). So with DS rule and with combination of 2 BBA's, the product $m_1(X_1)m_2(X_2)$ is transferred to $X_1\cap X_2=X$ only, no matter how the ratio between the cardinality of X and $X_1\cup X_2$ varies. This DS principle of redistribution has been questioned by Zhang in [7] and Fixsen and Malher in [8] because it does not discriminate the case where $X_1\cup X_2$ is large but $X_1\cap X_2$ is small with respect to the case where $X_1\cup X_2$ is small but $X_1\cap X_2$ is large. To palliate this problem, Zhang proposed in 1994 a modified version of DS rule [7] including a measure of degree of intersection of focal elements. The general formula of this modified DS rule is defined by $m^D_{1,\ldots,s}(\emptyset)=0$ and for all $X\neq\emptyset$ in 2^Θ

$$m^D_{1,\ldots,s}(X) \triangleq \frac{1}{K^D_{1,\ldots,s}}\cdot \sum_{\substack{X_1,\ldots,X_s\in 2^\Theta\\ X_1\cap\ldots\cap X_s=X}} D(X_1,\ldots,X_s)\prod_{i=1}^{s} m_i(X_i) \tag{4}$$

where $D(X_1,\ldots,X_s)$ denotes a measure of the degree of intersection between the focal elements X_1, X_2, $\ldots X_s$, and where $K^D_{1,\ldots,s}$ is a normalization constant allowing to get $\sum_{X\in 2^\Theta} m^D_{1,\ldots,s}(X)=1$. Because the measure of degree of intersection $D(X_1,\ldots,X_s)$ can be defined in different ways, this yields to different versions of the modified DS rule above. In [7], Zhang suggested to define $D(X_1,\ldots,X_s)$ as

$$D^Z(X_1,\ldots,X_s) \triangleq \frac{|X_1\cap X_2\cap\ldots\cap X_s|}{|X_1|\cdot|X_2|\cdot\ldots\cdot|X_s|} \tag{5}$$

where $|X_1\cap X_2\cap\ldots\cap X_s|$ is the cardinality of the intersection of the focal elements X_1, X_2,..., X_s, and $|X_1|$, $|X_2|$, $\ldots|X_s|$ their cardinalities. Replacing $D(X_1,\ldots,X_s)$ by $D^Z(X_1,\ldots,X_s)$ in the formula (4) defines Zhang's Center Rule (ZCR) of combination [7], denoted $m^{ZCR}_{1,\ldots,s}(.)$ in the sequel. The normalization constant of ZCR is denoted $K^{ZCR}_{1,\ldots,s}$.

If we use Jaccard's index as measure of the degree of intersection [9] which is defined by

$$D^J(X_1,\ldots,X_s) \triangleq \frac{|X_1\cap X_2\cap\ldots\cap X_s|}{|X_1\cup X_2\cup\ldots\cup X_s|} \tag{6}$$

then we obtain Jaccard's center rule (JCR) of combination, and we denote it $m^{JCR}_{1,\ldots,s}(.)$, in replacing $D(X_1,\ldots,X_s)$ by $D^J(X_1,\ldots,X_s)$ in the formula (4). The normalization constant of JCR is denoted $K^{JCR}_{1,\ldots,s}$. ZCR and JCR rules are particular instances of Modified DS rule (MDS) proposed by Fixsen and Mahler in [8]. ZCR and JCR are commutative but not idempotent. It can be proved that Zhang's degree is associative that is $D^Z(X_1,X_2,\ldots,X_s)=D^Z(X_1,D^Z(X_2,\ldots,X_s))$, whereas Jaccard's degree is not associative. If one combines three (or more) BBA's and there is no conflicting mass, then ZCR is associative, whereas JCR is not associative. If there is conflicting masses, then ZCR is still associative, but JCR is not associative. Zhang's and Jaccard's degrees pose a problem because ZCR and JCR become strictly equivalent with DS rule when the cardinality is 1 for all relevant sets, or when $|X_1\cap X_2\cap\ldots\cap X_s|=|X_1|\cdot|X_2|\cdot\ldots\cdot|X_s|$ in the circumstance of conflicting evidence. Therefore, it inherits the same limitations as DS rule – see example 2 in Section V.

The doubts of the validity of DS rule has been discussed by Zadeh in 1979 [10]–[12] based on a very simple example with two highly conflicting sources of evidences. Since 1980's, many criticisms have been done about the behavior and the justification of such DS rule. More recently, Dezert et al. in [3], [4], [18] have put in light other counter-intuitive behaviors of DS rule even in low conflicting cases and showed serious flaws in logical foundations of DST [5]. To overcome the limitations and problems of DS rule of combination, a new family of PCR rules have been developed in DSmT framework [2]. In PCR rules, we transfer the conflicting mass only to the elements involved in the conflict and proportionally to their individual masses, so that the specificity of the information is entirely preserved. The general principle of PCR consists: 1) to apply the conjunctive rule, 2) to calculate the total or partial conflicting masses; 3) then redistribute the (total or partial) conflicting mass proportionally on non-empty sets according to the integrity constraints one has for the frame Θ. Because the proportional transfer can be done in different ways, there exist several versions of PCR rules of combination. PCR6 fusion rule has been proposed by Martin and Osswald in [2] Vol. 2, Chap. 2, as a serious alternative to PCR5 fusion rule proposed originally by Smarandache and Dezert in [2] Vol. 2, Chap. 1. When only two BBA's are combined, PCR6 and PCR5 fusion rules coincide, but they differ in general as soon as more than two sources have to be combined altogether. Recently, it has been proved in [6] that only PCR6 rule is consistent with the averaging fusion rule which allows to estimate the empirical (frequentist) probabilities involved in a discrete random experiment, and that is why we recommend to use it in applications when possible. For Shafer's model of FoD[4], the PCR6[5] combination of two BBA's $m_1(.)$ and $m_2(.)$ is defined by $m^{PCR5/6}_{1,2}(\emptyset)=0$ and for all $X\neq\emptyset$ in 2^Θ

$$m^{PCR5/6}_{1,2}(X) = \sum_{\substack{X_1,X_2\in 2^\Theta\\ X_1\cap X_2=X}} m_1(X_1)m_2(X_2) + \sum_{\substack{Y\in 2^\Theta\setminus\{X\}\\ X\cap Y=\emptyset}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X)+m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X)+m_1(Y)}\right] \tag{7}$$

[4]that is when $G^\Theta=2^\Theta$, and assuming all elements exhaustive and exclusive.

