About on Integral of Legendre

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Abstract

In this note we show some formulas related with: Legendre Integral and Number Pi : π=3.14159...

Keyword:Legendre integral,Number Pi

I. Introduction: Legendre Integral

Recordamos una integral dada por Adrien Marie Legendre (1752-1833) :

$$\int_0^1 \frac{1}{\sqrt{1-x^8}}\,dx = \frac{1}{\sqrt{2}}\,K\left(\sqrt{2}-1\right) \tag{1}$$

donde

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2 x}}\,dx = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx \ , \ \ 0<k<1 \tag{2}$$

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(1/2)_n}{n!}\right)^2 k^{2n} \ , \ \ 0<k<1 \tag{3}$$

II. Formulas

La integral (1) se puede escribir como:

$$\int_0^1 \frac{1}{\sqrt{1-x^8}}\,dx = 1 + \int_1^{\infty}\left(1-\sqrt[8]{1-x^{-2}}\right)dx \tag{4}$$

de (4) se tiene:

$$\int_1^{\infty}\left(1-\sqrt[8]{1-x^{-2}}\right)dx = \frac{1}{\sqrt{2}}\,K\left(\sqrt{2}-1\right)-1 \tag{5}$$

para $0<z<1$, se tiene :

$$\int_0^1 \frac{1}{\sqrt{1-x^8}}\,dx = \int_0^z \frac{1}{\sqrt{1-x^8}}\,dx + \int_z^1 \frac{1}{\sqrt{1-x^8}}\,dx = \int_0^z \frac{1}{\sqrt{1-x^8}}\,dx + \frac{1-z}{\sqrt{1-z^8}} + \int_{1/\sqrt{1-z^8}}^{\infty}\left(1-\sqrt[8]{1-x^{-2}}\right)dx \tag{6}$$

Desarrollando en serie las integrales de (6) y usando (3) se tiene:

$$\frac{1}{\pi}=\left(\left(1/2\sqrt{2}\right)\sum_{n=0}^{\infty}((1/2)_n/n!)^2\left(\sqrt{2}-1\right)^{2n}\right)\Big/$$
$$\left(\frac{1-z}{\sqrt{1-z^8}}+\sum_{n=0}^{\infty}\left((1/2)_n z^{8n+1}/n!\,(8n+1)\right)-\frac{1}{\sqrt{1-z^8}}\sum_{n=1}^{\infty}\left((-1/8)_n\left(1-z^8\right)^n/n!\,(2n-1)\right)\right),\ 0<z<1 \tag{7}$$

Para $z^8=1/2$, se tiene :

$$\frac{1}{\pi}=\left(\sum_{n=0}^{\infty}((1/2)_n/n!)^2\left(\sqrt{2}-1\right)^{2n}\right)\Big/\left(2\left(2-2^{7/8}\right)+2\sqrt[8]{8}\sum_{n=0}^{\infty}((1/2)_n 2^{-n}/n!\,(8n+1))-4\sum_{n=1}^{\infty}((-1/8)_n 2^{-n}/n!\,(2n-1))\right) \tag{8}$$

Otras representaciónes de la integral (1) son:

$$\int_0^1\frac{1}{\sqrt{1-x^8}}\,dx=\frac{1}{2\sqrt{2}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{k+m}\binom{2n}{n}\binom{2k}{k}\binom{2m}{m}2^{-3n-3k-3m}f(n,k,m) \tag{9}$$

donde

$$f(n,k,m)=\int_0^1 x^{n-(1/2)}\left((1-x)^2-1\right)^k\left((1-x)^4-1\right)^m\,dx \tag{10}$$

y

$$\int_0^1\frac{1}{\sqrt{1-x^8}}\,dx=\frac{1}{2\sqrt{2}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k}(-1)^{k+m}\binom{2n-2k-2m}{n-k-m}\binom{2k}{k}\binom{2m}{m}2^{-3n}g(n,k,m) \tag{11}$$

donde

$$g(n,k,m)=\int_0^1 x^{n-k-m-(1/2)}\left((1-x)^2-1\right)^k\left((1-x)^4-1\right)^m\,dx \tag{12}$$

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