Prove Collatz Conjecture by Mathematical Induction via the Two-Way Operations

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Introduction: The Collatz conjecture is also well-known variously as the 3n+1 conjecture, the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, or the Syracuse problem, etc.

Yet it is still both unproved and un-negated a conjecture ever since named after Lothar Collats in 1937.

AMS subject classification: 11×××, 00A05.

Abstract

If every positive integer is able to be operated to 1 by the set operational rule of the Collatz conjecture, then begin with 1, we can get all positive integers after make infinitely many operations on the contrary of the set operational rule. In this article, we shall prove that the Collatz conjecture by the mathematical induction via the two-way operations is tenable.

Keywords: mathematical induction; the two-way operational rules; classify positive integers; the bunch of integers' chains; operational routes

Basic Concepts

The Collatz conjecture states that take any positive integer n, if n is an even number, then divide n by 2 to obtain an integer; if n is an odd number, then multiply n by 3 and add 1 to obtain an even number. Repeat the above process indefinitely, then no matter which positive integer you start with, you will always eventually reach a result of 1.

We consider the way of aforesaid two steps as leftward operational rule for any positive integer. Also consider operations on the contrary of the leftward operational rule as the rightward operational rule for any positive integer. Taken one with another, we consider such each other's- opposed operational rules as the two-way operational rules, and that operations by the two-way operational rules are called the two-way operations.

The rightward operational rule stipulates that for any positive integer n, multiply n by 2 to obtain an even number. Additionally, when n is an even number, if divide the difference of n minus 1 by 3 and obtain an odd number, then must operate this step otherwise, and proceed from here to continue to operate; if it is not such, then don't operate this step.

Begin with any positive integer to operate by either operational rule continuously, manifestly each operational result is a positive integer, then we consider a string of such consecutive positive integers on an identical operational direction and arrowheaded signs inter se as an operational route. Each of operational results comes only from preceding an adjacent positive integer at an identical operational route.

If any positive integer P exists at a certain operational route, then may term the operational route an operational route of P. Two operational routes of P branch from a positive integer certainly.

Begin with 1 to operate positive integers got successively by the rightward operational rule, so it forms a bunch of operational routes automatically. We term such a bunch of operational routes “a bunch of integers’ chains”. Manifestly whole a bunch of integers’ chains must consist of infinite many operational routes.

Since a direct origin of each of positive integers is unique, thus each of positive integers except for 1 is unique at the bunch of integers’ chains.

Comparatively speaking, inside greater limits, positive integers on the left side of the bunch of integers’ chains are smaller, yet positive integers on the right side are larger. Overall, from left to right positive integers at the bunch of integers’ chains are getting both more and more absolutely, and greater and greater relatively. Please, see a beginning of the bunch of integers’ chains as the follows.

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                        …        …    452→…
                      168↑     680↑  226↑→75→…
                     84↑     340↑→113↑   227→…
                   42↑       170↑      682↑→…
      1            21↑      85↑      341↑
1→2→4↑→8→16↓→32→64↑→128→256↑→512→1024↑→2048→…
           5→10↓→20→40↓→80→160↓→320→…
               3↓     13↓    53→106↓→212→…
                6↓     26↓       35→70↓→140→…
                 12↓    52↓→104→…    23→46↓→15→…
                  24↓    17↓               92→…
                   …      34↓→11→22↓→7→14→28↓→56…
                           68↓    44→88↓→29…9→18→…
                            136↓→…    29↓
                              45→…     58↓→19→…
                                         …   …
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First Illustration

Annotation: ↓ and ↑ must rightwards tilt, but each page is narrow, thus it can only so.

No matter which positive integer, it is surely at the bunch of integers’ chains so long as it is able to be operated to 1 by the leftward operational rule. Likewise, the converse proposition holds water.

That is to say, positive integers at the bunch of integers' chains and positive integers which are able to be operated to 1 by the leftward operational rule are one-to-one correspondence.

Thus it can be seen, whether a positive integer suits the conjecture, it needs merely us to determine whether the positive integer exists at the bunch of integers' chains. If every positive integer is able to be operated to 1 by the leftward operational rule, then there are all positive integers at the bunch of integers' chains.

Correspondingly, if we can prove that all positive integers exist at the bunch of integers' chains, then every positive integer is able to be operated to 1 by the leftward operational rule.

Because of this, we shall prove that the bunch of integers' chains contains all positive integers by mathematical induction.

If we divide the bunch of integers' chains into many one-way operational routes according to un-operated smallest odd number got as most left one of headmost operating row, then a beginning of the bunch of integers' chains is dismembered into many operational routes as follows.

