Prove Collatz Conjecture by Mathematical Induction via the Two-Way Operations

Zhand Tianshu
Zhanjiang city, Guangdong province, China
Email: chinazhingtianshu@126.com
china.zhingtianshu@tom.com

Introduction: The Collatz conjecture is also well-known variously as the 3n+1 conjecture, the Ulam conjecture, Kakutani’s problem, the Thwaites conjecture, Hasse’s algorithm, or the Syracuse problem, etc. Yet it is still both unproved and un-negated a conjecture ever since named after Lothar Collats in 1937.

AMS subject classification: 11×××, 00A05.

Abstract
If every positive integer is able to be operated to 1 by the set operational rule of the Collatz conjecture, then begin with 1, we can get all positive integers after make infinitely many operations on the contrary of the set operational rule. In this article, we shall prove that the Collatz conjecture by the mathematical induction via the two-way operations is tenable.

Keywords: mathematical induction; the two-way operational rules; classify positive integers; the bunch of integers’ chains; operational routes

Basic Concepts
The Collatz conjecture states that take any positive integer n, if n is an even number, then divide n by 2 to obtain an integer; if n is an odd number, then multiply n by 3 and add 1 to obtain an even number. Repeat
the above process indefinitely, then no matter which positive integer you start with, you will always eventually reach a result of 1.

We consider the way of aforesaid two steps as leftward operational rule for any positive integer. Also consider operations on the contrary of the leftward operational rule as the rightward operational rule for any positive integer. Taken one with another, we consider such each other’s- opposed operational rules as the two-way operational rules, and that operations by the two-way operational rules are called the two-way operations.

The rightward operational rule stipulates that for any positive integer n, multiply n by 2 to obtain an even number. Additionally, when n is an even number, if divide the difference of n minus 1 by 3 and obtain an odd number, then must operate this step otherwise, and proceed from here to continue to operate; if it is not such, then don’t operate this step.

Begin with any positive integer to operate by either operational rule continuously, manifestly each operational result is a positive integer, then we consider a string of such consecutive positive integers on an identical operational direction and arrowheaded signs inter se as an operational route. Each of operational results comes only from preceding an adjacent positive integer at an identical operational route.

If any positive integer P exists at a certain operational route, then may term the operational route an operational route of P. Two operational routes of P branch from a positive integer certainly.
Begin with 1 to operate positive integers got successively by the rightward operational rule, so it forms a bunch of operational routes automatically. We term such a bunch of operational routes “a bunch of integers’ chains”. Manifestly whole a bunch of integers’ chains must consist of infinite many operational routes.

Since a direct origin of each of positive integers is unique, thus each of positive integers except for 1 is unique at the bunch of integers’ chains.

Comparatively speaking, inside greater limits, positive integers on the left side of the bunch of integers’ chains are smaller, yet positive integers on the right side are larger. Overall, from left to right positive integers at the bunch of integers’ chains are getting both more and more absolutely, and greater and greater relatively. Please, see a beginning of the bunch of integers’ chains as the follows.

```
...         ...  452→...
 168↑       680↑ 226↑75→...
 84↑        340↑→113↑ 227→...
 42↑        170↑ 682→...
 1          21↑ 85↑ 341↑
1→2→4↑→8→16↓→32→64↑→128→256↑→512→1024↑→2048→...
 5→10↓→20→40↓→80→160↓→320→...
 3↓          13↓ 53→106↓→212→...
 6↓          26↓ 35→70↓→140→...
 12↓         52↓→104→... 23→46↓→15→...
 24↓          17↓ 92→...
...          34↓→11→22↓→7→14→28↓→56...
 68↓          44→88↓→29...9→18→...
 136↓→...    29↓
 45→...      58↓→19→...
  ... ... ...
```

First Illustration

Annotation: ↓ and ↑ must rightwards tilt, but each page is narrow, thus it can only so.

No matter which positive integer, it is surely at the bunch of integers’
chains so long as it is able to be operated to 1 by the leftward operational rule. Likewise, the converse proposition holds water.

That is to say, positive integers at the bunch of integers’ chains and positive integers which are able to be operated to 1 by the leftward operational rule are one-to-one correspondence.

Thus it can be seen, whether a positive integer suits the conjecture, it needs merely us to determine whether the positive integer exists at the bunch of integers’ chains. If every positive integer is able to be operated to 1 by the leftward operational rule, then there are all positive integers at the bunch of integers’ chains.

Correspondingly, if we can prove that all positive integers exist at the bunch of integers’ chains, then every positive integer is able to be operated to 1 by the leftward operational rule.

Because of this, we shall prove that the bunch of integers’ chains contains all positive integers by mathematical induction.

If we divide the bunch of integers’ chains into many one-way operational routes according to un-operated smallest odd number got as most left one of headmost operating row, then a beginning of the bunch of integers’ chains is dismembered into many operational routes as follows.

1→2→4→8→16→32→64→128→256→512→1024→2048→4096→8192…

1 5 21 85 341 1365

5→10→20→40→80→160→320→640→1280→2560→5120→10240→…

3 13 53 213 853 3413

3→6→12→24→48→96→192→384→768→1536→3072→6144→12288→…
From listed-above rows, it is observed that excepting an odd number on most left side of every row, others, either all are even numbers, or all are odd numbers without arrowheads. On operations of the contrary i.e. by the leftward operational rule, we regard which multiply an odd number by 3 and add 1 to obtain an even number as which the operation upgrades a stair; also look upon which divide an even number by 2 to obtain an integer as which the operation goes a step leftwards, at above-listed
operational courses. Whether it upgraded a stair or gone a step leftwards, enable the operation to go a step further to approach final result of 1.

Moreover, be necessary to determine an axiom beforehand and prove a theorem, so that apply either of them to affirm an anticipative result that suits the conjecture after such a result arises at an operational route.

Axiom* Known that positive integers which are smaller than positive integer P suit the conjecture, if a positive integer which is smaller than positive integer P appears at an operational route of P, then P is proved to suit the conjecture. Illustrate with examples as follows:

(1) Let \( P = 31 + 3^2\eta \) with \( \eta \geq 0 \), from \( 27 + 2^3\eta \rightarrow 82 + 3*2^3\eta \rightarrow 41 + 3*2^2\eta \rightarrow 124 + 3^2*2^2\eta \rightarrow 62 + 3^2*2^2\eta \rightarrow 31 + 3^2\eta > 27 + 2^3\eta \), we get that \( 31 + 3^2\eta \) suits the conjecture.