[5]which turns to be equal to PCR5 formula in case of fusion of two BBA's only.

where all denominators in (7) are different from zero. If a denominator is zero, that fraction is discarded. All propositions/sets are in a canonical form [2]. Basic MatLab codes of PCR rules can be found in [2], [13] or from the toolboxes repository on the web [14]. The general and concise formula of PCR6 rule for combining $s > 2$ sources of evidences is

$$m^{PCR6}_{1,2,\ldots,s}(X) = m_{1,2,\ldots,s}(X) + CR^{PCR6}(X) \tag{8}$$

where $m_{1,2,\ldots,s}(X)$ corresponds to the conjunctive consensus on X between s sources of evidence, which is defined by

$$m_{1,2,\ldots,s}(X) \triangleq \sum_{\substack{X_1,\ldots X_s \in 2^\Theta \\ X_1\cap\ldots\cap X_s = X}} \prod_{i=1}^{s} m_i(X_i) \tag{9}$$

and where $CR^{PCR6}(X)$ is the part of the conflicting masses redistributed back to the focal element X according to PCR6 redistribution principle, that is

$$CR^{PCR6}(X) \triangleq \sum_{k=1}^{s-1} \sum_{\substack{X_{i_1},X_{i_2},\ldots,X_{i_k}\in G^\Theta\setminus X \\ (\bigcap_{j=1}^{k} X_{i_j})\cap X=\emptyset}} \sum_{(i_1,i_2,\ldots,i_k)\in\mathcal{P}^s(\{1,\ldots,s\})} [m_{i_1}(X)+m_{i_2}(X)+\ldots+m_{i_k}(X)]\cdot \cdot \frac{m_{i_1}(X)\ldots m_{i_k}(X)m_{i_{k+1}}(X_{i_{k+1}})\ldots m_{i_s}(X_{i_s})}{m_{i_1}(X)+\ldots+m_{i_k}(X)+m_{i_{k+1}}(X_{i_{k+1}})+\ldots+m_{i_s}(X_{i_s})} \tag{10}$$

In Eq.(10), $\mathcal{P}^s(\{1,\ldots,s\})$ is the set of all permutations of the elements $\{1,2,\ldots,s\}$. It should be observed that X_{i_1}, X_{i_2},...,X_{i_s} may be different from each other, or some of them equal and others different, etc. As discussed and justified in [6], we focus here and in the sequel on PCR6 rule of combination rather than PCR5, but the general formula of PCR5 rule can be found in [2], [6] with examples, and a concise PCR5 general formula similar to (11) is possible. Like the averaging fusion rule, the PCR5 and PCR6 fusion rules are commutative but not associative.

III. PCR6 rule with degrees of intersection

As presented in the previous section, the original versions of PCR5 or PCR6 rules of combination (as well as original DS rule) use only part of the whole information available (i.e. the values of the masses of belief only), because they do not exploit the cardinalities of focal elements entering in the fusion process. Because the cardinalities of focal elements are fully taken into account in the computation of the measure of degree of intersection between sets, we propose to improve PCR rules using this measure. The basic idea is to replace any conjunctive product by its discounted version thanks to the measure of degree of intersection D when the intersection of focal elements is not empty. The product of partial (or total) conflicting masses are not discounted by the measure of degree of intersection because the degree of intersection between two (or more) conflicting focal elements always equals zero, that is if $X \cap Y = \emptyset$, then $D(X,Y) = 0$. Because there are different ways to define degrees of intersection between set (here we consider only Zhang's and Jaccard' degrees), and there are different ways to make the normalization because of the weighted conjunctive product involved in formulas, we come up with several versions of modified PCR6 rule of combination. We consider in fact two main modified versions of PCR6. The first modified version uses a classical normalization step based on the division by a normalization factor. The second modified version uses a sophisticate normalization step as shown through the general modified PCR6 formulas.

A. Simplest modified PCR6 rule

The simplest modified PCR6 rule including the measure of degree of intersection between sets is defined for $s \geq 2$ BBA by $m^{DPCR6}_{1,2,\ldots,s}(\emptyset) = 0$ and for any non empty $X \in 2^\Theta$, by

$$m^{DPCR6}_{1,2,\ldots,s}(X) \triangleq \frac{1}{K^{DPCR6}_{1,2,\ldots,s}} \cdot [m^{D}_{1,2,\ldots,s}(X) + CR^{PCR6}(X)] \tag{11}$$

where $K^{DPCR6}_{1,2,\ldots,s}$ is a normalization constant allowing to get $\sum_{X\in 2^\Theta} m^{PCR6}_{1,2,\ldots,s}(X) = 1$; $CR^{PCR6}(X)$ is the part of the conflicting masses redistributed back to the focal element X according to PCR6 redistribution principle and defined by (10); and $m^{D}_{1,2,\ldots,s}(X)$ is the discounted conjunctive consensus by the measure of the degree of intersection, defined by

$$m^{D}_{1,2,\ldots,s}(X) \triangleq \sum_{\substack{X_1,\ldots X_s \in 2^\Theta \\ X_1\cap\ldots\cap X_s = X}} D(X_1,\ldots,X_s) \prod_{i=1}^{s} m_i(X_i) \tag{12}$$

A similar general formula holds for the modified PCR5 rule with degrees of intersection between focal elements. For the fusion of two BBA's $m_1(.)$ and $m_2(.)$, the modified PCR6 and PCR5 formulas coincide and reduce to the formula below

$$m^{DPCR5/6}_{1,2}(X) = \frac{1}{K^{DPCR5/6}_{1,2}}\cdot [\sum_{\substack{X_1,X_2\in 2^\Theta \\ X_1\cap X_2 = X}} D(X_1,X_2)m_1(X_1)m_2(X_2) + \sum_{\substack{Y\in 2^\Theta\setminus\{X\} \\ X\cap Y=\emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X)+m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X)+m_1(Y)}]] \tag{13}$$

Depending on the degree of intersection we take (either D^Z or D^J), we get two versions of this modified PCR6 rule. The result of the fusion for each version will be denoted $m^{ZPCR6}_{1,2,\ldots,s}(.)$ and $m^{JPCR6}_{1,2,\ldots,s}(.)$ in the sequel. ZPCR6 and JPCR6 rules[6] are commutative but not associative.

B. Sophisticate modified PCR6 rule

We propose here a more sophisticate modified PCR6 rule which does not use the normalization by the division with a normalization constant but which makes a proportional redistribution of the non conflicting mass missing from the discounted conjunctive rule (after including a degree of intersection). Before providing the general formula of this sophisticate modified PCR6 rule, let's explain how the redistribution that we propose is done in the two BBA's case at first for simplicity.