1→2→4↓→8→16↓→32→64↓→128→256↓→512→1024↓→2048→4096↓→8192…
1 5 21 85 341 1365
5→10↓→20→40↓→80→160↓→320→640↓→1280→2560↓→5120→10240↓→…
3 13 53 213 853 3413
3→6→12→24→48→96→192→384→768→1536→3072→6144→12288→…

13→26→52↓→104→208↓→416→832↓→1664→3328↓→6656→13312↓→ …
17   69   277   1109   4437
17→34↓→68→136↓→272→544↓→1088→2176↓→4352→8704↓→17408→…
11   45   181   725   2901
11→22↓→44→88↓→176→352↓→704→1408↓→2816→5632↓→11264→…
7   29   117   469   1877
7→14→28↓→56→112↓→224→448↓→896→1792↓→3584→7168↓→14336→…
9   37   149   597   2389
9→18→36→72→144→288→576→1152→2304→4608→9216→18432→…

21→42→84→168→336→672→1344→2688→5376→10752→21504→…

29→58↓→116→232↓→464→928↓→1856→3712↓→7424→14848↓→29696→…
19   77   309   1237   4949
19→38→76↓→152→304↓→608→1216↓→2432→4864↓→9728→19456↓→ …
25   101   405   1621   6485
25→50→100↓→200→400↓→800→1600↓→3200→6400↓→12800→25600↓→…
33   133   533   2133   8533
33→66→132→264→528→1056→2112→4224→8448→16896→33792→…

37→74→148↓→296→592↓→1184→2368↓→4736→9472↓→18944→37888↓→…
49   197   789   3157   12629
45→90→180→360→720→1440→2880→5760→11520→23040→46080→…

49→98→196↓→392→784↓→1568→3136↓→6272→12544↓→25088→50176↓→…
65   261   1045   4181   16725
53→106↓→212→424↓→848→1696↓→3392→6784↓→13568→27136↓→54272…
35   141   565   2261   9035

…

From listed-above rows, it is observed that excepting an odd number on most left side of every row, others, either all are even numbers, or all are odd numbers without arrowheads. On operations of the contrary i.e. by the leftward operational rule, we regard which multiply an odd number by 3 and add 1 to obtain an even number as which the operation upgrades a stair; also look upon which divide an even number by 2 to obtain an integer as which the operation goes a step leftwards, at above-listed operational courses. Whether it upgraded a stair or gone a step leftwards, enable the operation to go a step further to approach final result of 1. Moreover, be necessary to determine an axiom beforehand and prove a theorem, so that apply either of them to affirm an anticipative result that suits the conjecture after such a result arises at an operational route.

Axiom* Known that positive integers which are smaller than positive integer P suit the conjecture, if a positive integer which is smaller than positive integer P appears at an operational route of P, then P is proved to suit the conjecture. Illustrate with examples as follows:

(1) Let $P=31+3^2\eta$ with $\eta\geq0$, from $27+2^3\eta\rightarrow82+3*2^3\eta\rightarrow41+3*2^2\eta\rightarrow 124+3^2*2^2\eta\rightarrow 62+3^2*2\eta\rightarrow31+3^2\eta >27+2^3\eta$, we get that $31+3^2\eta$ suits the conjecture.

(2) Let $P=51+48\mu$ with $\mu\geq0$, from $51+48\mu\rightarrow154+144\mu\rightarrow77+72\mu\rightarrow 232+216\mu\rightarrow 116+108\mu\rightarrow58+54\mu\rightarrow29+27\mu<51+48\mu$, we get that $51+48\mu$ suits the conjecture.

This axiom is established in the two-way operational rules visibly. Or rather, let positive integer C<positive integer P, and C suits the conjecture. Then, at an operational route by leftward operational rule, when C is before P, operations of C via P was operated into 1 already; when C is behind P, operations of P can continue to pass C to 1. Like that, at an operational route by rightward operational rule, when C is before P, C comes from 1; when C is behind P, P comes from 1.

Theorem* If an operational route of P intersects an operational route of C, and C which is smaller than P suits the conjecture, then P suits the conjecture too, where P and C are positive integers.

Proof * Suppose that an operational route of P intersects an operational route of C at positive integer α, since α exists at an operational route of C, so α suits the conjecture according to the axiom. And that α exists at an operational route of P too, then P suits the conjecture according to the axiom. Give an example to explain it as follows.

Let $P=63+3*2^8\varphi$ with $\varphi\geq0$, from $63+3*2^8\varphi\to190+3^2*2^8\varphi\to95+3^2*2^7\varphi\to 286+3^3*2^7\varphi\to143+3^3*2^6\varphi\to430+3^4*2^6\varphi\to215+3^4*2^5\varphi\to646+3^5*2^5\varphi\to 323+3^5*2^4\varphi\to970+3^6*2^4\varphi\to485+3^6*2^3\varphi\to1456+3^7*2^3\varphi\to728+3^7*2^2\varphi \to 364+3^7*2\varphi\to 182+3^7\varphi\uparrow\to \ldots$

$\uparrow121+3^6*2\varphi\leftarrow242+3^6*2^2\varphi\leftarrow484+3^6*2^3\varphi\leftarrow161+3^5*2^3\varphi\leftarrow322+3^5*2^4\varphi \leftarrow107+3^4*2^4\varphi\leftarrow214+3^4*2^5\varphi\leftarrow71+3^3*2^5\varphi\leftarrow142+3^3*2^6\varphi\leftarrow47+3^2*2^6\varphi< 63+3*2^8\varphi$, we get that $63+3*2^8\varphi$ suits the conjecture.

Inference * If an operational route of P and an operational route of C are at an indirect concatenation, and C suits the conjecture, then P suits the conjecture. For example, an operational route of P intersects an operational route of B, the operational route of B intersects an operational route of D…the operational route of E intersects an operational route of C, and C suits the conjecture, then P suits the conjecture.