(2) Let \( P = 51 + 48\mu \) with \( \mu \geq 0 \), from \( 51 + 48\mu \rightarrow 154 + 144\mu \rightarrow 77 + 72\mu \rightarrow 232 + 216\mu \rightarrow 116 + 108\mu \rightarrow 58 + 54\mu \rightarrow 29 + 27\mu < 51 + 48\mu \), we get that \( 51 + 48\mu \) suits the conjecture.

This axiom is established in the two-way operational rules visibly. Or rather, let positive integer \( C \leq \text{positive integer} \ P \), and \( C \) suits the conjecture. Then, at an operational route by leftward operational rule, when \( C \) is before \( P \), operations of \( C \) via \( P \) was operated into 1 already; when \( C \) is behind \( P \), operations of \( P \) can continue to pass \( C \) to 1. Like that, at an operational route by rightward operational rule, when \( C \) is before \( P \), \( C \) comes from 1; when \( C \) is behind \( P \), \( P \) comes from 1.
**Theorem** If an operational route of P intersects an operational route of C, and C which is smaller than P suits the conjecture, then P suits the conjecture too, where P and C are positive integers.

**Proof** Suppose that an operational route of P intersects an operational route of C at positive integer $\alpha$, since $\alpha$ exists at an operational route of C, so $\alpha$ suits the conjecture according to the axiom. And that $\alpha$ exists at an operational route of P too, then P suits the conjecture according to the axiom. Give an example to explain it as follows.

Let $P=63+3\cdot2^8\varphi$ with $\varphi\geq0$, from $63+3\cdot2^8\varphi \rightarrow 190+3^2\cdot2^7\varphi \rightarrow 286+3^3\cdot2^7\varphi \rightarrow 143+3^3\cdot2^6\varphi \rightarrow 430+3^4\cdot2^6\varphi \rightarrow 215+3^4\cdot2^5\varphi \rightarrow 646+3^5\cdot2^5\varphi \rightarrow 323+3^5\cdot2^4\varphi \rightarrow 970+3^6\cdot2^4\varphi \rightarrow 485+3^6\cdot2^3\varphi \rightarrow 1456+3^7\cdot2^3\varphi \rightarrow 728+3^7\cdot2^2\varphi \rightarrow 364+3^7\cdot2\varphi \rightarrow 182+3^7\varphi \uparrow \rightarrow \ldots$

$\uparrow121+3^6\cdot2\varphi \leftarrow 242+3^6\cdot2^2\varphi \leftarrow 484+3^6\cdot2^3\varphi \leftarrow 161+3^5\cdot2^3\varphi \leftarrow 322+3^5\cdot2^4\varphi \leftarrow 107+3^4\cdot2^4\varphi \leftarrow 214+3^4\cdot2^5\varphi \leftarrow 71+3^3\cdot2^5\varphi \leftarrow 142+3^3\cdot2^6\varphi \leftarrow 47+3^2\cdot2^6\varphi$$< 63+3\cdot2^8\varphi$, we get that $63+3\cdot2^8\varphi$ suits the conjecture.

**Inference** If an operational route of P and an operational route of C are at an indirect concatenation, and C suits the conjecture, then P suits the conjecture. For example, an operational route of P intersects an operational route of B, the operational route of B intersects an operational route of D… the operational route of E intersects an operational route of C, and C suits the conjecture, then P suits the conjecture.

Actually, each and every positive integer at one another’s-linked
operational routes suits the conjecture, provided therein there is a positive integer which suits the conjecture.

**The Proof**

Let us set about the proof that the bunch of integers’ chains contains all positive integers by mathematical induction hereinafter.

1. From preceding first illustration, we can directly find that there are 24 consecutive positive integers $\geq 1$ within positive integers got. Especially indicate that 15 within them belongs in $15+12c$, and 19 within them belongs in $19+12c$, in which case $c=0$.

2. Suppose that after further operate positive integers got by the rightward operational rule, there are consecutive positive integers $\leq n$ within positive integers got at a bunch of integers’ chains, where $n \geq 24$.

3. Prove that after continue to operate positive integers got already by the rightward operational rule, we can get consecutive positive integers $\leq 2n$ within positive integers got at a bunch of integers’ chains extended.

First, let us divide limits of consecutive positive integers at the number axis into segments according to $2^Xn$ as greatest positive integer per segment, where $X \geq 0$ and $n \geq 24$, so as to accord with the proof by the mathematical induction. A simple segmenting illustration is as follows.

$1\rightarrow 2^0n\rightarrow 2^1n\rightarrow 2^2n\rightarrow 2^3n\rightarrow$

Second Illustration

**Proof** * Since there are consecutive positive integers$\leq n$ within positive
integers got at a bunch of integers’ chains, thus multiply each and every positive integer \( \leq n \) by 2 according to the rightward operational rule, then we get all even numbers between \( n \) and \( 2n+1 \) at a bunch of integers’ chains extended, irrespective of repeated even numbers \( \leq n \).

After that, we must seek an origin of each kind of odd numbers between \( n \) and \( 2n+1 \) by the two-way operational rules, and that each such origin is necessarily smaller than corresponding a kind of odd numbers.

First, let us divide all odd numbers between \( n \) and \( 2n+1 \) into two kinds, i.e. \( 5+4k \) and \( 7+4k \), where \( k \) is a natural number \( \geq 5 \), then any odd number between \( n \) and \( 2n+1 \) belongs to one in the two kinds certainly.

By now, we list the two kinds of odd numbers in correspondence with their variable \( k \) as follows.

\[
\begin{align*}
k: & \quad 5, \ 6, \ 7, \ 8, \ 9, 10, 11, 12, 13, 14, 15, 16 \ldots \\
5+4k: & \quad 25, 29, \ 33, 37, 41, 45, 49, 53, 57, 61, 65, 69 \ldots \\
7+4k: & \quad 27, 31, \ 35, 39, 43, 47, 51, 55, 59, 63, 67, 71 \ldots
\end{align*}
\]

From \( 5+4k \to 16+12k \to 8+6k \to 4+3k < 5+4k \), we obtain that \( 5+4k \) suits the conjecture according to the axiom.

