The Omega Constant : Part 1

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Abstract

In this note we show some formulas for omega constant

Introduction

La constante omega , irracional y trascendental, se define por la ecuación:

$$\Omega e^{\Omega} = 1$$

Lo que implica que:

$$\Omega = e^{-e^{-e^{-\cdots}}} = \left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)^{\cdots}}}$$

$$\Omega = 0.56714329040978 \ldots$$

En esta nota mostramos fórmulas que involucran a la constante Ω .

Keywords: constante omega,recurrencias,series,integrales,función de Lambert.

Formulas

1. Recurrencias

(1) $$x_{n+1} = e^{-x_n} \;, x_1 = 0 \;, x_n \to \Omega$$

(2) $$x_{n+1} = \frac{1 + x_n}{1 + e^{x_n}} \;, x_1 = 0 \;, x_n \to \Omega$$

(3) $$x_{n+1} = \frac{1}{3} x_n + \frac{2}{3} e^{-x_n} \;, x_1 = 0 \;, x_n \to \Omega$$

(4) $$x_{n+1} = \frac{x_n^2 + e^{-x_n}}{1 + x_n} \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(5) $$x_{n+1} = \frac{x_n(1 - \ln x_n)}{1 + x_n} \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(6) $$x_{n+1} = \sqrt{-x_n \ln x_n} \ \ , x_1 = \frac{1}{2} \ \ , x_n \to \Omega$$

(7) $$x_{n+1} = \sqrt{\frac{x_n}{\ln x_n}} \ \ , x_1 = \frac{7}{4} \ \ , x_n \to \Omega^{-1}$$

(8) $$x_{n+1} = \frac{3x_n^2 + 2x_n^4 + (2 + 8x_n + 4x_n^2)e^{-x_n} - (2 + x_n)e^{-2x_n}}{2(1 + x_n)^3} \ \ , \qquad x_1 = \frac{1}{2} \ \ , x_n \to \Omega$$

(9) $$x_{n+1} = \frac{1}{2\cosh x_n - x_n} \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(10) $$x_{n+1} = \frac{1 + x_n^2}{2\cosh x_n} \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(11) $$x_{n+1} = \sqrt{\frac{1 - \tanh x_n}{1 + \tanh x_n}} \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(12) $$x_{n+1} = \frac{1}{3}\left(\ln 2 - \frac{1}{2}\right) + \frac{1}{3}(2x_n - \ln(1 + 2x_n)) \ \ , x_1 = 0 \ \ , x_n \to \Omega - \frac{1}{2}$$

(13) $$x_{n+1} = \frac{1}{2} + \frac{1}{2}(x_n + \ln(1 - x_n)) \ \ , x_1 = 0 \ \ , x_n \to 1 - \Omega$$

(14) $$x_{n+1} = \tanh\left(\frac{1 - x_n}{2 + 2x_n}\right) \ \ , x_1 = 0 \ \ , x_n \to \frac{1 - \Omega}{1 + \Omega}$$

(15) $$x_{n+1} = \frac{1}{2} + \frac{1}{2}(x_n - 1 + e^{-x_n}) \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(16) $$x_{n+1} = \frac{1}{4}(4x_n + e - {x_n}^{x_n}) \ \ , x_1 = \frac{7}{4} \ \ , x_n \to \Omega^{-1}$$

(17) $$x_{n+1} = \frac{(2e^{-1/2} - 1)x_n}{(1 + 2x_n)e^{x_n} - 1} \ \ , x_1 = \frac{1}{20} \ \ , x_n \to \Omega - \frac{1}{2}$$

(18) $$x_{n+1} = \sinh^{-1}(\cosh x_n - x_n) \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(19) $$x_{n+1} = \frac{1 - x_n(2 - e^{1/2}) - x_n e^{x_n}}{e^{1/2} - 2x_n e^{x_n}} \ \ , x_1 = 1 \ \ , x_n \to \Omega$$

(20) $$x_{n+1} = \frac{1 + x_n(e - 1) - x_n e^{x_n}}{e - x_n e^{x_n}} \ \ , x_1 = 0 \ \ , x_n \to \Omega$$

(21) $$x_{n+1} = \frac{1}{3}\left(x_n + e^{-x_n} + e^{-e^{-x_n}}\right) \ , x_1 = 0 \, , x_n \to \Omega$$