Actually, each and every positive integer at one another's-linked operational routes suits the conjecture, provided therein there is a positive integer which suits the conjecture.

The Proof

Let us set about the proof that the bunch of integers' chains contains all positive integers by mathematical induction hereinafter.

1. From preceding first illustration, we can directly find that there are 24 consecutive positive integers ≥ 1 within positive integers got. Especially indicate that 15 within them belongs in 15+12c, and 19 within them belongs in 19+12c, in which case c=0.

2. Suppose that after further operate positive integers got by the rightward operational rule, there are consecutive positive integers $\leq$ n within positive integers got at a bunch of integers' chains, where n ≥ 24.

3. Prove that after continue to operate positive integers got already by the rightward operational rule, we can get consecutive positive integers $\leq$ 2n within positive integers got at a bunch of integers' chains extended.

First, let us divide limits of consecutive positive integers at the number axis into segments according to $2^X n$ as greatest positive integer per segment, where $X \geq 0$ and $n \geq 24$, so as to accord with the proof by the mathematical induction. A simple segmenting illustration is as follows.

1——$\mathbf{2}^0$n———2^1n——————$\mathbf{2}^2$n————————$\mathbf{2}^3$n———→

Second Illustration

Proof * Since there are consecutive positive integers$\leq$ n within positive integers got at a bunch of integers' chains, thus multiply each and every positive integer $\le n$ by 2 according to the rightward operational rule, then we get all even numbers between n and $2n+1$ at a bunch of integers' chains extended, irrespective of repeated even numbers $\le n$.

After that, we must seek an origin of each kind of odd numbers between n and $2n+1$ by the two-way operational rules, and that each such origin is necessarily smaller than corresponding a kind of odd numbers.

First, let us divide all odd numbers between n and $2n+1$ into two kinds, i.e. $5+4k$ and $7+4k$, where k is a natural number ≥ 5, then any odd number between n and $2n+1$ belongs to one in the two kinds certainly.

By now, we list the two kinds of odd numbers in correspondence with their variable k as follows.

k:	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16…
5+4k:	25,	29,	33,	37,	41,	45,	49,	53,	57,	61,	65,	69 …
7+4k:	27,	31,	35,	39,	43,	47,	51,	55,	59,	63,	67,	71 …

From $5+4k\to 16+12k\to 8+6k\to 4+3k<5+4k$, we obtain that $5+4k$ suits the conjecture according to the axiom.

For $7+4k$, let us again divide it into three kinds, i.e. $11+12c$, $15+12c$ and $19+12c$, where $c \ge 1$.

From $7+8c\to 22+24c\to 11+12c > 7+8c$, we obtain that $11+12c$ suits the conjecture according to the axiom.

Let us list $15+12c$ and $19+12c$ in correspondence with their variable c as follows.

c: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, …

15+12c: 27, 39, 51, 63, 75, 87, 99, 111, 123, 135, 147, 159 …

19+12c: 31, 43, 55, 67, 79, 91, 103, 115, 127, 139, 151, 163 …

Hereinafter, we shall operate respectively 15+12c and 19+12c by the leftward operational rule, moreover, discover and affirm certain of satisfactory results at some operational branches.

Firstly, let us operate 15+12c by the leftward operational rule below.

15+12c→46+36c→23+18c→70+54c→35+27c ♣

d=2e+1: 29+27e (**1**) e=2f: 142+486f→71+243f ♥
♣35+27c↓→c=2d+1: 31+27d↑→d=2e: 94+162e→47+81e↑→e=2f+1:64+81f (**2**)
c=2d: 106+162d→53+81d↓→d=2e+1:67+81e↓→e=2f+1:74+81f (**3**)
d=2e:160+486e♦ e=2f: 202+486f→101+243f ♠

g=2h+1: 200+243h (**4**) **…**
♥ 71+243f↓→f=2g+1:157+243g↑→g=2h: 472+1458h→236+729h↑→ **…**
f=2g: 214+1458g→107+729g↓→g=2h+1: 418+729h↓→**…**
g=2h: 322+4374h→**… …**

g=2h: 86+243h (**5**)
♠101+243f↓→f=2g+1:172+243g↑→g=2h+1:1246+1458h→**…**
f=2g: 304+1458g→152+729g ↓→**…**
…

…
♦160+486e→80+243e↓→e=2f+1: 970+1458f→485+729f↑→ **…** **…**
e=2f:40+243f↓→f=2g+1:850+1458g→ 425+729g↑→ **…**
f=2g: 20+243g↓→g=2h: 10+243h (**6**) **…**
g=2h+1:880+1458h→440+729h↑→ **…**

Annotation:
Each of letters c, d, e, f, g, h …etc in the above-listed operational routes expresses each of natural numbers plus 0, similarly hereinafter.
Also, there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

We conclude several branch's satisfactory operational results from above-listed the bunch of operational routes of 15+12c, and these satisfactory operational results are analyzed as follows, one by one.

From c=2d+1 and d=2e+1, we get c=2d+1=2(2e+1)+1=4e+3, and 15+12c =15+12(4e+3)=51+48e >29+27e where mark (1), so 15+12c with c=4e+3 suits the conjecture according to the axiom.

From c=2d+1, d=2e, and e=2f+1, we get c=2d+1=4e+1=4(2f+1)+1=8f+5, and 15+12c=15+12(8f+5)=75+96f > 64+81f where mark (2), so 15+12c with c=8f+5 suits the conjecture according to the axiom.

