Pi Formulas , Part 11

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abstract

In this note we give some formulas related to the constant Pi

π - CONSTANTE – FORMULAS

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Resumen. Se muestran algunas fórmulas que involucran la constante π.

1. INTRODUCCIÓN.

Teorema. Para $k \in \mathbb{N} = \{1,2,3,...\}$, y $0 < \theta < \frac{\pi}{3}$, se tiene:

$$(-1)^k \theta^{2k-1} = (2k-1)! \sum_{n_1 > n_2 > ... > n_{2k-1} > 0} \frac{Im\left(\left(1-e^{i\theta}\right)^{n_1}\right)}{n_1 \cdots n_{2k-1}}$$

$$(-1)^k \theta^{2k} = (2k)! \sum_{n_1 > n_2 > ... > n_{2k} > 0} \frac{Re\left(\left(1-e^{i\theta}\right)^{n_1}\right)}{n_1 \cdots n_{2k}}$$

2. FÓRMULAS.

2.1. Para $k \in \mathbb{N} = \{1,2,3,...\}$, se tiene:

$$(-1)^k \pi^{2k-1} = 2^{4k-2} (2k-1)! \sum_{n_1 > n_2 > ... > n_{2k-1} > 0} \frac{Im\left(\left(2-\sqrt{2}-i\sqrt{2}\right)^{n_1}\right)}{2^{n_1} n_1 \cdots n_{2k-1}}$$

$$(-1)^k \pi^{2k-1} = 2^{4k-2} (2k-1)! \sum_{n_1 > n_2 > ... > n_{2k-1} > 0} \frac{c_{n_1} + d_{n_1}\sqrt{2}}{2^{n_1} n_1 \cdots n_{2k-1}}$$

$$\left(-1\right)^k \pi^{2k} = 2^{4k}\left(2k\right)! \sum_{n_1>n_2>...>n_{2k}>0} \frac{Re\left(\left(2-\sqrt{2}-i\sqrt{2}\right)^{n_1}\right)}{2^{n_1} n_1 \cdots n_{2k}}$$

$$\left(-1\right)^k \pi^{2k} = 2^{4k}\left(2k\right)! \sum_{n_1>n_2>...>n_{2k}>0} \frac{a_{n_1} + b_{n_1}\sqrt{2}}{2^{n_1} n_1 \cdots n_{2k}}$$

donde para $n = 0,1,2,3,...$, se tiene:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 & 2 \\ -1 & 2 & 1 & 0 \\ 0 & -2 & 2 & -2 \\ -1 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} \quad , \quad \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -2 & 0 & 2 \\ -1 & 2 & 1 & 0 \\ 0 & -2 & 2 & -2 \\ -1 & 0 & -1 & 2 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.2. Para $k \in \mathbb{N} = \{1,2,3,...\}$, se tiene:

$$\left(-1\right)^k \pi^{2k-1} = 6^{2k-1}\left(2k-1\right)! \sum_{n_1>n_2>...>n_{2k-1}>0} \frac{Im\left(\left(2-\sqrt{3}-i\right)^{n_1}\right)}{2^{n_1} n_1 \cdots n_{2k-1}}$$

$$\left(-1\right)^k \pi^{2k-1} = 6^{2k-1}\left(2k-1\right)! \sum_{n_1>n_2>...>n_{2k-1}>0} \frac{c_{n_1} + d_{n_1}\sqrt{3}}{2^{n_1} n_1 \cdots n_{2k-1}}$$

$$(-1)^k \pi^{2k} = 6^{2k} (2k)! \sum_{n_1 > n_2 > \ldots > n_{2k} > 0} \frac{Re\left(\left(2 - \sqrt{3} - i \right)^{n_1} \right)}{2^{n_1} n_1 \cdots n_{2k}}$$

$$(-1)^k \pi^{2k} = 6^{2k} (2k)! \sum_{n_1 > n_2 > \ldots > n_{2k} > 0} \frac{a_{n_1} + b_{n_1} \sqrt{3}}{2^{n_1} n_1 \cdots n_{2k}}$$