For \( 7+4k \), let us again divide it into three kinds, i.e. \( 11+12c \), \( 15+12c \) and \( 19+12c \), where \( c \geq 1 \).

From \( 7+8c \to 22+24c \to 11+12c \) \( > 7+8c \), we obtain that \( 11+12c \) suits the conjecture according to the axiom.

Let us list \( 15+12c \) and \( 19+12c \) in correspondence with their variable \( c \) as
follows.

c: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...

15+12c: 27, 39, 51, 63, 75, 87, 99, 111, 123, 135, 147, 159 ...

19+12c: 31, 43, 55, 67, 79, 91, 103, 115, 127, 139, 151, 163 ...

Hereinafter, we shall operate respectively 15+12c and 19+12c by the leftward operational rule, moreover, discover and affirm certain of satisfactory results at some operational branches.

Firstly, let us operate 15+12c by the leftward operational rule below.

\[ 15+12c \rightarrow 46+36c \rightarrow 23+18c \rightarrow 70+54c \rightarrow 35+27c \]

\[ d=2e+1: \ 29+27e (1) \quad e=2f: \ 142+486f \rightarrow 71+243f \]

\[ \bullet 35+27c \downarrow \rightarrow c=2d+1: \ 31+27d \uparrow \rightarrow d=2e: \ 94+162e \rightarrow 47+81e \uparrow \rightarrow e=2f+1: \ 64+81f (2) \]

\[ c=2d: \ 106+162d \rightarrow 53+81d \downarrow \rightarrow d=2e+1: \ 67+81e \downarrow \rightarrow e=2f+1: \ 74+81f (3) \]

\[ d=2e: \ 160+486e \rightarrow e=2f: \ 202+486f \rightarrow 101+243f \]

\[ g=2h+1: \ 200+243h (4) \]

\[ \heartsuit 71+243f \downarrow \rightarrow f=2g+1: \ 157+243g \uparrow \rightarrow g=2h: \ 472+1458h \rightarrow 236+729h \uparrow \rightarrow \ldots \]

\[ f=2g: \ 214+1458g \rightarrow 107+729g \downarrow \rightarrow g=2h+1: \ 418+729h \downarrow \rightarrow \ldots \]

\[ g=2h: \ 322+4374h \rightarrow \ldots \]

\[ \diamondsuit 101+243f \downarrow \rightarrow f=2g+1: \ 172+243g \uparrow \rightarrow g=2h+1: \ 1246+1458h \rightarrow \ldots \]

\[ f=2g: \ 304+1458g \rightarrow 152+729g \downarrow \rightarrow \ldots \]

\[ g=2h+1: \ 880+1458h \rightarrow 440+729h \uparrow \rightarrow \ldots \]

\[ \heartsuit 71+243f \downarrow \rightarrow f=2g+1: \ 172+243g \uparrow \rightarrow g=2h+1: \ 1246+1458h \rightarrow \ldots \]

\[ f=2g: \ 304+1458g \rightarrow 152+729g \downarrow \rightarrow \ldots \]

\[ g=2h+1: \ 880+1458h \rightarrow 440+729h \uparrow \rightarrow \ldots \]

\[ e=2f: \ 970+1458f \rightarrow 485+729f \uparrow \rightarrow \ldots \]

\[ f=2g: \ 20+243g \rightarrow g=2h: \ 10+243h (6) \]

\[ g=2h+1: \ 880+1458h \rightarrow 440+729h \uparrow \rightarrow \ldots \]

Annotation:
Each of letters c, d, e, f, g, h ...etc in the above-listed operational routes expresses each of natural numbers plus 0, similarly hereinafter.
Also, there are ♦↔♦, ♥↔♥, ♠↔♠, and ♣↔♣.

We conclude several branch’s satisfactory operational results from above-listed the bunch of operational routes of 15+12c, and these
satisfactory operational results are analyzed as follows, one by one.

From $c=2d+1$ and $d=2e+1$, we get $c=2d+1=2(2e+1)+1=4e+3$, and $15+12c = 15+12(4e+3)=51+48e > 29+27e$ where mark (1), so $15+12c$ with $c=4e+3$ suits the conjecture according to the axiom.

From $c=2d+1$, $d=2e$, and $e=2f+1$, we get $c=2d+1=4e+1=4(2f+1)+1=8f+5$, and $15+12c=15+12(8f+5)=75+96f > 64+81f$ where mark (2), so $15+12c$ with $c=8f+5$ suits the conjecture according to the axiom.

From $c=2d$, $d=2e+1$ and $e=2f+1$, we get $c=2d= 4e+2= 4(2f+1)+2=8f+6$, and $15+12c=15+12(8f+6)=87+96f > 74+81f$ where mark (3), so $15+12c$ with $c=8f+6$ suits the conjecture according to the axiom.

From $c=2d+1$, $d=2e$, $e=2f$, $f=2g+1$ and $g=2h+1$, we get $c=2d+1=4e+1=8f+1=8(2g+1)+1=16g+9=16(2h+1)+9=32h+25$, and $15+12c=15+12(32h+25)=315+384h > 200+243h$ where mark (4), so $15+12c$ with $c=32h+25$ suits the conjecture according to the axiom.

From $c=2d$, $d=2e+1$, $e=2f$, $f=2g+1$ and $g=2h$, we get $c=2d=2(2e+1)=4e+2=8f+2=8(2g+1)+2=16g+10=32h+10$, and $15+12c=15+12(32h+10)=135+384h > 86+243h$ where mark (5), so $15+12c$ with $c=32h+10$ suits the conjecture according to the axiom.

From $c=2d$, $d=2e$, $e=2f$, $f=2g$ and $g=2h$, we get $c=2d=32h$, and $15+12c = 15+12(32h)=15+384h > 10+243h$ where mark (6), so $15+12c$ with $c=32h$ suits the conjecture according to the axiom.

