A novel approach to the discovery of ternary BBP-type formulas for polylogarithm constants*

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Abstract

Using a clear and straightforward approach, we prove new ternary (base 3) digit extraction BBP-type formulas for polylogarithm constants. Some known results are also rediscovered in a more direct and elegant manner. A previously unproved degree 4 ternary formula is also proved. Finally, a couple of ternary zero relations are established, which prove two known but hitherto unproved formulas.

Contents

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1 Introduction

BBP-type formulas are formulas of the form

$$\alpha = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^{n} a_j/(kn+j)^s$$

where s, b, n and a_j are integers, and α is some constant. Formulas of this type were first introduced in a 1996 paper [1], where a formula of this type for π was given. Such formulas have the remarkable property that they permit one to calculate base-b digits of the constant α beginning at an arbitrary starting position, by means of a simple algorithm that requires almost no memory and (depending on how many digits are required) without the need for multiple-precision arithmetic software [2]. Such formulas also have intriguing connections to the age-old problem of understanding why the digits of various transcendental constants appear "normal" – each string of m-long digits appears, in the limit, with frequency $1/b^m$ [2, 3, 4, 5].

While many binary BBP-type formulas are now known, only relatively few ternary (base-3) BBP-type formulas have been discovered. This present paper is concerned with the symbolic (that is, non-computer-search-based) discovery of ternary (base-3) BBP-type formulas for polylogarithm constants. The methods used here aim to complement the experimental approaches that have dominated the area. Through fundamental methods, a wide range of interesting formulas will be obtained. In most cases, the procedure for obtaining the ternary formulas shall consist mainly of evaluating a polylogarithm functional equation at indicated coordinates and noting the following identities for the real and imaginary parts of the polylogarithm function:

$$\begin{aligned} \mathrm{Re}\ \mathrm{Li}_s\left[pe^{ix}\right] &= \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k^s}\,, \\ \mathrm{Im}\ \mathrm{Li}_s\left[pe^{ix}\right] &= \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k^s}\,, \end{aligned} \tag{1.1}$$

for $p \in [0, 1]$, $x \in \mathbb{R}$ and $s \in \mathbb{Z}^+$. In the above equations, Li is the notation for the polylogarithm function defined by

$$\mathrm{Li}_s[z] = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| \le 1\,.$$

When $p = 1$ we have

$$\begin{aligned} \mathrm{Li}_{2n}[e^{ix}] &= \mathrm{Gl}_{2n}(x) + i\mathrm{Cl}_{2n}(x) \\ \mathrm{Li}_{2n+1}[e^{ix}] &= \mathrm{Cl}_{2n+1}(x) + i\mathrm{Gl}_{2n+1}(x)\,, \end{aligned}$$

where Gl and Cl are Clausen sums [6] defined, for $n \in \mathbb{Z}^+$ by

$$\begin{aligned} \mathrm{Cl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n}}, \quad \mathrm{Cl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n+1}} \\ \mathrm{Gl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n}}, \quad \mathrm{Gl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n+1}}\,. \end{aligned}$$

We shall find the following formulas useful:

$$\begin{aligned} \mathrm{Gl}_{2n}(x) &= (-1)^{1+[n/2]} 2^{n-1} \pi^n \mathrm{B}_n(x/2\pi)/n! \\ \frac{1}{m^{n-1}} \mathrm{Cl}_n(mx) &= \sum_{r=0}^{m-1} \mathrm{Cl}_n(x + 2\pi r/m)\,. \end{aligned}$$

Here $[n/2]$ denotes the integer part of $n/2$ and B_n are the Bernoulli polynomials defined by

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{\mathrm{B}_n(x) t^n}{n!}\,.$$

2 Degree 1 Ternary BBP-type Formulas

In reference [7], several degree 1 BBP-type formulas in general bases are proven. In many of the formulas, ternary formulas may be readily obtained by writing the base in each case as a power of 3.

Here we now present a couple of interesting degree 1 ternary formulas.

The following identities are easily verified:

$$\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]=\frac{1}{2}\ln 3+\frac{i\pi}{6}$$

and

$$\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]=\frac{1}{2}\ln 3-\ln 2+\frac{i\pi}{6}\,.$$

We therefore have the formulas:

$$\ln 2=\mathrm{Re}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]-\mathrm{Re}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]\,, \tag{2.1}$$

$$\ln 3=2\mathrm{Re}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] \tag{2.2}$$

and

$$\pi=6\,\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]=6\,\mathrm{Im}\,\mathrm{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]\,. \tag{2.3}$$

It is also straightforward to verify that:

$$\begin{aligned}&\ln 2=\mathrm{Li}_1\left[\frac{1}{3}\right]-\mathrm{Li}_1\left[-\frac{1}{3}\right]\\&\text{and}\\&\ln 3=2\mathrm{Li}_1\left[\frac{1}{3}\right]-\mathrm{Li}_1\left[-\frac{1}{3}\right]\,.\end{aligned} \tag{2.4}$$

Based on the above identities, we are now ready to derive explicit BBP-type formulas for $\ln 2$, $\ln 3$ and π.

