General covariance, a new paradigm for relativistic Quantum Theory.

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Abstract

We offer a new look on multiparticle theory which was initiated in a recent philosophical paper [1] of the author. To accomplish such feature, we start by a revision and extension of the single particle theory as well relativistically as nonrelativistically. Standard statistics gets an interpretation in terms of symmetry properties of the two point function and any reference towards all existing quantization schemes is dropped. As I have repeatedly stated and was also beautifully explained by Weinberg, there is no a priori rationale why quantum field theory should take the form it does in a curved spacetime; there is no reason why the straightforward generalizations of the Klein Gordon and Dirac theory should have something to do with the real world. Perhaps, if we were to look differently at the flat theory, a completely satisfactory class of relativistic quantum theories would emerge. These may not have anything to do with quantum fields at all except in some limit.

1 Introduction.

Essentially, we still look at the world in terms of a $3 + 1$ decomposition: at least quantum theory is framed in that way as we will study now, and this leads to a myriad of interpretational difficulties when trying to merge (general) relativity with quantum theory. When trying to find an answer to this problem, it is convenient to look at well known theories from distinct, non-conventional angles; such richness should enable one to generalize in *different* ways and it is here that a solution for our puzzle may very well reside. Keeping this possibly important lesson in mind, we start by generalizing the free particle in a Newtonian cosmology and see what we can learn from that, this might at least shape our mind for more important things to come. Indeed, it teaches us we should entirely focus upon the two point function of the theory or the Feynman propagator; as is well known, the latter is not uniquely defined in standard quantum field theory which is a very unsatisfying feature indeed. I somehow hoped that the definition of the Sorkin-Johnston state [6, 7, 5] would be some progress in that direction but alas the latter is only well defined for compact chuncks of spacetime and is not natural. Naturality roughly means that the two

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point function $\omega(\phi(x)\phi(y))$ coincides with the two point function given by any globally hyperbolic subneighborhood containing x and y (modulo some details). As Fewster and Verch [2, 3, 4] have shown in full generality, locally covariant theories satisfying some plausible technical details carry no natural state. In my view, this is an almost leathal objection against their construction given that one would expect a transition amplitude not to depend upon the wholeness of spacetime and indeed this is the picture we shall hunt after. Therefore, instead of turning to algebra to define a relativistic quantum theory, we turn our heads towards geometry as the dominant language for quantum theory; hence we must focus upon objects having a geometrical meaning. The only such quantity in the free theory is the two point function: so instead of endowing it with the usual functional analytic meaning in terms of Nevanlinna spaces, we give it a very beautiful and simple geometric representation. This allows one to couple it to gravity in an infinite number of ways; suffice it to say that not all candidates are suitable as the quantum causality constraint turns out to be quite powerful. So, what we propose is a *natural*, in a slightly weaker sense than Verch and Fewster, construction of the two point function; this fixes the free theory. We shall make an explicit construction for the theory of spin-0, $\frac{1}{2}, 1$ particles. Also, we treat in detail how one should define interacting theories between both types of particles. Obviously, to further deepen the part played by the observer one should couple the system to a measurement apparatus but at least all transition amplitudes have a spacetime significance meaning they transform as scalars under coordinate transformations something which does not happen in standard quantum field theory. In my view, this is a serious step forwards towards defining process physics and I am somewhat happy to see that the basic quantities in the theory do not have to be framed in the language of differential operators. Everything has a geometric significance which should make it not too difficult to lift the theory to discrete spacetimes such as causal sets for example.

The history of quantum field theory is a long one and I have never found the field viewpoint very compelling; the way it is explained the best is by Weinberg [8]. He derives it from a few physical principles including Poincaré covariance; now one may very well argue that Poincaré covariance should not be taken too seriously since it is tied to Minkowski spacetime but at least I know that the resulting theory on Minkowski is the correct one. Also, it shows that the field viewpoint may not be a paradigm at all but something which is tied to Minkowski. This is the point of view taken in this paper; but we do more than that. We *derive* the general point of view, which reduces to the Minkowskian one, from *new* physical principles which are in retrospect much more logical and general than those of Weinberg which are still tied to the operational $3+1$ picture of quantum mechanics. In other words, we develop a "nouvelle cuisine" to talk about those things which were previously thought to belong to quantum mechanics. We do not speak about wave functions, operators, path integrals, action principles at all, but we derive precisely the same thing from something which is much more elementary and easy to swallow than all these previous concepts. This is a considerable step forwards in our understanding of nature and the reader should think about it well; it is not as trivial as it looks. To give an example, spin is implemented by means of $Sl(2,\mathbb{C})$ bundles over spacetime which is the point of view of Fewster [4], this is not the point of view of Weinberg who regards the irreducible *unitary* representations of the Poincaré group as fundamental; it has been shown however [8] that both points of view are equivalent in Minkowski for massive particles. For massless particles, one needs to supplement the $Sl(2,\mathbb{C})$ representation with the kinematical principle of gauge invariance for both viewpoints to be equivalent since the little group for such particles is the two dimensional Euclidean one instead of $SU(2)$. It may be clear that the $Sl(2,\mathbb{C})$ viewpoint is directly suited for curved spacetimes while Weinberg's appears to hold only in Minkowski (unless one works with *local* Fock bundles and likewise representations of the Poincaré group). In any case, it is my viewpoint on spin and nothing extraordinary, we will derive gauge invariance from other principles.

