Solutions to the Gravitational Field Equations in Curved Phase-Spaces

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Abstract

After reviewing the basics of the geometry of the cotangent bundle of spacetime, via the introduction of nonlinear connections, we build an action and derive the generalized gravitational field equations in phase spaces. A nontrivial solution generalizing the Hilbert-Schwarzschild black hole metric in spacetime is found. The most relevant physical consequence is that the metric becomes momentum-dependent (observer dependent) which is what one should aim for in trying to *quantize* geometry (gravity) : the observer must play an important role in any measurement (observation) process of the spacetime he/she lives in.

Keywords : Gravity, Finsler Geometry, Born Reciprocity, Phase Space.

1 Introduction : Quantum Gravity and Curved Phase Space

In the first introduction to Quantum Mechanics we are exposed to the Weyl-Heisenberg algebra given by the commutators $[\mathbf{x}^i, \mathbf{p}_j] = i\hbar\delta^i_j$ of the coordinate and momentum operators, and which hold the key behind Heisenberg's uncertaintity principle via the relation $\Delta x \; \Delta p \;\; \geq \;\; \frac{1}{2} \mid < [\mathbf{x}, \mathbf{p}] > \mid$, after taking expectation values. Inspired from the results obtained in the very high energy limit of string scattering amplitudes [1], a lot of work has been devoted in the past two decades to deformations of the Weyl-Heisenberg algebra [17], and which is associated to a generalized uncertaintity principle (GUP) leading to the notion of a minimal length scale (of the order of the Planck length). The strings begin to grow in size when trans-Planckian energies are reached, rather than probing smaller and smaller distances.

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [2]. The principle of Born's reciprocal relativity [2] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality). The generalized velocity and acceleration boosts/rotations transformations (where $x^i, p_i; i = 1, 2, 3, 4$ are all boosted/rotated into each-other) and which leave invariant the infinitesimal line interval of the $8D$ flat phase space, were given by [3] based on the group $U(1, 3)$ and which is the Born version of the Lorentz group $SO(1, 3)$. The extension of these transformations to Noncommutative phase spaces was analyzed in [12]

We explored in [7] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, *six* specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

A discussion of Mach's principle within the context of Born Reciprocal Gravity in Phase Spaces was described in [14]. The Machian postulate states that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. It is also consistent with equating the maximal proper force $m_{Planck}(c^2/L_{Planck})$ to $M_{Universe}(c^2/R_{Hubble})$ and reflecting a maximal/minimal acceleration duality. By invoking Born's reciprocity between coordinates and momenta, a minimal Planck scale should correspond to a minimum momentum, and consequently to an upper scale given by the Hubble radius. Further details can be found in [14].

It is better understood now that the Planck-scale modifications of the particle dispersion relations can be encoded in the nontrivial geometrical properties of momentum space [16]. When both spacetime curvature and Planck-scale deformations of momentum space are present, it is expected that the nontrivial geometry of momentum space and spacetime get intertwined. The interplay between spacetime curvature and non-trivial momentum space effects was essential in the notion of "relative locality" and in the deepening of the relativity principle [16]. Recently the authors [18] described the Hamilton geometry of the phase space of particles whose motion is characterized by general dispersion relations. Explicit examples of two models for Planck-scale modified dispersion relations, inspired from the q-de Sitter and κ-Poincare quantum groups, were considered. In the first case they found the expressions for the momentum and position dependent curvature of spacetime and momentum space, while for the second case the manifold is flat and only the momentum space possesses a nonzero, momentum dependent curvature.

The purpose of this work is to study deeper the geometry of the cotangent bundle in order to derive the analog of Einstein's field equations in curved phase spaces, and construct specific solutions. The *curved* phase-space geometry of the cotangent bundle of spacetime can be explored via the introduction of nonlinear connections which are associated with certain *nonholonomic* modifications of Riemann–Cartan gravity within the context of Finsler geometry. The geometry of the $8D$ tangent bundle of $4D$ spacetime and the physics of a limiting value of the proper acceleration in spacetime [6] has been studied by Brandt [4]. Generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the $infinite$ acceleration limit. We must emphasize that the results found in this work are very different than those obtained earlier by us in [13] and by [4] [11], [9], among others.

In the next section we review the geometry of the cotangent bundle of spacetime, build an action and derive the generalized field equations. In the final section we find some nontrivial solutions generalizing the Hilbert-Schwarzschild black hole solution in spacetime. The most relevant consequence is that the metric becomes momentum-dependent (observer dependent) which is what one should aim for in trying to *quantize* geometry (gravity) : the observer must play an important role in any measurement (observation) process of the spacetime he/she lives in.

