Pi Formulas , Part 9

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abstract

We give some formulas for constant Pi :

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} = 3.1415926535\,...$$

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Introducción

En esta nota mostramos algunas fórmulas que involucran a la constante Pi , en las fórmulas (4),(8),(9),(10), aparecen los números:

$$a_k = \overbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}}^{k-\text{radicals}} \ , \ b_k = \overbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}}^{k-\text{radicals}} \ , \ k \in \mathbb{N}$$

en la fórmula (4) aparece la función hipergeometríca de Gauss:

$$F(a,\, b,\, c,\, x) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n!}\, x^n \ , \ -1 < x < 1$$

y los polinomios de Legendre:

$$P_n(x) = F\left(-n,\, n+1,\, 1,\, \frac{1-x}{2}\right)$$

Fórmulas

$$2 \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{8\,n^2-1}\right) + \log\left(\frac{\pi}{2}\right) = 4 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\left(\frac{1}{8\,m^2-1}\right)^{4\,n-1}}{4\,n-1} \tag{1}$$

$$2 \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{8\,n^2+8\,n+1}\right) + \log\left(\frac{4}{\pi}\right) = 4 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\left(\frac{1}{8\,n^2+8\,n+1}\right)^{4\,n-1}}{4\,n-1} \tag{2}$$

$$\frac{\pi}{4} = \tan^{-1}\left(\prod_{n=1}^{\infty} \frac{4\,n^2-1}{4\,n^2+1}\right) + \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{8\,n-2}{(4\,n(2\,n-1))^2+1}\right) \tag{3}$$

$$\frac{2\pi}{a_k} F\left(-\frac{1}{2^{k+1}},\, 1+\frac{1}{2^{k+1}},\, 1,\, \frac{1-x}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^{k+1}}{1-2^{k+1}\, n} - \frac{2^{k+1}}{1+2^{k+1}\,(n+1)}\right) P_n(x) \tag{4}$$

donde

$$k \in \mathbb{N},\ -1 < x < 1,\ P_0(x) = 1,\ P_1(x) = x,\ P_{n+1}(x) = \frac{(2\,n+1)\,x}{n+1}\, P_n(x) - \frac{n}{n+1}\, P_{n-1}(x)$$

$$\log\left(\frac{\pi}{2}\right) = 2 \sum_{m=1}^{\infty} \frac{\sum_{n=1}^{\infty} \left(\frac{1}{8\,n^2-1}\right)^{2\,m-1}}{2\,m-1} \tag{5}$$

$$\frac{1}{4}\,\pi \tan^{-1}\left(\frac{1}{7}\right) = -\tan^{-1}\left(\frac{1}{3}\right)^2 + \tan^{-1}\left(\frac{1}{4}\right)^2 + \tan^{-1}\left(\frac{2}{9}\right)^2 + 2\tan^{-1}\left(\frac{1}{4}\right)\tan^{-1}\left(\frac{2}{9}\right) \tag{6}$$

$$\frac{1}{4}\,\pi \tan^{-1}\left(\frac{1}{7}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^n \left(2\left(\frac{1}{4}\right)^{2\,m+1}\left(\frac{2}{9}\right)^{-2\,m+2\,n+1} + \left(\frac{1}{4}\right)^{2\,n+2} - \left(\frac{1}{3}\right)^{2\,n+2} + \left(\frac{2}{9}\right)^{2\,n+2}\right)}{(2\,m+1)\,(-2\,m+2\,n+1)} \tag{7}$$

$$\pi = 2^m\, a_m \left(1 + \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \frac{k^{2^{m+1}-2}(n^2-k^2)}{n^{2^{m+1}} + k^{2^{m+1}}}\right),\ m \in \mathbb{N} \tag{8}$$

$$\frac{\log(A_k)}{6} + \frac{\pi\left(\frac{1}{2^{k+1}} + \frac{1}{6}\right)}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\sqrt{3}\ a_k}{b_k} + 1\right)^{3\,n+1}}{(3\,n+1)\,2^{3\,n+1}} \tag{9}$$

donde

$$A_k = 2 + \frac{1}{2}\, b_{k-1} + \frac{\sqrt{3}}{2}\, a_{k-1}\ ,\ k = 2,\,3,\,4,\ \ldots$$

$$\pi \log\left(\frac{1}{s_k}\right) = 2^{k+1} \int_{s_k}^{1} \left(\frac{\tan^{-1}(x)}{x} + \frac{\log(x)}{1+x^2}\right) dx,\ s_k = \frac{a_k}{b_k},\ k \in \mathbb{N} \tag{10}$$

$$\pi \log 3 = 12 \sum_{n=0}^{\infty} (-1)^n\, q_n\left(\frac{1}{\sqrt{3}}\right) \tag{11}$$

donde

$$q_n(x) = \sum_{m=0}^{n} \sum_{j=0}^{n-m+1} \frac{(-1)^j \binom{n-m+1}{j} x^{n+m+2-j}}{(2\,m+1)\,(n-m+1)}\ ,\ 0 \le x \le 1$$