2 The nonrelativistic particle theory extended.

Sometimes, when you are stuck with a problem, it is good to broaden ones horizon and investigate more closely slight generalizations of what you know to be true; the lessons one can draw from such exercise might provide for the crucial insight on how to solve one's problem or even just to figure out if there is a problem in the first place at all. What we will do in this section, constitutes a small extension of the work performed in a previous article of this author [1], more in particular we shall highlight in a better way the conclusions to be drawn from that paper. In particular, we asked ourselves the question about the covariance of nonrelativistic quantum mechanics for point particles; more specifically we would like to have that the interpretation of the wave function Ψ is observer independent. For that Ψ need to transform as a scalar or density of factor $\frac{1}{2}$ under spatial coordinate transformations (see [1] for details). In the first case, we arrived at the conclusion that the correct momentum covector was given by the *covariant* derivative ∇^t_μ which is the Levi Civita derivative associated to the spatial metric. The latter may be time dependent and hence we have a time dependent scalar product given by

$$\langle \Psi_t | \Phi_t \rangle_t = \int \overline{\Psi_t(x)} \Phi_t(x) \sqrt{h_t(x)} dx$$

where $h_t(x)$ is the determinant of the spatial metric. The correct Hamiltonian is then given by

$$H_t(x) = -(h_t)^{\mu\nu}(x) \nabla^t_\mu \nabla^t_\nu$$

and one may want to extend the wave function to the complexified cotangent bundle. That is, one considers covariant tensors $\Psi_{\mu_1 \dots \mu_n}$ for arbitrary n and extends the scalar product in a trivial way, again see [1] for details. Now, obviously, the Hamiltonian $H_t(x)$ constitutes a Hermitian operator with respect to the scalar product $\langle \, | \, \rangle_t$ but in order for time evolution to preserve the norm one should consider the following Schrodinger equation

$$i \frac{d}{dt} \Psi_t(x) = \left(H_t(x) - i \frac{\dot{h}_t(x)}{4 h_t(x)} \right) \Psi_t(x)$$

and therefore, the Hamiltonian gets a non-Hermitian correction by means of a multiplication operator. Therefore, the real physical Hamiltonian $H'_t(x) = H_t(x) - i \frac{\dot{h}_t(x)}{4 h_t(x)}$ is a non-Hermitian operator and one can therefore not even speak anymore about the vacuum or lowest energy state, not even in a time dependent sense. In case one would only consider $H_t(x)$ one obtains time dependent lowest energy states; likewise, in order to define Hermitian momentum operators, deviations from standard wisdom should be considered and they are given by

$$P = -in^{\mu}\nabla^t_{\mu} - \frac{i}{2}\nabla^t_{\mu}n^{\mu}$$

where n is any vector field in space. The standard momentum operators which one gets from a straightforward quantization are not Hermitian and cannot provide for the right Hamiltonian, see [1]. The important lesson to be learned here is that standard Hermitian quantities in the classical theory do not need to correspond to Hermitian quantities in quantum theory and therefore, the entire rationale behind the Born rule and vacuum states dissapears. This strongly suggests that what we should hold dear are things which might be defined in an independent way from operators and vacuum states; for the free theory, these are the two point functions and finding out the raison d'etre behind those precisely constitutes the novel insight we are looking for. This is the content of the next section.

3 The free relativistic theory revisited.

We have learned by now that Hermitian operators do probably *not* constitute the right framework for quantum theory and the only real object of importance is the two point function $W(x,y)$; in the following, we shall give the latter a nice geometric interpretation which can be straightforwardly generalized to curved spacetime. The viewpoint explained in [1] was that $iW(x,y)$ can be regarded as a projection operator where $\overline{W(x,y)} = W(y,x)$ and $i(W - \overline{W}) = E$ where E is the Pauli-Jordan operator which can be seen as the identity operator on the space of solutions to the Klein-Gordon equation. This is a functional analytic point of view which is of considerable difficulty due to the indefinite character of the inner product and associated the arbitrareness in the choice of W. Here, we will take a very different turn and focus on the geometry of the two point function and put the probability interpretation on the second place meaning that our two point functions have no interpretation in terms of projection operators in a general curved spacetime. In Minkowski, the two point function is given by

$$W(x,y) = \int \frac{d^3p}{(2\pi)^3\, 2E_p} e^{ip.(x-y)}$$

where the signature of the metric is $(-+++)$ and $p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$. Another way to write it is

$$W(x,y) = \int \frac{d^4p}{(2\pi)^3} e^{ip.(x-y)} \delta(p^2 + m^2)\theta(p^0)$$

where $\theta(x) = 1$ if $x \geq 0$ and 0 otherwise. The delta and theta function by themselves are nothing mysterious, the factor $e^{ip.(x-y)}$ has so far been understood as something which was a unique feature of Minkowski spacetime being an eigenfunction of the momentum operators and a solution to the Klein Gordon equation. Actually, as Weinberg beautifully explains, the Klein-Gordon equation can be derived from these functions; this urges one to stop for a while and take those functions more seriously. $W(x, y)$ is supposed to be a relational quantity in the sense that it relates the creation of a particle at x to the annihilation of it at y so therefore one should look for such a way to define the exponential function. One can do this and it works as follows: let $\gamma(t)$ be any path between x and y, e_a be a globally defined vierbein which is future oriented (meaning e_0 defines a future oriented timelike vector), then one can freely switch between a Lorentz vector k^a at x and its spacetime counterpart $k^\mu(x) = k^a e_a^\mu(x)$. Now, let us define a function $\phi(x, k^a, y)$ where $k^a e_a(x) \in T\mathcal{M}_x$, hence $\phi : T^\star\mathcal{M} \times \mathcal{M} \to U(1)$ by $\phi(x, k^a, x) = 1$ and

$$\frac{d}{dt}\phi(x, k^a, \gamma(t)) = -i\dot{\gamma}^\mu(t)k_\mu(t)\phi(x, k^a, \gamma(t))$$

where $k_\mu(t)$ is the parallel transported covector of $k_\mu(x)$; that is

$$\frac{D}{dt}k_\mu(t) = 0$$

and $k_\mu(0) = k_\mu(x)$. Now, as one can show for Minkowski spacetime, $\phi(x, k^a, y)$ defined in this way equals $e^{ik(x-y)}$ and is therefore independent of the curve γ joining x and y. From now on, we work with a consistent family of curves, meaning that the curve joining y to x is the reverse of the curve joining x to y and there is precisely one curve joining every pair of points (and trivially, when $x = y$ the curve is just a point). We pose for now no differentiability or continuity property on the class of curves; from the definition, one has the following two properties

$$|\phi(x, k^a, y)|^2 = 1, \;\; \phi(y, k^a_\star, x)\phi(x, k^a, y) = 1$$

where $k^a_\star e^\mu_a(y) = k^\mu(1)$ where we will assume that $\gamma(1) = y$. Now, it is our intention to define the two point function in a general time orientable curved spacetime by means of

$$W_\gamma(x, y) = \int_{T^\star\mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\delta(k^2 + m^2)\theta(k^0)\phi(x, k^a, y).$$

