

# Kaos, Reality and the Kaotic Algebra

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**Abstract.** Philosophical considerations suggest a dimensional structure of reality based on infinite subdivision of singularity via Cayley Dickson doubling. An algebra, labelled the Kaotic algebra, is constructed based on algebras obtained by Cayley-Dickson doubling. It provides a natural basis for a non-associative geometry with links to the associative geometry of  $Cl(1, 10)$ . Subalgebras of the kaotic algebra have a phenomenology corresponding to that of the fermions of the standard model.

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## 1 Introduction

Philosophically, the structure of reality should be an elegant construction based on simple geometry. However, the observed universe displays a lot of asymmetry, and current models of its structure are based on odd combinations of dimensionality, such as four plus six for string theory[1] or four plus seven for M-theory[2].

It would be more elegant for the dimensionality of reality to be a number such as 0, 1, or  $\infty$ . A singularity at the origin of the universe would have zero dimensionality. For this to evolve into a dimensionality greater than 1 and for that evolution to stop at 11 seems arbitrary. It would be more satisfactory for that evolution to proceed ad infinitum. Infinite dimensionality is not apparent in reality. However, if the evolutionary process were to be one of subdivision rather than extension, then it could be thought of as generating a chaotic structure corresponding to an infinitely fractal quantum foam, which is a feature of quantum theory. The structure of reality as observed might then be found within that foam as a consequence of properties of substructures with some orderly properties. Several models have been proposed featuring emergent spacetime[3].

The Cayley-Dickson construction[4] generates algebras of increasing size which lose algebraic properties as it generates complex numbers from real numbers, quaternions from complex numbers, octonions from quaternions, and sedenions from octonions. After the sedenions there is no further loss of algebraic properties. Infinite repetition of the Cayley-Dickson construction could correspond to generation of a chaotic structure corresponding to an infinitely fractal quantum foam. Reality might then be found within that foam as a consequence of properties of its real, complex, quaternion and octonion subalgebras which have some orderly properties.

Given the T-duality property of string theories, subdivision into an infinitely fractal foam may be mathematically equivalent to divergence into infinitely many worlds. Philosophically, there is a difference, as an infinitely divided singularity would constitute a more elegant conceptual basis for reality than an infinitely diverging 11 dimensional manifold.

## 2 Algebras used in theories of reality

Theories of reality have been based on various algebras.

General relativity[5] is based on four-dimensional space-time, which can be represented by the Clifford algebras  $Cl(3, 1)$  or  $Cl(1, 3)$ . It accounts for gravitation at extensive scales, but does not account for other forces or quantum effects. It allows the possibility of a scalar field.

The original Kaluza-Klein model[6] is based on five dimensional space-time, which can be represented by the Clifford algebra  $Cl(1, 4)$  or  $Cl(4, 1)$ . The Kaluza-Klein model unifies general relativity with classical electromagnetism, but fails to account for quantum effects.

Yang-Mills theories[7] describe electroweak and strong forces, but not gravity. The Higgs mechanism[8] features a complex doublet.

String theories extend the Kaluza-Klein model by adding extra dimensions, using a background of one time-like dimension, three spatial dimensions and six compactified dimensions. For M theory an extra dimension is added. Ten and eleven dimensions can be represented by the Clifford algebras  $Cl(p, q)$ ,  $p + q = 10$  or  $11$ .

An octonion based approach using the tensor product  $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$  has been proposed by G.Dixon[9]. It has been suggested that the non-associative elements of  $\mathbb{O}$  could account for the probabilistic nature of quantum mechanics. It has been noted that there is a correspondence between Feynman vertices[10] which feature two spinors and a vector component, and the way in which a higher division algebra can be assembled from two copies of a division algebra and a third component[11].

Consider a sequence:

1. From a singularity, generate handings of reals - positive and negative, associate them with a scalar field.
2. From the reals, generate handings of complex numbers, associate them with the complex doublet in the Higgs mechanism.
3. From the complex numbers, generate handings of quaternions, whose product is isomorphic to the the Clifford algebra  $Cl(3, 1)$ , associate it with space-time.
4. From the quaternions generate handings of octonions, associate their product with the compactified dimensions of string and M-theories.
5. From the octonions generate handings of sedenions, associate them with events described by vertices in Feynman diagrams.
6. Associate further Cayley Dickson algebras with virtual particles and quantum foam.

This suggests a model based on a chaotic background represented by an algebra obtained by infinitely repeated Cayley-Dickson constructions, or kaos (naming it after the primordial void of greek mythology). Observed features of reality - the Higgs field, space-time, particles and events, would then be represented by sub-algebras with more orderly properties such as commutativity, associativity and alternativity.

In this paper an algebra with these qualities, the kaotic algebra, is presented. It has been assembled using a modified form of Moufang Loop construction[12].

### 3 The kaotic algebra

The kaotic algebra is constructed using a modified Moufang loop construction. If applied to a quaternion quartet, it generates a sedenion. When applied to unit multivector matrices for  $Cl(1, 4)$ , it generates the Kaotic algebra.

