Special Relativity & Newton's Second Law

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(2016) Buenos Aires

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In special relativity, this article shows that Newton's second law can be applied in any inertial reference frame.

Introduction

In special relativity, the linear momentum $\mathbf{P}$ of a particle with rest mass m_o is given by the following equation:

$$\mathbf{P} \doteq \frac{m_o\,\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The relationship between the net Einsteinian force $\mathbf{F}$ acting on the particle and the linear momentum $\mathbf{P}$ of the particle is given by:

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = m_o \left[\frac{\mathbf{a}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{(\mathbf{a}\cdot\mathbf{v})\,\mathbf{v}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \right]$$

Now, substituting $\mathbf{a} = \mathbf{1}\cdot\mathbf{a}$ ($\mathbf{1}$ is the unit tensor) and $(\mathbf{a}\cdot\mathbf{v})\,\mathbf{v} = (\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{a}$ ($\otimes$ is the tensor product or dyadic product) we obtain:

$$\mathbf{F} = m_o \left[\frac{\mathbf{1}\cdot\mathbf{a}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{(\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{a}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \right]$$

that is:

$$\mathbf{F} = m_o \left[\frac{\mathbf{1}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{(\mathbf{v}\otimes\mathbf{v})}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \right]\cdot\mathbf{a}$$

The tensor in brackets is defined as the Newton tensor, and the above equation can be rearranged as follows:

$$\left[\frac{\mathbf{1}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{(\mathbf{v} \otimes \mathbf{v})}{c^2 \left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right]^{-1} \cdot \mathbf{F} \; = \; m_o \, \mathbf{a}$$

Identifying the left-hand side as the net Newtonian force $\overline{\mathbf{F}}$ acting on the particle, then we finally obtain:

$$\overline{\mathbf{F}} \; = \; m_o \, \mathbf{a}$$

Therefore, the acceleration $\mathbf{a}$ of a particle is always in the direction of the net Newtonian force $\overline{\mathbf{F}}$ acting on the particle.

The Newtonian Dynamics

In special relativity, if we consider a particle with rest mass m_o then the linear momentum $\mathbf{P}$ of the particle, the net Newtonian force $\overline{\mathbf{F}}$ acting on the particle, the work W done by the net Newtonian force acting on the particle, and the kinetic energy K of the particle, for an inertial reference frame, are given by:

$$\mathbf{P} \; \doteq \; m_o \, \mathbf{v}$$

$$\overline{\mathbf{F}} \; = \; \frac{d\mathbf{P}}{dt} \; = \; m_o \, \mathbf{a}$$

$$\mathrm{W} \; \doteq \; \int_1^2 \overline{\mathbf{F}} \cdot d\mathbf{r} \; = \; \Delta \, \mathrm{K}$$

$$\mathrm{K} \; \doteq \; {}^1\!/_2 \; m_o \, (\mathbf{v} \cdot \mathbf{v})$$

where $(\, \mathbf{r}, \, \mathbf{v}, \, \mathbf{a} \,)$ are the position, the velocity and the acceleration of the particle relative to the inertial reference frame. $\overline{\mathbf{F}} \; = \; \mathbf{N}^{-1} \cdot \mathbf{F}$, where $\mathbf{N}$ is the Newton tensor and $\mathbf{F}$ is the net Einsteinian force acting on the particle.