Secondly, we operate $19+12c$ by the leftward operational rule as follows.
\[19+12c \rightarrow 58+36c \rightarrow 29+18c \rightarrow 88+54c \rightarrow 44+27c \]

<table>
<thead>
<tr>
<th>d = 2e: 11 + 27e ((\alpha))</th>
<th>e = 2f: 37 + 81f ((\beta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>♣ 44 + 27c \downarrow c = 2d: 22 + 27d \uparrow \rightarrow d = 2e + 1: 148 + 162e \rightarrow 74 + 81e \uparrow \rightarrow e = 2f + 1: 466 + 486f \downarrow</td>
<td>d = 2e + 1: 94 + 81e \downarrow \rightarrow e = 2f: 47 + 81f ((\gamma))</td>
</tr>
<tr>
<td>c = 2d + 1: 214 + 162d \rightarrow 107 + 81d \downarrow \rightarrow d = 2e + 1: 322 + 486e</td>
<td>e = 2f + 1: 516 + 486f</td>
</tr>
<tr>
<td>g = 2h: 129 + 243h ((\delta))</td>
<td>\ldots</td>
</tr>
<tr>
<td>f = 2g + 1: 258 + 243g \uparrow \rightarrow g = 2h + 1: 1504 + 1458h \rightarrow 752 + 729h \uparrow \rightarrow \ldots</td>
<td>466 + 486f \rightarrow 233 + 243f \uparrow \rightarrow f = 2g: 700 + 1458g \rightarrow 350 + 729g \downarrow \rightarrow g = 2h + 1: 3238 + 4374h \downarrow</td>
</tr>
<tr>
<td>g = 2h: 175 + 729h \downarrow \rightarrow \ldots</td>
<td>g = 2h + 1: 172 + 243h ((\epsilon))</td>
</tr>
<tr>
<td>e = 2f + 1: 202 + 243f \uparrow \rightarrow f = 2g: 101 + 243g \uparrow \rightarrow g = 2h: 304 + 1458h \rightarrow \ldots</td>
<td>e = 2f + 1: 202 + 243f \uparrow \rightarrow f = 2g + 1: 1336 + 1458g \rightarrow \ldots</td>
</tr>
<tr>
<td>♠ 322 + 486e \rightarrow 161 + 243e \uparrow \rightarrow e = 2f: 484 + 1458f \rightarrow \ldots</td>
<td>♠ 516 + 486f \rightarrow 258 + 243f \downarrow \rightarrow f = 2g + 1: 1504 + 1458g \rightarrow \ldots</td>
</tr>
<tr>
<td>f = 2g: 129 + 243g \downarrow \rightarrow g = 2h: 388 + 1458h \rightarrow \ldots</td>
<td>g = 2h + 1: 186 + 243h ((\zeta))</td>
</tr>
</tbody>
</table>

**Annotation:**
each of letters c, d, e, f, g, h \ldots etc in the above-listed operational routes expresses each of natural numbers plus 0, similarly hereinafter.

Also, there are ♣ ↔ ♣, ♥ ↔ ♥, ♠ ↔ ♠, and ♦ ↔ ♦.

Likewise, we conclude several branch’s satisfactory operational results from above-listed the bunch of operational routes of 19+12c, and these satisfactory operational results are analyzed as follows, one by one.

From c = 2d, d = 2e, we get c = 2d = 4e, and 19 + 12c = 19 + 12(4e) = 19 + 48e > 11 + 27e where mark (\(\alpha\)), so 19 + 12c with c = 4e suits the conjecture according to the axiom.

From c = 2d, d = 2e + 1 and e = 2f, we get c = 2d = 2(2e + 1) = 4e + 2 = 8f + 2, and 19 + 12c = 19 + 12(8f + 2) = 43 + 96f > 37 + 81f where mark (\(\beta\)), so 19 + 12c with c = 8f + 2 suits the conjecture according to the axiom.

From c = 2d + 1, d = 2e + 1, and e = 2f, we get c = 2d + 1 = 4e + 3 = 8f + 3, and
19+12c=19+12(8f+3)=96f+55>47+81f where mark (γ), so 19+12c with c=8f+3 suits the conjecture according to the axiom.
From c=2d, d=2e+1, e=2f+1, f=2g+1 and g=2h, we get c=2d=2(2e+1)=
4e+2= 4(2f+1)+2 = 8f+6= 8(2g+1)+6 = 16g+14 = 32h+14, and 19+12c =
19+12(32h+14) = 187+384h >129+243h where mark (δ), so 19+12c with
c=32h+14 suits the conjecture according to the axiom.
From c=2d+1, d=2e, e=2f+1, f=2g and g=2h+1, we get c=2d+1=4e+1=
4(2f+1)+1=8f+5=16g+5=16(2h+1)+5=32h+21, and 19+12c=19+12(32h
+21) =271+384h > 172+243h where mark (ε), so 19+12c with c=32h+21
suits the conjecture according to the axiom.
From c=2d+1, d=2e+1, e=2f+1, f=2g and g=2h+1, we get c=2d+1=2(2e+
1)+1=4e+3=4(2f+1)+3=8f+7=16g+7=16(2h+1)+7=32h+23, and 19+12c=
19+12(32h+23)=295+384h > 186+243h where mark (ζ), so 19+12c with
c=32h+23 suits the conjecture according to the axiom.
Listed above proven 51+48e, 75+96f, 87+96f, 315+384h, 135+384h,
15+384h; 19+48e, 43+96f, 55+96f, 187+384h, 271+384h and 295+384h
are transformed into 51+2^4×3e, 75+2^5×3f, 87+2^5×3f, 315+2^7×3h,
135+2^7×3h, 15+2^7×3h; 19+2^4×3e, 43+2^5×3f, 55+2^5×3f, 187+2^7×3h,
271+2^7×3h and 295+2^7×3h, therein each exponent of 2 is actually the
number of times that an integer’s expression divided by 2 at an
operational rule from 15+12c/19+12c to first integer’s expression which
is smaller than a kind of 15+12c/19+12c.
Let $\chi$ represents together variables d, e, f, g, h, y, k, w, q, s, f, v, u … etc within integer’s expressions at two bunches of operational routes of 15+12c plus 19+12c, but $\chi$ represents not c.