From the equality

$$\overline{\phi(x, k^a, y)} = \phi(y, k^a_\star, x)$$

and the fact that the mapping $\star(x, y); T^\star\mathcal{M}_x \to T^\star\mathcal{M}_y : k^a \to k^a_\star$ is an orthochronous Lorentz transformation, it follows that

$$\overline{W(x, y)} = W(y, x)$$

as it should. So far, our analysis does not depend upon the paths joining x to y; the following demand however leaves in general just one option open:

$$W(x, y) = W(y, x)$$

for all $x \sim y$ where $\sim$ stands for being spacelike related. This is our demand of quantum causality, it says that the amplitude for propagation of a particle between two spacelike separated points x and y does not depend upon the order of the points. We now show that if γ is a geodesic between x and y, then this demand is automatically satisfied. By definition this geodesic must be a spacelike geodesic (it may be possible for timelike separated points to be joined by a spacelike geodesic such as occurs on the timelike cylinder); hence

$$\phi(x, k^a, y) = e^{-ik_a w^a}$$

where $w^a w_a = 2\sigma(x, y)$, w^a is tangent to the geodesic at x and $\sigma(x, y)$ is Synge's function. Equivalently,

$$\phi(x, k^a, y) = e^{i\sigma(x,y)_{,\mu} e^{\mu}_{a}(x) k^a}$$

as the reader may show or $w^a = -e^{a\mu}(x)\sigma_{,\mu}(x, y)$. To prove that the associated two point function satisfies indeed quantum causality, consider the reflection around w^a, the latter is a Lorentz transformation, preserving the sign of k^0 if k^a is a causal vector and maps $k^a w_a$ to $-k^a w_a$; hence, $W(x, y) = \overline{W(x, y)}$ which proves our assertion. In the case of general paths, the reader may easily see that this reflection of k^a does not need to flip the sign of $w(t)(k(t))$ as this quantity is not preserved under general transport; the very preservation requires the geodesic equation to be fullfilled. One can now wonder to what extend the Klein Gordon equation still plays a roll; consider that $W(x, y) \equiv W(\sigma_{,\mu}(x, y))$ satisfies

$$\left(\Box' - m^2\right) W(x, y) = ig^{\alpha'\beta'}\sigma_{,\mu\beta'\alpha'}\frac{\partial}{\partial\sigma_{,\mu}}W(x, y) - m^2 W(x, y) - g^{\alpha'\beta'}\sigma_{,\mu\alpha'}\sigma_{,\nu\beta'}\frac{\partial^2}{\partial\sigma_{,\mu}\partial\sigma_{,\nu}}W(x, y)$$

where primed indices refer to y and unprimed to x and all derivatives of σ are covariant derivatives. The reader now notices that in the coincidence limit $y \to x$, we have that the left and right hand side reduce to zero where we use Synge's rule $[\sigma_{,\mu\beta'}] = -g_{\mu\beta}$ and $[\sigma_{,\mu\alpha'\beta'}] = 0$ where the square brackets indicate that the limit $y \to x$ is taken. Before we proceed, let us stress that our point of view is relational in the sense that it is the way we have build the two point function, the point of view of field operators was absolute in the sense that propagation is derived concept of composite entities whereas here the bifunction is fundamental. Notice also that the above formula gives our covariantization of the flat spacetime equation and as anticipated the right hand side is in general not zero; we will come to other, more substantial deviations later on. Let us also notice that globally we will define $W(x, y)$ by summing over all distinct geodesics joining x and y; this small caveat ensures that we have to be a bit careful with our statement regarding naturality. Our two point function is natural in the sense that it only depends upon the geodesics joining the two points which is as "local" as one may get. There is a useful information interpretation of our formula which is that the information of the creation of a particle travels on geodesics possibly exceeding the local speed of light: therefore, the interacting theory will be constructed as a theory of interacting information currents. We now define the Feynman propagator as $\Delta_F(x, y) = W(x, y)$ if $y \in J^+(x)$, $W(y, x)$ if $x \in J^+(y)$ and $W(x, y) = W(y, x)$ otherwise. It is obvious that the singularity structure of our two point function is of Hadamard type and therefore identical to the one of the standard Minkowski vacuum; this leads to infinite renormalizations which one would preferably avoid. There are several ways of doing this and let us for now concentrate on a physical way of doing so: that is consider the following modification of $W(x,y)$:

$$W(x,y) = \int_{T^\star \mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\, \delta(k^2+m^2)\theta(k^0)e^{-\kappa R_{\mu\nu}k^\mu k^\nu - \kappa R_{\mu'\nu'}k_\star^{\mu'}k_\star^{\nu'}}\,\phi(x,k^a,y)$$

if and only if $x \in J^\pm(y)$ and for $x \sim y$ our expression remains unchanged. Here, we use the fact that $k^\mu_{\star(x,y)\star(y,x)} = k^\mu$ to deduce that W still satisfies $\overline{W(x,y)} = W(y,x)$. Here we of course allude to an energy condition of the type $R_{\mu\nu}V^\mu V^\nu > 0$ for any non-spacelike vector and $\kappa > 0$ so that the above integral converges and hence we remove the singularity structure of the two point function. Physically, this is very appealing since one would expect the gravitational field to give an ultraviolet regulator for quantum physics which is exactly what the above formula tells you. Minkowski spacetime with its associated infinite renormalization difficulties may then be seen as a singular limit where the gravitational energy condition trivializes. Let me also comment on how the Heisenberg commutation relations are hidden in our formalism: $W(x,x)$ is an integral over all on shell momenta, each with an equal amplitude (in the non-modified version) of one which basically means that if you nail the particle at x, the momentum is going to be democratically uncertain. This is precisely the content of Heisenberg's commutation relation, the gravitationally modified two point function hence imposes corrections to that backbone of quantum theory. I think these gravitational modifications are certainly worthwhile studying as they constitute natural candidates regarding the equivalence principle. Also, our "equation of motion" for $\phi(x,k^a,y)$ can be thought of as a covariant substitute for the Schrodinger equation. Since the two point function $W(x,y) = W(y,x)$ for $x \sim y$, exchanging the role of source and receiver is totally symmetric for spacelike separated points, hence the exchange of two particles is which explains Bose statistics. Hence, to violate Bose statistics, it is sufficient for information to travel on different paths than geodesics which is the case for sure when an external force such as the one associated to an observer intervenes.