$$\pi = \frac{2\,c}{b} \left(\prod_{n=1}^{\infty} \frac{n^2((2\,n-1)^2 + (a/c)^2)}{(2\,n-1)^2\,(n^2 + (b/2\,c)^2)}\right) \tan\left(\int_0^{\infty} \frac{\cos(a\,x)\sin(b\,x)}{x\cosh(c\,x)}\, dx\right)\ ,\ a > 0,\ b > 0,\ c > 0 \tag{12}$$

$$\pi = 32 \int_0^{\alpha} \frac{3\cosh(3\,x) - 7\cosh(x)}{\cosh(6\,x) - 14\cosh(4\,x) + 63\cosh(2\,x) - 18}\, dx \tag{13}$$

donde

$$\alpha = \sinh^{-1}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}\right)$$

$$\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}=\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\ldots}}}$$

$$\pi = 16\int_0^\infty \frac{3\cosh(3x)-7\cosh(x)}{\cosh(6x)-14\cosh(4x)+63\cosh(2x)-18}\,dx \tag{14}$$

$$\pi = \sum_{n\in\mathbb{Z}} \tan^{-1}\left(\frac{\sqrt{2}}{(n+1+x)(n+x)+2}\right),\ \ x\geq 0 \tag{15}$$

$$\pi = 6\sum_{n=1}^{\infty}(-1)^{n-1}\tan^{-1}\left(\frac{3^{-n}}{\sqrt{3}}\right)+6\tan^{-1}\left(\sqrt{3}\,\frac{\sum_{n\in\mathbb{Z}}(-1)^n\left(3^{-4n^2-3n-1}-3^{-4n^2-5n-2}\right)}{\sum_{n\in\mathbb{Z}}(-1)^n\left(3^{-4n^2-n}-3^{-4n^2-7n-3}\right)}\right) \tag{16}$$

$$\pi = 6\sqrt{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{-n}}{(2n-1)(3^{2n-1}+1)}+6\tan^{-1}\left(\sqrt{3}\,\frac{\sum_{n\in\mathbb{Z}}(-1)^n\left(3^{-4n^2-3n-1}-3^{-4n^2-5n-2}\right)}{\sum_{n\in\mathbb{Z}}(-1)^n\left(3^{-4n^2-n}-3^{-4n^2-7n-3}\right)}\right) \tag{17}$$

$$\pi = 4ab\left(\sum_{n=0}^{\infty}\frac{(-1)^n(2a^2)^n}{b^{n+1}}f(a,b,c,n)+\sum_{n=0}^{\infty}\frac{(-1)^n b^{2n}}{(2a)^{n+1}}g(a,b,c,n)\right) \tag{18}$$

donde

$$a,\ b,\ c>0\ \ ,\ \ \frac{\left(\sqrt{3}-1\right)b}{2a}<c<\frac{\left(\sqrt{3}+1\right)b}{2a} \tag{19}$$

$$f(a,b,c,n)=\int_0^c \frac{x^{2n}}{(2ax+b)^{n+1}}\,dx \tag{20}$$

$$g(a,b,c,n)=\int_c^\infty \frac{1}{(ax^2+bx)^{n+1}}\,dx \tag{21}$$

Las funciones f , g se pueden expresar en términos de la función Beta incompleta :

$$f(a,b,c,n)=-\frac{b^n}{(2a)^{2n+1}}B\left(-\frac{2ac}{b},\ 2n+1,\ -n\right) \tag{22}$$

$$g(a,b,c,n)=-\frac{a^n}{b^{2n+1}}B\left(-\frac{b}{ac},\ 2n+1,\ -n\right) \tag{23}$$

donde

$$B(z,x,y)=\int_0^z t^{x-1}(1-t)^{y-1}\,dt \tag{24}$$

$$f(a,b,c,n)=\left(\frac{1}{2a}\right)^{2n+1}\left(\log\left(1+\frac{2ac}{b}\right)+\sum_{k=0,k\neq n}^{2n}\binom{2n}{k}\frac{(-b)^{2n-k}\left((2ac+b)^{k-n}-b^{k-n}\right)}{k-n}\right) \tag{25}$$

$$g(a,b,c,n)=\left(\frac{1}{b}\right)^{2n+1}\left(\log\left(1+\frac{b}{ac}\right)+\sum_{k=0,k\neq n}^{2n}\binom{2n}{k}\frac{(-a)^{2n-k}\left(\left(a+\frac{b}{c}\right)^{k-n}-a^{k-n}\right)}{k-n}\right) \tag{26}$$

$$\pi=\int_0^1\frac{4\,n!}{(1+x^2)_{n+1}}\,dx+\sum_{k=1}^{n}\int_0^1\frac{4\,k!}{k\,(1+x^2)_k}\,dx\ ,\ \ n\in\mathbb{N} \tag{27}$$

$$\log \pi + \gamma - \log a_k = (2\,k + 1) \log 2 - \log\left(2^{k+1} - 1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \log\left(\left(1 + \frac{1}{n\,2^{k+1}}\right) \left(1 + \frac{2^{k+1} - 1}{n\,2^{k+1}}\right) \right) \right), \quad k \in \mathbb{N} \tag{28}$$

donde

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n \right) = 0.5272 \ldots$$

Referencias

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