[6]ZPCR6 and JPCR6 denote the PCR6 rules modified with Zhang's and Jaccard's degrees of intersection respectively.

Let's suppose to have only two BBA's $m_1(.)$ and $m_2(.)$ defined on the same FoD Θ (assuming Shafer's model for simplicity). When $X_1 \cap X_2 = X$, then $(1 - D(X_1, X_2))m_1(X_1)m_2(X_2)$ will be transferred back to X_1 and X_2 proportionally with respect to their masses (following PCR5/6 principle), that is:

$$\frac{\alpha}{m_1(X_1)} = \frac{\beta}{m_2(X_2)} = \frac{(1 - D(X_1, X_2))m_1(X_1)m_2(X_2)}{m_1(X_1) + m_2(X_2)}$$

whence,

$$\alpha = (1 - D(X_1, X_2)) \cdot \frac{m_1^2(X_1)m_2(X_2)}{m_1(X_1) + m_2(X_2)}$$

$$\beta = (1 - D(X_1, X_2)) \cdot \frac{m_1(X_1)m_2^2(X_2)}{m_1(X_1) + m_2(X_2)}$$

The formula of this sophisticate modified combination rule, denoted[7] SDPCR5/6, is given by $m_{1,2}^{SDPCR5/6}(\emptyset) = 0$ and by

$$\begin{aligned} m_{1,2}^{SDPCR5/6}(X) &\triangleq \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} D(X_1, X_2)m_1(X_1)m_2(X_2) \\ &+ \sum_{\substack{Y \in 2^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}] \\ &+ \sum_{\substack{Y \in 2^\Theta \setminus \{X\} \\ X \cap Y \neq \emptyset}} (1 - D(X, Y))[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} \\ &+ \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}] \end{aligned} \tag{14}$$

The third sum of Eq.(14) represents the non-conflicting mass missing from the conjunctive rule including a degree of intersection. As for ZPCR6 or JPCR6 rules, we can choose Zhang's or Jaccard's degrees (or any other measures of degree of intersection if preferred). The generalization of this principle of redistribution of missing discounting conjunctive masses yields the following general sophisticate modified PCR6 rule of combination.

$$m_{1,2,\ldots,s}^{SDPCR6}(X) = m_{1,2,\ldots,s}^{D}(X) + CR^{PCR6}(X) + MR^{PCR6}(X) \tag{15}$$

where $MR^{PCR6}(X)$ is the part of the missing conjunctive masses due to discounting back to the focal element involved in the conjunction which is redistributed according to PCR6 redistribution principle. $MR^{PCR6}(X)$ is defined by

$$\begin{aligned} MR^{PCR6}(X) \triangleq \sum_{k=1}^{s-1} \sum_{\substack{X_{i_1},X_{i_2},\ldots,X_{i_k} \in 2^\Theta \setminus X \\ (\bigcap_{j=1}^{k} X_{i_j}) \cap X \neq \emptyset}} \sum_{(i_1,i_2,\ldots,i_k) \in \mathcal{P}^s(\{1,\ldots,s\})} \\ (1 - D(X, \ldots, X, X_{i_{k+1}}, \ldots, X_{i_s})) \cdot \sum_{j=1}^{k} m_{i_j}(X) \\ \cdot \frac{m_{i_1}(X) \ldots m_{i_k}(X) m_{i_{k+1}}(X_{i_{k+1}}) \ldots m_{i_s}(X_{i_s})}{m_{i_1}(X) + \ldots + m_{i_k}(X) + m_{i_{k+1}}(X_{i_{k+1}}) + \ldots + m_{i_s}(X_{i_s})} \end{aligned} \tag{16}$$

SZPCR6 and SJPCR6 rules[8] are commutative but not associative.

[7] S letter in this acronym stands for *Sophisticate*.

[8] SZPCR6 and SJPCR6 denote the PCR6 rules modified with Zhang's and Jaccard's degrees of intersection respectively.

IV. Analysis of the neutrality of VBBA

When there is no conflict between BBA's, DS, PCR5 or PCR6 rules reduce to the conjunctive rule which preserves the neutrality of VBA. When there is conflict between BBA's only DS preserves neutrality of VBA because DS is associative. In general, PCR5 and PCR6 do not preserve the neutrality of the VBA if more than two conflicting BBA's (including the VBA) are combined altogether[9]. In general, the VBA $m_v(.)$ is not a neutral element for the conjunctive rule of combination discounted with Jaccard's degree of intersection when combining two (or more) BBA's as shown in the following counter-example. If we take $\Theta = \{A, B\}$, with $A \cap B = \emptyset$, and $m_1(.)$ defined as $m_1(A) = 0.5$, $m_1(B) = 0.3$ and $m_v(A \cup B) = 0.2$. Then the result of the JCR fusion is $m_{1v}^{JCR}(A) \approx 0.4167$, $m_{1v}^{JCR}(B) = 0.25$ and $m_{1v}^{JCR}(A \cup B) \approx 0.3333$, which shows that $m_{1v}^{JCR}(.) \neq m_1(.)$. The VBA $m_v(.)$ is a neutral element for the ZCR combination of $m_1(.)$ with the VBA $m_v(.)$, because the discounted conjunctive mass for any focal element X is $m_{1v}(X) = \frac{|X \cap \Theta|}{|X| \cdot |\Theta|} m_1(X) m_v(\Theta) = \frac{|X|}{|X| \cdot |\Theta|} \cdot m_1(X) \cdot 1 = \frac{1}{n} m_1(X)$, where $n = |\Theta|$. The normalization constant equals $K_{1,v}^{ZCR} = \sum_X \frac{1}{n} m_1(X) = 1/n$. Therefore, after dividing by $K_{1,v}^{ZCR}$, we always gets $m_{1v}^{ZCR}(X) = m_1(X)$ for any focal element X of $m_1(.)$. Same property holds if we combine three (or more) BBA's with the VBA and even if these BAA's are in conflict or not. Because $D^Z(X_1, \ldots, X_n, \Theta) = D^Z(X_1, \ldots, X_n)/|\Theta|$ and $m_v(\Theta) = 1$, the constant $|\Theta|$ always simplifies in normalization step of ZCR and because conjunctive rule and Zhang's degree are associative. In the general case, ZPCR6, SZPRC6, JPRC6 and SJPCR6 do not preserve the neutrality of the VBA. This can be verified using the simple example of the footnote no 9. More precisely, the combination $[m_1 \oplus m_2 \oplus \ldots \oplus m_n \oplus m_v](.)$ is not equal to $[m_1 \oplus m_2 \oplus \ldots \oplus m_n](.)$. In the very specific case when there is no conflict between the BBA's, only ZPCR6 rule preserves the neutrally of VBA because it coincides with ZCR.