2 Field Equations in Curved Phase Spaces

In the first part of this section we shall review the geometry of the cotangent bundle case T^*M (phase space) following the monographs by [11]. Readers familiar with this material can proceed to the second part of this section where we derive the field equations. A classical treatise on the Geometry of Phase Spaces can be found in [8]. In the case of the cotangent space of a d-dim manifold T^*M_d the metric can be equivalently rewritten in the block diagonal form [11] as

$$(ds)^2 \ = \ g_{ij}(x^k, p_a) \ dx^i d \ x^j \ + \ h^{ab}(x^k, p_c) \ \delta p_a \ \delta p_b \ =$$

$$g_{ij}(x^k, p_a) \ dx^i d \ x^j \ + \ h_{ab}(x^k, p_c) \ \delta p^a \ \delta p^b \qquad (2.1)$$

$i, j, k = 1, 2, 3, ...d$, $a, b, c = 1, 2, 3, ...d$, if instead of the standard coordinate basis one introduces the following anholonomic frames (non-coordinate basis)

$$\delta_i \ = \ \delta/\delta x^i \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial^a \ = \ \partial_{x^i} \ + \ N_{ia} \ \partial_{p_a}; \ \ \partial^a \ \equiv \ \partial_{p_a} \ = \ \frac{\partial}{\partial p_a} \qquad (2.2)$$

One should note the *key* position of the indices that allows us to distinguish between derivatives with respect to x^i and those with respect to p_a. The dual basis of $(\delta_i = \delta/\delta x^i; \partial^a = \partial/\partial p_a)$ is

$$dx^i, \ \ \delta p_a \ = \ dp_a \ - \ N_{ja} \ dx^j, \ \ \delta p^a \ = \ dp^a \ - \ N^a_j \ dx^j \qquad (2.3)$$

where the N–coefficients define a nonlinear connection, N–connection structure.

An N-linear connection D on T^*M can be uniquely represented in the adapted basis in the following form

$$D_{\delta_j}(\delta_i) \;=\; H^k_{ij}\;\delta_k;\;\; D_{\delta_j}(\partial^a) \;=\; -\;H^a_{bj}\;\partial^b; \tag{2.4a}$$

$$D_{\partial^a}(\delta_i) \;=\; C^{ka}_i\;\delta_k;\;\; D_{\partial^a}(\partial^b) \;=\; -\;C^{ba}_c\;\partial^c \tag{2.4b}$$

where $H^k_{ij}(x,p), H^a_{bj}(x,p), C^{ka}_i(x,p), C^{ba}_c(x,p)$ are the connection coefficients. Our notation for the derivatives is

$$\partial^a = \partial/\partial p_a,\;\;\; \partial_i \;=\; \partial_{x^i},\;\;\; \delta_i \;=\; \delta/\delta x^i \;=\; \partial_{x^i} \;+\; N_{ia}\;\partial^a \tag{2.4c}$$

The N–connection structures can be naturally defined on (pseudo) Riemannian spacetimes and one can relate them with some anholonomic frame fields (vielbeins) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W^\gamma_{\alpha\beta}\delta_\gamma$. The only nontrivial (nonvanishing) nonholonomy coefficients are

$$W_{ija} \;=\; R_{ija};\;\; W^a_{\;\;jb} \;=\; \partial^a N_{jb} \;=\; -\;W_j{}^a{}_b \tag{2.5a}$$

and

$$R_{ija} \;=\; \delta_j N_{ia} \;-\; \delta_i N_{ja} \tag{2.5b}$$

is the nonlinear connection curvature (N–curvature).

Imposing a zero nonmetricity condition of $g_{ij}(x,p), h^{ab}(x,p)$ along the horizontal and vertical directions, respectively, gives

$$D_i g_{jk} \;=\; \delta_i\;g_{gk}\;\; - H^l_{ij}\;g_{lk}\;\; - H^l_{ik}\;g_{jl} \;=\; 0, \tag{2.6a}$$

$$D^a h^{bc} \;=\; \partial^a\;h^{bc} \;+\; C^{ab}_d\;h^{dc} \;+\; C^{ac}_d\;h^{bd} \;=\; 0 \tag{2.6b}$$

Performig a cyclic permutation of the indices in eqs-(2.6), followed by linear combination of the equations obtained yields the irreducible (horizontal, vertical) h-v-components for the connection coefficients

$$H^i_{\;\;jk} \;=\; \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right) \tag{2.7}$$

$$C^{ab}_c \;=\; \frac{1}{2}\;h_{cd}\;\left(\;\partial^b h^{ad} \;+\; \partial^a h^{bd} \;-\; \partial^d h^{ab}\;\right) \tag{2.8}$$

The additional conditions $D_i h^{ab} = 0, D^a g_{ij} = 0$, yield the *mixed* components of the connection coefficients

$$H^a_{bj} \;=\; \frac{1}{2}\;\left(\;h^{ac}\;\delta_j h_{bc} \;-\; h^{ac}\;h_{bd}\;\partial^d N_{jc} \;+\; \partial^a N_{jb}\;\right) \tag{2.9}$$

$$C^{ja}_i \;=\; \frac{1}{2}\;g^{jk}\;\partial^a g_{ik} \tag{2.10}$$

For any N-linear connection D with the above coefficients the torsion 2-forms are

$$\Omega^i \;=\; \frac{1}{2}T^i_{jk}\; dx^j \wedge dx^k \;+\; C^{ia}_j\; dx^j \wedge \delta p_a \tag{2.11a}$$

$$\Omega_a \;=\; \frac{1}{2}R_{jka}\; dx^j \wedge dx^k \;+\; P^b_{aj}\; dx^j \wedge \delta p_b \;+\; \frac{1}{2}S^{bc}_a\; \delta p_b \wedge \delta p_c \tag{2.11b}$$

and the curvature 2-forms are

$$\Omega^i_j \;=\; \frac{1}{2}R^i_{jkm}\; dx^k \wedge dx^m \;+\; P^{ia}_{jk}\; dx^k \wedge \delta p_a \;+\; \frac{1}{2}S^{iab}_j\; \delta p_a \wedge \delta p_b \tag{2.12}$$

$$\Omega^a_b \;=\; \frac{1}{2}R^a_{bkm}\; dx^k \wedge dx^m \;+\; P^{ac}_{bk}\; dx^k \wedge \delta p_c \;+\; \frac{1}{2}S^{acd}_b\; \delta p_c \wedge \delta p_d \tag{2.13}$$

where one must recall that the dual basis of $\delta_i = \delta/\delta x^i,\ \partial^a = \partial/\partial p_a$ is given by $dx^i,\ \delta p_a = dp_a - N_{ja}dx^j$.