(22) $$x_{n+1} = \frac{x_n \ \pi}{4 \ \tan^{-1}(x_n e^{x_n})} \ , x_1 = \frac{1}{2} \ , x_n \to \Omega$$

(23) $$x_{n+1} = x_n - \frac{5}{8} + \frac{1}{8}\sqrt{57 - 32 x_n e^{x_n}} \ , x_1 = \frac{1}{2} \, , x_n \to \Omega$$

(24) $$x_{n+1} = \sqrt{x_n \ e^{-x_n}} \ , x_1 = 1 \, , x_n \to \Omega$$

(25) $$x_{n+1} = \frac{2 + 4x_n + x_n^2 + x_n^3 \ e^{x_n}}{2 + x_n + (2 + 2x_n + x_n^2)e^{x_n}} \ , x_1 = 0 \, , x_n \to \Omega$$

(26) $$x_{n+1} = 1 - \frac{1}{e} e^{x_n} \ , x_1 = \frac{1}{2} \, , x_n \to 1 - \Omega$$

(27) $$x_{n+1} = \sqrt{x_n (1 + \ln(1 - x_n))} \ , x_1 = \frac{1}{2} \, , x_n \to 1 - \Omega$$

Algunas de las recurrencias más eficientes son: (2),(4),(5),(8),(25).

2. Integrales

(28) $$\pi = 2 \int_0^\infty \frac{\cos x}{x^2 + \Omega^2} dx$$

(29) $$\pi \ \Omega^{k-1} = 2 \int_0^\infty \frac{\cos(k \ x)}{x^2 + \Omega^2} dx \ , k \in \mathbb{N} \cup \{0\}$$

(30) $$\pi \ \Omega = 2 \int_0^\infty \frac{x \sin x}{x^2 + \Omega^2} dx$$

(31) $$\pi = 4 \int_0^\infty \frac{\Omega^{-x} \sin(\Omega \ x)}{x} dx$$

(32) $$\sqrt{\pi} = 2 \int_0^\infty e^{-\left((\Omega x)^2 + (2x)^{-2}\right)} dx$$

(33) $$\gamma = \Omega - \Omega \int_0^\infty \Omega^x \ln x \, dx$$

(34) $$\gamma = 1 + \Omega - \Omega^2 \int_0^\infty \Omega^x x \ln x \, dx$$

(35) $$\gamma = \Omega + \int_0^\Omega \frac{1 - \cos x}{x} dx - \int_\Omega^\infty \frac{\cos x}{x} dx$$

En las fórmulas (33),(34),(35):

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = 0.5772 \ldots$$

(36) $$\pi = 4 \int_\Omega^1 \frac{1 - \ln x}{x^2 + (\ln x)^2} dx$$

(37) $$\int_0^{2\pi} \frac{e^{2ix + e^{ix}}}{e^{ix} - \Omega} \, dx = 2\pi$$

(38) $$\Omega = 1 - exp\left(-\frac{i}{2\pi} \int_0^\infty \frac{1}{1+x} \ln\left(\frac{x - \ln x - i\pi}{x - \ln x + i\pi}\right) dx\right)$$

(39) $$\Omega = 1 - exp\left(-\frac{1}{\pi} \int_0^\infty \frac{1}{1+x} \tan^{-1}\left(\frac{\pi}{x - \ln x}\right) dx\right)$$

(40) $$\frac{1}{1+\Omega} = \int_{-\infty}^\infty \frac{1}{(e^x - x)^2 + \pi^2} dx$$

(41) $$\frac{2\pi}{1+\Omega} = \int_0^{2\pi} \frac{e^{ix}}{e^{ix} - e^{-e^{ix}}} dx$$

(42) $$\frac{2\pi}{1+\Omega} = \int_0^{2\pi} \frac{1 - e^{-\cos x}\cos(x + \sin x)}{1 - 2e^{-\cos x}\cos(x + \sin x) + e^{-2\cos x}} dx$$

(43) $$\frac{\pi}{1+\Omega} = \int_0^{\pi} \frac{1 - e^{-\cos x}\cos(x + \sin x)}{1 - 2e^{-\cos x}\cos(x + \sin x) + e^{-2\cos x}} dx$$

(44) $$\frac{2\pi}{1+\Omega} = \int_0^{2\pi} \frac{1 - e^{-\sin x}\sin(x - \cos x)}{1 - 2e^{-\sin x}\sin(x - \cos x) + e^{-2\sin x}} dx$$