V. Examples

Here we analyze the behavior of the different rules (DS, PCR6, ZCR, JCR, ZPCR6, JPCR6, SZPCR6 and SJPCR6) in emblematic examples to determinate which one presents the best interest for the combination of BBA's.

Example 1: (No conflicting case)

Let's consider the FoD $\Theta = \{A_1, A_2, \ldots, A_{10}\}$ with Shafer's model, and the following two BBA's to combine $m_1(A_1) = 0.9$, $m_1(\Theta) = 0.1$, $m_2(X) = 0.9$ and $m_2(\Theta) = 0.1$ where the focal element X of $m_2(.)$ can take the values A_1, $A_1 \cup A_2$, $A_1 \cup A_2 \cup A_3$, …, or Θ.

In this case, the DS and PCR5/6 rules coincide with the conjunctive rule of combination because there is no conflicting mass to redistribute because $m_{1,2}(\emptyset) = 0$. If

[9] For example, if one considers $\Theta = \{A, B\}$ with Shafer's model, and the BBA's $\{m_1(A) = a_1, m_1(B) = b_1, m_1(\Theta) = c_1\}$, $\{m_2(A) = a_2, m_2(B) = b_2, m_2(\Theta) = c_2, m_v(\Theta) = 1\}$. Then $[m_1 \oplus m_2](.) \neq [m_1 \oplus m_2 \oplus m_v](.)$ (where $\oplus$ denotes the PCR5 or PCR6 fusion rule) because in $m_1 \oplus m_2$ nothing from the redistribution of the conflicting mass goes to ignorance, contrarily to what happens in $[m_1 \oplus m_2 \oplus m_v](.)$.

$X = A_1$, then $m_{1,2}^{DS}(A_1) = m_{1,2}^{PCR6}(A_1) = m_1(A_1)m_2(A_1) + m_1(A_1)m_2(\Theta) + m_1(\Theta)m_2(A_1) = 0.99$ and $m_{1,2}^{DS}(\Theta) = m_{1,2}^{PCR6}(\Theta) = m_1(\Theta)m_2(\Theta) = 0.01$, which is a reasonable result since the belief in A_1 is reinforced because each source does strongly support the same hypothesis A_1. When $X \supset A_1$ and $|X| > 1$, the behavior of the conjunctive rule becomes questionable because one always gets

$$m_{1,2}^{DS}(A_1) = m_{1,2}^{PCR5/6}(A_1) = m_1(A_1)(m_2(X) + m_2(\Theta)) = 0.9$$
$$m_{1,2}^{DS}(X) = m_{1,2}^{PCR5/6}(X) = m_1(\Theta)m_2(X) = 0.09$$
$$m_{1,2}^{DS}(\Theta) = m_{1,2}^{PCR5/6}(\Theta) = m_1(\Theta)m_2(\Theta) = 0.01$$

When $X \to \Theta$, $m_2(.)$ tends to become a fully ignorant source of evidence, and the combination of $m_1(.)$ with $m_2(.)$ tends towards $m_1(.)$ because $m_2(.)$ brings none useful information at all in this limit case. This behavior of conjunctive rule is then conform with what we intuitively expect. However, when $|X|$ decreases from $r = 10$ to $r = 2$, the behavior of conjunctive rule (and in this case DS and PCR6 rules also) is not very satisfactory, because we obtain same results on the mass of A_1 whatever the cardinality of X is. In fact, it is rather intuitively expected that after the combination, the mass of A_1 should substantially increase if the cardinality of X decreases because $m_2(.)$ becomes more and more specific (and focused towards A_1). When $m_2(.)$ is more in agreement with $m_1(.)$, the combination of $m_1(.)$ with $m_2(.)$ should reinforce the belief on A_1 when $|X|$ decreases, which is not what happens with the pure (strict) conjunctive rule.

Let's examine how ZCR, JCR rules work in this example. Let $|X| = r \geq 1$, and $r \leq 10$. Also $|\Theta| = |A_1 \cup A_2 \cup \ldots \cup A_{10}| = 10$. If we compute the (unnormalized) discounted conjunctive fusion with Zhang's degree of intersection, we get

$$\begin{aligned}
m_{1,2}^{Z}(A_1) &= \frac{|A_1 \cap X|}{|A_1| \cdot |X|} m_1(A_1)m_2(X) + \frac{|A_1 \cap \Theta|}{|A_1| \cdot |\Theta|} m_1(A_1)m_2(\Theta) \\
&= \frac{1}{r}(0.9)(0.9) + \frac{1}{10}(0.9)(0.1) = \frac{0.81}{r} + 0.009 \\
m_{1,2}^{Z}(X) &= \frac{|\Theta \cap X|}{|\Theta| \cdot |X|} m_1(\Theta)m_2(X) = \frac{1}{10}(0.1)(0.9) = 0.009 \\
m_{1,2}^{Z}(\Theta) &= \frac{|\Theta \cap \Theta|}{|\Theta| \cdot |\Theta|} m_1(\Theta)m_2(\Theta) = \frac{1}{10}(0.1)(0.1) = 0.001
\end{aligned}$$

If we compute the (unnormalized) discounted conjunctive fusion with Jaccard's degree of intersection, we get

$$\begin{aligned}
m_{1,2}^{J}(A_1) &= \frac{|A_1 \cap X|}{|A_1 \cup X|} m_1(A_1)m_2(X) + \frac{|A_1 \cap \Theta|}{|A_1 \cup \Theta|} m_1(A_1)m_2(\Theta) \\
&= \frac{1}{r}(0.9)(0.9) + \frac{1}{10}(0.9)(0.1) = \frac{0.81}{r} + 0.009 \\
m_{1,2}^{J}(X) &= \frac{|\Theta \cap X|}{|\Theta \cup X|} m_1(\Theta)m_2(X) = \frac{r}{10}(0.1)(0.9) = 0.009 \cdot r \\
m_{1,2}^{J}(\Theta) &= \frac{|\Theta \cap \Theta|}{|\Theta \cup \Theta|} m_1(\Theta)m_2(\Theta) = \frac{10}{10}(0.1)(0.1) = 0.01
\end{aligned}$$