donde para $n = 0,1,2,3,\ldots$, se tiene:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 2 & -3 \\ 0 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} \quad , \quad \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 2 & -3 \\ 0 & -1 & -1 & 2 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.3. Para $k \in \mathbb{N} = \{1,2,3,\ldots\}$, se tiene:

$$(-1)^k \pi^{2k-1} = 12^{2k-1} (2k-1)! \sum_{n_1 > n_2 > \ldots > n_{2k-1} > 0} \frac{Im\left(\left(4 - \sqrt{2} - \sqrt{6} - i\left(\sqrt{6} - \sqrt{2} \right) \right)^{n_1} \right)}{2^{2n_1} n_1 \cdots n_{2k-1}}$$

$$(-1)^k \pi^{2k-1} = 12^{2k-1}(2k-1)! \sum_{n_1>n_2>\ldots>n_{2k-1}>0} \frac{A_{n_1} + B_{n_1}\sqrt{2} + C_{n_1}\sqrt{3} + D_{n_1}\sqrt{6}}{2^{2n_1} n_1 \cdots n_{2k-1}}$$

$$(-1)^k \pi^{2k} = 12^{2k}(2k)! \sum_{n_1>n_2>\ldots>n_{2k}>0} \frac{Re\left(\left(4-\sqrt{2}-\sqrt{6}-i\left(\sqrt{6}-\sqrt{2}\right)\right)^{n_1}\right)}{2^{2n_1} n_1 \cdots n_{2k}}$$

$$(-1)^k \pi^{2k} = 12^{2k}(2k)! \sum_{n_1>n_2>\ldots>n_{2k}>0} \frac{a_{n_1} + b_{n_1}\sqrt{2} + c_{n_1}\sqrt{3} + d_{n_1}\sqrt{6}}{2^{2n_1} n_1 \cdots n_{2k}}$$

donde para $n = 0,1,2,3,\ldots$, se tiene:

$$X_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ A_n \\ B_n \\ C_n \\ D_n \end{pmatrix} \quad , \quad X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad Y = \begin{pmatrix} 4 & -2 & 0 & -6 & 0 & -2 & 0 & 6 \\ -1 & 4 & -3 & 0 & -1 & 0 & 3 & 0 \\ 0 & -2 & 4 & -2 & 0 & 2 & 0 & -2 \\ -1 & 0 & -1 & 4 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -6 & 4 & -2 & 0 & -6 \\ 1 & 0 & -3 & 0 & -1 & 4 & -3 & 0 \\ 0 & -2 & 0 & 2 & 0 & -2 & 4 & -2 \\ -1 & 0 & 1 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$$

$$X_{n+1} = Y X_n \quad , \quad X_n = Y^n X_0$$

3. OTRAS FÓRMULAS PARA π^{2m-1} Y π^{2m}.

3.1. Para $m = 1,2,3,\ldots$, se tiene:

$$\frac{\pi^{2m-1}}{\sqrt{3}} = 2^{2m-1} 3^{m-1} (2m-1)! \times$$
$$\times \sum_{n_1=0}^{\infty} \cdots \sum_{n_{2m-1}=0}^{\infty} \frac{(-3)^{-(n_1+\ldots+n_{2m-1})}}{(2n_1+1)(2n_1+2n_2+2)\ldots(2n_1+\ldots+2n_{2m-1}+2m-1)}$$

3.2. Para $m = 1,2,3,\ldots$, se tiene:

$$\pi^{2m} = 2^{2m} 3^m (2m)! \times$$
$$\times \sum_{n_1=0}^{\infty} \cdots \sum_{n_{2m}=0}^{\infty} \frac{(-3)^{-(n_1+\ldots+n_{2m})}}{(2n_1+1)(2n_1+2n_2+2)\ldots(2n_1+\ldots+2n_{2m}+2m)}$$

4. REFERENCIAS.

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