Before we go over to the interacting theory, let us show in detail how Fermions are included in the formalism. Again, we will almost completely abandon the field viewpoint and concentrate almost exclusively on our story of propagators; as is the case for the free field, it is just no good to simply take the results of the free theory and generalize those. That would be too easy, we aim further than that and try do derive the results of the free theory in flat Minkowski without ever speaking about Hamiltonians, field operators, action principles and so on. So, what I propose is a nouvelle cuisine for quantum theory: a purely geometrical framework with a realist ontology. Since spin enters now the scene and is a degree of freedom of the particle, we must look at finite dimensional irreducible representations of $Sl(2,\mathbb{C})$ such a representation is given by the Dirac representation and one has the γ^a matrices satisfying

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}$$

and $(\gamma^a)^\dagger = \eta^{aa}\gamma^a$ with a special role for γ^0 since

$$\gamma^0(\gamma^a)^\dagger\gamma^0 = \gamma^a.$$

The generators of spin rotations $\mathcal{J}^{ab}$ is given by

$$\mathcal{J}^{ab} = \frac{-i}{4}\gamma^{[a}\gamma^{b]}$$

and the reader may verify that the spin connection is given by

$$\omega^k_{\mu j} = i\omega_{\mu ab}(\mathcal{J}^{ab})^k_j$$

where the $k, j : 0 \ldots 3$ denote spinor indices. Since every component is complex, this signifies there are 6 real degrees of freedom implying one has two particles each with three real degrees of spin. Hence, the spin covariant derivative looks like

$$\nabla^s_\mu = \nabla_\mu + \omega^a_{\mu b} + i\omega_{\mu ab}(\mathcal{J}^{ab})^k_l$$

where $\omega^a_{\mu b}$ is given by

$$\omega^a_{\mu b} = -e^\nu_b \nabla_\mu e^a_\nu$$

and one may directly verify the antisymmetry property

$$\omega_{\mu ab} = -\omega_{\mu ba}.$$

Coming back to the main line of our story, we would like to introduce a function $\phi_m(x, k^a, y)^i_{j'}$ where primed indices again refer to y and m is the mass of the particle such that

$$W(x,y)^i_{j'} = \int_{T^\star \mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\delta(k^2+m^2)\theta(k^0)\phi_m(x,k^a,y)^i_{j'}$$

denotes some propagator. Upper indices refer to spin properties of the vector, lower indices to those of a covector and moreover, annihilation and creation always go in a vector-covector pair. We *agree* that particle creation corresponds to a covector while antiparticle creation corresponds to a vector. So, the above propagator signifies the amplitude for an antiparticle to propagate from x, with spin component i, towards y with spin component j'. Likewise, we should have an amplitude $\psi_m(x, k^a, y)^{j'}_i$ to denote the propagation of a particle from x, with spin i towards y with spin j'. Again, we will proceed in the same way as before, arguing what the coincidence limit $\phi_m(x, k^a, x)$ should look like and then solve for the entire spacetime by using the Schrodinger equation associated to (geodesic) paths γ:

$$\frac{D'^s}{dt}\phi(x,k^a,\gamma(t))^i_{j'} = -i\dot{\gamma}^\mu(t)k_\mu(t)\phi(x,k^a,\gamma(t))^i_{j'}.$$

Indeed, the latter is our replacement for the Dirac equation and we will study its solution later on. Let us start by the most straightforward principles of which the first does not necessarily need to be satisfied in a general curved spacetime but it is for sure true in Minkowski due to spatial homogeneity. That is, the coincidence limit $\phi_m(x, k^a, x)^i_j$ does not depend upon x and it transforms in the adjoint representation of $SL(2, \mathbb{C})$; both taken together imply that our only building blocks are $k_a\gamma^a$ and $m1$ and since we only work with on shell momenta, $\phi_m(x, k^a, x)$ may be chosen of the form $\alpha(-ik_a\gamma^a + \beta m1)$ where α and β are complex numbers (the mass dimension should be zero so that the limit of zero mass gives a nonvanishing result) and the $-i$ has been inserted for future convenience. Now, we arrive at our third and most important principle which says that the creation and annihilation of both a particle and antiparticle with the same four momentum should give a vanishing amplitude on shell, that is:

$$\phi_m(x,k^a,x)\psi_m(x,k^a,x) = \psi_m(x,k^a,x)\phi_m(x,k^a,x) \sim (k^2+m^2).$$

This gives that $\phi_m(x, k^a, x) = \alpha(-ik_a\gamma^a \pm m1)$ and $\psi_m(x, k^a, x) = \alpha'(-ik_a\gamma^a \mp m1)$. Finally, we have our fourth condition which I call the positive energy condition, which says that

$$\frac{1}{4}\mathrm{Tr}(i\gamma^0\phi_m(x, k^a, x)) = k^0 = \frac{1}{4}\mathrm{Tr}(i\gamma^0\psi_m(x, k^a, x))$$

which states that the energy of a particle equals the zero'th component of its momentum vector. This further limits $\alpha = \alpha' = 1$; so we are left with

$$\phi_m(x, k^a, x) = (-ik_a\gamma^a \pm m1), \quad \psi_m(x, k^a, x) = (-ik_a\gamma^a \mp m1)$$

and we now *agree* that the particle propagator $\psi_m(x, k^a, x)$ should come with positive mass. This ends our discussion of the coincidence limit; now, we come to the integration of the Schrodinger equation. The latter is easy and natural and before giving its solution, denote by $(\Lambda^{\frac{1}{2}}(x, y))_i^{j'}$ the *spin* holonomy attached to the preferred geodesic(s) form x to y and similarly for $(\Lambda(x, y))_a^{b'}$ the associated Lorentz holonomy. Thus given our initial conditions, the solutions to the equations read

$$\phi_m(x, k^a, y)^i_{j'} = (-ik_a(\gamma^a)^i_r - m\delta^i_r)((\Lambda^{\frac{1}{2}}(x, y))^{-1})^r_{j'}\phi(x, k^a, y)$$

and

$$\psi_m(x, k^a, y)_i^{j'} = (\Lambda^{\frac{1}{2}}(x, y))_r^{j'}(-ik_a(\gamma^a)^r_i + m\delta^r_i)\phi(x, k^a, y).$$

