The Polynomial : $P(x)=x^6-21x^4+35x^2-7$

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Abstract

We presents some ralations which involving the polynomial : $P(x)=x^6-21x^4+35x^2-7$

1. Introduction

We exibit some formulas related with the polynomial : $P(x)=x^6-21x^4+35x^2-7$

2. Roots

$$P(x)=x^6-21x^4+35x^2-7=$$
$$=\left(x^2-\left(\tan\left(\frac{\pi}{7}\right)\right)^2\right)\left(x^2-\left(\tan\left(\frac{2\pi}{7}\right)\right)^2\right)\left(x^2-\left(\tan\left(\frac{3\pi}{7}\right)\right)^2\right)$$

For $x^2=y$ we have $Q(y)=P\left(\sqrt{y}\right)$

$$Q(y)=y^3-21y^2+35y-7=$$
$$=\left(y-\left(\tan\left(\frac{\pi}{7}\right)\right)^2\right)\left(y-\left(\tan\left(\frac{2\pi}{7}\right)\right)^2\right)\left(y-\left(\tan\left(\frac{3\pi}{7}\right)\right)^2\right)$$

$$P(x)=0 \Rightarrow x=\begin{cases} \pm\left(\tan\left(\frac{\pi}{7}\right)\right) \\ \pm\left(\tan\left(\frac{2\pi}{7}\right)\right) \\ \pm\left(\tan\left(\frac{3\pi}{7}\right)\right) \end{cases}$$

3. Trigonometric Identities

$$\left(\tan\left(\frac{\pi}{7}\right)\right)^2+\left(\tan\left(\frac{2\pi}{7}\right)\right)^2+\left(\tan\left(\frac{3\pi}{7}\right)\right)^2=21$$

$$\left(\tan\left(\frac{\pi}{7}\right)\right)^2\left(\tan\left(\frac{2\pi}{7}\right)\right)^2+\left(\tan\left(\frac{2\pi}{7}\right)\right)^2\left(\tan\left(\frac{3\pi}{7}\right)\right)^2+$$
$$+\left(\tan\left(\frac{\pi}{7}\right)\right)^2\left(\tan\left(\frac{3\pi}{7}\right)\right)^2=35$$

$$\left(\tan\left(\frac{\pi}{7}\right)\right)^2\left(\tan\left(\frac{2\pi}{7}\right)\right)^2\left(\tan\left(\frac{3\pi}{7}\right)\right)^2=7$$

$$\left(\tan\left(\frac{\pi}{7}\right)\right)^2+\left(\tan\left(\frac{2\pi}{7}\right)\right)^2+7\left(\cot\left(\frac{\pi}{7}\right)\cot\left(\frac{2\pi}{7}\right)\right)^2=21$$

$$\left(\tan\left(\frac{\pi}{7}\right)\right)^2+\left(\tan\left(\frac{3\pi}{7}\right)\right)^2+7\left(\cot\left(\frac{\pi}{7}\right)\cot\left(\frac{3\pi}{7}\right)\right)^2=21$$

$$\left(\tan\left(\frac{2\pi}{7}\right)\right)^2+\left(\tan\left(\frac{3\pi}{7}\right)\right)^2+7\left(\cot\left(\frac{2\pi}{7}\right)\cot\left(\frac{3\pi}{7}\right)\right)^2=21$$

$$\left(\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\right)^2+7\left(\left(\cot\left(\frac{\pi}{7}\right)\right)^2+\left(\cot\left(\frac{2\pi}{7}\right)\right)^2\right)=35$$

$$\left(\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right)\right)^2+7\left(\left(\cot\left(\frac{\pi}{7}\right)\right)^2+\left(\cot\left(\frac{3\pi}{7}\right)\right)^2\right)=35$$

$$\left(\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right)\right)^2+7\left(\left(\cot\left(\frac{2\pi}{7}\right)\right)^2+\left(\cot\left(\frac{3\pi}{7}\right)\right)^2\right)=35$$