2.1 Ternary formulas for $\ln 2$

Using the first equality of (1.1), we note that

$$\begin{aligned}\operatorname{Re}\operatorname{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] &= \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^k\frac{\cos(k\pi/6)}{k}\\ &= \frac{1}{2\cdot 3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^6}{12k+1}+\frac{3^5}{12k+2}\right.\\ &\quad -\frac{3^4}{12k+4}-\frac{3^4}{12k+5}-\frac{2\cdot 3^3}{12k+6}-\frac{3^3}{12k+7}\\ &\quad \left.-\frac{3^2}{12k+8}+\frac{3}{12k+10}+\frac{3}{12k+11}+\frac{2}{12k+12}\right]\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\operatorname{Re}\ \operatorname{Li}_1\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] &= \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[-\frac{3^5}{12k+2}+\frac{3^4}{12k+4}\right.\\ &\quad \left.-\frac{3^3}{12k+6}+\frac{3^2}{12k+8}-\frac{3}{12k+10}+\frac{2}{12k+12}\right].\end{aligned} \tag{2.6}$$

Subtracting (2.6) from (2.5) in accordance with (2.1), we obtain the following ternary BBP-type formula for $\ln 2$:

$$\begin{aligned}\ln 2 = \frac{1}{2\cdot 3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\frac{3^5}{12k+1}+\frac{3^5}{12k+2}\right.\\ &-\frac{3^4}{12k+4}-\frac{3^3}{12k+5}-\frac{3^2}{12k+7}-\frac{3^2}{12k+8}\\ &\left.+\frac{3}{12k+10}+\frac{1}{12k+11}\right].\end{aligned} \tag{2.7}$$

Note that an alternating version of (2.7), using the same scheme, is

$$\ln 2 = \frac{1}{18}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{6k+1}+\frac{9}{6k+2}-\frac{3}{6k+4}-\frac{1}{6k+5}\right].$$

From the first equality of (2.4), we can obtain yet another ternary formula for $\ln 2$. We first note that

$$\mathrm{Li}_1\left[\frac{1}{3}\right] = \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{3}{2k+1}+\frac{1}{2k+2}\right] \tag{2.8}$$

and

$$\mathrm{Li}_1\left[-\frac{1}{3}\right] = \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[-\frac{3}{2k+1}+\frac{1}{2k+2}\right]. \tag{2.9}$$

Subtracting (2.9) from (2.8) in accordance with the first equality of (2.4), we obtain

$$\ln 2 = \frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{1}{2k+1}\right], \tag{2.10}$$

which is listed in the BBP Compendium as formula (64).

2.2 Ternary formulas for $\ln 3$

From (2.5) and (2.2), we obtain the following ternary BBP-type formula for $\ln 3$:

$$\begin{aligned}\ln 3 = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\frac{3^6}{12k+1}+\frac{3^5}{12k+2}\\ &-\frac{3^4}{12k+4}-\frac{3^4}{12k+5}-\frac{2\cdot 3^3}{12k+6}-\frac{3^3}{12k+7}\\ &-\frac{3^2}{12k+8}+\frac{3}{12k+10}+\frac{3}{12k+11}+\frac{2}{12k+12}\Bigg].\end{aligned} \tag{2.11}$$

An alternating version of the above formula is

$$\ln 3 = \frac{1}{27}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{27}{6k+1}+\frac{9}{6k+2}-\frac{3}{6k+4}-\frac{3}{6k+5}-\frac{2}{6k+6}\right].$$

Combining (2.8) and (2.9) according to the second equality of (2.4) we obtain another ternary BBP-type formula for $\ln 3$ as :

$$\ln 3 = \frac{1}{9}\sum_{k=0}^{\infty}\frac{1}{9^k}\left[\frac{9}{2k+1}+\frac{1}{2k+2}\right], \tag{2.12}$$

which is Formula (67) of the Compendium.

2.3 Ternary formulas for $\pi\sqrt{3}$

From (2.3), we immediately obtain the ternary BBP-type formulas

$$\begin{aligned}\pi\sqrt{3} = \frac{1}{3^4}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\bigg[&\frac{3^5}{12k+1}+\frac{3^5}{12k+2}+\frac{2\cdot 3^4}{12k+3}\\ &+\frac{3^4}{12k+4}+\frac{3^3}{12k+5}-\frac{3^2}{12k+7}-\frac{3^2}{12k+8}\\ &-\frac{2\cdot 3}{12k+9}-\frac{3}{12k+10}-\frac{1}{12k+11}\bigg]\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\pi\sqrt{3} = \frac{2}{3^4}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\bigg[&\frac{3^5}{12k+1}-\frac{3^4}{12k+3}+\frac{3^3}{12k+5}\\ &-\frac{3^2}{12k+7}+\frac{3}{12k+9}-\frac{1}{12k+11}\bigg].\end{aligned} \tag{2.14}$$

An alternating version of (2.14) is

$$\pi\sqrt{3} = 6\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k\left[\frac{1}{2k+1}\right]. \tag{2.15}$$

2.4 Ternary Zero Relations

Identity (2.10) may be rewritten in base 3^6, length 12 as

$$\ln 2 = \frac{4}{3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^4}{12k+2}+\frac{3^2}{12k+6}+\frac{1}{12k+10}\right]. \tag{2.16}$$

Subtracting (2.16) from (2.7), we obtain the following ternary zero relation:

$$
\begin{aligned}
0 = \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \Bigg[&\frac{3^5}{12k+1} - \frac{5 \cdot 3^4}{12k+2} - \frac{3^4}{12k+4} \\
&- \frac{3^3}{12k+5} - \frac{2^3 \cdot 3^2}{12k+6} - \frac{3^2}{12k+7} - \frac{3^2}{12k+8} \\
&- \frac{5}{12k+10} + \frac{1}{12k+11} \Bigg] .
\end{aligned} \tag{2.17}
$$