The distinguished torsion tensors are

$$T^i_{jk} \;=\; H^i_{jk} \;-\; H^i_{kj};\;\; S^{ab}_c \;=\; C^{ab}_c \;-\; C^{ba}_c;\;\; T^{ia}_j \;=\; C^{ia}_j \;=\; -\; T^{ia}{}_j$$

$$P_b{}^a{}_j \;=\; H^a_{bj} \;-\; \partial^a N_{jb},\;\; P_b{}^a{}_j \;=\; -\; P_{bj}{}^a$$

$$R_{ija} \;=\; \frac{\delta N_{ja}}{\delta x^i} \;-\; \frac{\delta N_{ia}}{\delta x^j} \tag{2.14}$$

The distinguished tensors of the curvature are

$$R^i_{kjh} \;=\; \delta_h H^i_{kj} \;-\; \delta_j H^i_{kh} \;+\; H^l_{kj}\; H^i_{lh} \;-\; H^l_{kh}\; H^i_{lj} \;-\; C^{ia}_k\; R_{jha} \tag{2.15}$$

$$P^{ab}_{cj} \;=\; \partial^a\; H^b_{cj} \;+\; C^{ad}_c\; P^b_{dj} \;-\; \left(\delta_j\; C^{ab}_c \;+\; H^b_{dj}\; C^{da}_c \;+\; H^a_{dj}\; C^{bd}_c \;-\; H^d_{cj}\; C^{ab}_d\right) \tag{2.16}$$

$$P^{ak}_{ij} \;=\; \partial^a\; H^k_{ij} \;+\; C^{al}_i\; T^k_{lj} \;-\; \left(\delta_j\; C^{ak}_i \;+\; H^a_{bj}\; C^{bk}_i \;+\; H^k_{lj}\; C^{al}_i \;-\; H^l_{ij}\; C^{ak}_l\right) \tag{2.17}$$

$$S^{abc}_d \;=\; \partial^c\; C^{ab}_d \;-\; \partial^b\; C^{ac}_d \;+\; C^{eb}_d\; C^{ac}_e \;-\; C^{ec}_d\; C^{ab}_e; \tag{2.18}$$

$$S^{ibc}_j \;=\; \partial^c C^{bi}_j \;-\; \partial^b C^{ci}_j \;+\; C^{bh}_j\; C^{ci}_h \;-\; C^{ch}_j\; C^{bi}_h \tag{2.19}$$

$$R^a_{bjk} \;=\; \delta_k H^a_{bj} \;-\; \delta_j H^a_{bk} \;+\; H^c_{bj}\; H^a_{ck} \;-\; H^c_{bk}\; H^a_{cj} \;-\; C^{ca}_b\; R_{jkc} \tag{2.20}$$

Equipped with these curvature tensors one can perform suitable contractions involving g^{ij}, h_{ab} to obtain two curvature scalars of the $\mathcal{R}, \mathcal{S}$ type

$$\mathcal{R} \ = \ \delta^j_i \ R^i_{kjl} \ g^{kl}; \ \ \mathcal{S} \ = \ \delta^d_b \ S^{abc}_d \ h_{ac} \tag{2.21}$$

and construct a $2d$-dim gravitational phase space action involving a linear combination of the curvature scalars

$$\mathbf{S} \ = \ \frac{1}{2\kappa'^2} \int d^dx \ d^dp \ \sqrt{|det \ g|} \ \sqrt{|det \ h|} \ (\ c_1 \ \mathcal{R} \ + \ c_2 \ \mathcal{S} \ ) \tag{2.22}$$

where c_1, c_2 are real-valued numerical coefficients (with the appropriate physical units) and κ'^2 is the analog of gravitational coupling constant in phase space. We shall fix $c_1 = c_2 = 1$. The metric in the nonholonomic (non-coordinate basis) is *block* diagonal as described by eq-(2-1). As a result the determinant factorizes into a product giving for measure $\sqrt{|det \ g|} \ \sqrt{|det \ h|}$. Other measures of integration are possible as well as many more actions besides (2.22). For instance, one may add curvature and torsion squared terms. In natural units $\hbar = c = 1$, the physical units are fixed by ensuring the action is dimensionless and such that $c_1\mathcal{R}$ has the same physical units as $c_2\mathcal{S}$.