From c=2d, d=2e+1 and e=2f+1, we get c=2d= 4e+2= 4(2f+1)+2=8f+6, and 15+12c=15+12(8f+6)=87+96f >74+81f where mark (3), so 15+12c with c=8f+6 suits the conjecture according to the axiom.

From c=2d+1, d=2e, e=2f, f=2g+1 and g=2h+1, we get c=2d+1=4e+1= 8f+1=8(2g+1)+1=16g+9=16(2h+1)+9=32h+25, and 15+12c=15+12(32h +25)=315+384h >200+243h where mark (4), so 15+12c with c=32h+25 suits the conjecture according to the axiom.

From c=2d, d=2e+1, e=2f, f=2g+1 and g=2h, we get c=2d=2(2e+1)=4e+2 =8f+2=8(2g+1)+2=16g+10=32h+10, and 15+12c=15+12(32h+10)= 135+ 384h > 86+243h where mark (5), so 15+12c with c=32h+10 suits the conjecture according to the axiom.

From c=2d, d=2e, e=2f, f=2g and g=2h, we get c=2d =32h, and 15+12c = 15+12(32h)=15+384h > 10+243h where mark (6), so 15+12c with c=32h suits the conjecture according to the axiom.

Secondly, we operate 19+12c by the leftward operational rule as follows.

19+12c→58+36c→ 29+18c→ 88+54c→ 44+27c ♣

d=2e: 11+27e (**α**) e=2f:37+81f (**β**)
♣ 44+27c↓→c=2d: 22+27d↑→d=2e+1:148+162e→74+81e↑→e=2f+1:466+486f ♥
c=2d+1: 214+162d→107+81d↓→d=2e:322+486e ♠
d=2e+1:94+81e↓→e=2f:47+81f (**γ**)
e=2f+1:516+486f ♦

g=2h: 129+243h (**δ**) …
f=2g+1:258+243g↑→g=2h+1:1504+1458h→752+729h↑→…
♥466+486f→233+243f↑→f=2g:700+1458g→350+729g↓→g=2h+1:3238+4374h↓
g=2h: 175+729h↓→… …
…

g=2h+1:172+243h (**ε**)
f=2g: 101+243g↑→g=2h: 304+1458h→…
e=2f+1:202+243f↑→f=2g+1:1336+1458g→…
♠322+486e→161+243e↑→e=2f:484+1458f→…

♦516+486f→258+243f↓→f=2g+1:1504+1458g→…
f=2g: 129+243g↓→g=2h: 388+1458h→…
g=2h+1:186+243h (**ζ**)

Annotation:
Each of letters c, d, e, f, g, h …etc in the above-listed operational routes expresses each of natural numbers plus 0, similarly hereinafter.
Also, there are ♣↔♣, ♥↔♥, ♠↔♠, and ♦↔♦.

Likewise, we conclude several branch's satisfactory operational results from above-listed the bunch of operational routes of 19+12c, and these satisfactory operational results are analyzed as follows, one by one.

From c=2d, d=2e, we get c=2d=4e, and 19+12c=19+12(4e) =19+48e > 11+27e where mark (α), so 19+12c with c=4e suits the conjecture according to the axiom.

From c=2d, d=2e+1 and e=2f, we get c=2d = 2(2e+1) = 4e+2 = 8f+2, and 19+12c=19+12(8f+2) = 43+96f >37+81f where mark (β), so 19+12c with c=8f+2 suits the conjecture according to the axiom.

From c=2d+1, d=2e+1, and e=2f, we get c=2d+1=4e+3=8f+3, and

$19+12c=19+12(8f+3)=96f+55>47+81f$ where mark (γ), so $19+12c$ with $c=8f+3$ suits the conjecture according to the axiom.

From $c=2d$, $d=2e+1$, $e=2f+1$, $f=2g+1$ and $g=2h$, we get $c=2d=2(2e+1)=4e+2= 4(2f+1)+2 = 8f+6= 8(2g+1)+6 = 16g+14 = 32h+14$, and $19+12c = 19+12(32h+14) = 187+384h >129+243h$ where mark (δ), so $19+12c$ with $c=32h+14$ suits the conjecture according to the axiom.

From $c=2d+1$, $d=2e$, $e=2f+1$, $f=2g$ and $g=2h+1$, we get $c=2d+1=4e+1=4(2f+1)+1=8f+5=16g+5=16(2h+1)+5=32h+21$, and $19+12c=19+12(32h+21) =271+384h > 172+243h$ where mark (ε), so $19+12c$ with $c=32h+21$ suits the conjecture according to the axiom.

From $c=2d+1$, $d=2e+1$, $e=2f+1$, $f=2g$ and $g=2h+1$, we get $c=2d+1=2(2e+1)+1=4e+3=4(2f+1)+3=8f+7=16g+7=16(2h+1)+7=32h+23$, and $19+12c=19+12(32h+23)=295+384h > 186+243h$ where mark (ζ), so $19+12c$ with $c=32h+23$ suits the conjecture according to the axiom.