Naturally the oddity of a part of integer’s expressions which contain variable $\chi$ is still indeterminate. That is to say, for every such integer’s expression which contains variable $\chi$, both consider it as an odd number to operate, and consider it as an even number to operate. Thus, let us label such integer’s expressions “odd-even expressions”.

For any odd-even expression at a bunch of operational routes of 15+12c/19+12c, two operations synchronize according as $\chi$ expresses both an odd number and an even number. Namely, both operate any odd-even expression as an odd number into threefold itself and add 1, and operate the odd-even expression as an even number into a half of itself.

Evidently, after an odd-even expression as an odd number to operate, we get a greater operational result, yet, after the odd-even expression as an even number to operate, we get a smaller operational result.

If you begin with any odd-even expression to do continuously operations by the leftward operational rule, then such an operational route via consecutive greater operational results will elongate infinitely.

Begin with 15+12c/19+12c to operate continuously by the leftward operational rule, if a newborn operational result is smaller than a kind of 15+12c/19+12c, this manifests that the kind of 15+12c/19+12c suits the
conjecture according to the axiom, so operations of the branch may stop at the here. If enable a kind of 15+12c/19+12c via operations to reach the eventual result of 1, then these operations must pass some smaller operational results.

Now that there are both a greater operational result due to \( \chi \) as an odd number to operate and a smaller operational result due to \( \chi \) as an even number to operate after every odd-even expression operates once by the leftward operational rule, then not only greater operational results at an operational route all are greater than their own common origin, but also consecutive greater operational results are getting greater and greater along with the continuation of operations, up to infinity.

Since greater operational results operate continuously, have to cause that odd-even expressions are getting more and more, up to infinite many, and smaller operational results which accompany greater operational results in synchronisms are getting more and more too, up to infinite many.

In other words, on the one hand, begin with any odd-even expression, two kinds’ operations progress and branch always endlessly due to the oddity of variable \( \chi \), up to arise infinitely more progress and branch. Of course, odd-even expressions are getting more and more, and that the more rear, the greater values, up to engender infinitely many infinities theoretically.

On the other hand, uninterruptedly stop operations of part branches at operational routes increased ceaselessly, in the case each such branch is
operated to a result which is smaller than a kind of $15+12c/19+12c$, and that there are infinitely many such results likewise, because so long as operations along consecutive smaller operational results at an operational route to proceed, would come into being such a result inevitably.

Thus it can be seen, operations of $15+12c/19+12c$ will proceed infinitely. Judging from this, $15+12c$ and $19+12c$ must be divided respectively into infinite many kinds, just sufficiently enable every kind of them to be operated by the leftward operational rule to suit the conjecture.

This notwithstanding, what we need is merely that prove each and every odd number of $15+12c$ plus $19+12c$ between $n$ and $2n+1$ to suit the conjecture, yet it is not all of $15+12c$ plus $19+12c$. Undoubtedly odd numbers of $15+12c/19+12c$ between $n$ and $2n+1$ are smaller and /or smallest within unproved kindred odd numbers because values which $c$ like the ordinal takes are 0, 1, 2 etc. smaller positive integers in the case.

We have known that consecutive 24 concrete positive integers $\geq 1$ suit the conjecture. In addition, supposed consecutive positive integers $\leq n$ suit the conjecture according to step 2 of the mathematical induction where $n \geq 24$.

Such being the case, if $n$ is the infinity, then it means that every positive integer $\geq 24$ suits the conjecture, so we need not to prove it.

If $n$ is a concrete positive integer inside finite limits, then $2n$ is a concrete positive integer inside finite limits too, of course, every odd number of $15+12c/19+12c$ between $n$ and $2n+1$ is a concrete positive odd numbers,
and that the number of them is finite, so the number of their kinds is finite. Let $15+12c=2n+1$, figure out $c=-(n-7)/6$; again let $19+12c=2n+1$, figure out $c=-(n-9)/6$. Namely the number of kinds of $15+12c$ between $n$ and $2n+1$ is smaller than $(n-7)/6$, and the number of kinds of $19+12c$ between $n$ and $2n+1$ is smaller than $(n-9)/6$.

Hereinabove, we have spoken that odd-even expressions at a bunch of operational routes of $15+12c/19+12c$ are getting greater and greater along with the continuation of operations by the leftward operational rule, actually, it is precisely that coefficients of $\chi$ and constant terms of odd-even expressions are getting greater and greater concurrently along with the continuation of operations, yet variable $\chi$ expresses 0 and natural numbers $\geq 1$ throughout, no matter which variable $\chi$ represents.

As is well-known, all kinds of $15+12c/19+12c$ are embodied at itself of $15+12c/19+12c$ collectively, nothing but pass the evaluations of $c$ to distinguish each of them. In addition to this, you ought to notice that each and every kind of $15+12c/19+12c$ can be expressed by an integer’s expression which contains variable $\chi$, for examples, hereinabove listed $51+48e$ i.e. $15+12c$ with $c=4e+3$, and $19+48e$ i.e. $19+12c$ with $c=4e$.

Once an emerging integer’s expression whose coefficient of $\chi$ and constant term are respectively smaller than the coefficient of $\chi$ and the constant term of a kind of $15+12c/19+12c$ appears at an operational route of $15+12c/19+12c$, then it means that the kind of $15+12c/19+12c$ is
proved to suit the conjecture according to the axiom.

If each of many integer’s expressions which contain variable \( \chi \) at operational routes of \( 15+12c/19+12c \) is smaller than a kind of \( 15+12c/19+12c \), then foregoing derivative kinds of \( 15+12c/19+12c \) which suit the conjecture have smaller coefficients of \( \chi \) and constant terms in most instances, relative to tail derivative kinds of \( 15+12c/19+12c \).

Therefore, after variable \( \chi \) of foregoing derivative kinds of \( 15+12c/19+12c \) which suit the conjecture is evaluated with 0 and 1, 2, 3, etc smaller natural numbers, we can get some smaller concrete positive odd numbers of \( 15+12c/19+12c \). Of course, these smaller concrete positive odd numbers of \( 15+12c/19+12c \) suit the conjecture too.