The reason one is not adding to the action the other curvature contractions involving the remaining components

$$\delta^c_a \ P^{ab}_{cj}, \ \ \delta^i_k \ P^{ak}_{ij}, \ \ \delta^b_a \ R^a_{bjk}, \ \ \delta^i_j \ S^{jbc}_i \tag{2.23}$$

is that the latter two curvature contractions are *antisymmetric* in the jk, bc indices, respectively. Thus a further contraction with g^{kl}, h_{bc} will be identically *zero*, so one will not be able to include these curvature components into an action linear in the curvature unless there are *antisymmetric* components to the metrics. One could introduce antisymmetric metrics but for the moment this would be the subject of a future investigation. The first two curvature contractions in (2.23) could be contracted further if one had at our disposal a second rank tensor with *mixed* upper and lower indices, but such tensor is not available. One cannot use $N_{ia}h^{ab}$ because the nonlinear connection does not transform as a tensor.

Given the action linear in curvatures, the vacuum field equations associated with the geometry of the cotangent bundle are

$$\frac{\delta \mathbf{S}}{\delta g_{ij}} = 0, \ \ \frac{\delta \mathbf{S}}{\delta h^{ab}} = 0, \ \ \frac{\delta \mathbf{S}}{\delta N_{ia}} = 0 \tag{2.24}$$

When $i, j = 1, 2, \ldots, d$, and $a, b = 1, 2, \ldots, d$ the number of field equations is

$$\frac{1}{2}d(d+1) \ + \ \frac{1}{2}d(d+1) \ + \ d^2 \ = \ \frac{2d(2d+1)}{2} \tag{2.25}$$

which match the number of independent degrees of freedom of a metric g_{MN} in $2d$-dimensions. One should emphasize that there is *no* mathematical equivalence of the above eqs-(2.24) with the ordinary Einstein vacuum field equations in a Riemannian spacetime of $2d$-dimensions

$$\mathbf{R}_{MN}(X) \; - \; \frac{1}{2} \; \mathbf{g}_{MN}(X) \; \mathbf{R}(X) \; = 0; \quad M, N = 1, 2, 3, ..., 2d \qquad (2.26)$$

one of the reasons being is the nontrivial presence of the nonlinear connection $N_{ia}(x,p)$.

The variation of the action $\delta\mathbf{S}$ is more complicated than usual due to the fact that the variation does not commute with the elongated derivatives : the commutator $[\delta, \; \frac{\delta}{\delta x^i}](\cdots) \sim (\delta N_{ia}) \frac{\partial}{\partial p_a}(\cdots) \neq 0$.

Furthermore, the integral $\int \; d^dx \; d^dp \; \sqrt{|g| \; |h|} \; D_M(\mathcal{J}^M)$ is *no* longer a total derivative leading to boundary terms which can be dropped when the fields vanish fast enough at infinity. The reason being that the covariant horizontal derivative operator D_i is defined in terms of the elongated noncommuting derivatives $\frac{\delta}{\delta x^i} = \partial_{x^i} + N_{ia} \frac{\partial}{\partial p_a}$. For these reasons we shall bypass the more complicated variational procedure of eqs-(2.24) and instead recur to the Bianchi identities in order to derive the field equations.

The Bianchi identities in the *absence* of torsion for the horizontal and vertical curvature tensors are [11]

$$(D_i\mathcal{R})^m{}_{jkl} \; + \; (D_k\mathcal{R})^m{}_{jli} \; + \; (D_l\mathcal{R})^m{}_{jik} \; = \; 0 \qquad (2.27)$$

$$(D_a\mathcal{S})^m{}_{bcd} \; + \; (D_c\mathcal{S})^m{}_{bda} \; + \; (D_d\mathcal{S})^m{}_{bac} \; = \; 0 \qquad (2.28)$$

In the presence of torsion the Bianchi identities are modified by the inclusion of torsion-curvature terms in the right hand side, and the field equations are more complicated. From the Bianchi identities in the *absence* of torsion (and when the nonmetricity is *zero*) one can retrieve Einstein's tensor, as usual, by performing two successive contractions of the indices in eqs-(2.27,2.28) giving

$$D^i(\; 2\mathcal{R}_{ij} \; - \; g_{ij} \; \mathcal{R} \;) \; = \; 0 \; \Rightarrow \; D^i(\; \mathcal{R}_{ij} \; - \frac{1}{2} \; g_{ij} \; \mathcal{R} \;) \; = \; 0 \qquad (2.29)$$

$$D^a(\; 2\mathcal{S}_{ab} \; - \; h_{ab} \; \mathcal{S} \;) \; = \; 0 \; \Rightarrow \; D^a(\; \mathcal{S}_{ab} \; - \frac{1}{2} \; h_{ab} \; \mathcal{S} \;) \; = \; 0 \qquad (2.30)$$