Listed above proven $51+48e$, $75+96f$, $87+96f$, $315+384h$, $135+384h$, $15+384h$; $19+48e$, $43+96f$, $55+96f$, $187+384h$, $271+384h$ and $295+384h$ are transformed into $51+2^4\times3e$, $75+2^5\times3f$, $87+2^5\times3f$, $315+2^7\times3h$, $135+2^7\times3h$, $15+2^7\times3h$; $19+2^4\times3e$, $43+2^5\times3f$, $55+2^5\times3f$, $187+2^7\times3h$, $271+2^7\times3h$ and $295+2^7\times3h$, therein each exponent of 2 is actually the number of times that an integer's expression divided by 2 at an operational rule from $15+12c/19+12c$ to first integer's expression which is smaller than a kind of $15+12c/19+12c$.

Let χ represents together variables d, e, f, g, h, y, k, w, q, s, f, v, u … etc within integer's expressions at two bunches of operational routes of 15+12c plus 19+12c, but χ represents not c.

Naturally the odevity of a part of integer's expressions which contain variable χ is still indeterminate. That is to say, for every such integer's expression which contains variable χ, both consider it as an odd number to operate, and consider it as an even number to operate. Thus, let us label such integer's expressions "odd-even expressions".

For any odd-even expression at a bunch of operational routes of 15+12c/19+12c, two operations synchronize according as χ expresses both an odd number and an even number. Namely, both operate any odd-even expression as an odd number into threefold itself and add 1, and operate the odd-even expression as an even number into a half of itself. Evidently, after an odd-even expression as an odd number to operate, we get a greater operational result, yet, after the odd-even expression as an even number to operate, we get a smaller operational result.

If you begin with any odd-even expression to do continuously operations by the leftward operational rule, then such an operational route via consecutive greater operational results will elongate infinitely.

Begin with 15+12c/19+12c to operate continuously by the leftward operational rule, if a newborn operational result is smaller than a kind of 15+12c/19+12c, this manifests that the kind of 15+12c/19+12c suits the conjecture according to the axiom, so operations of the branch may stop at the here. If enable a kind of 15+12c/19+12c via operations to reach the eventual result of 1, then these operations must pass some smaller operational results.

Now that there are both a greater operational result due to χ as an odd number to operate and a smaller operational result due to χ as an even number to operate after every odd-even expression operates once by the leftward operational rule, then not only greater operational results at an operational route all are greater than their own common origin, but also consecutive greater operational results are getting greater and greater along with the continuation of operations, up to infinity.

Since greater operational results operate continuously, have to cause that odd-even expressions are getting more and more, up to infinite many, and smaller operational results which accompany greater operational results in synchronisms are getting more and more too, up to infinite many.

In other words, on the one hand, begin with any odd-even expression, two kinds' operations progress and branch always endlessly due to the odevity of variable χ, up to arise infinitely more progress and branch. Of course, odd-even expressions are getting more and more, and that the more rear, the greater values, up to engender infinitely many infinities theoretically.

On the other hand, uninterruptedly stop operations of part branches at operational routes increased ceaselessly, in the case each such branch is operated to a result which is smaller than a kind of 15+12c/19+12c, and that there are infinitely many such results likewise, because so long as operations along consecutive smaller operational results at an operational route to proceed, would come into being such a result inevitably.

Thus it can be seen, operations of 15+12c/ 19+12c will proceed infinitely.

Judging from this, 15+12c and 19+12c must be divided respectively into infinite many kinds, just sufficiently enable every kind of them to be operated by the leftward operational rule to suit the conjecture.

This notwithstanding, what we need is merely that prove each and every odd number of 15+12c plus 19+12c between n and 2n+1 to suit the conjecture, yet it is not all of 15+12c plus 19+12c. Undoubtedly odd numbers of 15+12c/19+12c between n and 2n+1 are smaller and /or smallest within unproved kindred odd numbers because values which c like the ordinal takes are 0, 1, 2 etc. smaller positive integers in the case.

We have known that consecutive 24 concrete positive integers ≥ 1 suit the conjecture. In addition, supposed consecutive positive integers $\leq n$ suit the conjecture according to step 2 of the mathematical induction where $n \geq 24$.

Such being the case, if n is the infinity, then it means that every positive integer ≥ 24 suits the conjecture, so we need not to prove it.

If n is a concrete positive integer inside finite limits, then 2n is a concrete positive integer inside finite limits too, of course, every odd number of 15+12c/ 19+12c between n and 2n+1 is a concrete positive odd numbers, and that the number of them is finite, so the number of their kinds is finite. Let 15+12c=2n+1, figure out c= (n-7)/6; again let 19+12c=2n+1, figure out c= (n-9)/6. Namely the number of kinds of 15+12c between n and 2n+1 is smaller than (n-7)/6, and the number of kinds of 19+12c between n and 2n+1 is smaller than (n-9)/6.

Hereinabove, we have spoken that odd-even expressions at a bunch of operational routes of 15+12c/19+12c are getting greater and greater along with the continuation of operations by the leftward operational rule, actually, it is precisely that coefficients of χ and constant terms of odd-even expressions are getting greater and greater concurrently along with the continuation of operations, yet variable χ expresses 0 and natural numbers ≥ 1 throughout, no matter which variable χ represents.