We will now prove a remarkable property which shows that quantum causality, as it is usually understood, holds for this propagator. Indeed, the very structure of our formulae suggests that there may be a relationship between $\psi_m(x, k^a, y)$ and $\phi_m(y, k_\star^{a'}, x)$ where as before, $k_\star^{a'} = (\Lambda(x, y))_b^{a'}k^b$. Indeed, a small calculation reveals that

$$\begin{aligned}\phi_m(y, k_\star^{a'}, x)_i^{j'} &= (-ik_b((\Lambda(x, y))^{-1})^b_{a'}(\gamma^{a'})^{j'}_{k'} - m\delta^{j'}_{k'})(\Lambda(x, y)^{\frac{1}{2}})^{k'}_i\phi(y, k_\star^{a'}, x) \\ &= (\Lambda^{\frac{1}{2}}(x, y))_l^{j'}\left(-ik_b(\gamma^b)^l_i - m\delta^l_i\right)\overline{\phi(x, k^a, y)}\end{aligned}$$

where we have used on the first line that $\Lambda^{\frac{1}{2}}(x, y) = (\Lambda^{\frac{1}{2}}(y, x))^{-1}$; in the second line, we used covariance of the gamma matrices under joint spin and Lorentz transformations as well as the previous established formula for $\phi(x, k^a, y)$. Now, the way in which this formula becomes useful is by means of the particle and antiparticle propagators:

$$W_p(x, y)_i^{j'} = \int_{T^\star\mathcal{M}_x}\frac{d^4k}{(2\pi)^3}\delta(k^2 + m^2)\theta(k^0)\psi_m(x, k^a, y)_i^{j'}$$

and

$$W_a(x, y)^i_{j'} = \int_{T^\star\mathcal{M}_x}\frac{d^4k}{(2\pi)^3}\delta(k^2 + m^2)\theta(k^0)\phi_m(x, k^a, y)^i_{j'}.$$

Indeed,

$$\begin{aligned}W_a(y, x)_i^{j'} &= \int_{T^\star\mathcal{M}_y}\frac{d^4k_\star}{(2\pi)^3}\delta(k_\star^2 + m^2)\theta(k_\star^0)\phi_m(y, k_\star^{a'}, x)_i^{j'} \\ &= (\Lambda^{\frac{1}{2}}(x, y))_l^{j'}\int_{T^\star\mathcal{M}_x}\frac{d^4k}{(2\pi)^3}\delta(k^2 + m^2)\theta(k^0)\left(-ik_b(\gamma^b)^l_i - m\delta^l_i\right)\overline{\phi(x, k^a, y)}\end{aligned}$$

and we concentrate now on points $x \sim y$ which are exclusively connected by spacelike geodesics. In that case, we could write

$$\phi(x, k^a, y) = e^{-ik_a w^a}$$

where w^a is the spacelike tangent at x to the geodesic connecting x with y. Choosing now a Lorentz frame at x such that the vector w is parallel to the three axis e_3, we perform, as before, a reflection around w given by $k^3 \to -k^3$ to obtain

$$W_a(y,x)_i^{j'} = (\Lambda^{\frac{1}{2}}(x,y))_l^{j'} \int_{T^\star \mathcal{M}_x} \frac{d^4k}{(2\pi)^3} \delta(k^2+m^2)\theta(k^0) \left(-ik_b(\gamma^b)_i^l + 2ik_3(\gamma^3)_i^l - m\delta_i^l\right) e^{-ik_3 w^3}$$

where $e^{-ik_3 w^3} = \phi(x, k^a, y)$. Summing this formula with $W_p(x,y)_i^{j'}$ gives

$$W_p(x,y)_i^{j'} + W_a(y,x)_i^{j'} = (\Lambda^{\frac{1}{2}}(x,y))_l^{j'} \int_{T^\star \mathcal{M}_x} \frac{d^4k}{(2\pi)^3} \delta(k^2+m^2)\theta(k^0) \left(-2i \sum_{j=0\ldots 2} k_j(\gamma^j)_i^l\right) e^{-ik_3 w^3}$$

which is immediately seen, due to the antisymmetry of some part of the integrand under $k_1, k_2 \to -k_1, -k_2$, to reduce to

$$(\Lambda^{\frac{1}{2}}(x,y))_l^{j'} (\gamma^0)_i^l i \int_{T^\star \mathcal{M}_x} \frac{d^3k}{(2\pi)^3} e^{-ik_3 w^3}$$

which equals $\delta^3(w^a)$ which proves that

$$W_p(x,y)_i^{j'} + W_a(y,x)_i^{j'} = 0.$$

This is the well known statement that the amplitude for a particle with spin i to travel form x to y and be annihilated with spin j' equals *minus* the amplitude for an antiparticle with spin j' to travel from y to x where it is annihilated with spin i. The very minus sign reveals that spin-$\frac{1}{2}$ particles are fermions, meaning that exchanging two particles comes with a minus sign; this constitutes the proof of the spin statistics theorem in our setting at least for spin-0 and spin-$\frac{1}{2}$ particles. As before, we can now define the Feynman propagator for particle propagation $\Delta_{F,p}(x,y)_i^{j'} = W_p(x,y)_i^{j'}(x,y)$ if $y \notin J^-(x)$ and $-W_a(y,x)_i^{j'}$ otherwise; note that this definition could be framed more democratically when $x \sim y$. We also could define a Feynman propagator for anti-particle propagation as $\Delta_{F,a}(x,y)_{j'}^i = W_a(x,y)_{j'}^i$ if $y \notin J^-(x)$ and $-W_p(y,x)_{j'}^i$ otherwise. The reader however immediately notices that $\Delta_{F,a}(x,y)_{j'}^i = -\Delta_{F,p}(y,x)_{j'}^i$ as is the case in Minkowski quantum field theory. This concludes our discussion of the free Fermi theory and the reader notices that all salient features of the standard Minkowski theory have been saved. We can now, as in the previous case suggest gravitational modifications of the two point function for causally related points such that causality remains valid but the singularity structure of the propagator changes. The way to do this is exactly identical to the one suggested before for the scalar two point function and therefore, we do not have to discuss this further on here. Evidently, our propagator does not satisfy the Dirac equation anymore and the reader is invited to investigate if the latter would still hold in the coincidence limit $y \to x$ just as the Klein Gordon equation did for the scalar two point function. We now turn to the investigation of spin-1 particles.