Therefore, the field equations consistent with the Bianchi identities in the absence of torsion (and for zero nonmetricity) are given by

$$\mathcal{R}_{ij} \; - \frac{1}{2} \; g_{ij} \; \mathcal{R} \; = \; T^{(H)}_{ij}, \;\; \mathcal{S}_{ab} \; - \frac{1}{2} \; h_{ab} \; \mathcal{S} \; = \; T^{(V)}_{ab} \qquad (2.31)$$

where $T^{(H)}_{ij}, T^{(V)}_{ab}$ are the conserved energy-momentum tensors in the horizontal and vertical space, respectively. The vacuum field equations are then

$$\mathcal{R}_{ij} \; - \frac{1}{2} \; g_{ij} \; \mathcal{R} \; = \; 0, \;\; \mathcal{S}_{ab} \; - \frac{1}{2} \; h_{ab} \; \mathcal{S} \; = \; 0 \qquad (2.32)$$

and which are equivalent to the Ricci flat conditions obtained after taking the trace of eqs-(2.32)

$$\mathcal{R}_{ij} \ = \ 0, \ \ \mathcal{S}_{ab} \ = \ 0 \tag{2.33}$$

We must supplement the above equations with the *vanishing* torsion conditions

$$\begin{gathered}
T^i_{jk} \ = \ H^i_{jk} \ - \ H^i_{kj} \ = \ 0; \ \ S^{ab}_c \ = \ C^{ab}_c \ - \ C^{ba}_c \ = \ 0, \\
T^{ia}_j \ = \ C^{ia}_j \ = \ - \ T^{ia}{}_j \ = \ 0 \\
P_b{}^a{}_j \ = \ H^a_{bj} \ - \ \partial^a \ N_{jb} \ = \ - \ P_{bj}{}^a \ = \ 0 \\
R_{ija} \ = \ \frac{\delta N_{ja}}{\delta x^i} \ - \ \frac{\delta N_{ia}}{\delta x^j} \ = \ 0
\end{gathered} \tag{2.34}$$

3 Some Solutions to Gravity in Curved Phase Spaces

Next we shall find some solutions to the above eqs-(2.33) for the metric fields $g_{ij}(x,p), h_{ab}(x,p)$ in the absence of torsion (2.34) (and zero nonmetricity). In the particular case when the nonlinear connection is $N_{ia} = 0$, and the metric components depend solely on x and p, respectively, $g_{ij}(x), h_{ab}(p)$, while the connection coefficients are given in eqs-(2.7-2.10), one can verify that all the torsion components (2.34) are zero. Thus, a static spherically symmetric metric, consistent with Born's reciprocity principle ($x \leftrightarrow p$), and which is a solution of the field equations (2.33) in the $8D$ cotangent bundle T^*M_4 associated with a $4D$ spacetime is

$$\begin{gathered}
(ds)^2 \ = \ - \, (1 - \frac{2GM}{r}) \ (dt)^2 \ + \ (1 - \frac{2GM}{r})^{-1} \ (dr)^2 \ + r^2 \ (d\Omega_{(x)})^2 \ + \\
\kappa^{-4} \ \left(-(1 - \frac{2M}{p_r}) \ (dE)^2 \ + \ (1 - \frac{2M}{p_r})^{-1} \ (dp_r)^2 \ + (p_r)^2 \ (d\Omega_{(p)})^2 \right)
\end{gathered} \tag{3.1}$$

the spacetime and momentum space infinitesimal solid angle elements are respectively

$$\begin{gathered}
(d\Omega_{(x)})^2 \ = \ r^2 \ (\ sin^2\theta_{(x)} \ (d\phi_{(x)})^2 \ + \ (d\theta_{(x)})^2 \ ), \\
(d\Omega_{(p)})^2 \ = \ (p_r)^2 \ (\ sin^2\theta_{(p)} \ (d\phi_{(p)})^2 \ + \ (d\theta_{(p)})^2 \ )
\end{gathered} \tag{3.2}$$

care must be taken not to confuse the angles. In particular, for the momentum variables in spherical coordinates one has $p_r \ = \ \sqrt{(p_x)^2 + (p_y)^2 + (p_z)^2}$ and

$$p_z \ = \ p_r \ cos(\theta_{(p)}), \ \ p_x \ = \ p_r \ sin(\theta_{(p)}) \ cos(\phi_{(p)}), \ \ p_y \ = \ p_r \ sin(\theta_{(p)}) \ sin(\phi_{(p)}) \tag{3.3}$$

κ has mass units and can be equated to the Planck's mass ; i.e. inverse of the Planck scale L. The Newtonian gravitational coupling $G = L^2$. The above metric is the phase space counterpart of the Hilbert-Schwarzschild metric. If we were to set $\kappa = \infty$, the pre-factor in front of the second line in eq-(3.1) collapses to zero and the momentum-space metric components would degenerate to *zero*. To avoid a degenerate metric in the momentum space would require a cutoff $\kappa \neq \infty$, and consequently $L \neq 0$. This requirement is compatible with the minimal Planck length postulate of the literature [15], [16]. The metric solution (3.3) is in a sense trivial since there is no *entanglement* among x and p. The solution is simply a "diagonal" sum of a spacetime and momentum metric. As expected, one finds

(i) when $p_r = 0$, eq-(3-3) yields a *singularity* in momentum space, while $r = \infty$ leads to an asymptotically flat spacetime metric. The interesting feature is that low values of the momentum correspond to the *interior* region (inside the momentum horizon) of the momentum space. Meaning that the location of the momentum horizon signals a natural infrared cutoff in the values of the momentum .

(ii) when $r = 0$, eq-(3-3) yields a black hole *singularity* in the underlying spacetime, while $p_r = \infty$ leads to an asymptotically flat momentum space metric.