As is well-known, all kinds of 15+12c/19+12c are embodied at itself of 15+12c/19+12c collectively, nothing but pass the evaluations of c to distinguish each of them. In addition to this, you ought to notice that each and every kind of 15+12c/19+12c can be expressed by an integer's expression which contains variable χ, for examples, hereinabove listed 51+48e i.e. 15+12c with c=4e+3, and 19+48e i.e. 19+12c with c=4e.

Once an emerging integer's expression whose coefficient of χ and constant term are respectively smaller than the coefficient of χ and the constant term of a kind of 15+12c/19+12c appears at an operational route of 15+12c/19+12c, then it means that the kind of 15+12c/ 19+12c is proved to suit the conjecture according to the axiom.

If each of many integer's expressions which contain variable χ at operational routes of 15+12c/19+12c is smaller than a kind of 15+12c/19+12c, then foregoing derivative kinds of 15+12c/19+12c which suit the conjecture have smaller coefficients of χ and constant terms in most instances, relative to tail derivative kinds of 15+12c/19+12c.

Therefore, after variable χ of foregoing derivative kinds of 15+12c/ 19+12c which suit the conjecture is evaluated with 0 and 1, 2, 3, etc smaller natural numbers, we can get some smaller concrete positive odd numbers of 15+12c/ 19+12c. Of course, these smaller concrete positive odd numbers of 15+12c/ 19+12c suit the conjecture too.

Let us respectively give concrete odd numbers after foregoing 6 evaluations of foresaid several kinds of 15+12c/ 19+12c plus individually operated odd numbers to explain the proposition. Foresaid 6 kinds of 15+12c after foregoing 6 evaluations of χ are listed below.

χ:	0,	1,	2,	3,	4,	5
51+48e:	51,	99,	147,	195,	243,	291
75+96f:	75,	171,	267,	363,	459,	555
87+96f:	87,	183,	279,	375,	471,	567
315+384h:	315,	699,	1083	1467,	1851,	2235
135+384h:	135,	519,	903,	1287,	1671,	2055
15+384h:	15,	399,	783,	1167,	1551,	1935

As listed above, from small to large odd numbers of 15+12c under positive integer 200 have 15, 51, 75, 87, 99, 135, 147, 171, 183 and 195.

From small to large odd numbers of 15+12c under positive integer 200 are 15, 27, 39, 51, 63, 75, 87, 99, 111, 123, 135, 147, 159, 171, 183 and 195, therein underlined odd numbers are absent odd numbers in listed above 6 kinds of 15+12c.

Here, the reason that absent a few of odd numbers of 15+12c under integer 200, is due to be unable to show continued operations at operational routes of 15+12c, because if do so, it will cause an overlong operational route.

Let us operate absent each odd number all alone to suit the conjecture and point out the belongingness of each of them.

From 27→82→41→124→62→31[&]→94→47→142→71→214→107→322→161→484→242→121→364*→182→91[#]→274→137→412→206→103[##]→310→155→466→233→700→350→175[###]→526→263→790→395→1186→593→1780→890→445→1336→668→334**→167→502→251→754→377→1132→566→283→850→425→1276→638→319→958→479→1438→719→2158→1079→3238→1619→4858→2429→7288→3644→1822***→911→2734→1367→4102→2051→6154→3077→9232→4616→2308→1154→577→1732→866→433→1300→650→325→976→488→244→122→61→184→92→46→23<27, get that 27 suits the conjecture according to the axiom. Also, odd number 27 belongs within $27+2^{59}\times3y$.

In addition, several signs except arrowheads at the operational route of 27 will be applied by latter operations.

From 39→118→59→178→89→268→134→67→202→101→304→152 →76→38 < 39, get that 39 suits the conjecture according to the axiom. Also, odd number 39 belongs within $39+2^{8}\times3k$.

From 63→190→95→286→143→430→215→646→323→970→485→ 1456→728→364*—connect to preceding operational route of 27→…→ 61< 63, get that 63 suits the conjecture according to the theorem. Also odd number 63 belongs within $63+2^{54}\times3w$.

From 111→334**—connect to preceding operational route of 27→… → 61 < 111, get that 111 suits the conjecture according to the theorem. Also odd number 111 belongs within $111+2^{31}\times3q$.

From 123→370→185→556→278→139→418→209→628→314→157 → 472→236→118 < 123, get that 123 suits the conjecture according to the axiom. Also odd number 123 belongs within $123+2^{8}\times3k$.

From 159→478→239→718→359→1078→539→1618→809→2428→ 1214→607→1822***—connect to preceding operational route of 27 →… → 122 < 159, get that 159 suits the conjecture according to the theorem. Also odd number 159 belongs within $159+2^{21}\times3s$.

Foresaid 6 kinds of 19+12c after foregoing 6 evaluations of χ are listed below.

χ:	0,	1,	2,	3,	4,	5
19+48e:	19,	67,	115,	163,	211,	259
43+96f:	43,	139,	235,	331,	427,	523
55+96f:	55,	151,	247,	343,	449,	535
187+384h:	187,	571,	955,	1339,	1723,	2107
271+384h:	271,	655,	1039,	1423,	1807,	2191
295+384h:	295,	679,	1063,	1447,	1831,	2215

As listed above, from small to large odd numbers of 19+12c under positive integer 200 have 19, 43, 55, 67, 115, 139, 163 and 187.