(45) $$\frac{\Omega}{1+\Omega} = \frac{1}{\pi}\int_0^{\pi} \frac{\sin x}{\sin x + x\, e^{-x/\tan x}}\, dx$$

(46) $$\Omega = \frac{1}{\pi}\int_0^{\pi} \ln\left(1 + \frac{e^{x/\tan x}\ \sin x}{x}\right) dx$$

(47) $$\Omega = 1 - \ln \pi + \frac{1}{\pi}\int_0^{\pi} \ln\left(x + e^{x/\tan x} \sin x\right) dx$$

(48) $$\frac{1}{1+\Omega} = \frac{1}{\pi}\int_0^{\pi} \frac{x}{x + e^{x/\tan x} \sin x}\, dx$$

(49) $$\Omega^{-1} = 1 + \frac{1}{\pi}\int_0^{\pi} \frac{(x \sin x)^2 + (\sin x - x\cos x)^2}{x(x + e^{x/\tan x} \sin x)}\, dx$$

(50) $$\Omega = \int_0^{1} \frac{x-1}{\pi^2 + (x - \ln x)^2}\, dx + \frac{2}{\pi}\int_0^{\pi} \frac{\cos(x/2) + x\sin(3x/2)}{1 + 2x\sin x + x^2} \cos(x/2)\, dx$$

(51) $$\Omega = \frac{\int_0^\infty \frac{x}{(1+x^2)(\pi^2 + (x - \ln x)^2)}\, dx}{\frac{2}{2+\pi} - \int_0^\infty \frac{1}{(1+x^2)(\pi^2 + (x - \ln x)^2)}\, dx}$$

(52) $$\Omega = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{2ix}\left(1 + e^{-e^{ix}}\right)}{e^{ix} - e^{-e^{ix}}}\, dx$$

(53) $$\Omega = \frac{1}{8\pi}\int_0^{2\pi} \frac{\left(1 + e^{ix}\right)\left(3 + e^{ix}\right)e^{ix}}{1 + e^{ix} - 2e^{-\left(1+e^{ix}\right)/2}}\, dx$$

(54) $$\frac{\pi\Omega^2}{2(1+\Omega)} = \int_0^{2\pi} \frac{e^{ix}}{2e^{2e^{ix}} - e^{-ix}}\, dx$$

(55) $$\frac{2\pi e^{-1}}{1+\Omega} = \int_0^{2\pi} \frac{e^{ix}}{(1+e^{ix})^{1+e^{ix}} - e}\, dx$$

(56) $$2\pi\Omega = \int_0^{2\pi} \frac{e^{-\cos x}(\cos(\sin x) - \Omega\cos(x - \sin x))}{1 - 2\Omega\cos x + \Omega^2}\, dx$$

(57) $$\Omega = \frac{2}{\pi}\int_0^{\pi/2} \frac{(x^2+(1+x\tan x)^2)xe^{x\tan x}\sin x}{(\cos x)^2+x^2e^{2x\tan x}}dx$$

(58) $$\Omega = 1 - exp\left(-\int_0^{\infty} \frac{(x-1)\ln(1+x)}{x(\pi^2+(x-\ln x)^2)}dx\right)$$

(59) $$\Omega = \frac{1}{2\pi}\int_0^{\pi} \ln(1+e^{-2\cos x}-2e^{-\cos x}\cos(x+\sin x))\,dx$$

(60) $$\Omega = \frac{1}{\pi}\int_0^1 \frac{1+e^x(\cos 1+(1+x+x^2)\sin 1)}{1+e^x(2\sin 1-2x\cos 1)+(1+x^2)e^{2x}}dx + \frac{1}{2\pi}\int_{-1}^{1} \frac{A+Bx^2-(C+Dx^2)x^2\cos x+ex^2\cos 2x+(E+Dx^2+2e\cos x)x\sin x}{(1+2x\sin x+x^2)\big(1+2e(x\sin x-\cos x)+e^2(1+x^2)\big)}dx$$

donde

(61) $A=-1+e^2, B=-2+e+e^2, C=1-3e+e^2, D=-e+e^2, E=-3+e+e^2$

(62) $$\Omega = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{e^{2\cos x}\cos x-\cos 2x+2e^{\cos x}\sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}+\sin x\right)}{1-2e^{\cos x}\cos(x+\sin x)+e^{2\cos x}}dx - \frac{1}{2\pi}\int_{-1}^{1} \frac{x(x+x\cos x+\sin x)}{1+2x\sin x+x^2}dx$$