One can construct a *nontrivial* phase space metric solution to the vacuum field equations, and which is given in the *block diagonal* form described by the second line of eq-(2.1), as follows

$$(ds)^2 \ = \ -(1 - \frac{2GM}{r})\ (dt)^2 \ + \ (1 - \frac{2GM}{r})^{-1}\ (dr)^2 \ + r^2\ (d\Omega_{(x)})^2 \ +$$

$$\kappa^{-4} \left(\ -\ (\ 1 - \frac{2M}{\rho(r,p_r)}\ )\ (dE)^2 \ + \ \frac{(\partial_{p_r} \rho(r,p_r))^2}{(\ 1\ -\ \frac{2M}{\rho(r,p_r)}\ )}\ (\ dp_r \ - \ N_r^{p_r}(r,p_r)\ dr\ )^2 \right) +$$

$$\kappa^{-4}\ \rho(r,p_r)^2\ (d\Omega_{(p)})^2 \qquad (3.4)$$

where one has modified the phase space metric by introducing the nonlinear connection $N_r^{p_r}(r,p_r) \neq 0$ (after setting *all* the other components of $N_i^a = 0$) and inserting the function $\rho(r,p_r)$ which plays the role of the area radial-momentum function. The area radial-momentum function $\rho(r,p_r)$ can be determined in addition to the nonlinear connection $N_r^{p_r}(r,p_r)$ by solving eqs- (2.34) as follows.

Firstly, an entire Appendix is devoted to show explicitly that the momentum space metric components $h_{ab}(x,p)$ solve the Ricci flat $\mathcal{S}_{ab} = 0$ flat eqs-(2.33). The $g_{ij}(x)$ components (Hilbert-Schwarzschild solution) solve the Ricci flat $\mathcal{R}_{ij} = 0$ eqs-(2.33). The next step is to impose the zero torsion conditions which will allow to determine $\rho(r,p_r)$ and $N_i^a(r,p_r)$. Setting $N_r^{p_r}(r,p_r) \neq 0$, and *all* the other components $N_i^a = 0$, simplifies dramatically the zero torsion conditions (2.34). The nontrivial equations turn out to be

$$P_{p_r\ r}^{\ p_r} \ = \ \frac{1}{2}\ h^{p_r p_r}\ (\frac{\partial}{\partial r} \ + \ N_r^{p_r} \frac{\partial}{\partial p_r}\ )\ h_{p_r p_r} \ - \ \frac{\partial N_r^{p_r}}{\partial p_r} \ = \ 0 \qquad (3.5)$$

$$P_{p_t\ r}^{\ p_t} \ = \ \frac{1}{2}\ h^{p_tp_t}\ (\frac{\partial}{\partial r}\ +\ N_r^{p_r}\frac{\partial}{\partial p_r}\ )\ h_{p_tp_t}\ = \ 0 \tag{3.6}$$

Therefore one ends up with the above two differential equations for the sought-after two functions $\rho(r,p_r)$ and $N_r^{p_r}(r,p_r)$ after substituting

$$h_{p_tp_t} \ = \ -\ \kappa^{-4}\ (1-\frac{2M}{\rho(r,p_r)}),\quad h^{p_tp_t} = \frac{1}{h_{p_tp_t}} \tag{3.7}$$

$$h_{p_rp_r} \ = \ \kappa^{-4}\ \frac{(\partial_{p_r}\rho(r,p_r))^2}{(1-\frac{2M}{\rho(r,p_r)})},\quad h^{p_rp_r} = \frac{1}{h_{p_rp_r}} \tag{3.8}$$

into eqs-(3.5,3.6). The naive factorization condition $\rho(r,p) = \Psi(r)\ \Xi(p);\ N_r^{p_r}(r,p_r) = p_rF(r)$ allows us to integrate the second equation (3.6) giving

$$\rho(r,p_r)\ \sim\ p\ exp\ \left(-\int\ dr\ F(r)\right),\quad N_r^{p_r}(r,p_r)\ =\ p_r\ F(r) \tag{3.9}$$

Inserting these solutions into the first equation (3.5) yields after some algebra $F(r) = 0$ and one ends up with the trivial solution $N_r^{p_r} = 0$ and $\rho(r,p) = p$. Therefore, the naive factorization $\rho(r,p) = \Psi(r)\ \Xi(p);\ N_r^{p_r}(r,p_r) = p_rF(r)$ leads to the trivial solutions $N_r^{p_r} = 0$ and $\rho(r,p) = p$. For this reason, one must discard them and look for other *non* factorizable solutions; i.e. meaning when there is a true *entanglement* of r and p_r. Nontrivial solutions to the field equations in the four-dim tangent bundle of a two-dim spacetime have been found by [10]. They require arbitrary integrating functions and generating functions.