From small to large odd numbers of 19+12c under positive integer 200 are 19, <u>31</u>, 43, 55, 67, <u>79</u>, <u>91</u>, <u>103</u>, 115, 127, 139, <u>151</u>, 163, <u>175</u>, 187 and <u>199</u>, therein underlined odd numbers are absent odd numbers in listed above 6 kinds of 19+12c.

Here, the reason that absent a few of odd numbers of 19+12c under positive integer 200, is due to be unable to show continued operations at operational routes of 19+12c, because if do so, it will cause an overlong operational route. So, we operate each of them all alone to suit the conjecture, and point out the belongingness of each of them.

From $31^{\&}$— connect to preceding operational route of 27 →… →23< 31, get that 31 suits the conjecture according to the axiom. Also odd number 31 belongs within $31+2^{56}\times 3x$.

From 79→238→119→358→179→538→269→808→404→202←67<79, get that 79 suits the conjecture according to the theorem. Also odd number 79 belongs within $79+2^5\times 3f$.

From $91^{\#}$— connect to preceding operational route of 27→…→ 61 < 91, get that 91 suits the conjecture according to the axiom. Also odd number 91 belongs within $91+2^{45}\times 3v$.

From $103^{\#\#}$—connect to preceding operational route of 27→…→ 61< 103, get that 103 suits the conjecture according to the axiom. Also odd number 103 belongs within $103+2^{42}\times 3u$.

From 151→454→227→682→341→1024→512 →256 → 128 < 151, get that 151 suits the conjecture according to the axiom. Also odd number 151 belongs within $151+2^5\times 3f$.

From $175^{\#\#\#}$— connect to preceding operational route of 27→…→167 < 175, get that 175 suits the conjecture according to the axiom. Also odd number 175 belongs within $175+2^8\times 3k$.

From 199→598→299→898→449→1348→674→337→1012→506→ 253→760→380→190<199, get that 199 suits the conjecture according to the axiom. Also odd number 199 belongs within $199+2^8\times 3k$.

Variables y, x, k, w, q, s, f, v and u within above-mentioned integer's expressions, each of them expresses 0 and natural numbers ≥1 likewise.

Above-deduced $27+2^{59}\times 3y$, $39+2^8\times 3k$, $31+2^{56}\times 3x$, $63+2^{54}\times 3w$, $79+2^5\times 3f$, $91+2^{45}\times 3v$, $103+2^{42}\times 3u$, $111+2^{31}\times 3q$, $123+2^8\times 3k$, $159+2^{21}\times 3s$, $151+2^5\times 3f$, $175+2^8\times 3k$, $199+2^8\times 3k$ is respectively a kind of 15+12c plus 19+12c too.

Whether derivative a kind of 15+12c/ 19+12c come from either bunch of operational routes, or deduced a kind of 15+12c/ 19+12c come from a single operational route, it is possessed of coequal qualification as one another's irreplaceable a kind of 15+12c/ 19+12c.

By this token, after variable χ of foregoing emerging kinds of 15+12c/ 19+12c is bestowed with 0 and 1, 2, 3, etc smaller natural numbers, not only can get some smaller concrete positive odd numbers of 15+12c/ 19+12c, but also can get smaller concrete consecutive positive odd numbers of 15+12c/19+12c, such as consecutive positive odd numbers of 15+12c/19+12c under positive integer 200. Without doubt, these consecutive positive odd numbers of 15+12c/19+12c suit the conjecture like foregoing emerging kinds of 15+12c/ 19+12c.

Thereinafter, let us quote aforementioned concerned conclusions once more to prove further that odd numbers of 15+12c/19+12c between n and 2n+1 suit the conjecture.

Generally speaking, 15+12c/19+12c between n and 2n+1 are concrete smaller positive odd numbers, that is to say, each of them is the front-end or smaller odd number of a kind of 15+12c/19+12c because large or small 15+12c/19+12c depend exactly their own variable c like the ordinal.

Moreover, 15+12c/19+12c embodied every kind of itself intensively, and that operations of 15+12c/19+12c proceed endlessly.

On the one hand, ceaselessly stop operations of part branches at operational routes of 15+12c/19+12c, because the coefficient of χ and the constant term of an integer's expression at each such branch are smaller than the coefficient of χ and the constant term of a kind of 15+12c/19+12c respectively, and derive the kind of 15+12c/19+12c from the integer's expression to suit the conjecture according to the axiom.

For frontally derived kinds of 15+12c/19+12c, since their coefficients of χ and constant terms are relatively smaller, so after their variable χ is evaluated with 0 and 1, 2, 3 etc smaller natural numbers, hereby get concrete smaller odd numbers of 15+12c/ 19+12c. Naturally each such odd number suits the conjecture.

On the other hand, constant terms and coefficients of χ of integer's expressions at operational routes of 15+12c/19+12c are getting greater and greater along with the continuation of operations, accordingly constant terms and coefficients of χ of derived kinds of 15+12c/19+12c from these integer's expressions are getting greater and greater too.

Since odd numbers of 15+12c/ 19+12c between n and 2n+1 are finite, and each of them belongs within a kind of 15+12c/19+12c, additionally any two kinds of 15+12c/ 19+12c have not an identical constant term.