4 Spin one “gauge” particles.

In contrast to what one may expect, the two point function for massless spin-1 particles is extremely easy to guess, even when they carry another charge such as is the case for non-abelian gauge theories. We do not speak anymore in terms of gauge transformations which were necessitated by the quantum field viewpoint [8] but we derive the main formula for the two point function and the Feynman propagator from two simple demands. The reader should appreciate the plain simplicity of the construction as the computation of the two point function for non-abelian gauge fields in standard QFT is a matter of laborious work, the proof that gauge particles satisfy bosonic statistics is evident. Hence, we are interested in computing a quantitity

$$W^{\alpha\beta'}_{\mu\nu'}(x,y) = \int_{T^{\star}\mathcal{M}_x} \frac{d^4k}{(2\pi)^3}\delta(k^2)\theta(k^0)\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'}$$

and again, we derive the correct form of the two point function. Note here that our group transformations are global transformations and therefore do *not* depend upon the spacetime point; so, the indices α, β' stands for the adjoint representation of the compact simple Lie group, see Weinberg [8], whose algebra is defined by

$$[t_\alpha, t_\beta] = i f^{\gamma}_{\alpha\beta} t_\gamma$$

where $f_{\alpha\beta\gamma}$ is totally antisymmetric and the positive definite invariant Cartan metric is given by $g_{\alpha\beta}$. The fact that we do not make any distinction between covariant and contravariant vectors is due to the possibility to raise and lower indices with both metrics $g_{\mu\nu}$ and $g_{\alpha\beta}$. Let us study the coincidence limit $y \to x$ of $\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'}$ first. Since there is no mass parameter, the only object of mass dimension zero which we can write down is a multiple of $g_{\mu\nu}g^{\alpha\beta}$, the only other term one can write down on shell has mass dimension squared and is given by a multiple of $k_\mu k_\nu g^{\alpha\beta}$. So here, we make our first law, $\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'}$ has mass dimension zero which is logical since it concerns a particle of zero mass. We can absorb any positive, real constant in the definition of the Cartan metric, so we obtain that

$$\psi(x,k^a,x)^{\alpha\beta}_{\mu\nu} = g_{\mu\nu}g^{\alpha\beta}.$$

Writing out our Schrodinger equation is extremely easy

$$\frac{D'}{dt}\psi(x,k^a,\gamma(t))^{\alpha\beta'}_{\mu\nu'} = -i\left[\dot{\gamma}(k)\right](t)\psi(x,k^a,\gamma(t))^{\alpha\beta'}_{\mu\nu'}$$

and when $\gamma(t)$ is a geodesic, the solution is given by

$$\psi(x,k^a,y)^{\alpha\beta'}_{\mu\nu'} = g_{\mu\nu'}(x,y)\phi(x,k^a,y)g^{\alpha\beta'}$$

where $g_{\mu\nu'}(x,y)$ denotes the parallel transport of the metric along the geodesic. The latter can be written as a composition of the Van Vleck matrix with Synge's function and since the metric is covariantly constant one has that $g_{\mu\nu'}(x,y) = g_{\nu'\mu}(y,x)$. Inserting this into the formula for the two point function gives

$$W^{\alpha\beta'}_{\mu\nu'}(x,y) = g_{\mu\nu'}(x,y)g^{\alpha\beta'}W(x,y)$$

which shows that the two point[1] function for spin-1 particles transforming under a global, compact symmetry group is determined by the two point function of the scalar theory, a transporter and the Cartan metric. From our previous results and the symmetry of the transporter as well as the Cartan metric it follows that

$$W^{\alpha\beta'}_{\mu\nu'}(x,y) = W^{\beta'\alpha}_{\nu'\mu}(y,x)$$

for $x \sim y$ so that our theory satifies quantum causality and has bosonic exchange properties. Clearly, massless spin-1 particles are their own antiparticles as there exists only one two point function and not two. Let us better understand the magic which happened here; instead of following the quantization procedure of a theory with a local gauge symmetry and impose a gauge, we simply took the transformation group of the quantum numbers to be a global one. This is a meaningful point of view since those numbers themselves do not correspond to any force field, they are attributes of particles which is something different. It is possible to introduce classical gauge fields and introduce a dynamical gauge bundle so that we have to use the holonomies associated to this gauge field. This would be new physics and I hold it entirely possible that the future may lead us there; for now, we obtain on one sheet of paper a result which can be found in every textbook and which requires a long introduction to derive. As mentioned in the previous section, the structure constants $f_{\alpha\beta\gamma}$ and Cartan metric $g_{\alpha\beta}$ will be used to build interactions, everything is perfectly consistent with QCD and QED. The Feynman propagator $\Delta^{\alpha\beta'}_{F\,\mu\nu'}(x,y)$ is given by

$$\Delta^{\alpha\beta'}_{F\,\mu\nu'}(x,y) = g_{\mu\nu'}(x,y) g^{\alpha\beta'} \Delta_F(x,y)$$

which concludes the discussion for spin one particles. We now come to the discussion of ghosts; first, let us ask ourselves why we insist upon spin-1 particles to transform in the adjoint representation and spin-$\frac{1}{2}$ in the defining one. The general reason is that it allows us to write down intertwiners of the kind

$$(\gamma^a)^i_j e^\mu_a(x)(t_\alpha)^m_n$$

and as the reader may verify, this is the only way to couple spin-1 and spin-$\frac{1}{2}$ particles. This leaves us with the question of coupling spin-0 particles to spin-1, the relevant intertwiner is given by

$$f_{\alpha\beta\gamma}\nabla^\mu$$

where the derivative acts on the gauge boson propator only and therefore these spin-0 particles should transform as a vector in the adjoint representation. Such particles could be coupled to ordinary spin-$\frac{1}{2}$ matter though by means of the intertwiner

$$f_{\alpha\beta\gamma}(t^\alpha)^m_n \delta^i_j$$

and it is very easy to derive the unique propagator having the correct transformation properties

$$W^{\alpha\beta}(x,y) = g^{\alpha\beta} W(x,y)$$

which proves that such particles, if they would exist, should behave like massless bosons. What the standard quantization of gauge theories tells you is that

[1]The fact that we need the Cartan metric for the construction of the two point function is precisely the reason why the Lie group had to be compact and simple in the first place.

one should use “fermionic rules” for them and that they should not couple to matter at all. In the next chapter, I will try to seek for a physical principle leading to such conclusion as we will still need some novel input to correctly relate the coupling constants of the different intertwiners. The correct Feynman propagator reads

$$\Delta_F^{\alpha\beta}(x,y) = g^{\alpha\beta}\Delta_F(x,y)$$

where we use the massless propagator and the reader should notice that we have provided a reason originating from propagator considerations as to why these particles should exist as opposed to a technical one originating from some gauge fixing procedure within the Feynman path integral framework.