Let us respectively give concrete odd numbers after foregoing 6 evaluations of foresaid several kinds of \( 15+12c/19+12c \) plus individually operated odd numbers to explain the proposition. Foresaid 6 kinds of \( 15+12c \) after foregoing 6 evaluations of \( \chi \) are listed below.

<table>
<thead>
<tr>
<th>( \chi ):</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>51+48e:</td>
<td>51</td>
<td>99</td>
<td>147</td>
<td>195</td>
<td>243</td>
<td>291</td>
</tr>
<tr>
<td>75+96f:</td>
<td>75</td>
<td>171</td>
<td>267</td>
<td>363</td>
<td>459</td>
<td>555</td>
</tr>
<tr>
<td>87+96f:</td>
<td>87</td>
<td>183</td>
<td>279</td>
<td>375</td>
<td>471</td>
<td>567</td>
</tr>
<tr>
<td>315+384h:</td>
<td>315</td>
<td>699</td>
<td>1083</td>
<td>1467</td>
<td>1851</td>
<td>2235</td>
</tr>
<tr>
<td>135+384h:</td>
<td>135</td>
<td>519</td>
<td>903</td>
<td>1287</td>
<td>1671</td>
<td>2055</td>
</tr>
<tr>
<td>15+384h:</td>
<td>15</td>
<td>399</td>
<td>783</td>
<td>1167</td>
<td>1551</td>
<td>1935</td>
</tr>
</tbody>
</table>
As listed above, from small to large odd numbers of $15+12c$ under positive integer 200 have 15, 51, 75, 87, 99, 135, 147, 171, 183 and 195. From small to large odd numbers of $15+12c$ under positive integer 200 are 15, 27, 39, 51, 63, 75, 87, 99, 111, 123, 135, 147, 159, 171, 183 and 195, therein underlined odd numbers are absent odd numbers in listed above 6 kinds of $15+12c$.

Here, the reason that absent a few of odd numbers of $15+12c$ under integer 200, is due to be unable to show continued operations at operational routes of $15+12c$, because if do so, it will cause an overlong operational route.

Let us operate absent each odd number all alone to suit the conjecture and point out the belongingness of each of them.

From $27 \rightarrow 82 \rightarrow 41 \rightarrow 124 \rightarrow 62 \rightarrow 31 \& \rightarrow 94 \rightarrow 47 \rightarrow 142 \rightarrow 71 \rightarrow 214 \rightarrow 107 \rightarrow 322 \rightarrow 161 \rightarrow 484 \rightarrow 242 \rightarrow 121 \rightarrow 364 \* \rightarrow 182 \rightarrow 91 \# \rightarrow 274 \rightarrow 137 \rightarrow 412 \rightarrow 206 \rightarrow 103 \## \rightarrow 310 \rightarrow 155 \rightarrow 466 \rightarrow 233 \rightarrow 700 \rightarrow 350 \rightarrow 175 \### \rightarrow 526 \rightarrow 263 \rightarrow 790 \rightarrow 395 \rightarrow 1186 \rightarrow 593 \rightarrow 1780 \rightarrow 890 \rightarrow 445 \rightarrow 1336 \rightarrow 668 \rightarrow 334 \* \rightarrow 167 \rightarrow 502 \rightarrow 251 \rightarrow 754 \rightarrow 377 \rightarrow 1132 \rightarrow 566 \rightarrow 283 \rightarrow 850 \rightarrow 425 \rightarrow 1276 \rightarrow 638 \rightarrow 319 \rightarrow 958 \rightarrow 479 \rightarrow 1438 \rightarrow 719 \rightarrow 2158 \rightarrow 1079 \rightarrow 3238 \rightarrow 1619 \rightarrow 4858 \rightarrow 2429 \rightarrow 7288 \rightarrow 3644 \rightarrow 1822 \* \rightarrow 911 \rightarrow 2734 \rightarrow 1367 \rightarrow 4102 \rightarrow 2051 \rightarrow 6154 \rightarrow 3077 \rightarrow 9232 \rightarrow 4616 \rightarrow 2308 \rightarrow 1154 \rightarrow 577 \rightarrow 1732 \rightarrow 866 \rightarrow 433 \rightarrow 1300 \rightarrow 650 \rightarrow 325 \rightarrow 976 \rightarrow 488 \rightarrow 244 \rightarrow 122 \rightarrow 61 \rightarrow 184 \rightarrow 92 \rightarrow 46 \rightarrow 23 \* \rightarrow 27$, get that 27 suits the conjecture according to the axiom. Also, odd number 27
belongs within $27+2^{59} \times 3y$.

In addition, several signs except arrowheads at the operational route of 27 will be applied by latter operations.

From $39 \rightarrow 118 \rightarrow 59 \rightarrow 178 \rightarrow 89 \rightarrow 268 \rightarrow 134 \rightarrow 67 \rightarrow 202 \rightarrow 101 \rightarrow 304 \rightarrow 152 \rightarrow 76 \rightarrow 38 < 39$, get that 39 suits the conjecture according to the axiom.
Also, odd number 39 belongs within $39+2^8 \times 3k$.

From $63 \rightarrow 190 \rightarrow 95 \rightarrow 286 \rightarrow 143 \rightarrow 430 \rightarrow 215 \rightarrow 646 \rightarrow 323 \rightarrow 970 \rightarrow 485 \rightarrow 1456 \rightarrow 728 \rightarrow 364^* \rightarrow$ connect to preceding operational route of $27 \rightarrow \ldots \rightarrow 61 < 63$, get that 63 suits the conjecture according to the theorem. Also odd number 63 belongs within $63+2^{54} \times 3w$.

From $111 \rightarrow 334^{**} \rightarrow$ connect to preceding operational route of $27 \rightarrow \ldots \rightarrow 61 < 111$, get that 111 suits the conjecture according to the theorem. Also odd number 111 belongs within $111+2^{31} \times 3q$.