After normalization of $m_{1,2}^{Z}(.)$ by $K_{1,2}^{Z} = \frac{0.81}{r} + 0.019$, and $m_{1,2}^{J}(.)$ by $K_{1,2}^{J} = \frac{0.81}{r} + 0.009 \cdot r + 0.010$ we get the result of ZCR and JCR rules, which are

$$\begin{aligned}
m_{1,2}^{ZCR}(A_1) &= [\frac{0.81}{r} + 0.009]/K_{1,2}^{Z} & m_{1,2}^{JCR}(A_1) &= [\frac{0.81}{r} + 0.009]/K_{1,2}^{J} \\
m_{1,2}^{ZCR}(X) &= 0.009/K_{1,2}^{Z} & m_{1,2}^{JCR}(X) &= 0.009 \cdot r/K_{1,2}^{J} \\
m_{1,2}^{ZCR}(\Theta) &= 0.001/K_{1,2}^{Z} & m_{1,2}^{JCR}(\Theta) &= 0.01/K_{1,2}^{J}
\end{aligned}$$

In the limit case when $r = 1$ we get

$$\begin{aligned}
m_{1,2}^{ZCR}(A_1) &= 0.988 & m_{1,2}^{JCR}(A_1) &= 0.988 \\
m_{1,2}^{ZCR}(\Theta) &= 0.012 & m_{1,2}^{JCR}(\Theta) &= 0.012
\end{aligned}$$

In the limit case when $r = 10$ we get

$$\begin{aligned}
m_{1,2}^{ZCR}(A_1) &= 0.90 & m_{1,2}^{JCR}(A_1) &= 0.4337 \\
m_{1,2}^{ZCR}(\Theta) &= 0.10 & m_{1,2}^{JCR}(\Theta) &= 0.5263
\end{aligned}$$

Clearly, one sees that both ZCR and JCR have now a good expected behavior when $|X|$ decreases, but only ZCR provides also a good behavior when $r = 10$ because in this case one gets $m_{1,2}^{ZCR}(.) = m_1(.)$ which is normal because $m_2(.)$ is the VBA (fully ignorant source). With JCR, the result we obtain when $|X| = r = 10$ is not good because $m_{1,2}^{ZCR}(.) \neq m_1(.)$. Because there is no conflict, ZPCR6 rule coincides with ZCR rule in this example, and JPCR6 rule coincides with JCR rule. Therefore, JPCR6 rule does not work well (at least for this example) as explained previously. The evaluation of masses of A_1 and of Θ after the combination of $m_1(.)$ with $m_2(.)$ for the different rules is shown in Fig. 1 and Fig. 2 respectively and for different values of $r = |X|$.

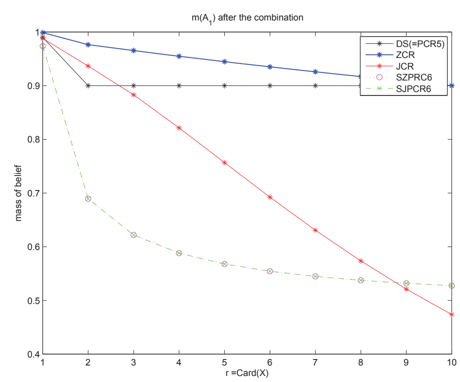

Figure 1. $m(A_1)$ after combination of $m_1(.)$ with $m_2(.)$.

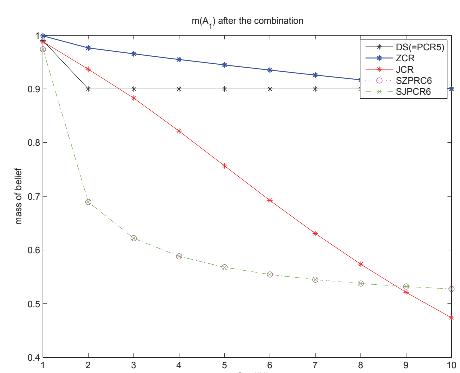

Figure 2. $m(\Theta)$ after combination of $m_1(.)$ with $m_2(.)$.

If we apply sophisticate normalization procedures we obtain[10] with SZPCR6 and SJPCR6

$$m_{1,2}^{SZPCR6}(A_1) = 0.0819 + 0.81 \cdot \frac{1}{r} + 0.405 \cdot \frac{r-1}{r}$$
$$m_{1,2}^{SZPCR6}(X) = 0.0819 + 0.405 \cdot \frac{r-1}{r}$$
$$m_{1,2}^{SZPCR6}(\Theta) = 0.0262$$

$$m_{1,2}^{SJPCR6}(A_1) = 0.0819 + 0.81 \cdot \frac{1}{r} + 0.405 \cdot \frac{r-1}{r}$$
$$m_{1,2}^{SJPCR6}(X) = 0.009 \cdot r + 0.405 \cdot \frac{r-1}{r} + (10-r) \cdot 0.0081$$
$$m_{1,2}^{SJPCR6}(\Theta) = 0.0181 + (10-r) \cdot 0.0081$$

In the limit case, when $r = 1$ we get

$$m_{1,2}^{SZPCR6}(A_1) = 0.9738 \qquad m_{1,2}^{SJPCR6}(A_1) = 0.9738$$
$$m_{1,2}^{SZPCR6}(\Theta) = 0.0262 \qquad m_{1,2}^{SJPCR6}(\Theta) = 0.0262$$

In the limit case, when $r = 10$ we get

$$m_{1,2}^{SZPCR6}(A_1) = 0.5274 \qquad m_{1,2}^{SJPCR6}(A_1) = 0.5274$$
$$m_{1,2}^{SZPCR6}(\Theta) = 0.4726 \qquad m_{1,2}^{SJPCR6}(\Theta) = 0.4726$$

This result shows clearly that SZPCR6 and SJPCR6 rules behave better than conjunctive rule (and so better than DS and PCR6 rules) in the limit case when $X = A_1$ because after the combination the mass committed to A_1 is reinforced (as it is naturally expected). But the reinforcement of mass of A_1 is lower than with ZPCR6 or JPCR6 rules[11] based on simple normalization because the sophisticate normalization procedure degrades the specificity of the information. In the other limit case when $r = 10$, (i.e. $X = \Theta$, and $m_2(.)$ equals the VBA) SZPCR6 and SJPCR6 rules do not work well because clearly one has $m_{1,2}^{SZPCR6}(.) \neq m_1(.)$ and $m_{1,2}^{SJPCR6}(.) \neq m_1(.)$ also. So we at least have shown one example where SZPCR6 and SJPCR6 are not very efficient and consequently, we do not recommend to use them. In summary, only ZCR and ZPCR6 (equivalent to ZCR in this example) allow to get an acceptable behavior for combining the two BBA's $m_1(.)$ and $m_2(.)$ for any focal element $X \supseteq A_1$.