#### 3.1 Moufang Loop construction for octonions

For the Cayley-Dickson construction, octonions are represented by quaternion pairs, where, for  $p = (a, b)$ ,  $q = (c, d)$ , the product rule is:

$$pq = (ac - db^*, a^*d + cb) \text{ or, equivalently: } pq = (ac - d^*b, da + bc^*)$$

For Moufang loop construction, a dis-association operator,  $\mu$ , is assigned to the second quaternion pair, and the product rule is, for  $p = (p, \mu p)$  and  $q = (q, \mu q)$ :

$$\begin{aligned} p \cdot q &= (pq) \\ p \cdot \mu q &= \mu(p^{-1}q) \\ \mu p \cdot q &= \mu(qp) \\ \mu p \cdot \mu q &= -(qp^{-1}) \end{aligned}$$

#### 3.2 Modified Moufang Loop construction for sedenions

For sedenions, a similar construction can be assembled, based on quaternion quartets. Dis-association operators,  $\mu$ ,  $\nu$  and  $\lambda$ , are assigned to three of the quaternions, with rules imposed for products of  $p = (p, \mu p, \nu p, \lambda p)$  and  $q = (q, \mu q, \nu q, \lambda q)$  as follows:

$$\begin{aligned} p \cdot q &= (pq) \\ p \cdot \mu q &= \mu(p^{-1}q) \\ p \cdot \nu q &= \nu(p^{-1}q) \\ p \cdot \lambda q &= \lambda(p^{-1}q) \\ \mu p \cdot q &= \mu(qp) \\ \nu p \cdot q &= \nu(qp) \\ \lambda p \cdot q &= \lambda(qp) \\ \mu p \cdot \mu q &= -(qp^{-1}) \\ \nu p \cdot \nu q &= -(qp^{-1}) \\ \lambda p \cdot \lambda q &= -(p^{-1}q) \\ \mu p \cdot \lambda q &= \nu(pq) \\ \mu p \cdot \nu q &= -\lambda(pq) \\ \nu p \cdot \mu q &= \lambda(qp) \\ \nu p \cdot \lambda q &= -\mu(qp) \\ \lambda p \cdot \nu q &= \mu(pq) \\ \lambda p \cdot \mu q &= -\nu(qp) \end{aligned}$$

These rules generate anti-commuting products whose sign matrix is a Hadamard matrix, as is required for Cayley-Dickson sedenions. Note that there are differences between the rules for  $\mu$ ,  $\nu$  and  $\lambda$ .

A feature of this construction is that, if the signs of products with dissimilar dis-association operators are reversed, the sign matrix is not a Hadamard matrix.

### 3.3 Construction of the kaotic algebra

To generate the table of products for the Kaotic algebra, the multiplication rules used in the modified Moufang Loop construction for sedenions are applied to the 32 real and imaginary  $4 \times 4$  unit matrices for unit multivector components for the Clifford algebra  $Cl(1,4)$  factored by 1,  $\mu$ ,  $\nu$ , or  $\lambda$ , generating an algebra with 128 unit elements.

### 3.4 Notation used to label basis elements for the kaotic algebra

In order to allow compact presentation basis elements, letters have been used to label unit  $4 \times 4$  matrices as shown in table 1. A prescript is added to denote a dis-association operator and  $i$  is added for imaginary matrices, resulting in labels such as:  ${}^\nu iX$ . When referring to several unit matrices with a common dis-association operator, and/or common real/imaginary status, matrices are enclosed in square brackets, e.g.:  ${}^\nu [iXiYiZT]$  and  ${}^\lambda i[PQFD]$ .

TABLE 1. Notation used to label  $4 \times 4$  unit matrices

$$\begin{array}{ccc}
 S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & & \\
 \\
 R = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\
 \\
 Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
 \\
 D = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} & X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} & N = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \\
 \\
 F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} & Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
 \\
 U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} & V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
 \end{array}$$

Note that the positive and negative forms of matrices  $R, Q, L, X, Y, Z$  are opposite to those used in a previous paper by this author, The Pattern of Reality[13], for consistency with a convention that matrices with a +1 entry in the first row are the positive form.

## 4 The Kaotic algebra and Clifford algebras

### 4.1 Dimensionality

For Clifford algebras, the concept of dimensionality is clear cut, with unit vectors assigned to orthogonal directions for each dimension. The unit vectors all anti-commute with each other. For M-theory, if its eleven dimensions are separated into a  $Cl(1,3)$  world manifold and a seven dimensional compactified tangent bundle, the compactified dimensions have to have distinct unit vectors for an associated Clifford algebra, so  $Cl(1,10)$  is used as the algebra. A  $Cl(1,10)$  multivector has 2048 components.

For the kaotic algebra, dimensionality is less clear cut. Sets of four unit elements that have no dis-association operator, and which anti-commute with each other can be found. But, also, sets of seven unit elements that have dis-association operators and which anti-commute with each other can be found, such that they all anti-commute with four elements having no dis-association operator. However, having only 128 unit components, the equivalent of a multivector for the kaotic algebra should only be equivalent to a Clifford algebra for seven dimensions.