Let $R127$ denote the right hand side of Compendium formula 127 and $R128$ the right hand side of Compendium formula 128. We note that

$$
\text{equation (2.17)} \equiv R128 - R127 = 0\,. \tag{2.18}
$$

Subtracting (2.13) from (2.14), we obtain the zero relation:

$$
\begin{aligned}
0 = \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \Bigg[&\frac{243}{12k+1} - \frac{243}{12k+2} - \frac{324}{12k+3} \\
&- \frac{81}{12k+4} + \frac{27}{12k+5} - \frac{9}{12k+7} + \frac{9}{12k+8} \\
&+ \frac{12}{12k+9} + \frac{3}{12k+10} - \frac{1}{12k+11} \Bigg] .
\end{aligned} \tag{2.19}
$$

Note also that

$$
\text{equation (2.19)} \equiv R128 + R127 = 0 \tag{2.20}
$$

Equations (2.18) and (2.20) therefore establish that

$$
\begin{gathered}
R127 = 0 \\
\text{and} \\
R128 = 0\,.
\end{gathered}
$$

Thus the hitherto unproved formulas (127) and (128) in the BBP Compendium are now proved.

3 Degree 2 Ternary BBP-type Formulas

The dilogarithm reflection formula (equation A.2.1.7 of [6]) is

$$\frac{\pi^2}{6} - \ln x \ln(1-x) = \mathrm{Li}_2[x] + \mathrm{Li}_2[1-x]\,.$$

Putting $x = -\exp(i\pi/3)$ in the above identity and taking real and imaginary parts we find

$$\frac{5\pi^2}{72} - \frac{1}{8}\ln^2 3 = \mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] \tag{3.1}$$

and

$$\frac{2}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi\ln 3}{12} = \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]\,. \tag{3.2}$$

A two-variable functional equation for dilogarithms, due to Kummer (equation A.2.1.19 of [6]) is

$$\begin{aligned}\mathrm{Li}_2\left[\frac{x(1-y)^2}{y(1-x)^2}\right] = \mathrm{Li}_2\left[-\frac{x(1-y)}{(1-x)}\right] + \mathrm{Li}_2\left[-\frac{(1-y)}{y(1-x)}\right] \\ + \mathrm{Li}_2\left[\frac{x\,(1-y)}{y\,(1-x)}\right] + \mathrm{Li}_2\left[\frac{1-y}{1-x}\right] + \frac{1}{2}\ln^2 y\,.\end{aligned} \tag{3.3}$$

Choosing $x = -\exp(i\pi/3)$ and $y = \exp(i\pi/3)$ in (3.3) gives

$$\frac{\pi^2}{12} - \frac{\ln^2 3}{4} = \mathrm{Li}_2\left[\frac{1}{3}\right] - \frac{1}{2}\mathrm{Li}_2\left[-\frac{1}{3}\right]\,. \tag{3.4}$$

Note that the choice of $x = -1$ and $y = 1/3$ gives the same result.

Another two-variable functional equation for dilogarithms, due to Abel (equation A.2.1.16 of [6]) is

$$\begin{aligned}\mathrm{Li}_2\left[\frac{x}{1-x}\cdot\frac{y}{1-y}\right] = \mathrm{Li}_2\left[\frac{x}{(1-y)}\right] + \mathrm{Li}_2\left[\frac{y}{(1-x)}\right] \\ - \mathrm{Li}_2\left[x\right] - \mathrm{Li}_2\left[y\right] - \ln(1-x)\ln(1-y)\,.\end{aligned} \tag{3.5}$$

Choosing $x = -\exp(i\pi/3)$ and $y = \exp(-i\pi/3)$ in (3.5) and taking imaginary parts, we obtain

$$\frac{5}{2}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\pi\ln 3}{4} = 3\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]\,. \tag{3.6}$$

3.1 Ternary Formula for π^2

Solving (3.1) and (3.4) for π^2, we obtain

$$\pi^2 = 36\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] - 18\mathrm{Li}_2\left[\frac{1}{3}\right] + 9\mathrm{Li}_2\left[-\frac{1}{3}\right]\,. \tag{3.7}$$

Writing

$$\begin{aligned}\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = \frac{1}{2\cdot 3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\frac{3^6}{(12k+1)^2} + \frac{3^5}{(12k+2)^2}\right.\\ &- \frac{3^4}{(12k+4)^2} - \frac{3^4}{(12k+5)^2} - \frac{3^3}{(12k+6)^2}\\ &- \frac{3^3}{(12k+7)^2} - \frac{3^2}{(12k+8)^2}\\ &\left.+\frac{3}{(12k+10)^2} + \frac{2}{(12k+12)^2}\right]\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\mathrm{Li}_2\left[\pm\frac{1}{3}\right] = \frac{2^2}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}&\left[\pm\frac{3^5}{(12k+2)^2} + \frac{3^4}{(12k+4)^2}\right.\\ &\pm \frac{3^3}{(12k+6)^2} + \frac{3^2}{(12k+8)^2}\\ &\left.\pm\frac{3}{(12k+10)^2} + \frac{1}{(12k+12)^2}\right]\,,\end{aligned} \tag{3.9}$$

and combining them according to (3.7) we establish a ternary BBP-type formula for π^2:

$$\begin{aligned}\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\frac{3^5}{(12k+1)^2}-\frac{5\cdot 3^4}{(12k+2)^2}\\ &-\frac{3^4}{(12k+4)^2}-\frac{3^3}{(12k+5)^2}-\frac{2^3\cdot 3^2}{(12k+6)^2}\\ &-\frac{3^2}{(12k+7)^2}-\frac{3^2}{(12k+8)^2}\\ &-\frac{5}{(12k+10)^2}+\frac{1}{(12k+11)^2}\Bigg]\,.\end{aligned} \tag{3.10}$$

Incidentally, (3.10) is Formula (73) of the Compendium.