From $123 \rightarrow 370 \rightarrow 185 \rightarrow 556 \rightarrow 278 \rightarrow 139 \rightarrow 418 \rightarrow 209 \rightarrow 628 \rightarrow 314 \rightarrow 157 \rightarrow 472 \rightarrow 236 \rightarrow 118 < 123$, get that 123 suits the conjecture according to the axiom. Also odd number 123 belongs within $123+2^8 \times 3k$.

From $159 \rightarrow 478 \rightarrow 239 \rightarrow 718 \rightarrow 359 \rightarrow 1078 \rightarrow 539 \rightarrow 1618 \rightarrow 809 \rightarrow 2428 \rightarrow 1214 \rightarrow 607 \rightarrow 1822^{***} \rightarrow$ connect to preceding operational route of $27 \rightarrow \ldots \rightarrow 122 < 159$, get that 159 suits the conjecture according to the theorem. Also odd number 159 belongs within $159+2^{21} \times 3s$.

Foresaid 6 kinds of 19+12c after foregoing 6 evaluations of $\chi$ are listed below.
\[
\chi: \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5
\]

19+48e: 19, 67, 115, 163, 211, 259
43+96f: 43, 139, 235, 331, 427, 523
55+96f: 55, 151, 247, 343, 449, 535
187+384h: 187, 571, 955, 1339, 1723, 2107
271+384h: 271, 655, 1039, 1423, 1807, 2191
295+384h: 295, 679, 1063, 1447, 1831, 2215

As listed above, from small to large odd numbers of 19+12c under positive integer 200 have 19, 43, 55, 67, 115, 139, 163 and 187.

From small to large odd numbers of 19+12c under positive integer 200 are 19, 31, 43, 55, 67, 79, 91, 103, 115, 127, 139, 151, 163, 175, 187 and 199, therein underlined odd numbers are absent odd numbers in listed above 6 kinds of 19+12c.

Here, the reason that absent a few of odd numbers of 19+12c under positive integer 200, is due to be unable to show continued operations at operational routes of 19+12c, because if do so, it will cause an overlong operational route. So, we operate each of them all alone to suit the conjecture, and point out the belongingness of each of them.

From 31 — connect to preceding operational route of 27 →… →23< 31, get that 31 suits the conjecture according to the axiom. Also odd number 31 belongs within 31+2^56×3x.

From 79→238→119→358→179→538→269→808→404→202←67<79,
get that 79 suits the conjecture according to the theorem. Also odd number 79 belongs within $79 + 2^{5} \times 3f$.

From 91 — connect to preceding operational route of $27 \rightarrow \ldots \rightarrow 61 < 91$, get that 91 suits the conjecture according to the axiom. Also odd number 91 belongs within $91 + 2^{45} \times 3v$.

From 103 — connect to preceding operational route of $27 \rightarrow \ldots \rightarrow 61 < 103$, get that 103 suits the conjecture according to the axiom. Also odd number 103 belongs within $103 + 2^{42} \times 3u$.

From 151 — connect to preceding operational route of $27 \rightarrow \ldots \rightarrow 61 < 151$, get that 151 suits the conjecture according to the axiom. Also odd number 151 belongs within $151 + 2^{5} \times 3f$.

From 175 — connect to preceding operational route of $27 \rightarrow \ldots \rightarrow 61 < 175$, get that 175 suits the conjecture according to the axiom. Also odd number 175 belongs within $175 + 2^{8} \times 3k$.

Variables $y$, $x$, $k$, $w$, $q$, $s$, $f$, $v$ and $u$ within above-mentioned integer’s expressions, each of them expresses 0 and natural numbers $\geq 1$ likewise.

Above-deduced $27 + 2^{59} \times 3y$, $39 + 2^{8} \times 3k$, $31 + 2^{56} \times 3x$, $63 + 2^{54} \times 3w$, $79 + 2^{5} \times 3f$, $91 + 2^{45} \times 3v$, $103 + 2^{42} \times 3u$, $111 + 2^{31} \times 3q$, $123 + 2^{8} \times 3k$, $159 + 2^{21} \times 3s$, $151 + 2^{5} \times 3f$, $175 + 2^{8} \times 3k$, $199 + 2^{8} \times 3k$ is respectively a kind of $15 + 12c$ plus $19 + 12c$ too.
Whether derivative a kind of $15+12c/19+12c$ come from either bunch of operational routes, or deduced a kind of $15+12c/19+12c$ come from a single operational route, it is possessed of coequal qualification as one another’s irreplaceable a kind of $15+12c/19+12c$.

By this token, after variable $\chi$ of foregoing emerging kinds of $15+12c/19+12c$ is bestowed with 0 and 1, 2, 3, etc smaller natural numbers, not only can get some smaller concrete positive odd numbers of $15+12c/19+12c$, but also can get smaller concrete consecutive positive odd numbers of $15+12c/19+12c$, such as consecutive positive odd numbers of $15+12c/19+12c$ under positive integer 200. Without doubt, these consecutive positive odd numbers of $15+12c/19+12c$ suit the conjecture like foregoing emerging kinds of $15+12c/19+12c$.

Thereinafter, let us quote aforementioned concerned conclusions once more to prove further that odd numbers of $15+12c/19+12c$ between $n$ and $2n+1$ suit the conjecture.

Generally speaking, $15+12c/19+12c$ between $n$ and $2n+1$ are concrete smaller positive odd numbers, that is to say, each of them is the front-end or smaller odd number of a kind of $15+12c/19+12c$ because large or small $15+12c/19+12c$ depend exactly their own variable $c$ like the ordinal.

Moreover, $15+12c/19+12c$ embodied every kind of itself intensively, and that operations of $15+12c/19+12c$ proceed endlessly.

On the one hand, ceaselessly stop operations of part branches at
operational routes of 15+12c/19+12c, because the coefficient of $\chi$ and the constant term of an integer’s expression at each such branch are smaller than the coefficient of $\chi$ and the constant term of a kind of 15+12c/19+12c respectively, and derive the kind of 15+12c/19+12c from the integer’s expression to suit the conjecture according to the axiom.

For frontally derived kinds of 15+12c/19+12c, since their coefficients of $\chi$ and constant terms are relatively smaller, so after their variable $\chi$ is evaluated with 0 and 1, 2, 3 etc smaller natural numbers, hereby get concrete smaller odd numbers of 15+12c/19+12c. Naturally each such odd number suits the conjecture.