This suggests the hypothesis that, for a model based on the kaotic algebra, the equivalent of the compactified dimensions in M-theory would not be true dimensions, but could be thought of as subdivisions of dimensions, or pseudo-dimensions, allowing multivector components equivalent to unit vectors for true dimensions to be products of three or more unit vector equivalents for pseudo-dimensions. This could provide a basis for a model based on emergent spacetime, a feature of several attempts to find a fundamental basis for reality[3].

In order to relate physics based on the kaotic algebra to physics based on  $Cl(1,10)$ , it is useful to map the true dimensions of a  $Cl(1,10)$  manifold into a combination of true and pseudo-dimensions for a kaotic algebra manifold. To describe this map, a compact form of notation for  $32 \times 32$  matrices used to represent unit elements of a  $Cl(1,10)$  multivector is needed.

### 4.2 Notation for $Cl(1,10)$ unit matrices

$32 \times 32$  unit matrices can be assembled as a nesting of a group of  $2 \times 2$  unit matrices entered into a group of  $4 \times 4$  unit matrices entered into a second group of  $4 \times 4$  unit matrices. In this paper  $4 \times 4$  unit matrices are labelled as shown in table 1 (in section 3), and  $2 \times 2$  matrices are labelled:

$$s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

This allows a real  $32 \times 32$  matrix to be labelled using a three letter combination such as:  ${}^g Y X$ , and an imaginary  $32 \times 32$  matrix to be labelled using a four letter combination such as:  ${}^{hD} i F$ . Using this notation, products for two matrices can be obtained by multiplying the nested matrices separately. For instance:

$${}^g Y X \times {}^{hD} F = {}^{g.h Y.D} X.F$$

As for kaotic algebra labels, sets of several matrices with similar nested extension matrices can be put in square brackets, such as  ${}^{iS} [TXYZ]$ .

### 4.3 Matrices chosen to represent unit vectors for $Cl(1, 10)$

${}^{sS} [iVTXYZ]$  all anticommute with each other. If  ${}^{sS} [TXYZ]$  are assigned to represent unit vectors for space-time dimensions,  $[iV]$  could be assigned to the fifth dimension used in the original Kaluza-Klein theory. Then, to expand that theory to eleven dimensions for use in M-theory, it would be logical to select eleven matrices to represent unit vectors such as:

$${}^{sS} [TXYZ] \quad {}^g [TUV]_V \quad {}^h [LMN]_V \quad {}^j S_i V$$

Then:

${}^{sS} [TXYZ]$  can be assigned to space-time dimensions,

and

${}^g [TUV]_V \quad {}^h [LMN]_V \quad {}^j S_i V$  can be assigned to compactified dimensions.

### 4.4 Mapping between $Cl(1, 10)$ and the kaotic algebra

Space-time dimensions can be regarded as being generated using  $\mathbb{H} \otimes \mathbb{H}$ , the product of a right and a left handed quaternion. This suggests using the product of a right and a left handed octonion to generate compactified dimensions. Consider unit elements from the kaotic algebra such as:

$${}^\mu [TUV] \quad {}^\nu [LMN] \quad {}^\lambda i S$$

If sign is ignored, the products for these elements have the same latin square as for  ${}^g [TUV]_V \quad {}^h [LMN]_V \quad {}^j S_i V$

This suggests that an equivalent of the Clifford algebra multivector for the compactified dimensions of M-theory can be identified for the kaotic algebra elements.

The algebra multivector components for  ${}^g [L V^M V^N V] \quad {}^h [T V^U V^V V] \quad {}^i j^S V$  are:

Scalar:  ${}^S S$

Vector:  ${}^g [TUV]_V, {}^h [LMN]_V, {}^j S_i V$

Bivector:  ${}^{[TUVLMN]}_V, {}^{j[XYZPQRDEF]}_V, {}^h [TUV]_i V, {}^g [LMN]_i V$

3-vector:  ${}^j [TUVLMN]_i V, {}^g [SXYZPQRDEF]_i V, {}^h [SXYZPQRDEF]_i V, [XYZPQRDEF]_i V$

4-vector:  ${}^j [TUVLMN]_i V, {}^g [SXYZPQRDEF]_i V, {}^h [SXYZPQRDEF]_i V, [XYZPQRDEF]_i V$

5-vector:  ${}^{[TUVLMN]}_i V, {}^{j[XYZPQRDEF]}_i V, {}^h [TUV]_i V, {}^g [LMN]_i V$

6-Vector:  ${}^g [TUV]_i V, {}^h [LMN]_i V, {}^j S_i V$

Pseudoscalar:  ${}^S i V$

Equivalents of  $Cl(0,7)$  unit multivector components for the kaotic algebra elements can be identified as:

Scalar:  $S$

Vector:  ${}^{\mu}[TUV], {}^{\nu}[LMN], {}^{\lambda}iS$

Bivector:  $[TUVLMN], {}^{\lambda}[XYZPQRDEF], {}^{\nu}i[TUV], {}^{\mu}i[LMN]$

3-vector:  ${}^{\lambda}i[TUVLMN], {}^{\mu}[SXYZPQRDEF], {}^{\nu}[SXYZPQRDEF], i[XYZPQRDEF]$

4-vector:  ${}^{\lambda}[TUVLMN], {}^{\mu}i[SXYZPQRDEF], {}^{\nu}i[SXYZPQRDEF], [XYZPQRDEF]$

5-vector:  $i[TUVLMN], {}^{\lambda}i[XYZPQRDEF], {}^{\nu}[TUV], {}^{\mu}[LMN]$

6-Vector:  ${}^{\mu}i[TUV], {}^{\mu}i[LMN], {}^{\lambda}S$

Pseudoscalar:  $iS$

## 5 The kaotic algebra and the standard model

For the equivalents of  $Cl(0,7)$  unit multivector components for the kaotic algebra elements as identified above, real elements are the equivalent of  $Cl(0,6)$  products for the six elements:  ${}^{\mu}[TUV] {}^{\nu}[LMN]$ , and the remaining elements are their imaginary counterparts. This allows representation of phase.

For the products of the six elements  ${}^{\mu}[TUV] {}^{\nu}[LMN]$ , the even components are:

Scalar:  $S$

Bivector:  $[TUVLMN], {}^{\lambda}[XYZPQRDEF]$

4-vector:  ${}^{\lambda}[TUVLMN], [XYZPQRDEF]$

6-vector:  ${}^{\lambda}S$

These exclude two of the disassociation operators.

This suggests consideration of similar arrangements:

Scalar:  $S$

Bivector:  $[TUVLMN], {}^{\mu}[XYZPQRDEF]$

4-vector:  ${}^{\mu}[TUVLMN], [XYZPQRDEF]$

6-vector:  ${}^{\mu}S$

and

Scalar:  $S$

Bivector:  $[TUVLMN], {}^{\nu}[XYZPQRDEF]$

4-vector:  ${}^{\nu}[TUVLMN], [XYZPQRDEF]$

6-vector:  ${}^{\nu}S$

A combination such as  $SLMN^{\mu}[SLMN]$  or  $SLMN^{\nu}[SLMN]$  is octonionic, but a combination such as  $SLMN^{\lambda}[SLMN]$  is not. However, subalgebras of  $SLMN^{\lambda}[SLMN]$  such as  $SL^{\lambda}[SL]$  are quaternionic.

A combination such as  $SLMN, {}^{\mu}[SLMN], {}^{\nu}[SLMN], {}^{\lambda}[SLMN]$  is sedenionic. It can represent a combination of two octonions with  $[SLMN]$  elements in common -  $SLMN^{\mu}[SLMN]$  and  $SLMN^{\nu}[SLMN]$  together with a non-octonionic subalgebra -  $SLMN^{\lambda}[SLMN]$  assembled as a combination of two quaternions -



$SL^\lambda[SL] \times SM^\lambda[SM]$ . This would allow a sedenion to represent a vertex in a Feynman diagram, featuring two spinorial components and one component assembled from two vectorial components.

Having related elements of the kaotic algebra to multivector components for an associative compactified manifold, to assemble a model of reality requires the algebra to be also related to a  $Cl(1,3)$  or  $Cl(3,1)$  algebra for space-time. One way to do this would be to assemble a tensor product of the kaotic algebra with  $Cl(1,3)$  or  $Cl(3,1)$ . An alternative scheme would be to redefine the concept of dimensionality, such that the non-associative manifold is one of sub-divided or pseudo-dimensions, and postulating that products of pseudo-dimensional vectors generate true dimensions for an emergent space-time.

For this scheme, unit elements of the kaotic algebra such as  $[VTiXiYiZ]$  or  $[MNiXiPiD]$  can be found, all of which anti-commute with each other and anti-commute with all elements of the non-associative manifold. However, these elements have signature  $(- - - -)$ , so, to assign an element to time requires a the imaginary counterpart of one element to be used, such as for  $[iT]$ .

For  $[iTXiYiZ]$ ,  $iX$ ,  $iY$  and  $iZ$  all anti-commute with all elements of the non-associative manifold, but  $iT$  commutes with all elements of the non-associative manifold. One effect of this would be to create an arrow of time with respect to phenomena associated with the non-associative manifold.

The choice of elements to represent space-time such as  $[iTXiYiZ]$  is arbitrary,  $[iNiXiPiD]$  would serve equally well. However, once that choice is made, it breaks a symmetry of the kaotic algebra, as the relationship to  $[iTXiYiZ]$  for the octonion ( $SLMN \mu[SLMN]$ ) is a different to that for the octonion ( $STUV \mu[STUV]$ ). This symmetry breakage is equivalent to that used to associate permutations of subalgebras of  $Cl(p,q)$ ,  $p+q=5$  with fermions described in a previous paper, The Pattern of Reality[13], a rearranged version of which is shown in Appendix B, table 11. The same permutations of subalgebras can be extended into octonionic combinations from the kaotic algebra to create a similar phenomenology.