Example 2 (Zadeh [10], [12]): (Conflicting case)

Let's $\Theta = \{A, B, C\}$ with Shafer's model, and the two BBA's to combine $m_1(A) = 0.9$, $m_1(C) = 0.1$, $m_2(B) = 0.9$ and $m_2(C) = 0.1$.

In this case, Shafer's conflict is $m_{1,2}(\emptyset) = m_1(A)(m_2(B) + m_2(C)) + m_1(C)m_2(B) = 0.9 + 0.1 \cdot 0.9 = 0.99$. If we use DS rule (2), we get $m_{1,2}^{DS}(C) = 1$. The discounted conjunctive consensus $D(C,C)m_1(C)m_2(C)$ (with Zhang's or Jaccard's degree) is always equal to the un-discounted conjunctive consensus $m_1(C)m_2(C) = 0.01$ because $D^Z(C,C) = \frac{|C \cap C|}{|C| \cdot |C|} = 1$ and $D^J(C,C) = \frac{|C \cap C|}{|C \cup C|} = 1$. Therefore the degree of intersection does not impact the conjunctive combination result and ZCR and JCR rules (4) give same counter-intuitive result as DS rule, that is $m_{1,2}^{ZCR}(C) = m_{1,2}^{JCR}(C) = m_{1,2}^{DS}(C) = 1$.

Because the degree of intersection does not impact the conjunctive combination part of PCR6 rule in this example, modified PCR6 rules (ZPCR6, JPCR6, SZPCR6 and SJPCR6) give the same result as PCR6 rule which is $m_{1,2}^{PCR5/6}(A) = 0.486$, $m_{1,2}^{PCR5/6}(B) = 0.486$ and $m_{1,2}^{PCR5/6}(C) = 0.028$.

In summary, ZCR and JCR rules do not help to modify the result obtained by DS rule in Zadeh's example and cannot be viewed as real alternatives to DS rule for this example. Conversely, ZPCR6, JPCR6, SZPCR6 and SJPCR6 rule (which coincide with PCR6 rule in this example) remain good alternatives to DS rule.

Example 3 (Voorbraak [15]): (Conflicting case)

Let's consider the FoD $\Theta = \{A, B, C\}$ with Shafer's model, and the following two BBA's to combine $m_1(A) = 0.5$, $m_1(B \cup C) = 0.5$, $m_2(C) = 0.5$, and $m_2(A \cup B) = 0.5$.

One has $m_{1,2}(\emptyset) = m_1(A)m_2(C) = 0.25$, and DS rule gives $m_{1,2}^{DS}(A) = m_{1,2}^{DS}(B) = m_{1,2}^{DS}(C) = 1/3$. As reported by Voorbraak [15], this result is counterintuitive, since intuitively B seems to *share* twice a probability mass of 0.5, while both A and C only have to share once 0.5 with B and are once assigned 0.5 individually. This counterintuitive result comes from the fact that DS rule implicitly assumes that all possible pairs of focal elements are equally confirmed by the combined evidence, while intuitively, in this example $B = (B \cup C) \cap (A \cup B)$ is less confirmed than $A = A \cap (A \cup B)$ and $C = (B \cup C) \cap C$. With ZCR and JCR rules, we get

$$m_{1,2}^{ZCR}(A) = 0.40 \qquad m_{1,2}^{JCR}(A) = 0.375$$
$$m_{1,2}^{ZCR}(B) = 0.20 \qquad m_{1,2}^{JCR}(B) = 0.250$$
$$m_{1,2}^{ZCR}(C) = 0.40 \qquad m_{1,2}^{JCR}(C) = 0.375$$

Contrarily to DS rule, with ZCR or JCR rules one sees that the mass committed to B is less than of A and of C which is a more reasonable result. In applying PCR6 rule, we also circumvent this problem because we get from Eq. (13), $m_{1,2}^{PCR6}(A) = 0.375$, $m_{1,2}^{PCR6}(B) = 0.25$ and $m_{1,2}^{PCR6}(C) = 0.375$ (same as with JCR results for this particular example).

With ZPCR6 rule, we compute at first the following (unnormalized) discounted conjunctive masses added with proportional conflict redistribution

$$m_{1,2}^{Z}(A) = \frac{|A \cap (A \cup B)|}{|A| \cdot |A \cup B|} m_1(A)m_2(A \cup B) + \frac{1}{2} m_{1,2}(\emptyset) = 0.25$$
$$m_{1,2}^{Z}(B) = \frac{|(B \cup C) \cap (A \cup B)|}{|B \cup C| \cdot |A \cup B|} m_1(B \cup C)m_2(A \cup B) = 0.0625$$
$$m_{1,2}^{Z}(C) = \frac{|(B \cup C) \cap C|}{|B \cup C| \cdot |C|} m_1(B \cup C)m_2(C) + \frac{1}{2} m_{1,2}(\emptyset) = 0.25$$

After a simple normalization (dividing by $K_{1,2}^{Z} = 0.25 + 0.0625 + 0.25 = 0.5625$), we get finally

$$m_{1,2}^{ZPCR6}(A) = 0.25/0.5625 \approx 0.4444$$
$$m_{1,2}^{ZPCR6}(B) = 0.0625/0.5625 \approx 0.1112$$
$$m_{1,2}^{ZPCR6}(C) = 0.25/0.5625 \approx 0.4444$$

[10] Here there is no conflicting mass to redistribute which makes the derivation more easier.

[11] which coincide here with ZCR and JCR rule because there is no conflicting mass to redistribute.

Similarly, if we apply JPCR6 rule based on Jaccard's index and simple normalization step, we will get the following result

$$m_{1,2}^{JPCR6}(A) = [(0.25/2) + 0.125]/K_{1,2}^{J} \approx 0.4286$$
$$m_{1,2}^{JPCR6}(B) = (0.25/3)/K_{1,2}^{J} = 0.1428$$
$$m_{1,2}^{JPCR6}(C) = [(0.25/2) + 0.125]/K_{1,2}^{J} \approx 0.4286$$

where the normalization factor equals $K_{1,2}^{J} = (0.25/2) + 0.125 + (0.25/3) + (0.25/2) + 0.125 \approx 0.5833$.

These results show that ZPCR6 and JPCR6 rules diminish substantially the mass committed to B (as expected) and reinforce more strongly the masses of A and C than with ZCR, JCR or PCR6 rules.