3.2 Ternary Formula for $\ln^2 3$

Solving (3.1) and (3.4) for $\ln^2 3$, we obtain

$$\ln^2 3 = 12\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] - 10\,\mathrm{Li}_2\left[\frac{1}{3}\right] + 5\,\mathrm{Li}_2\left[-\frac{1}{3}\right]\,. \tag{3.11}$$

Using (3.8) and (3.9) above in (3.11), we obtain the ternary BBP-type formula for $\ln^2 3$ as

$$\begin{aligned}\ln^2 3 = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\frac{2\cdot 3^7}{(12k+1)^2}-\frac{2\cdot 3^8}{(12k+2)^2}\\ &-\frac{2\cdot 13\cdot 3^4}{(12k+4)^2}-\frac{2\cdot 3^5}{(12k+5)^2}-\frac{2^3\cdot 3^5}{(12k+6)^2}\\ &-\frac{2\cdot 3^4}{(12k+7)^2}-\frac{2\cdot 13\cdot 3^2}{(12k+8)^2}\\ &-\frac{2\cdot 3^4}{(12k+10)^2}+\frac{2\cdot 3^2}{(12k+11)^2}-\frac{8}{(12k+12)^2}\Bigg]\,,\end{aligned} \tag{3.12}$$

which is Formula (74) of the Compendium.

3.3 Ternary Formula for $\pi\sqrt{3}\ln 3$

Solving (3.2) and (3.6) for $\pi \ln 3$ gives

$$\pi \ln 3 = 48 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{2}\right)\right] - 60 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6}\right)\right]. \tag{3.13}$$

Now

$$\begin{aligned}\operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6}\right)\right] = \frac{\sqrt{3}}{54} \sum_{k=0}^{\infty} \left(-\frac{1}{27}\right)^k & \left[\frac{9}{(6k+1)^2} + \frac{9}{(6k+2)^2}\right. \\ & \left. + \frac{6}{(12k+3)^2} + \frac{3}{(12k+4)^2} + \frac{1}{(6k+5)^2}\right]\end{aligned} \tag{3.14}$$

and

$$\operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{2}\right)\right] = \frac{\sqrt{3}}{27} \sum_{k=0}^{\infty} \left(-\frac{1}{27}\right)^k \left[\frac{9}{(6k+1)^2} - \frac{3}{(6k+3)^2} + \frac{1}{(6k+5)^2}\right] \tag{3.15}$$

Using (3.14) and (3.15) in (3.13) leads to the ternary BBP-type formula

$$\begin{aligned}\pi\sqrt{3}\ln 3 = 2 \sum_{k=0}^{\infty} \left(-\frac{1}{27}\right)^k & \left[\frac{9}{(6k+1)^2} - \frac{15}{(6k+2)^2} - \frac{18}{(6k+3)^2}\right. \\ & \left. - \frac{5}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right].\end{aligned}$$

3.4 Ternary Formula for $\sqrt{3}\operatorname{Cl}_2(\pi/3)$

Solving (3.2) and (3.6) for $\operatorname{Cl}_2(\pi/3)$ gives

$$\operatorname{Cl}_2\left(\frac{\pi}{3}\right) = 6 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{2}\right)\right] - 6 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6}\right)\right]. \tag{3.16}$$

Using (3.14) and (3.15) in (3.16) leads to the ternary BBP-type formula

$$\sqrt{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{1}{3}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{(6k+1)^2} - \frac{9}{(6k+2)^2} - \frac{12}{(6k+3)^2} - \frac{3}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right].$$

4 Degree 3 Ternary BBP-type Formulas

A functional identity for trilogarithms (equation A.2.6.10 of [6]) is

$$\begin{aligned}\mathrm{Li}_3\left[\frac{1-x}{1+x}\right] - \mathrm{Li}_3\left[\frac{x-1}{x+1}\right] &= 2\,\mathrm{Li}_3\left[1-x\right] + 2\,\mathrm{Li}_3\left[\frac{1}{1+x}\right] \\ &\quad - \frac{1}{2}\,\mathrm{Li}_3\left[1-x^2\right] - \frac{7}{4}\zeta(3) \\ &\quad + \frac{\pi^2}{6}\ln(1+x) - \frac{1}{3}\ln^3(1+x)\,.\end{aligned}$$

The use of $x = 2$ in the above equation gives

$$\frac{13}{6}\zeta(3) - \frac{1}{6}\pi^2\ln 3 + \frac{1}{6}\ln^3 3 = 2\,\mathrm{Li}_3\left[\frac{1}{3}\right] - \mathrm{Li}_3\left[-\frac{1}{3}\right].$$

Putting $x = \exp i\pi/3$ in the functional equation and taking real and imaginary parts gives

$$\frac{13}{18}\zeta(3) - \frac{5}{144}\pi^2\ln 3 + \frac{1}{48}\ln^3 3 = \mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] \tag{4.1}$$

and

$$\begin{aligned}&\frac{29}{1296}\pi^3 - \frac{1}{48}\pi\ln^2 3 \\ &\quad = 4\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] - 5\,\mathrm{Im}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right].\end{aligned} \tag{4.2}$$

Using

$$\mathrm{Li}_3\left[\pm\frac{1}{3}\right] = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\pm\frac{3^5}{(6k+1)^3} + \frac{3^4}{(6k+2)^3} \pm \frac{3^3}{(6k+3)^3} + \frac{3^2}{(6k+4)^3} \pm \frac{3}{(6k+5)^3} + \frac{1}{(6k+6)^3}\right]$$

leads to the ternary BBP-type formula

$$
\begin{aligned}
&13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 \\
&\quad = \frac{2}{3^5} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^6}{(6k+1)^3} + \frac{3^4}{(6k+2)^3} + \frac{3^4}{(6k+3)^3} \right. \\
&\qquad \left. + \frac{3^2}{(6k+4)^3} + \frac{3^2}{(6k+5)^3} + \frac{1}{(6k+6)^3} \right] .
\end{aligned}
$$

A shorter version (length 2) of the above formula is

$$
13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 = \frac{2}{3} \sum_{k=0}^{\infty} \frac{1}{9^k} \left[\frac{9}{(2k+1)^3} + \frac{1}{(2k+2)^3} \right] .
$$

The ternary BBP-type formula that results from (4.1) is discussed elsewhere [8].