The feature of the kaotic algebra, that, if the signs of products with dissimilar dis-association operators are reversed, the sign matrix is not a Hadamard matrix could account for the matter/anti-matter asymmetry observed in the universe.

## 6 Discussion

### 6.1 Introduction

Assembly and analysis of grand unification theories requires an understanding of complex mathematics and physics. However, the goal of such theories is to find a relatively simple algebra at their heart. Physicists have searched exhaustively for that algebra with limited success, suggesting that there may be some element of conventional physics that misdirects the search. This paper presents the result of an attempt to find an algebra that achieves the goal by naively focussing on algebraic patterns, reducing the possibility of any such misdirection. The result, the kaotic algebra, appears to have features that make it a candidate for the algebra of reality. A simplistic analysis of these features is presented in this paper. Further work is required, but if the kaotic algebra is fundamental, it will serve science better for it to be brought to the attention of the physics community now, rather than in a more rigorous way in the distant future.

### 6.2 Development of the Kaotic algebra

Twenty years after attending university, and twenty years ago, recollection of a remark made by a university friend about the dimensionalities which allow a cross product, aroused my curiosity. That led me to an article by A. Eddington[14], which related the metric properties of spacetime to those of a group.

For that group, I assigned letters to matrices representing unit multivector components for  $Cl(3,1)$  relating them to space-time dimensions, and arranged their multiplication table in sets of anti-commuting pentads as shown in table 2. This table suggested association of one pentad with gravity, two with electroweak forces, and three with the strong nuclear force.

TABLE 2.  $4 \times 4$  unit matrix products

	$S$	$V$	$T$	$X$	$Y$	$Z$
$S$	$S$	$V$	$T$	$X$	$Y$	$Z$
$V$	$V$	$-S$	$U$	$P$	$Q$	$-R$
$T$	$T$	$-U$	$-S$	$-D$	$E$	$-F$
$X$	$X$	$-P$	$D$	$S$	$N$	$-M$
$Y$	$Y$	$-Q$	$-E$	$-N$	$S$	$-L$
$Z$	$Z$	$R$	$F$	$M$	$L$	$S$

The description of the electron using even components of the multivector for  $Cl(3,1)$  by D. Hestenes[15] suggested searching for a pattern corresponding to the phenomenology of fermions for sub-algebras of higher Clifford algebras. This suggested the scheme of symmetry and symmetry breakage described in a paper, The Pattern of Reality[13].

That paper identified a pattern of subalgebras of  $Cl(p, q), p + q = 5$  as corresponding to the phenomenology of the fermions of the standard model together with hypothetical fermions for dark matter. The absence of observation of dark matter fermions made assessment of the validity of the correspondence problematic, as the known components only constitute a part of the pattern. For further development, I attempted to find a geometric basis for the Higgs mechanism. In this endeavour, I have found some properties of matrices that generate an expression with the form of the Higgs potential, as set out in Appendix C, but have yet to relate it to the kaotic algebra.

Thinking about bosons led me to consider how to model their interactions with fermions. The nature of protons and neutrons, being combinations of three quarks, suggested investigating combinations of 8-dimensional algebras, such as the octonions. The Moufang loop construction enables assembly of octonions using matrix representations of quaternions together with dis-association operators. Generation of a matrix representation of unit multivectors for spacetime from the direct product of matrix representations of left and right handed quaternions suggested seeking a similar process for octonions. Octonions are not associative so a group direct product is not defined. One possible product is the tensor product of two octonions, the octo-octonions, but their algebra does not include that of sedenions, which I had been hoping to use to represent combinations of three quarks.

This focussed my efforts on trying to find a way to generate sedenions from octonions using matrix representations of quaternions and dis-association operators, using direct calculation of matrix products for different multiplication rules on a spreadsheet, searching for the right commutation properties for products, and checking that the sign matrix for products was a Hadamard matrix.

After a great deal of trial and error, I found the construction described in section 3. Using it, a sedenion is generated from two octonions with a common quaternionic subalgebra, together with a third 8 dimensional algebra having the same quaternionic subalgebra that is not octonionic. I realised that it made more sense to have a sedenion represent two fermions and a boson combining at a Feynman vertex than to have it represent a proton or neutron.

Applying the construction to matrices representing unit multivector components of  $Cl(1, 4)$  resulted in the kaotic algebra. Its manner of construction and number of elements (128) suggests a relationship with the octonionic projective plane  $\mathbb{O}P^2$ . The differences between the product rules for the three dis-association operators creates the possibility of symmetry breakage resulting in the distinction between fermions and bosons. The handing of the multiplication rule for elements with dissimilar dis-association operators accounts for the handedness of the universe.