If we apply the sophisticate normalization for SZPCR6, the lost discounted mass $(1 - \frac{|A\cap(A\cup B)|}{|A|\cdot|A\cup B|})m_1(A)m_2(A\cup B) = 0.125$ is redistributed to A and to $A\cup B$ proportionally[12] to $m_1(A) = 0.5$ and $m_2(A\cup B) = 0.5$. Similarly, the second lost discounted mass $(1-\frac{|(B\cup C)\cap(A\cup B)|}{|B\cup C|\cdot|A\cup B|})m_1(B\cup C)m_2(A\cup B) = 0.1875$ is redistributed to $B\cup C$ and to $A\cup B$ proportionally to $m_1(B\cup C) = 0.5$ and $m_2(A\cup B) = 0.5$, and the third lost discounted mass $(1 - \frac{|(B\cup C)\cap C|}{|B\cup C|\cdot|C|})m_1(B\cup C)m_2(C) = 0.125$ is redistributed to $B\cup C$ and to C proportionally to $m_1(B\cup C) = 0.5$ and $m_2(C) = 0.5$. Similar computations are done for SJPCR6 in replacing Zhang's degree by Jaccard's degree of intersection. Finally we obtain with SZPCR6 and SJPRC6 the following combined masses:

$$m_{1,2}^{SZPCR6}(A) = 0.3125 \qquad m_{1,2}^{SZPCR6}(A\cup B) = 0.15625$$
$$m_{1,2}^{SZPCR6}(B) = 0.0625 \qquad m_{1,2}^{SZPCR6}(B\cup C) = 0.15625$$
$$m_{1,2}^{SZPCR6}(C) = 0.3125$$

and

$$m_{1,2}^{SJPCR6}(A) = 0.3125 \qquad m_{1,2}^{SJPCR6}(A\cup B) \approx 0.14585$$
$$m_{1,2}^{SJPCR6}(B) \approx 0.0833 \qquad m_{1,2}^{SJPCR6}(B\cup C) \approx 0.14585$$
$$m_{1,2}^{SJPCR6}(C) = 0.3125$$

Of course, these results are a bit less specific than with ZPCR6 and JPCR6, which is normal. As shown, SZPCR6 and SJPCR6 rules diminish also the mass committed to B (as expected) but reinforce less strongly the masses of A and C because the specificity of the result is degraded because one gets positive masses committed to new uncertainties $A\cup B$ and $B\cup C$. For this example, ZPCR6 and JPCR6 are the most interesting rules for combining BBA's $m_1(.)$ and $m_2(.)$.

Example 4 (Dezert et al. [3]): (Conflicting case)

This emblematic example is very interesting to analyze because for in this case the DS rule does not respond to level of conflict between the sources. This *anomaly* has been analyzed and discussed in details in [3].

Let's consider the FoD $\Theta = \{A, B, C\}$ with Shafer's model, and the following two BBA's to combine

$$m_1(A) = 0.9, \qquad m_1(A\cup B) = 0.1$$
$$m_2(A\cup B) = 0.1 \qquad m_2(C) = 0.7, \qquad m_2(A\cup B\cup C) = 0.2$$

In this example, the two sources are not vacuous (they are truly informative), they are in conflict because $m_{1,2}(\emptyset) = 0.7$ but DS rule does not respond to the level of conflict because one gets $m_{1,2}(.) = m_1(.)$. In fact, the second source has no impact in the DS fusion as if it is equivalent to the VBA.

If we apply PCR6 rule of combination the first partial conflict $m_1(A)m_2(C) = 0.72$ is redistributed to A and C proportionally to $m_1(A)$ and $m_2(C)$, and the second conflict $m_1(A\cup B)m_2(C) = 0.08$ is redistributed to $A\cup B$ and to C proportionally to $m_1(A\cup B)$ and $m_2(C)$. So with PCR6 rule (7), we obtain $m_{1,2}^{PCR6}(A) = 0.6244$, $m_{1,2}^{PCR6}(A\cup B) = 0.0388$ and $m_{1,2}^{PCR6}(C) = 0.3369$. One sees that the PCR6 fusion result now reacts with the value of second sources because $m_{1,2}^{PCR6}(.) \neq m_1(.)$ which makes sense if both sources are equireliable, truly informative and in some disagreement.

In discounting with Zhang's degree, one gets the (unnormalized) discounted conjunctive BBA

$$m_{1,2}^{Z}(A) = \frac{1}{2}(0.9)(0.1) + \frac{1}{3}(0.9)(0.2) = 0.1050$$
$$m_{1,2}^{Z}(A\cup B) = \frac{2}{2\cdot 2}(0.1)(0.1) + \frac{2}{2\cdot 3}(0.1)(0.2) \approx 0.0117$$

After the normalization by the factor $K_{1,2}^{Z} = 0.1050 + 0.0117 = 0.1167$, we get finally $m_{1,2}^{ZCR}(A) = 0.1050/0.1167 \approx 0.9$ and $m_{1,2}^{ZCR}(A\cup B) \approx 0.0117/0.1167 \approx 0.1$. Therefore as with DS rule, we get same behavior with ZCR rule that is $m_{1,2}^{ZCR}(.) = m_1(.)$ as if the second informative source does not count in the fusion process, which is abnormal.

If we use Jaccard's degree, one gets

$$m_{1,2}^{J}(A) = \frac{1}{2}(0.9)(0.1) + \frac{1}{3}(0.9)(0.2) = 0.1050$$
$$m_{1,2}^{J}(A\cup B) = \frac{2}{2}(0.1)(0.1) + \frac{2}{3}(0.1)(0.2) \approx 0.0233$$

After the normalization by the factor $K_{1,2}^{J} = 0.1050 + 0.0233 = 0.12833$, we get finally $m_{1,2}^{JCR}(A) \approx 0.8182$ and $m_{1,2}^{ZCR}(A\cup B) \approx 0.1818$. One sees that JCR fusion result is not equal to the BBA $m_1(.)$, which means that $m_2(.)$ has had some impact in the fusion process with JCR (as expected). However, it is not clear why such JCR result will really make sense or not. Because we have already shown in Example 1, that it can happen than JCR does not work well, we have serious doubt on the interest of using JCR result in such emblematic example.