Next we obtain the ternary BBP-type formula that results from (4.2).

$$
\begin{aligned}
\operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{2} \right) \right] &= \frac{\sqrt{3}}{3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^5}{(12k+1)^3} - \frac{3^4}{(12k+3)^3} \right. \\
&\quad \left. + \frac{3^3}{(12k+5)^3} - \frac{3^2}{(12k+7)^3} + \frac{3}{(12k+9)^3} - \frac{1}{(12k+11)^3} \right]
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
\operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{\sqrt{3}} \exp\left(\frac{i\pi}{6} \right) \right] &= \frac{\sqrt{3}}{2 \cdot 3^6} \sum_{k=0}^{\infty} \frac{1}{3^{6k}} \left[\frac{3^5}{(12k+1)^3} + \frac{3^5}{(12k+2)^3} + \frac{2 \cdot 3^4}{(12k+3)^3} \right. \\
&\quad + \frac{3^4}{(12k+4)^3} + \frac{3^3}{(12k+5)^3} - \frac{3^2}{(12k+7)^3} \\
&\quad \left. - \frac{3^2}{(12k+8)^3} - \frac{2 \cdot 3}{(12k+9)^3} - \frac{3}{(12k+10)^3} - \frac{1}{(12k+11)^3} \right]
\end{aligned} \tag{4.4}
$$

Combining (4.3) and (4.4) according to the prescription of (4.2), we arrive at

$$\begin{aligned}
&\frac{29}{1296}\pi^3\sqrt{3} - \frac{1}{48}\pi\sqrt{3}\ln^2 3 \\
&\quad = \frac{1}{2\cdot 3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^6}{(12k+1)^3} - \frac{5\cdot 3^5}{(12k+2)^3}\right. \\
&\qquad - \frac{2\cdot 3^6}{(12k+3)^3} - \frac{5\cdot 3^4}{(12k+4)^3} + \frac{3^4}{(12k+5)^3} - \frac{3^3}{(12k+7)^3} \\
&\qquad\quad \left. + \frac{5\cdot 3^2}{(12k+8)^3} + \frac{2\cdot 3^3}{(12k+9)^3} + \frac{5\cdot 3}{(12k+10)^3} - \frac{3}{(12k+11)^3}\right].
\end{aligned}$$

5 Degree 4 Ternary BBP-type Formulas

A two-variable functional equation for degree 4 polylogarithms (equation A.2.7.40 of [6]) reads

$$\begin{aligned}
&\mathrm{Li}_4\left[-\frac{x^2y\eta}{\xi}\right] + \mathrm{Li}_4\left[-\frac{y^2x\xi}{\eta}\right] + \mathrm{Li}_4\left[\frac{x^2y}{\eta^2\xi}\right] + \mathrm{Li}_4\left[\frac{y^2x}{\xi^2\eta}\right] \\
&= 6\ \mathrm{Li}_4[xy] + 6\ \mathrm{Li}_4\left[\frac{xy}{\eta\,\xi}\right] + 6\ \mathrm{Li}_4\left[-\frac{xy}{\eta}\right] + 6\ \mathrm{Li}_4\left[-\frac{xy}{\xi}\right] \\
&+ 3\ \mathrm{Li}_4[x\eta] + 3\ \mathrm{Li}_4[y\xi] + 3\ \mathrm{Li}_4\left[\frac{x}{\eta}\right] + 3\ \mathrm{Li}_4\left[\frac{y}{\xi}\right] + 3\ \mathrm{Li}_4\left[-\frac{x\eta}{\xi}\right] \\
&+ 3\ \mathrm{Li}_4\left[-\frac{y\xi}{\eta}\right] + 3\ \mathrm{Li}_4\left[-\frac{x}{\eta\,\xi}\right] + 3\ \mathrm{Li}_4\left[-\frac{y}{\eta\,\xi}\right] - 6\ \mathrm{Li}_4[x] \\
&- 6\ \mathrm{Li}_4[y] - 6\ \mathrm{Li}_4\left[-\frac{x}{\xi}\right] - 6\ \mathrm{Li}_4\left[-\frac{y}{\eta}\right] + 3/2\ \ln^2\xi\ln^2\eta\,,
\end{aligned} \tag{5.1}$$

where $\xi = 1-x$, $\eta = 1-y$.

Putting $x = -\exp(i\pi/3)$ and $y = \exp(i\pi/3)$ in (5.1), simplifying and taking real and imaginary parts, we obtain

$$\begin{aligned}
&-12\ \mathrm{Re}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/2}\right] - 3\ \mathrm{Re}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right] \\
&\quad + \mathrm{Li}_4\left[\frac{1}{3}\right] + \frac{1}{4}\mathrm{Li}_4\left[-\frac{1}{3}\right] \\
&\qquad = -\frac{127\pi^4}{10368} + \frac{1}{64}\pi^2\ln^2 3 - \frac{5}{384}\ln^4 3
\end{aligned} \tag{5.2}$$

and

$$
\begin{aligned}
&-12\ \mathrm{Im}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/2}\right]+15\ \mathrm{Im}\ \mathrm{Li}_4\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]\\
&\quad=\frac{29}{864}\pi^3\ln 3-\frac{1}{96}\pi\ln^3 3-\frac{11}{3}\mathrm{Cl}_4\left(\frac{\pi}{3}\right).
\end{aligned}
\tag{5.3}
$$

First we proceed to obtain the BBP-type formula invoked by (5.2).