(63) $$\Omega = -\frac{1}{\pi}+\frac{1}{2\pi}\int_{-1}^{1}\left(\frac{i-x}{i+xe^{ix}}+\frac{(1+2ix)(-1-ix+x^2)\big(e+e^{-ix+x^2}\big)}{e(-1-ix+x^2)+e^{-ix+x^2}}\right)dx$$

3. Fórmulas con arcotangentes

(64) $$\pi = 4\ \tan^{-1}(\Omega)+4\ \tan^{-1}\left(\tanh\frac{\Omega}{2}\right)$$

(65) $$\pi = 4\ \tan^{-1}(\Omega)+4\ \tan^{-1}\left(\tanh\frac{\Omega}{4}\right)+4\ \tan^{-1}\left(\frac{\sqrt{\Omega}-\Omega}{1+\Omega\sqrt{\Omega}}\right)$$

Sea $x_1 = 1$, $x_{n+1} = e^{-x_n}$,entonces:

(66) $$\pi = 4\tan^{-1}(e^{-x_n})+4\sum_{k=1}^{n}\tan^{-1}\left(\frac{x_k-e^{-x_k}}{1+x_k\ e^{-x_k}}\right)\ ,\qquad n\in\mathbb{N}$$

4. Producto infinito para la constante omega

(67) $$\Omega = e^{-1} \cdot e^{1-e^{-1}} \cdot e^{e^{-1}-e^{-e^{-1}}} \cdot e^{e^{-e^{-1}}-e^{-e^{-e^{-1}}}} \cdots = \prod_{n=1}^{\infty} p_n$$

donde

(68) $$p_n = e^{x_n - x_{n+1}} \ , x_{n+1} = e^{-x_n} \ , x_1 = 0$$

5. Series

(69) $$\pi = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \Omega^{2n+1} \left(2 + \frac{E_n}{(2n)!}\right)$$

$\{E_n : n \in \mathbb{N} \cup \{0\}\} = \{1,1,5,61,1385,\dots\}$, números de Euler

(70) $$e = 4 \sum_{n=0}^{\infty} (-2)^n \left(\Omega - \frac{1}{2}\right)^n \sum_{k=0}^{n} \frac{k+1}{(n-k)!} = 4 \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \left(\Omega - \frac{1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} (k+1)!$$

(71) $$e^{-1} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{2^n (n^2 + n + 1)}{n!} \left(\Omega - \frac{1}{2}\right)^n$$

(72) $$e = \sum_{n=0}^{\infty} (1-\Omega)^n \sum_{k=0}^{n} \frac{1}{k!}$$

(73) $$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n (1-\Omega)^n (n+1)}{n!}$$

(74) $$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n c_n f_n$$

donde

(75) $$c_0 = 1, c_1 = 0, c_n = -\sum_{k=2}^{n} \frac{2^{k-2}}{3(k-2)!} c_{n-k} \ , n \geq 2$$

(76) $$c_n = \left\{1,0,-\frac{1}{3},-\frac{2}{3},-\frac{5}{9},0,\frac{17}{27},\frac{118}{135},\frac{203}{405},\dots\right\}$$

(77) $$f_n = \gamma(n+2,-\Omega) - \gamma(n+1,-\Omega)$$

(78) $$\gamma(n,-\Omega) = \int_{0}^{-\Omega} t^{n-1} e^{-t} \, dt = (n-1)! \left(1 - e^{\Omega} \sum_{k=0}^{n-1} \frac{(-\Omega)^k}{k!}\right) , n \in \mathbb{N}$$

(79) $$\ln 2 = \frac{1}{2} + 3\left(\Omega - \frac{1}{2}\right) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 2^n}{n} \left(\Omega - \frac{1}{2}\right)^n$$

6. Series (base 2)

Sean

(80) $$f(x) = xe^x, s_n = \sum_{k=1}^{n} 2^{-k}\, y(k)\, , n \in \mathbb{N}, y(n) = \begin{cases} 1 & n = 1 \\ 1 & f(s_{n-1} + 2^{-n}) < 1 \\ 0 & f(s_{n-1} + 2^{-n}) > 1 \end{cases}$$