Then, after operations go beyond some limits, cause that a constant term of every derived kind of 15+12c/19+12c which suits the conjecture is not smaller than 2n+1. That is to say, even if let their variable χ be equal to 0, smallest odd number of each such kind of 15+12c/19+12c is not smaller than 2n+1 either. In this situation, an integer's expression on the terminal of each and every branch which stopped already operations at operational routes of 15+12c/ 19+12c is smaller than a kind of 15+12c/ 19+12c. We can obtain each and every odd number of 15+12c/19+12c between n and 2n+1 from odd numbers after variable χ of derived some kinds of 15+12c/19+12c from such integer's expressions is evaluated with 0 and 1, 2, 3, etc. smaller natural numbers. Go without saying, got odd numbers like so suit the conjecture.

To state succinctly, any odd number of 15+12c/19+12c between n and 2n+1 is able to pass operations of 15+12c/ 19+12c by the leftward operational rule to first get an integer's expression which is smaller than a kind of 15+12c/ 19+12c which contains the odd number, then the kind of 15+12c/ 19+12c is proved to suit the conjecture according to the axiom. After that, pass the evaluations of variable χ of the kind of 15+12c/ 19+12c to obtain such an odd number, so the odd number is proven to suit the conjecture like the kind of 15+12c/ 19+12c.

Follow such a set pattern, each and every odd number of 15+12c/19+12c between n and 2n+1 is proven to suit the conjecture.

Excepting the above-mentioned deductive inference, we also can apply straightway the theorem or the deduction to prove odd number of 15+12c/ 19+12c between n and 2n+1 to suit the conjecture. Comparatively speaking, this method is rather too simplicity, vide infra.

We known that 15+12c/19+12c have infinite-many strips of operational routes, and that we can regard them as which begin with 15+12c/ 19+12c, or regard them as infinite-many branches of the bunch of operational routes. When regard them as which begin with 15+12c/ 19+12c, all operational routes of 15+12c/19+12c intersect at 35+27c/ 44+27c. When regard them as infinite-many branches of the bunch of operational routes, all operational routes of 15+12c/ 19+12c are at indirect concatenations.

Also known that 15+12c/19+12c have infinite-many kinds, nothing but where their each operational route extends to an integer's expression which is smaller than a kind of 15+12c/19+12c, the kind of 15+12c/ 19+12c is educed by the integer's expression just.

By this token, each and every operational route of 15+12c/19+12c implies the emergence of a kind of 15+12c/19+12c, and that there is affirmatively an integer's expression which is smaller than a kind of 15+12c/19+12c at every operational route of 15+12c/19+12c from preceding analyses.

Hereinbefore, we have operated out certain of satisfactory integer's expressions whose each is smaller than a kind of 15+12c/19+12c at stopped operational routes of 15+12c/ 19+12c, for examples, 29+27e, 64+82f, 11+27e and 37+81f etc, and that these smaller integer's expressions suit the conjecture like derivative kind of 15+12c/19+12c too.

Consequently, unsighted those satisfactory integer's expressions whose each is smaller than a kind of 15+12c/19+12c at better operational routes of

15+12c/ 19+12c suit the conjecture certainly according to the theorem or the inference.

Accordingly derived kinds of 15+12c/19+12c from unsighted those satisfactory integer's expressions whose each is smaller than a kind of 15+12c/19+12c suit the conjecture, according to the axiom.

After variable χ of foregoing derivative kinds of 15+12c/ 19+12c is evaluated with 0 and 1, 2, 3, etc. smaller natural numbers, we can get all odd numbers of 15+12c/ 19+12c between n and 2n+1. Beyond doubt, odd numbers got by this way suit the conjecture.

Altogether, we have proven that odd numbers between n and 2n+1 suit the conjecture by the leftward operational rule, so they all exist at the bunch of integers' chains.

To sum up, we have proven that all even numbers and all odd numbers between n and 2n+1 exist at a bunch of integers' chains extended by two-way operational rules. Namely all positive integers between n and 2n+1 are proven by us to suit the conjecture.

Thus far, we have proven that positive integers $\leq 2^1n$ suit the conjecture by consecutive positive integers $\leq$ n, like that, we can too prove that positive integers $\leq 2^2n$ suit the conjecture by consecutive positive integers $\leq 2^1n$ according to the foregoing way of doing.

At the beginning of the proof, we have spoken that divide limits of all consecutive positive integers into segments according to greatest positive integer $2^X n$ per segment, where $X \geq 0$, and $n \geq 24$.

After we proven that positive integers between $2^{X-1}n$ and $2^X n$ suit the conjecture by consecutive proven positive integers$\leq 2^{X-1}n$, in the same old way, we likewise are able to prove that positive integers between $2^X n$ and $2^{X+1}n$ suit the conjecture by consecutive proven positive integers$\leq 2^X n$.

For greatest positive integer $2^X n$ at each segment, X begins with 0, next, it is evaluated with 1, 2, 3, etc. natural numbers in proper order. Along with which values of X are getting greater and greater, consecutive positive integers $\leq 2^X n$ are getting more and more, and that emerging positive integers are getting greater and greater. If X is equal to 0 plus every natural number, then all positive integers are proven to suit the conjecture, namely every positive integer is proven to suit the conjecture.

Heretofore, the Collatz conjecture is proven by us at long last integrally. The proof was thus brought to a close. As a consequence, the Collatz conjecture holds water.