4. Recurrences , $n\in\mathbb{N}\cup\{0\}$

$$y_{n+1}=\frac{7+21y_n^2-y_n^3}{35}\quad ,y_0=0\quad ,y_n\to\left(\tan\left(\frac{\pi}{7}\right)\right)^2$$

$$y_{n+1}=\frac{7}{35-21y_n+y_n^2}\quad ,y_0=0\quad ,y_n\to\left(\tan\left(\frac{\pi}{7}\right)\right)^2$$

$$y_{n+1}=\frac{7+21y_n^2}{35+y_n^2}\quad ,\begin{cases} y_0=0\quad ,y_n\to\left(\tan\left(\frac{\pi}{7}\right)\right)^2 \\ y_0=19\quad ,y_n\to\left(\tan\left(\frac{3\pi}{7}\right)\right)^2 \end{cases}$$

$$y_{n+1}=\frac{-7+35y_n+y_n^3}{21y_n}\quad ,y_0=1\quad ,y_n\to\left(\tan\left(\frac{2\pi}{7}\right)\right)^2$$

$$y_{n+1}=\frac{7-35y_n+21y_n^2}{y_n^2}\quad ,y_0=19\quad ,y_n\to\left(\tan\left(\frac{3\pi}{7}\right)\right)^2$$

5. Faster Recurrences , $n\in\mathbb{N}\cup\{0\}$

$$x_{n+1}=\frac{5x_n^6-63x_n^4+35x_n^2+7}{6x_n^5-84x_n^3+70x_n}\quad ,\begin{cases} x_0=1/2\,,x_n\to\tan\left(\frac{\pi}{7}\right) \\ x_0=1\,,x_n\to\tan\left(\frac{2\pi}{7}\right) \\ x_0=4\,,x_n\to\tan\left(\frac{3\pi}{7}\right) \end{cases}$$

$$y_{n+1}=\frac{2y_n^3-21y_n^2+7}{3y_n^2-42y_n+35}\quad ,\begin{cases} y_0=0\,,\,y_n\to\left(\tan\left(\frac{\pi}{7}\right)\right)^2 \\ y_0=2\,,\,y_n\to\left(\tan\left(\frac{2\pi}{7}\right)\right)^2 \\ y_0=19\,,\,y_n\to\left(\tan\left(\frac{3\pi}{7}\right)\right)^2 \end{cases}$$

6. Radicals

For $s=\frac{2}{3}\sqrt[3]{756+i84\sqrt{3}}$, $t=\frac{2}{3}\sqrt[3]{756-i84\sqrt{3}}$ we have

$$\tan\left(\frac{\pi}{7}\right)=\sqrt{7-\frac{s+t}{2}+\frac{i\sqrt{3}(s-t)}{2}}$$

$$\tan\left(\frac{2\pi}{7}\right)=\sqrt{7-\frac{s+t}{2}-\frac{i\sqrt{3}(s-t)}{2}}$$

$$\tan\left(\frac{3\pi}{7}\right)=\sqrt{7+s+t}$$

7. Continued Radicals

$$\left(\tan\left(\frac{3\pi}{7}\right)\right)^2=7+\sqrt[3]{448+112\sqrt[3]{448+112\sqrt[3]{448+....}}}$$

8. Linear Recurrences

$$\begin{cases} a_{n+3}=21a_{n+2}-35a_{n+1}+7a_n \quad ,a_0=1,a_1=21,a_2=406 \\ \frac{a_{n+1}}{a_n}\to\left(\tan\left(\frac{3\pi}{7}\right)\right)^2 \end{cases}$$

$$\begin{cases} a_{n+3}=35a_{n+2}-147a_{n+1}+49a_n \quad ,a_0=1,a_1=35,a_2=1078 \\ \frac{7a_n}{a_{n+1}}\to\left(\tan\left(\frac{\pi}{7}\right)\right)^2 \end{cases}$$