Now,

$$
\begin{aligned}
\mathrm{Re}\,\mathrm{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right]=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&-\frac{3^5}{(12k+2)^4}+\frac{3^4}{(12k+4)^4}\\
&-\frac{3^3}{(12k+6)^4}+\frac{3^2}{(12k+8)^4}-\frac{3}{(12k+10)^4}+\frac{1}{(12k+12)^4}\Bigg]
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
\mathrm{Re}\,\mathrm{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right]=\frac{1}{2\cdot 3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\frac{3^6}{(12k+1)^4}+\frac{3^5}{(12k+2)^4}\\
&-\frac{3^4}{(12k+4)^4}-\frac{3^4}{(12k+5)^4}-\frac{2\cdot 3^3}{(12k+6)^4}\\
&-\frac{3^3}{(12k+7)^4}-\frac{3^2}{(12k+8)^4}+\frac{3}{(12k+10)^4}\\
&+\frac{2}{(12k+12)^4}\Bigg].
\end{aligned}
\tag{5.5}
$$

Also

$$
\begin{aligned}
\mathrm{Li}_4\left[\pm\frac{1}{3}\right]=\frac{2^4}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\pm\frac{3^5}{(12k+2)^4}+\frac{3^4}{(12k+4)^4}\\
&\pm\frac{3^3}{(12k+6)^4}+\frac{3^2}{(12k+8)^4}\pm\frac{3}{(12k+10)^4}\\
&+\frac{1}{(12k+12)^4}\Bigg].
\end{aligned}
\tag{5.6}
$$

Combining (5.4), (5.5) and (5.6) according to (5.2), we obtain the following degree 4 ternary BBP-type formula:

$$\begin{aligned}\frac{127\pi^4}{5184} - \frac{\pi^2\ln^2 3}{32} + \frac{5\ln^4 3}{192} = \frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\Bigg[&\frac{3^7}{(12k+1)^4} - \frac{5\cdot 3^7}{(12k+2)^4} \\ &- \frac{19\cdot 3^4}{(12k+4)^4} - \frac{3^5}{(12k+5)^4} - \frac{2\cdot 3^6}{(12k+6)^4} \\ &- \frac{3^4}{(12k+7)^4} - \frac{19\cdot 3^2}{(12k+8)^4} - \frac{5\cdot 3^3}{(12k+10)^4} \\ &+ \frac{3^2}{(12k+11)^4} - \frac{10}{(12k+12)^4}\Bigg].\end{aligned}$$

Next we obtain the BBP-type formula invoked by (5.3).

Writing

$$\begin{aligned}&\operatorname{Im}\operatorname{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{2}\right)\right] \\ &\quad = \frac{\sqrt{3}}{27}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\left[\frac{9}{(6k+1)^4} - \frac{3}{(6k+3)^4} + \frac{1}{(6k+5)^4}\right]\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\operatorname{Im}\operatorname{Li}_4\left[\frac{1}{\sqrt{3}}\exp\left(\frac{i\pi}{6}\right)\right] = \frac{\sqrt{3}}{54}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\Bigg[&\frac{9}{(6k+1)^4} + \frac{9}{(6k+2)^4} \\ &+ \frac{6}{(6k+3)^4} + \frac{3}{(6k+4)^4} + \frac{1}{(6k+5)^4}\Bigg],\end{aligned}$$

and combining these according to (5.3), we obtain the following degree 4 BBP-type formula:

$$\begin{aligned}&\frac{1}{\sqrt{3}}\left(11\operatorname{Cl}_4\left(\frac{\pi}{3}\right) - \frac{29}{288}\pi^3\ln 3 + \frac{\pi\ln^3 3}{32}\right) \\ &\quad = \frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{27}\right)^k\Bigg[\frac{9}{(6k+1)^4} - \frac{15}{(6k+2)^4} \\ &\qquad - \frac{18}{(6k+3)^4} - \frac{5}{(6k+4)^4} + \frac{1}{(6k+5)^4}\Bigg].\end{aligned} \tag{5.8}$$

It is interesting to remark that (5.8) was first obtained by Broadhurst [9], using the PSLQ Algorithm. We have thus found its formal proof for the first time, through (5.3)!

6 Degree 5 Ternary BBP-type Formulas

The following degree 5 polylogarithm identity is derived in [10]