In general, when the nonlinear connection is $N_{ia}(x,p)\neq 0$, but still obeys the *zero* torsion conditions (2.34), the metric components $g_{ij}(x,p), h_{ab}(x,p)$ obeying the field equations will have more flexibility and freedom to depend on *both* the x^i, p_a coordinates of phase space. A special class of solutions consistent with *all* the *zero* torsion conditions (2.34) when $N_{ia}(x,p)\neq 0$ are of the form $g_{ij}(x),\ h_{ab}(x,p)$. The *physical relevance* of the solutions (3.4) is that when one works with holonomic coordinates, the metric is no longer block diagonal as in eq-(2.1), but instead is given by

$$(ds)^2\ =\ \big(\ g_{ij}(x)\ +\ h^{ab}(x,p)\ N_{ia}(x,p)\ N_{jb}(x,p)\ \big)\ dx^i\ dx^j\ +$$
$$h^{ab}(x,p)\ dp_a\ dp_b\ -\ N_{ib}(x,p)\ h^{ab}(x,p)\ dx^i\ dp_a\ -\ N_{ja}(x,p)\ h^{ab}(x,p)\ dx^j\ dp_b \tag{3.10}$$

Consequently the effective spacetime metric is now momentum-dependent; i.e. observer dependent

$$g_{ij}^{eff}(x,p)\ =\ g_{ij}(x)\ +\ h^{ab}(x,p)\ N_{ia}(x,p)\ N_{jb}(x,p)\ =$$
$$g_{ij}(x)\ +\ h_{ab}(x,p)\ N_i^a(x,p)\ N_j^b(x,p) \tag{3.11}$$

In particular, the metric (3.4) leads to momentum-dependent (observer dependent) modifications of the radial components of the Hilbert-Schwarzschild solution

$$g_{rr}^{eff}(r,p_r) \;=\; g_{rr}(r) \;+\; h_{p_r p_r}(r,p_r)\; N_r^{p_r}(r,p_r)\; N_r^{p_r}(r,p_r) \tag{3.12}$$

where $h_{p_r p_r}(r,p_r)$ is given by eq-(3.8) and $N_r^{p_r}(r,p_r)$, $\rho(r,p_r)$ are solutions of eqs-(3.5,3.6) . The possibility that the underlying spacetime geometry might become *observer dependent* was envision also by Gibbons and Hawking long ago.

We should note that this curved phase-space procedure is *not* the same as the Rainbow gravity approach proposed in the literature after inserting, by hand, extra momentum-dependent scalar factors $f(E,p)$ into the spacetime metric components in order to modify the energy-momentum dispersion relations. The immediate extension of this work is to introduce matter. Brandt [5] has studied the wave equations of scalar fields $\Phi(x^\mu, v^\mu)$ in the spacetime tangent bundle and found that by *complexifying* the coordinates $z^\mu = x^\mu + iL\, v^\mu$ a natural UV (ultraviolet) regulator L in the space of solutions of the wave equations exists at the Planck scale. The possibility that a fundamental physical theory might provide a physical cutoff for field theory was speculated long ago by Landau and others. The regularization of quantum fields in complex spacetimes have been studied in particular by [19].

String and p-branes propagating in spacetime tangent bundle backgrounds were briefly studied as well by Brandt [5]. Accelerated strings in tangent bundle backgrounds were studied in further detail by [20]. The worldsheet associated with those accelerated open strings envisages a continuum family of worldlines of accelerated points. It is when one embeds the two-dim string worldsheet into the tangent bundle TM background (associated with a uniformly accelerated observer in spacetime) that the effects of the maximal acceleration are manifested. The induced worldsheet metric as a result of this embedding has a *null* horizon. It is the presence of this null horizon that limits the acceleration values of the points inside string. If the string crosses the null horizon some of its points will exceed the maximal acceleration value c^2/L and that portion of the string will become causally disconnected from the rest of string outside the horizon. We also found a modified Rindler metric which has a true curvature singularity at the location of the null horizon due to a finite maximal acceleration c^2/L. This might have important physical consequences in the construction of generalized QFT in accelerated frames and the black hole information paradox.

APPENDIX : Momentum Space Schwarzschild-like solutions in $D > 3$

We will show below that the momentum space metric components of eq-(3.4) obey the momentum space Ricci flat conditions $\mathcal{S}_{ab} = 0$. In this Appendix we shall find the most general static spherically symmetric *vacuum* solutions to the momentum space Ricci flat $\mathcal{S}_{ab} = 0$ equations ( ) in any momentum space of dimension $D > 3$. The phase space is $2D$-dim. Let us start with the momentum space line element with signature $(-, +, +, +, ..., +)$

$$(ds)^2_{(p)} \;=\; -\; e^{\mu(r,p_r)}\;(dE)^2 \;+\; e^{\nu(r,p_r)}(dp_r)^2 \;+\; \rho(r,p_r)^2\;\tilde{h}_{ab}\;d\xi^a\;d\xi^b. \quad (A.1)$$

where the area radial-momentum function is $\rho(r,p_r)$. The metric components $\tilde{h}_{ab}$ correspond to a $D-2$-dim homogeneous space. The indices of $\tilde{h}_{ab}$ span $a, b = 3, 4, ..., D-2$, whereas the temporal and radial indices are denoted by $1, 2$ respectively. The only non-vanishing Christoffel symbols in momentum space are given in terms of the following partial derivatives with respect to the radial momentum p_r variable, and are denoted with a prime. These derivatives must *not* be confused with derivatives with respect to the radial variable r in spacetime. Therefore our notation here is such that $\rho'(r,p_r) \equiv \partial_{p_r}\rho(r,p_r)$; $\mu' \equiv \partial_{p_r}\mu(r,p_r)$, etc ... Eq-(2.8) yields