With ZPCR6 rule of combination, we obtain $m_{1,2}^{ZPCR6}(A) = \frac{1}{K_{1,2}^{ZPCR6}}[0.1050 + x(A)] = 0.56250$, $m_{1,2}^{ZPCR6}(A\cup B) = \frac{1}{K_{1,2}^{ZPCR6}}[0.0117 + x(A\cup B)] = 0.0250$, and $m_{1,2}^{ZPCR6}(C) = \frac{1}{K_{1,2}^{ZPCR6}}[x_1(C) + x_2(C)] = 0.4125$, where $K_{1,2}^{ZPCR6}$ is the normalization constant, and where $x(A) = m_1(A)\frac{m_1(A)m_2(C)}{m_1(A)+m_2(C)} = 0.354375$ is the part of the conflicting mass $m_1(A)m_2(C) = 0.63$ transferred to A; $x_1(C) = m_1(C)\frac{m_1(A)m_2(C)}{m_1(A)+m_2(C)} = 0.275625$ is the part of the conflicting mass $m_1(A)m_2(C) = 0.63$ transferred to C; $x(A\cup B) = m_1(A\cup B)\frac{m_1(A\cup B)m_2(C)}{m_1(A\cup B)+m_2(C)} = 0.00875$ is the part of the conflicting mass $m_1(A\cup B)m_2(C) = 0.07$ transferred to $A\cup B$; and $x_2(C) = m_1(C)\frac{m_1(A\cup B)m_2(C)}{m_1(A\cup B)+m_2(C)} = 0.06125$ is the part of the conflicting mass $m_1(A\cup B)m_2(C) = 0.07$ transferred to C.

[12] equally in fact in this case.

With JPCR6 rule of combination, we obtain $m_{1,2}^{JPCR6}(A) \approx 0.55458$, $m_{1,2}^{JPCR6}(A \cup B) = 0.03873$ and $m_{1,2}^{JPCR6}(C) = 0.40669$, which is close to ZPCR6 result. Comparatively to PCR6, we diminish the mass of belief committed to A and to $A \cup B$ and we reinforce the mass committed to C using ZPCR6 and JPCR6 rules. We do not give results with SZPCR6 and SJPRC6 due to space constraint and because we know that these rules do not perform so well as shown in the previous examples.

Example 5 (Sebbak [16]): (Conflicting case with 3 sources) Let's consider the FoD $\Theta = \{A, B, C\}$ with Shafer's model, and the following three BBA's to combine

$$\begin{aligned} m_1(A) &= 0.8, & m_1(A \cup B \cup C) &= 0.2 \\ m_2(A) &= 0.1, & m_2(C) &= 0.9 \\ m_3(A) &= 0.4, & m_3(A \cup B \cup C) &= 0.6 \end{aligned}$$

The conjunctive rule gives

$$\begin{aligned} m_{1,2,3}(A) &= m_1(A)m_2(A)m_3(A) + m_1(A)m_2(A)m_3(\Theta) \\ &\quad + m_1(\Theta)m_2(A)m_3(\Theta) + m_1(\Theta)m_2(A)m_3(A) = 0.10 \\ m_{1,2,3}(C) &= m_1(\Theta)m_2(C)m_3(\Theta) = 0.108 \end{aligned}$$

with the total conflicting mass

$$\begin{aligned} m_{1,2,3}(\emptyset) &= m_1(A)m_2(C)m_3(A) + m_1(A)m_2(C)m_3(\Theta) \\ &\quad + m_1(\Theta)m_2(C)m_3(A) = 0.792 \end{aligned}$$

With DS rule we get $m_{12}^{DS}(A) \approx 0.4808$ and $m_{12}^{DS}(C) \approx 0.5192$, and With PCR5 and PCR6 rules [17]

$$\begin{aligned} m_{1,2,3}^{PCR5}(A) &= 0.3450 & m_{1,2,3}^{PCR6}(A) &= 0.4340 \\ m_{1,2,3}^{PCR5}(C) &= 0.5327 & m_{1,2,3}^{PCR6}(C) &= 0.4437 \\ m_{1,2,3}^{PCR5}(\Theta) &= 0.1223 & m_{1,2,3}^{PCR6}(\Theta) &= 0.1223 \end{aligned}$$

Note that with PCR5 one gets $0.4247/0.7920 \approx 53.62\%$ of the total conflicting mass redistributed to C, but not *almost all conflicting mass*. Using PCR6, C actually gained from the total conflicting mass only $0.3357/0.7920 \approx 42.3864\%$, not even half of it, not *almost all of the conflicting mass (the majority)* as the authors wrongly claimed in [16].
With ZCR and JCR rules, one gets

$$\begin{aligned} m_{1,2,3}^{ZCR}(A) &= 0.8125 & m_{1,2,3}^{JCR}(A) &= 0.6032 \\ m_{1,2,3}^{ZCR}(\Theta) &= 0.1875 & m_{1,2,3}^{JCR}(\Theta) &= 0.3968 \end{aligned}$$

With ZPCR6, JPCR6, SZPCR6 and SJPCR6 rules[13] one gets

$$\begin{aligned} m_{1,2,3}^{ZPCR6}(A) &= 0.4511 & m_{1,2,3}^{JPCR6}(A) &= 0.4405 \\ m_{1,2,3}^{ZPCR6}(C) &= 0.4061 & m_{1,2,3}^{JPCR6}(C) &= 0.4210 \\ m_{1,2,3}^{ZPCR6}(\Theta) &= 0.1428 & m_{1,2,3}^{JPCR6}(\Theta) &= 0.1385 \end{aligned}$$

$$\begin{aligned} m_{1,2,3}^{SZPCR6}(A) &= 0.4102955 & m_{1,2,3}^{SJPCR6}(A) &= 0.412699 \\ m_{1,2,3}^{SZPCR6}(C) &= 0.3984240 & m_{1,2,3}^{SJPCR6}(C) &= 0.409718 \\ m_{1,2,3}^{SZPCR6}(\Theta) &= 0.1912805 & m_{1,2,3}^{SJPCR6}(\Theta) &= 0.177616 \end{aligned}$$

[13]The derivations are not included in this paper due to space restriction.

One sees that C gained $(0.4061-0.108)/0.7920 \approx 37.64\%$ using ZPCR6, $(0.4210 - 0.108)/0.7920 \approx 39.52\%$ using JPCR6, $(0.398424-0.108)/0.7920 \approx 36.67\%$ using SZPCR6, and $(0.409718 - 0.108)/0.7920 \approx 38.10\%$ using SJPCR6.

VI. CONCLUSIONS

The modifications of the PCR6 rule of combination presented exploit judiciously Zhang's and Jaccard's degrees of intersections of focal elements. Our analysis shows that ZPCR6 rule is in fact the most interesting modified PCR6 rule because it behaves well in all emblematic examples contrarily to other rules. SZPCR6 and SJPCR6 rules are more complicate to implement and they increase the non-specificity of the result in general which is not good for helping the decision-making. So we do not recommend them for applications. All these rules are not associative and do not preserve the neutrality of VBA when some sources are in conflict.

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