For the kaotic algebra, matrices can be assigned to represent unit vectors for space-time, but the matrix assigned to represent time commutes with matrices assigned to represent compactified dimensions. This would have the effect of creating a difference between positive time and negative time for phenomena associated with compact dimensions, providing a basis for the arrow of time. The choice of matrices to be associated with space-time creates differences between subalgebras which corresponds to symmetry breakage distinguishing the electroweak forces from the strong nuclear force.

In parallel with the search for an algebra incorporating sedenions, I was also looking at Clifford algebras for 10 and 11 dimensional space-times. The form of notation adopted links the Kaotic algebra with Clifford algebras in a way that suggests that the kaotic algebra can be compatible with many of the results obtained using string/M-theories.

Being no mathematician or physicist, my analysis of the correspondence between the kaotic algebra and reality is likely to be flawed, but provides an indication of the potential of the algebra. It is presented in the hope that others will be able to use it to develop a viable theory of reality.

## **7 Acknowledgements**

Many papers available on arXiv and elsewhere on the internet contributed to the development of this paper, but most use higher mathematical methods, which make them less accessible for a non-mathematician and making it hard to acknowledge particular contributions. I thank Wikipedia and its contributors and John Baez for information presented on the internet on a more accessible level.

### 8 Appendix A - Products of unit elements of the kaotic algebra

An overview of the commutation properties of products of the kaotic algebra is shown in condensed form in figure 1, where products that anti-commute have an A in their entry. The multiplication table for the kaotic algebra is shown in tables 3 to 10.

FIGURE 1. Pattern of anticommuting products for the kaotic algebra

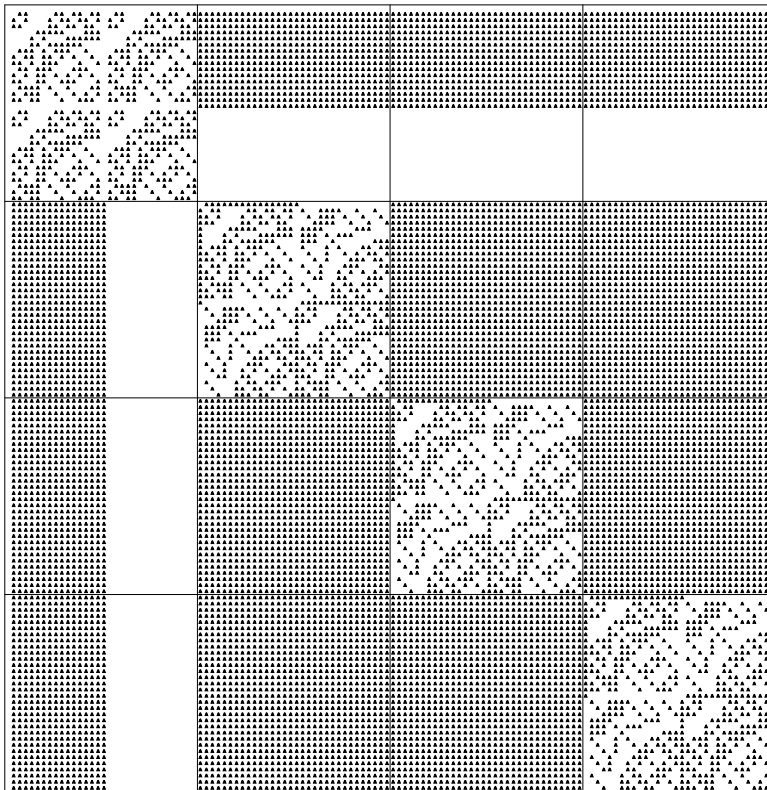




TABLE 4. Kaotic Algebra multiplication table, part A2

	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
S	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
L	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
M	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
N	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
T	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
U	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
V	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iX	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iY	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iZ	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iP	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iQ	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iR	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iD	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iE	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iF	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iS	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iL	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iM	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iN	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iT	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iU	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
iV	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
X	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
Y	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
Z	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
P	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
Q	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
R	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
D	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
E	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F
F	S	L	M	N	T	U	V	iX	iY	iZ	iP	iQ	iR	iD	iE	iF	iS	iL	iM	iN	iT	iU	iV	X	Y	Z	P	Q	R	D	E	F





TABLE 6. Kaotic Algebra multiplication table, part B2

Table with 48 columns and 48 rows, representing a multiplication table for the Kaotic Algebra. The columns and rows are labeled with algebraic elements, each consisting of a base symbol (e.g., s, L, M) and a superscripted symbol (e.g., -S, +L, \*M). The table contains a dense grid of algebraic expressions representing the product of the corresponding elements.