5 Towards a definition of the interacting theory.

This last section differs from all others in the sense that it is not complete but we work our way towards a fuller comprehension of the theory; rougly speaking, we aim to define the theory by means of Feynman diagrams. One might call this a perturbative definition albeit there is no real reason to do that: the point of view which will be explained here is the one of “integrated” processes and some of the very ideas behind it have been explained in [1]. What we shall *not* do is simply to try to mimic the definition coming from the perturbative Minkowski approach; indeed, as before, we will try to give a rationale for *why* things are the way they are. Let me point out one important lesson already: I have so far not commented upon the definition of the Feynman propagator, so let us do this now. The latter concept reveals that information cannot propagate to the relativistic past and that it does not matter in which direction it propagates if the points are spacelike separated. So, in the real world, information does travel faster than light, it just cannot travel into the relativistic past; this is my version of the quantum causality condition. The reader should note that this has nothing to do with operators anymore and with commutativity at spacelike distances; that requirement is stronger but roughly equivalent to the one we are going to set up once we treat n-point functions. Before we proceed, let us stress again how remarkable it is that so far we managed to sidestep any language of action principles, path integrals, wave equations, operators, Grassmann numbers, gauge invariance, Hilbert spaces, Hamiltonians and have still been able to reach the same conclusions from self-evident covariant Schrodinger equations. Also, we partially found already a rationale for ghost particles without ever speaking about expressing some determinant as a functional integral over Grassmann fields transforming in the adjoint representation. Obviously, we will need a novel justification for *local* gauge covariance and it is this amongst others we shall be looking for in this section. Actually, we will derive local gauge covariance from two logical principles involving the coupling of particles to themselves and the gravitational field.

As is usual, we will assume that information interacts in points and we now start with the task to write down all allowed irreducible intertwiners for two point functions at those vertices. Let us therefore generalize our results and allow for spin-0, $\frac{1}{2}$ particles to carry extra quantum numbers m, n in the defining representation of a compact simple Lie group with an associated spin-1 particle carrying a quantum number α, β in the adjoint representation; as usual,

μ, ν denote spacetime indices a, b Lorentz indices and i, j spin indices. We argue that any interaction term between two identical particles must vanish since they belong to the free theory; in particular the coupling

$$g^{\mu\nu}(x) g_{ab}$$

between two gauge particles is forbidden, it just produces a process of annihilation and recreation of the same particle, with the same quantum numbers, which is not an interaction but something which indeed belongs to the realm of the free theory. Moreover, it has only mass dimension of two instead of four so that we would need a coupling constant of dimension mass squared in a theory of massless particles. We have already seen the interaction term

$$\tilde{g} (\gamma^a)^i_j e^\mu_a(x) (t_\alpha)^m_n$$

between two spin-$\frac{1}{2}$ particles and a gauge particle where $\tilde{g}$ defines the coupling constant of the theory, this is the only irreducible matter-gauge coupling. For the same reason as before do we expell matter-matter couplings of the kind

$$\delta^m_n \delta^i_j$$

they are already included in the free theory and have mass dimension three instead of four; there are still two other matter-matter coupling terms given by

$$(\gamma^a)^i_j \eta_{ab} (\gamma^b)^k_l (t_\alpha)^m_n (t_\beta)^o_p g^{\alpha\beta}, \;\; (\gamma^a)^i_j \eta_{ab} (\gamma^b)^k_l \delta^m_n \delta^o_p$$

both of which we exclude by demanding that matter can only interact by means of gauge particles, they are the transmitters of information; moreover, these terms have mass dimension 6 instead of 4. Now, we arrive at the self-interaction terms for non-abelian gauge bosons: we consider the following two irreducible candidates

$$-a \tilde{g} f_{\alpha\beta\gamma} \left(\partial_\kappa \left(\begin{array}{c} \alpha \\ \mu \end{array} \right) \right) \left(\begin{array}{c} \beta \\ \nu \end{array} \right) \left(\begin{array}{c} \gamma \\ \lambda \end{array} \right) g^{\nu\kappa}(x) g^{\mu\lambda}(x)$$

and

$$-\frac{1}{4} b \tilde{g}^2 f_{\alpha\beta\gamma} f^{\alpha}{}_{\beta'\gamma'} g_{\mu\nu}(x) g_{\kappa\lambda}(x)$$