On the other hand, constant terms and coefficients of $\chi$ of integer’s expressions at operational routes of 15+12c/19+12c are getting greater and greater along with the continuation of operations, accordingly constant terms and coefficients of $\chi$ of derived kinds of 15+12c/19+12c from these integer’s expressions are getting greater and greater too.

Since odd numbers of 15+12c/19+12c between n and 2n+1 are finite, and each of them belongs within a kind of 15+12c/19+12c, additionally any two kinds of 15+12c/19+12c have not an identical constant term.

Then, after operations go beyond some limits, cause that a constant term of every derived kind of 15+12c/19+12c which suits the conjecture is not smaller than 2n+1. That is to say, even if let their variable $\chi$ be equal to 0, smallest odd number of each such kind of 15+12c/19+12c is not smaller
than $2n+1$ either. In this situation, an integer’s expression on the terminal of each and every branch which stopped already operations at operational routes of $15+12c/19+12c$ is smaller than a kind of $15+12c/19+12c$. We can obtain each and every odd number of $15+12c/19+12c$ between $n$ and $2n+1$ from odd numbers after variable $\chi$ of derived some kinds of $15+12c/19+12c$ from such integer’s expressions is evaluated with 0 and 1, 2, 3, etc. smaller natural numbers. Go without saying, got odd numbers like so suit the conjecture.

To state succinctly, any odd number of $15+12c/19+12c$ between $n$ and $2n+1$ is able to pass operations of $15+12c/19+12c$ by the leftward operational rule to first get an integer’s expression which is smaller than a kind of $15+12c/19+12c$ which contains the odd number, then the kind of $15+12c/19+12c$ is proved to suit the conjecture according to the axiom. After that, pass the evaluations of variable $\chi$ of the kind of $15+12c/19+12c$ to obtain such an odd number, so the odd number is proven to suit the conjecture like the kind of $15+12c/19+12c$.

Follow such a set pattern, each and every odd number of $15+12c/19+12c$ between $n$ and $2n+1$ is proven to suit the conjecture.

Excepting the above-mentioned deductive inference, we also can apply straightway the theorem or the deduction to prove odd number of $15+12c/19+12c$ between $n$ and $2n+1$ to suit the conjecture. Comparatively speaking, this method is rather too simplicity, vide infra.
We known that $15+12c/19+12c$ have infinite-many strips of operational routes, and that we can regard them as which begin with $15+12c/19+12c$, or regard them as infinite-many branches of the bunch of operational routes. When regard them as which begin with $15+12c/19+12c$, all operational routes of $15+12c/19+12c$ intersect at $35+27c/44+27c$. When regard them as infinite-many branches of the bunch of operational routes, all operational routes of $15+12c/19+12c$ are at indirect concatenations.

Also known that $15+12c/19+12c$ have infinite-many kinds, nothing but where their each operational route extends to an integer’s expression which is smaller than a kind of $15+12c/19+12c$, the kind of $15+12c/19+12c$ is educed by the integer’s expression just.

By this token, each and every operational route of $15+12c/19+12c$ implies the emergence of a kind of $15+12c/19+12c$, and that there is affirmatively an integer’s expression which is smaller than a kind of $15+12c/19+12c$ at every operational route of $15+12c/19+12c$ from preceding analyses.

Hereinbefore, we have operated out certain of satisfactory integer’s expressions whose each is smaller than a kind of $15+12c/19+12c$ at stopped operational routes of $15+12c/19+12c$, for examples, $29+27e$, $64+82f$, $11+27e$ and $37+81f$ etc, and that these smaller integer’s expressions suit the conjecture like derivative kind of $15+12c/19+12c$ too.

Consequently, unsighted those satisfactory integer’s expressions whose each is smaller than a kind of $15+12c/19+12c$ at better operational routes of
15+12c/ 19+12c suit the conjecture certainly according to the theorem or the inference.

Accordingly derived kinds of 15+12c/19+12c from unsighted those satisfactory integer’s expressions whose each is smaller than a kind of 15+12c/19+12c suit the conjecture, according to the axiom.

After variable $\chi$ of foregoing derivative kinds of 15+12c/ 19+12c is evaluated with 0 and 1, 2, 3, etc. smaller natural numbers, we can get all odd numbers of 15+12c/ 19+12c between n and 2n+1. Beyond doubt, odd numbers got by this way suit the conjecture.

Altogether, we have proven that odd numbers between n and 2n+1 suit the conjecture by the leftward operational rule, so they all exist at the bunch of integers’ chains.

To sum up, we have proven that all even numbers and all odd numbers between n and 2n+1 exist at a bunch of integers’ chains extended by two-way operational rules. Namely all positive integers between n and 2n+1 are proven by us to suit the conjecture.

Thus far, we have proven that positive integers $\leq 2^1n$ suit the conjecture by consecutive positive integers $\leq n$, like that, we can too prove that positive integers $\leq 2^2n$ suit the conjecture by consecutive positive integers $\leq 2^1n$ according to the foregoing way of doing.

At the beginning of the proof, we have spoken that divide limits of all consecutive positive integers into segments according to greatest positive
integer $2^X n$ per segment, where $X \geq 0$, and $n \geq 24$.

After we proven that positive integers between $2^{X-1} n$ and $2^X n$ suit the conjecture by consecutive proven positive integers $\leq 2^{X-1} n$, in the same old way, we likewise are able to prove that positive integers between $2^X n$ and $2^{X+1} n$ suit the conjecture by consecutive proven positive integers $\leq 2^X n$.

For greatest positive integer $2^X n$ at each segment, $X$ begins with 0, next, it is evaluated with 1, 2, 3, etc. natural numbers in proper order. Along with which values of $X$ are getting greater and greater, consecutive positive integers $\leq 2^X n$ are getting more and more, and that emerging positive integers are getting greater and greater. If $X$ is equal to 0 plus every natural number, then all positive integers are proven to suit the conjecture, namely every positive integer is proven to suit the conjecture.

Heretofore, the Collatz conjecture is proven by us at long last integrally. The proof was thus brought to a close. As a consequence, the Collatz conjecture holds water.