## 9 Appendix B - Subalgebras with phenomenology corresponding to fermions

In a previous paper, The Pattern of Reality[13], it was found that the phenomenology of the fermions of the standard model could be associated with that of some  $SU(2) \times U(1)$  subalgebras of  $SL(4, C)$ . It was postulated that other  $SU(2) \times U(1)$  subalgebras of  $SL(4, C)$  could describe the phenomenology of fundamental particles for dark matter, labelled varks. These subalgebras were represented as the product of matrices isomorphic to the quaternions with matrices isomorphic to the complex numbers, as shown in table 2. The same pattern can be assembled with the quaternions extended into octonions, generating right and left handed versions. For example, the entry  $SLMN \times SV$  can be extended to  $([SLMN], {}^\mu[SLMN]) \times [SV]$  or to  $([SLMN], {}^\nu[SLMN]) \times [SV]$ .

TABLE 11. Fermions

Family	Texture	Flavor	Color		
Electron		$e$	$SLMN \times SV$		
		$\mu$	$SLMN \times ST$		
		$\tau$	$SLMN \times SU$		
Neutrino		$\nu_e$	$SV(TU) \times SL$	$SV(TU) \times SM$	$SV(TU) \times SN$
		$\nu_\mu$	$ST(UV) \times SL$	$ST(UV) \times SM$	$ST(UV) \times SN$
		$\nu_\tau$	$SU(VT) \times SL$	$SU(VT) \times SM$	$SU(VT) \times SN$
			<i>red</i>	<i>blue</i>	<i>green</i>
Up Quark		up	$SiEiFL \times SV$	$SiDiFM \times SV$	$SiDiEN \times SV$
		charm	$SiQiRL \times ST$	$SiPiRM \times ST$	$SiPiQN \times ST$
		bottom	$SiYiZL \times SU$	$SiXiZM \times SU$	$SiXiYN \times SU$
Down Quark		down	$SiXiPV \times SL$	$SiYiQV \times SM$	$SiZiRV \times SN$
		strange	$SiDiXT \times SL$	$SiEiYT \times SM$	$SiFiZT \times SN$
		top	$SiPiDU \times SL$	$SiQiEU \times SM$	$SiRiFU \times SN$
			<i>magenta</i>	<i>yellow</i>	<i>cyan</i>
Sharp vark	Rough	hard	$STiEiY \times SiP$	$STiFiZ \times SiQ$	$STiDiX \times SiR$
		bright	$SUiQiE \times SiX$	$SUiRiF \times SiY$	$SUiPiD \times SiZ$
		cool	$SViYiQ \times SiD$	$SViZiR \times SiE$	$SViXiP \times SiF$
	Smooth	hard	$STiFiZ \times SiP$	$STiDiX \times SiQ$	$STiEiY \times SiR$
		bright	$SUiRiF \times SiX$	$SUiPiD \times SiY$	$SUiQiE \times SiZ$
		cool	$SViRiZ \times SiD$	$SViXiP \times SiE$	$SViYiQ \times SiF$
Round vark	Rough	soft	$SiELiF \times SiP$	$SiFMiD \times SiQ$	$SiDNiE \times SiR$
		dull	$SiQLiR \times SiX$	$SiRMiP \times SiY$	$SiPNiQ \times SiZ$
		warm	$SiYLiZ \times SiD$	$SiZMiX \times SiE$	$SiXNiY \times SiF$
	Smooth	soft	$SiZLiY \times SiP$	$SiXMiZ \times SiQ$	$SiYNiX \times SiR$
		dull	$SiFLiE \times SiX$	$SiDMiF \times SiY$	$SiENiD \times SiZ$
		warm	$SiRLiQ \times SiD$	$SiPMiR \times SiE$	$SiQNiP \times SiF$

### 10 Appendix C - The Higgs mechanism

It is not obvious how to generate the phenomenology of the Higgs mechanism from simple mathematical structures. Considerations such as the requirement for bosons to have symmetric wave functions and the application of Heisenberg groups to their physics suggest investigation of subgroups of Heisenberg groups.

A simple symmetric matrix with four components is the  $4 \times 4$  diagonal matrix. It's entries could be set to  $e^{i\theta_n}$ , generating the matrix:

$$\begin{bmatrix} e^{i\alpha} & 0 & 0 & 0 \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & 0 & e^{i\gamma} & 0 \\ 0 & 0 & 0 & e^{i\delta} \end{bmatrix}$$

If  $-\delta$  is set equal to  $\alpha + \beta + \gamma$ , the matrix has unit determinant.

For multiplication, a matrix with similar properties can be found as a subgroup of the Heisenberg group  $H3$ :

$$\begin{bmatrix} 1 & a & b & d+ab \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Substituting  $c = (d+ab)$  this matrix can be written in terms of unit  $4 \times 4$  matrices as:

$$S + a(T + Y)/2 + b(M + R)/2 + c(D + U + N + Q)/4$$

For matrices rows and columns can be manipulated whilst maintaining overall properties. In terms of the unit  $4 \times 4$  matrices, an equivalent manipulation is one which preserves commutation relationships, interchanging rows and columns for an array such as:

<i>S</i>	<i>Z</i>	<i>R</i>	<i>M</i>	<i>L</i>	<i>F</i>
<i>Z</i>	<i>S</i>	<i>V</i>	<i>X</i>	<i>Y</i>	<i>T</i>
<i>R</i>	<i>V</i>	<i>S</i>	<i>P</i>	<i>Q</i>	<i>U</i>
<i>M</i>	<i>X</i>	<i>P</i>	<i>S</i>	<i>N</i>	<i>D</i>
<i>L</i>	<i>Y</i>	<i>Q</i>	<i>N</i>	<i>S</i>	<i>E</i>
<i>F</i>	<i>T</i>	<i>U</i>	<i>D</i>	<i>E</i>	<i>S</i>

For this array,  $[TY]$  anticommute with  $[ZE]$ ,  $[MR]$  anticommute with  $[ZP]$ ,  $[DUNQ]$  anticommute with  $[PE]$ . If rows and columns of the array are interchanged to:

$S$	$V$	$T$	$X$	$Y$	$Z$
$V$	$S$	$U$	$P$	$Q$	$R$
$T$	$U$	$S$	$D$	$E$	$F$
$X$	$P$	$D$	$S$	$N$	$M$
$Y$	$Q$	$E$	$N$	$S$	$L$
$Z$	$R$	$F$	$M$	$L$	$S$

For this array,  $[QR]$  anticommute with  $[VL]$ ,  $[TX]$  anticommute with  $[VD]$ ,  $[EFMN]$  anticommute with  $[LD]$ , and matrices with similar properties to:

$$S + a(T + Y)/2 + b(M + R)/2 + c(D + U + N + Q)/4$$

can be assembled such as:

$$S + a(Q + R)/2 + b(T + X)/2 + c(E + F + M + N)/4$$

Another possibility is:

$$S + a(U + P)/2 + b(Y + Z)/2 + c(E + F + M + N)/4$$

These properties include having unit determinant and forming an abelian group isomorphic to that formed by:

$$\begin{bmatrix} 1 & a & b & d + ab \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, for:  $[S] + [QR] + [TX] + [EFMN]$  and  $[S] + [UP] + [YZ] + [EFMN]$ , a matrix with unit determinant can also be represented by:

$$\sqrt{|(1 \pm 2a^2)|} [S] + a[Q + R]/2 + b[V + L]/2 + c[E + F + M + N]/4$$

This is also the case for  $[S] + [UP] + [YZ] + [EFMN]$  and for any similar combination for which the commuting elements diagonally adjacent to the leading diagonal of the array have signature  $(-+-)$ , including complex combinations such as  $[iZPiE]$ .

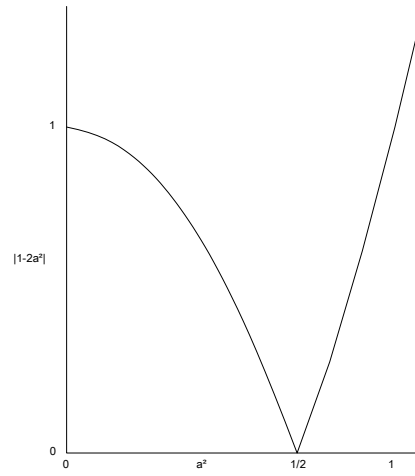
For:  $\sqrt{|(1 - a^2/2)|} [S] + a[Q + R]/2 + b[V + L]/2 + c[E + F + M + N]/4$ , the term in  $S$  is:

$$\begin{bmatrix} \sqrt{|(1 - a^2/2)|} & 0 & 0 & 0 \\ 0 & \sqrt{|(1 - a^2/2)|} & 0 & 0 \\ 0 & 0 & \sqrt{|(1 - a^2/2)|} & 0 \\ 0 & 0 & 0 & \sqrt{|(1 - a^2/2)|} \end{bmatrix}$$

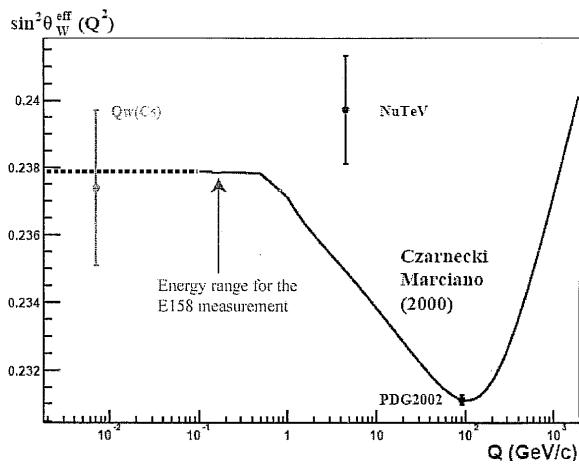
The determinant for this matrix =  $1 - 2a^2 + a^4$  which has the form of the Higgs potential.

A plot of  $\sqrt{|(1 - a^2/2)|}$  against  $a$  in the range 0 to 2 is shown below:





This has a similar form to a plot of  $\sin^2\theta_W$  against momentum transfer as shown below.



At this stage I have not investigated the consequences of using octonionic elements instead of matrices.

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