where the factor $\frac{1}{4}$ has been included for symmetry reasons. Note that all of our accepted irreducible interactions come with a mass dimension of four, one for every gauge boson (the propagator has mass dimension two) and three for every pair of spin-$\frac{1}{2}$ particles (the propagator has mass dimension three), so $\tilde{g}, a, b$ are all dimensionless. These two candidates do not include all irreducible possibilities, however, and the reader may want to write down the remaining terms of mass dimension four. Those, however, do not consist out of antisymmetric pairs of μ, ν indices given that our above two terms are build out of "commutators". This constitutes a part of the principle we are looking for: only self interaction terms build from "commutator" terms are allowed for. They reflect that gauge particles can only couple to the Lie algebra generators t_α and the derivative ∇_ν and that all interactions must be constructed from commutators of these terms. These facts are natural and have nothing to do with gauge invariance: saying that ∇_μ couples only antisymmetrically to the gauge particles in interaction vertices boils down to saying that the gravitational force does not couple to the gauge field during interactions. The commutator terms of the Lie algebra couplings are the only thing one could write down since the latter constitutes the intrinsic operation of the symmetry algebra. The previous principle does not explain yet the values of a, b but it does say that $a^2 = b$ so we can get away with them by absorbing a in the definition of the structure constants $f_{\alpha\beta\gamma}$. Therefore, we have derived the correct couplings from the demands of (a) mass dimension four (b) constructed from couplings to the Lie algebra generators and no coupling to the gravitational connection. So, just like Weinberg argued, "gauge invariance" is a consequence of deeper underlying physical principles and this is precisely what I was aiming for. Remains to write down the interaction vertices for the "ghosts" as well as the scalar Brout-Englert-Higgs particles. The interaction terms for the ghost particles have been explained in the previous section and we need to find a rationale as to why the coupling term to spin-$\frac{1}{2}$ particles should not exist as well as why the ghosts are fermions. Honestly speaking, I have no good physical reason for this and neither does the standard derivation in the path integral framework; there one simply observes that ghosts carry "the wrong statistics" and hence should not be observed in nature. In other words, one declares rather by fiat that one should not compute ghost-ghost and ghost-gauge boson amplitudes. Let us agree from the beginning that *if* we knew why ghosts should be unobservable and have the wrong statistics, then the coupling to ordinary spin-$\frac{1}{2}$ particles would vanish since fermions do not interact directly amongst one another. Let me propose a trick here which attaches to a scalar particle its "ghost" Fermi-complement, that is define

$$W_p^{\alpha\beta}(x, y) = \theta(x)\overline{\theta}(y)g^{\alpha\beta}W(x, y)$$

and

$$W_a^{\alpha\beta}(x, y) = \overline{\theta}(x)\theta(y)g^{\alpha\beta}W(x, y)$$

with $\theta, \overline{\theta}$ forming the standard complex Grassmann algebra, then the word ghost is rather well chosen since the square (but not the modulus squared) of these amplitudes vanish. Moreover, if $x \sim y$, then

$$W_p^{\alpha\beta}(x, y) + W_a^{\beta\alpha}(y, x) = 0$$

which reveals fermionic statistics. This reasoning is moreover perfectly fine since, as the reader may verify,

$$W_p^{\alpha\beta}(x, y)W_a^{\gamma\delta}(y, x) = 0 = W_p^{\alpha\beta}(x, y)\overline{W_a^{\gamma\delta}(x, y)}$$

where the conjugation also applies to the Grassmann algebra[2], implying that the amplitude for instantaneous particle-anti particle pair creation and annihilation vanishes and therefore our novel principle explained in section three is satisfied. This would answer our previous questions: we explained why ghosts are unobservable which is connected to them having the wrong statistics. This does not explain yet *why* we should include ghosts of scalar particles transforming in the adjoint representation in our interaction framework. Strictly

[2] In case the conjugation did not apply to the Grassmann algebra, we would obtain the result that $W_p^{\alpha\beta}(x, y)\overline{W_p^{\gamma\delta}(x, y)} = 0$ which is paramount to saying that the ordinary modulus squared of the amplitude vanishes.

speaking, the fact that for gauge invariant theories, one needs to fix the gauge before quantization is something which has been added to the framework of quantum mechanics due to Dirac's analysis of how to deal with first and second class constraints in Hamiltonian systems. In the meantime, one has developped quantization methods which preserve manifest gauge invariance but give up on manifest Poincaré covariance due to the choice of a grid for spacetime. I did not enter into that representation of affairs since I find my point of view somewhat more universal and applicable to a wide range of circumstances. Therefore, I feel free to add a novel principle to quantum theory which is the first nontrivial one, which is that any theory of gauge spin-1 bosons should include the according spin-0 *ghost* particles transforming in the adjoint representation. For quantum electrodynamics, this means one has strictly speaking one ghost, but it doesn't couple to anything since the structure constants vanish; it is only at the level of non-abelian gauge theory that ghosts will really play a part. In the light of the above, I leave the rather trivial discussion of spin-0 Higgs particles transforming in the defining representation of the gauge group as an exercise for the reader.

This finishes for now our analysis of the structure of interaction vertices; we did not impose yet any restriction upon the allowed diagrams and neither did we say anything about the domain of integration for the intermediate vertices, nor on how to apply the correct statistics rules. However, except for the domain of integration, these matters are all fairly standard and we leave the detailed discussion for the future.

6 Conclusions.

Albeit this paper is rather short, we did succeed in a couple of things: we derived the spin-$0, \frac{1}{2}, 1$ theories from scratch and provided one with a fully covariant interpretation of the two point function. The spin statistics theorem came freely and chiefly relies on a positive energy criterion as well as some natural constraint regarding simultaneous particle-anti particle creation. This is a great advance and definetly a stronger result regarding this theorem than the one obtained by Fewster who relied upon the flat spacetime result by conventional techniques. I have shown how gravity may serve as a regulator to tame the UV divergencies in the formalism and have defined, again from first principles, the interacting theory. It is noteworthy that our theory is not a locally covariant theory in any sense as Brunetti, Fredenhagen, Verch and Fewster intend to construct but there is a sense in which our construction is natural and as "local" as possible albeit global winding of geodesics on a closed space for example may spoil the construction. Very few, if almost no limitations, on spacetime exist; the only demand being that it must be almost everywhere time orientable and that no closed timelike curves exist which would spoil the definiton of the Feynman propagator. This is certainly a much better situation than the one occuring in attempts to generalize field theory to curved spacetime; here, global hyperbolicity is almost a paradigm. Our construction is intrinsically four dimensional and all reference towards an observer, even in the choice of state, has been removed; this was the prime rationale for the author to consider alternative constructions. The very simplicity and naturality of the construction is extremely encouraging; to reach all results obtained in this paper, Weinberg

[8] needs around 300 pages and this only for the flat theory, he does not reveal how his results can be generalized to the case of a general gravitational field and neither does he mention how the gravitational field may serve as an UV regulator to obtain finite perturbative results. In my opinion, this means we did something good here and explicit computations in a general curved background satisfying appropriate energy conditions should follow.

7 Acknowledgements.

I remind a discussion some 15 years ago where Rafael Sorkin tried to convince me of the importance of two point functions in the construction of quantum field theory where I was someone more insisting upon the operator picture. It is somehow amusing to see that this viewpoint also dismisses the path integral, something which Sorkin did not anticipate.

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