9. Series

$$\left(\tan\left(\frac{\pi}{7}\right)\right)^2 = 7 - \frac{4}{9}\sqrt[3]{28}\sum_{n=0}^{\infty}\frac{(-1)^n\left(-1/3\right)_{2n}(6n+5)}{(2n+1)!\,3^{3n}}$$

$$\left(\tan\left(\frac{2\pi}{7}\right)\right)^2 = 7 - \frac{16}{9}\sqrt[3]{28}\sum_{n=0}^{\infty}\frac{(-1)^n\left(-1/3\right)_{2n}(3n+1)}{(2n+1)!\,3^{3n}}$$

$$\left(\tan\left(\frac{3\pi}{7}\right)\right)^2 = 7 + 4\sqrt[3]{28}\sum_{n=0}^{\infty}\frac{(-1)^n\left(-1/3\right)_{2n}}{(2n)!\,3^{3n}}$$

10. General Identities

For $x = \tan\left(\frac{\pi}{7}\right), \tan\left(\frac{2\pi}{7}\right), \tan\left(\frac{3\pi}{7}\right)$ we have

$$x^{2n}\left(A_n + B_n x^2 + C_n x^4\right) = 7^n \quad , n \in \mathbb{N}$$

where

$$\begin{cases} A_{n+1} = 35A_n + 7B_n \\ B_{n+1} = -21A_n + 7C_n \\ C_{n+1} = A_n \\ A_1 = 35, B_1 = -21, C_1 = 1 \end{cases}$$

$$x^{2n} = \frac{A_n + B_n\, x^2 + C_n\, x^4}{D_n + E_n\, x^2 + F_n\, x^4} \quad , n \in \mathbb{N}$$

where

$$\begin{cases} A_{n+1} = 7A_n + 147B_n + 3087C_n \\ B_{n+1} = -728B_n - 15288C_n \\ C_{n+1} = 21A_n + 441B_n + 8533C_n \\ A_1 = 7, B_1 = 0, C_1 = 21 \end{cases}$$

$$\begin{cases} D_{n+1} = 35D_n + 7E_n + 147F_n \\ E_{n+1} = -728F_n \\ F_{n+1} = D_n + 21E_n + 441F_n \\ D_1 = 35, E_1 = 0, F_1 = 1 \end{cases}$$

$$x^{2n+4} = A_n + B_n x^2 + C_n x^4 \quad , n \in \mathbb{N}$$

where

$$(*)\begin{cases} A_{n+1} = 7C_n \\ B_{n+1} = A_n - 35C_n \\ C_{n+1} = B_n + 21C_n \\ A_1 = 7, B_1 = -35, C_1 = 21 \end{cases}$$

11. Pi Formula

For $m \in \mathbb{N}, x = \tan\left(\frac{\pi}{7}\right)$ we have

$$\frac{\pi}{7} = x(1+\alpha_m) + x^3\left(\beta_m - \frac{1}{3}\right) + x^5\left(\gamma_m + \frac{1}{5}\right) + \sum_{n=m+1}^{\infty} \frac{(-1)^n x^{2n+5}}{2n+5}$$

where

$$\alpha_m = \sum_{n=1}^{m} \frac{(-1)^n A_n}{2n+5} \, , \beta_m = \sum_{n=1}^{m} \frac{(-1)^n B_n}{2n+5} \, , \gamma_m = \sum_{n=1}^{m} \frac{(-1)^n C_n}{2n+5}$$

and A_n, B_n, C_n are defined by $(*)$

12. Arctangents Formulas

$$\frac{3\pi}{4} + \tan^{-1}\left(\frac{5}{12}\right) =$$
$$= \tan^{-1}\left(\left(\tan\left(\frac{\pi}{7}\right)\right)^2\right) + \tan^{-1}\left(\left(\tan\left(\frac{2\pi}{7}\right)\right)^2\right) + \tan^{-1}\left(\left(\tan\left(\frac{3\pi}{7}\right)\right)^2\right)$$

$$\frac{3\pi}{4} - \tan^{-1}\left(\frac{5}{12}\right) =$$
$$= \tan^{-1}\left(\left(\cot\left(\frac{\pi}{7}\right)\right)^2\right) + \tan^{-1}\left(\left(\cot\left(\frac{2\pi}{7}\right)\right)^2\right) + \tan^{-1}\left(\left(\cot\left(\frac{3\pi}{7}\right)\right)^2\right)$$

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