$$
\begin{aligned}
&\mathrm{Li}_5\left[\frac{x\alpha}{y\beta}\right] + \mathrm{Li}_5\left[x\alpha\, y\eta\right] + \mathrm{Li}_5\left[\frac{x\alpha\,\beta}{\eta}\right] + \mathrm{Li}_5\left[x\xi\, y\beta\right] + \mathrm{Li}_5\left[\frac{x\xi}{y\eta}\right] \\
&+ \mathrm{Li}_5\left[\frac{x\xi\,\eta}{\beta}\right] + \mathrm{Li}_5\left[\frac{\alpha\, y\beta}{\xi}\right] + \mathrm{Li}_5\left[\frac{\alpha}{\xi\, y\eta}\right] + \mathrm{Li}_5\left[\frac{\alpha\,\eta}{\xi\,\beta}\right] \\
&- 9\,\mathrm{Li}_5\left[xy\right] - 9\,\mathrm{Li}_5\left[x\beta\right] - 9\,\mathrm{Li}_5\left[x\eta\right] - 9\,\mathrm{Li}_5\left[\frac{x}{y}\right] - 9\,\mathrm{Li}_5\left[\frac{x}{\beta}\right] \\
&- 9\,\mathrm{Li}_5\left[\frac{x}{\eta}\right] - 9\,\mathrm{Li}_5\left[\alpha\, y\right] - 9\,\mathrm{Li}_5\left[\alpha\,\beta\right] - 9\,\mathrm{Li}_5\left[\alpha\,\eta\right] \\
&- 9\,\mathrm{Li}_5\left[\frac{\alpha}{y}\right] - 9\,\mathrm{Li}_5\left[\frac{\alpha}{\beta}\right] - 9\,\mathrm{Li}_5\left[\frac{\alpha}{\eta}\right] - 9\,\mathrm{Li}_5\left[\xi\, y\right] - 9\,\mathrm{Li}_5\left[\xi\,\beta\right] \\
&- 9\,\mathrm{Li}_5\left[\xi\,\eta\right] - 9\,\mathrm{Li}_5\left[\frac{y}{\xi}\right] - 9\,\mathrm{Li}_5\left[\frac{\beta}{\xi}\right] - 9\,\mathrm{Li}_5\left[\frac{\eta}{\xi}\right] \\
&+ 18\,\mathrm{Li}_5\left[x\right] + 18\,\mathrm{Li}_5\left[\alpha\right] + 18\,\mathrm{Li}_5\left[\xi\right] + 18\,\mathrm{Li}_5\left[y\right] + 18\,\mathrm{Li}_5\left[\beta\right] \\
&+ 18\,\mathrm{Li}_5\left[\eta\right] - 18\,\zeta(5) = 3/10\ (\ln\xi)^5 + 3/4\ (\ln y - \ln x)\,(\ln\xi)^4 \\
&+ 3/2\ (3\ \ln y - \ln\eta)\,(\ln\eta)^2\,(\ln\xi)^2 + 1/2\,\pi^2\,(\ln\xi - 3\ \ln\eta)\,(\ln\xi)^2 + 1/5\,\pi^4\ln\xi\,.
\end{aligned}
$$

Here $\xi = 1 - x$, $\eta = 1 - y$, $\alpha = -x/\xi$ and $\beta = -y/\eta$.

Putting $x = -\exp(i\pi/3)$ and $y = \exp(i\pi/3)$ in (6) and simplifying, gives

$$
\begin{aligned}
&\frac{1}{64}\pi^2\ln^3 3 - \frac{127}{3456}\pi^4\ln 3 - \frac{1}{128}\ln^5 3 + \frac{1573}{144}\zeta(5) \\
&= \frac{3}{2}\,\mathrm{Li}_5\left[-\frac{1}{3}\right] - 3\,\mathrm{Li}_5\left[\frac{1}{3}\right] + 9\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right] + 9\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{-i\pi/6}\right]\,.
\end{aligned}
\tag{6.1}
$$

On taking real parts

$$
\begin{aligned}
&\frac{1}{64}\pi^2\ln^3 3-\frac{127}{3456}\pi^4\ln 3-\frac{1}{128}\ln^5 3+\frac{1573}{144}\zeta(5)\\
&=18\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]+\frac{3}{2}\,\mathrm{Li}_5\left[-\frac{1}{3}\right]-3\,\mathrm{Li}_5\left[\frac{1}{3}\right]\,.
\end{aligned}
\tag{6.2}
$$

Now

$$
\begin{aligned}
&18\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{3}}e^{i\pi/6}\right]=18\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{3}}\right)^k\frac{1}{k^5}\cos\left(\frac{k\pi}{6}\right)\\
&=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^8}{(12k+1)^5}+\frac{3^7}{(12k+2)^5}-\frac{3^6}{(12k+4)^5}\right.\\
&-\frac{3^6}{(12k+5)^5}-\frac{2\cdot 3^5}{(12k+6)^5}-\frac{3^5}{(12k+7)^5}-\frac{3^4}{(12k+8)^5}\\
&\left.+\frac{3^3}{(12k+10)^5}+\frac{3^3}{(12k+11)^5}+\frac{2\cdot 3^2}{(12k+12)^5}\right]\,,
\end{aligned}
\tag{6.3}
$$

$$
\begin{aligned}
&\frac{3}{2}\,\mathrm{Li}_5\left[-\frac{1}{3}\right]=\frac{3}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k\frac{1}{k^5}\\
&=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{-3^6\cdot 2^4}{(12k+2)^5}+\frac{3^5\cdot 2^4}{(12k+4)^5}-\frac{3^4\cdot 2^4}{(12k+6)^5}\right.\\
&\left.+\frac{3^3\cdot 2^4}{(12k+8)^5}-\frac{3^2\cdot 2^4}{(12k+10)^5}+\frac{3\cdot 2^4}{(12k+12)^5}\right]
\end{aligned}
\tag{6.4}
$$

and

$$
\begin{aligned}
&3\,\mathrm{Li}_5\left[\frac{1}{3}\right]=3\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^k\frac{1}{k^5}\\
&=\frac{1}{3^6}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^6\cdot 2^5}{(12k+2)^5}+\frac{3^5\cdot 2^5}{(12k+4)^5}+\frac{3^4\cdot 2^5}{(12k+6)^5}\right.\\
&\left.+\frac{3^3\cdot 2^5}{(12k+8)^5}+\frac{3^2\cdot 2^5}{(12k+10)^5}+\frac{3\cdot 2^5}{(12k+12)^5}\right]\,.
\end{aligned}
\tag{6.5}
$$