$$\begin{array}{lll} C^1_{21} = \frac{1}{2}\mu', & C^2_{22} = \frac{1}{2}\nu', & C^2_{11} = \frac{1}{2}\mu' e^{\mu-\nu}, \\ C^2_{ab} = -e^{-\nu}\rho\rho'\tilde{h}_{ab}, & C^a_{2b} = \frac{\rho'}{\rho}\delta^a_b, & C^a_{bc} = \tilde{C}^a_{bc}(\tilde{h}_{ab}), \end{array} \quad (A.2)$$

and the only nonvanishing momentum space curvature tensor components are

$$\begin{array}{ll} \mathcal{S}^1_{212} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\nu'\mu', & \mathcal{S}^1_{a1b} = -\frac{1}{2}\mu' e^{-\nu}\rho\rho'\tilde{h}_{ab}, \\ \mathcal{S}^2_{121} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\nu'\mu'), & \mathcal{S}^2_{a2b} = e^{-\nu}(\frac{1}{2}\nu'\rho\rho' - \rho\rho'')\tilde{h}_{ab}, \\ \mathcal{S}^a_{bcd} = \tilde{R}^a_{bcd} - \rho'^2 e^{-\nu}(\delta^a_c\tilde{h}_{bd} - \delta^a_d\tilde{h}_{bc}). & \end{array} \quad (A.3)$$

The vacuum field equations are

$$\mathcal{S}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{\rho'}{\rho}) = 0, \quad (A.4)$$

$$\mathcal{S}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{\rho'}{\rho} - \frac{\rho''}{\rho}) = 0, \quad (A.5)$$

and

$$\mathcal{S}_{ab} = \frac{e^{-\nu}}{\rho^2}\left(\frac{1}{2}(\nu' - \mu')\rho\rho' - \rho\rho'' - (D-3)\rho'^2\right)\tilde{h}_{ab} + \frac{k}{\rho^2}(D-3)\tilde{h}_{ab} = 0, \quad (A.6)$$

where $k = \pm 1$, depending if $\tilde{h}_{ab}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}\mathcal{S}_{11} + \mathcal{S}_{22} = 0$ we get

$$\mu' + \nu' = \frac{2\rho''}{\rho'}. \quad (A.7)$$

The solution of this equation is

$$\mu + \nu \;=\; \ln \rho'^2 \;+\; C, \tag{A.8}$$

where C is an integration constant that one can set to *zero* if one wishes to recover the flat momentum space metric in the asymptotic region $p_r \to \infty$.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu}\;(\;\nu'\rho\rho' - 2\rho\rho'' - (D-3)\rho'^2\;)\;=\;-\,k(D-3) \tag{A.9}$$

or

$$\gamma'\rho\rho' + 2\gamma\rho\rho'' + (D-3)\gamma\rho'^2 = k(D-3), \tag{A.10}$$

where

$$\gamma = e^{-\nu}. \tag{A.11}$$

The solution of (A.10) corresponding to a $D-2$-dim homogeneous momentum space of positive curvature ($k=1$) can be written as

$$\gamma = (1 - \frac{\beta_D M}{(D-2)\;\Omega_{D-2}\;\rho(r,p_r)^{D-3}})\;(\frac{\partial\rho}{\partial p_r})^{-2} \Rightarrow$$

$$h_{p_r p_r} = e^{\nu} = (1 - \frac{\beta_D M}{(D-2)\;\Omega_{D-2}\;\rho(r,p_r)^{D-3}})^{-1}\;(\frac{\partial\rho}{\partial p_r})^2. \tag{A.12}$$

where Ω_{D-2} is the appropriate momentum space solid angle in $D-2$-dim and β_D is a suitable constant.

For the most general $D-2$-dim homogeneous momentum space we may write

$$-\nu \;=\; \ln(k - \frac{\beta_D M}{(D-2)\;\Omega_{D-2}\;\rho^{D-3}}) \;-\; 2\;\ln\rho' \tag{A.13}$$

Thus, according to (A.8) we get

$$\mu \;=\; \ln(k - \frac{\beta_D M}{(D-2)\;\Omega_{D-2}\;\rho^{D-3}}) \;+\; constant. \tag{A.14}$$

we can set the constant to zero, and this means the momentum space line element (A.1) can be written as

$$(ds)^2_{(p)} = -(k - \frac{\beta_D M}{(D-2)\;\Omega_{D-2}\;\rho^{D-3}})(dE)^2 \;+\; \frac{(\partial_{p_r}\rho)^2}{(k - \frac{\beta_D M}{(D-2)\;\Omega_{D-2}\;\rho^{D-3}})}(dp_r)^2 \;+$$

$$\rho^2(r,p_r)\;\tilde{h}_{ab}\;d\xi^a\;d\xi^b \tag{A.15}$$

In the case of a $D-2$-dim sphere in momentum space we have $k=1$, and the angular part of (A.15) is simply $\rho(r,p_r)^2(d\Omega_{(p)})^2$. When $D=4$, one has $\beta_D = 16\pi$ so that $\beta_D M/(D-2)\Omega_{D-2}\;\rho(r,p_r)^{D-3} \Rightarrow 2M/\rho(r,p_r)$. It is interesting to observe that the only effect of the homogeneous metric $\tilde{h}_{ab}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim momentum space. $k = 0$ corresponds to a spatially flat $D-2$-dim section. Concluding, we have shown in the static spherically symmetric case, that the momentum space metric components in eq-(3-4) obey the momentum space Ricci flat conditions $\mathcal{S}_{ab} = 0$.

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