Using (6.3), (6.4) and (6.5) in (6.2), we obtain the ternary BBP-type formula

$$\begin{aligned}
&\frac{1}{64}\pi^2\ln^3 3-\frac{127}{3456}\pi^4\ln 3-\frac{1}{128}\ln^5 3+\frac{1573}{144}\zeta(5)\\
&=\frac{1}{3^5}\sum_{k=0}^{\infty}\frac{1}{3^{6k}}\left[\frac{3^7}{(12k+1)^5}-\frac{5\cdot 3^7}{(12k+2)^5}-\frac{19\cdot 3^4}{(12k+4)^5}\right.\\
&-\frac{3^5}{(12k+5)^5}-\frac{2\cdot 3^6}{(12k+6)^5}-\frac{3^4}{(12k+7)^5}\\
&\left.-\frac{19\cdot 3^2}{(12k+8)^5}-\frac{5\cdot 3^3}{(12k+10)^5}+\frac{3^2}{(12k+11)^5}-\frac{10}{(12k+12)^5}\right].
\end{aligned}$$

7 Conclusion

Using a fairly straightforward method, we have obtained several ternary BBP-type formulas, which can now be added to the literature. In particular we proved the following formulas (written in the now standard BBP notation [3]).

$\ln 2=1/(2\cdot 3^5)\mathrm{P}((1,3^6,12,(3^5,3^5,0,-3^4,-3^3,0,-3^2,-3^2,0,3,1,0)))$

$\ln 3=1/3^6\mathrm{P}((1,3^6,12,(3^6,3^5,0,-3^4,-3^4,-2\cdot 3^3,-3^3,-3^2,0,3,3,2)))$

$\ln 3=1/27\mathrm{P}((1,-27,6,(27,9,0,-3,-3,-2)))$

$\pi\sqrt{3}=1/3^4\mathrm{P}((1,3^6,12,(3^5,3^5,2\cdot 3^4,3^4,3^3,0,-3^2,-3^2,-6,-3,-1,0)))$

$\pi\sqrt{3}=2/3^4\mathrm{P}((1,3^6,12,(3^5,0,-3^4,0,3^3,0,-3^2,0,3,0,-1,0)))$

$\pi\sqrt{3}=6\mathrm{P}((1,-3,2,(1,0)))$

$\pi^2=2/27\mathrm{P}((2,3^6,12,(3^5,-5\cdot 3^4,0,-3^4,-3^3,-2^3\cdot 3^2,-3^2,-3^2,0,-5,1,0)))$

$\ln^2 3=1/3^6\mathrm{P}((2,3^6,12,(2\cdot 3^7,-2\cdot 3^8,0,-2\cdot 13\cdot 3^4,-2\cdot 3^5,-2^3\cdot 3^5,-2\cdot 3^4,$
$-2\cdot 13\cdot 3^2,0,-2\cdot 3^4,2\cdot 3^2,-8)))$

$\pi\sqrt{3}\ln 3=2\mathrm{P}((2,-27,6,(9,-15,-18,-5,1,0)))$

$\sqrt{3}\mathrm{Cl}_2(\pi/3) = 1/3\mathrm{P}((2, -27, 6, (9, -9, -12, -3, 1, 0)))$

$13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 = 2/3^5\mathrm{P}((3, 3^6, 6, (3^6, 3^4, 3^4, 3^2, 3^2, 1)))$

$13\zeta(3) - \pi^2 \ln 3 + \ln^3 3 = 2/3\mathrm{P}((3, 9, 2, (9, 1)))$

$(29\pi^3/1296 - \pi \ln^2 3/48)/\sqrt{3} = 1/2/3^5\mathrm{P}((3, 3^6, 12, (3^6, -5 \cdot 3^5,$
$-2 \cdot 3^6, -5 \cdot 3^4, 3^4, 0, -3^3, 5 \cdot 3^2, 2 \cdot 3^3, 5 \cdot 3, -3, 0)))$

$127\pi^4/5184 - \pi^2 \ln^2 3/32 + 5 \ln^4 3/192 = 1/3^6\mathrm{P}((4, 3^6, 12, (3^7, -5 \cdot 3^7, 0,$
$-19 \cdot 3^4, -3^5, -2 \cdot 3^6, -3^4, -19 \cdot 3^2, 0, -5 \cdot 3^3, 3^2, -10)))$

$(11\mathrm{Cl}_4(\pi/3) - 29\pi^3 \ln 3/288 + \pi \ln^3 3/32)/\sqrt{3} = 1/2\mathrm{P}((4, -27, 6, (9, -15, -18, -5, 1, 0)))$

$\pi^2 \ln^3 3/64 - 127\pi^4 \ln 3/3456 - \ln^5 3/128 + 1573\zeta(5)/144 = 1/3^5\mathrm{P}((5, 3^6, 12, (3^7,$
$-5 \cdot 3^7, 0, -19 \cdot 3^4, -3^5, -2 \cdot 3^6, -3^4, -19 \cdot 3^2, 0, -5 \cdot 3^3, 3^2, -10)))$

We also proved the following ternary zero relations:

$0 = \mathrm{P}((1, 3^6, 12, (3^5, -5 \cdot 3^4, 0, -3^4, -3^3, -2^3 \cdot 3^2, -3^2, -3^2, 0, -5, 1, 0)))$

$0 = \mathrm{P}((1, 3^6, 12, (3^5, -3^5, -2^2 \cdot 3^4, -3^4, 3^3, 0, -3^2, 3^2, 3 \cdot 2^2, 3, -1, 0)))$

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