Entonces:

(81) $$y(n) = \{1,00,1,0,0,0,1,0,0,1,1,0, \dots\}$$

(82) $$\Omega = \sum_{k=1}^{\infty} 2^{-k}\, y(k)$$

Sean

(83) $$f(x) = xe^x, s_n = \sum_{k=1}^{n} 2^{-m(k)}\ , n \in \mathbb{N}, m(n) = \begin{cases} 1 & n = 1 \\ 1 & f(s_{n-1} + 2^{-n}) < 1 \\ \infty & f(s_{n-1} + 2^{-n}) > 1 \end{cases}$$

Entonces

(84) $$m(n) = \{1, \infty, \infty, 4, \infty, \infty, \infty, 8, \infty, \infty, 11,12, \dots\}$$

(85) $$\Omega = \sum_{k=1}^{\infty} 2^{-m(k)}$$

De las fórmulas anteriores se deduce el siguiente producto infinito:

(86) $$\Omega = \frac{1}{2} \cdot \frac{9}{8} \cdot \frac{145}{144} \cdot \frac{1161}{1160} \cdot \frac{2323}{2322} \cdot \frac{148673}{148672} \cdots$$

El producto infinito anterior se puede representar como sigue:

(87) $$\Omega = \frac{1}{2} \prod_{y(n)>1} \frac{y(n)}{y(n) - 1}$$

donde para $n \in \mathbb{N}$:

(88) $$y(n) = \begin{cases} [2^n \Omega] & si\ [2^n \Omega]\ es\ impar \\ 0 & en\ otro\ caso \end{cases}$$

(89) $$y(n) = \{1,0,0,9,0,0,0,145,0,0,1161,2323,0,0, \dots\}$$

7. Desigualdades para omega

(90) $$x_{n+1} = e^{-x_n} \;, x_1 = 0 \;\Rightarrow x_n = \left\{0,1, e^{-1}, e^{-e^{-1}}, \dots\right\}$$

(91) $$x_{2n-1} < \Omega < x_{2n} \;\;, n \in \mathbb{N}$$

(92) $$0 < a < \Omega < b < 1 \Rightarrow e^{-1} < e^{-b} < \Omega < e^{-a} < 1$$

(93) $$0 < a < \Omega < b < 1 \Rightarrow \frac{a + e^{-b}}{2} < \Omega < \frac{e^{-a} + b}{2}$$

(94) $$\Omega < a < 1 \Rightarrow e^{-a} < \Omega < e^{-e^{-a}}$$

(95) $$0 < a < \Omega \Rightarrow e^{-e^{-a}} < \Omega < e^{-a}$$

8. Algunas relaciones con polinomios

Sea $P_n(x)$ el polinomio definido como sigue:

(96) $$P_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} (n-k)!\, x^k - (n-1)! \;\;, n \in \mathbb{N}$$

son válidas las siguientes propiedades:

(97) $$P_n(0) < 0 \,, P_n(1) > 0 \,, n \in \mathbb{N}$$

(98) $$P_n'(x) > 0 \,, \forall x \in [0,1], n \in \mathbb{N}$$

(99) $$\exists!\; x_n \in [0,1] \text{ tal que } P_n(x_n) = 0$$

(100) $$x_1 = 1 \Leftrightarrow P_1(x_1) = 0$$

(101) $$x_{n+1} < x_n \,, \forall n \in \mathbb{N}$$

(102) $$x_n \to \Omega$$

Sea $Q_n(x)$ el polinomio definido como sigue:

(103) $$Q_n(x) = -n! + 2n!\, x + \sum_{k=2}^{n} \binom{n}{k} (n-k)!\, (k-1)!\, x^k \,, n \in \mathbb{N}$$

Son válidas las siguientes propiedades:

(104) $$Q_n(0) < 0 \,, Q_n(1) > 0 \,, n \in \mathbb{N}$$

(105) $$Q_n'(x) > 0 \,, \forall x \in [0,1], n \in \mathbb{N}$$

(106) $$\exists!\; x_n \in [0,1] \text{ tal que } Q_n(x_n) = 0$$

(107) $$x_1 = \frac{1}{2} \Leftrightarrow Q_1(x_1) = 0$$

(108) $$x_{n+1} < x_n \,, \forall\, n \in \mathbb{N}$$

(109) $$x_n \to 1 - \Omega$$

n	$x_n \,:\; P_n(x_n) = 0$	$x_n \,:\; Q_n(x_n) = 0$
1	1	0.5
2	0.6180339890…	0.4494897428…
3	0.5747430740…	0.4380261519…
4	0.5681469783…	0.4346298529…
5	0.5672540853…	0.4334976722…
6	0.5671536049…	0.4330959030…
7	0.5671441173…	0.4329478948…
8	0.5671433487…	0.4328920105…
9	0.5671432942…	0.4328705374…
10	0.5671432907…	0.4328621771…

Para $n \in \mathbb{N}$, sea $x_n \in (0,1)$ tal que:

(110) $$\left(1 - \frac{x_n}{n}\right)^n = x_n$$

Entonces se tiene:

(111) $$x_1 = \frac{1}{2} \,, x_n < x_{n+1} < \Omega \,, x_n \to \Omega$$

Para $n \in \mathbb{N}$, sea $x_n \in (0,1)$ tal que:

(112) $$x_n \left(1 + \frac{x_n}{n}\right)^n = 1$$

Entonces se tiene:

(113) $$x_1 = \frac{\sqrt{5} - 1}{2} \,, \Omega < x_{n+1} < x_n \,, x_n \to \Omega$$

Para $n \in \mathbb{N}$ sean u_n, v_n definidas por:

(114) $$u_n = \sqrt[\frac{n+1}{n}]{\frac{1}{n} + \sqrt[\frac{n+1}{n}]{\frac{1}{n} + \sqrt[\frac{n+1}{n}]{\frac{1}{n} + \cdots}}}$$

(115) $$v_1 = 2 \ , \qquad v_n = \frac{1}{n} + \sqrt[\frac{n}{n-1}]{\frac{1}{n} + \sqrt[\frac{n}{n-1}]{\frac{1}{n} + \sqrt[\frac{n}{n-1}]{\frac{1}{n} + \cdots}}} \ , n = 2,3,4, \ldots$$

Se tiene:

(116) $$u_n < u_{n+1} < \Omega^{-1} < v_{n+1} < v_n \ , n \in \mathbb{N}$$

9. Series para omega

(117) $$\Omega = \frac{1}{2} + \sum_{n=1}^{\infty} c_n \left(2e^{-1/2} - 1\right)^n$$

donde

(118) $$c_0 = \frac{1}{2} \ , c_1 = \frac{1}{3}$$

(119) $$c_{n+1} = -\frac{2}{3(n+1)}\left((n-1)c_n + \sum_{k=0}^{n-1}(k+1)c_{k+1}(c_{n-k-1} + c_{n-k})\right) , n = 1,2,3, \ldots$$

(120) $$\{c_n : n \geq 0\} = \left\{\frac{1}{2}, \frac{1}{3}, -\frac{5}{54}, \frac{1}{27}, -\frac{151}{8748}, \frac{862}{98415}, -\frac{4157}{885735}, \ldots\right\}$$

(121) $$\Omega = \sum_{n=1}^{\infty} c_n \, 2^{-n}$$

donde

(122) $$c_1 = 1 \ , c_{n+1} = \frac{1}{2(n+1)}\left(2nc_n - \sum_{k=0}^{n-1}(k+1)c_{k+1}c_{n-k}\right) , n = 1,2,3, \ldots$$

(123) $$c_n = \left\{1, \frac{1}{4}, \frac{1}{24}, -\frac{1}{192}, -\frac{13}{1920}, -\frac{47}{23040}, \ldots\right\}$$

(124) $$\Omega^{-1} = e^{1/2}\left(1 + \sum_{n=1}^{\infty} c_n \left(e^{-1/2} - \frac{1}{2}\right)^n\right)$$

donde

(125) $$c_1 = \frac{2}{3}\,, c_{n+1} = -\frac{2}{3(n+1)}\left((n-1)c_n + \sum_{k=0}^{n-1}(k+1)c_{k+1}c_{n-k}\right)\,, n = 1,2,3,\ldots$$

(126) $$c_n = \left\{\frac{2}{3}, -\frac{4}{27}, \frac{8}{81}, -\frac{184}{2187}, \frac{7952}{98415}, -\frac{73856}{885735}, \frac{335872}{3720087}, \ldots\right\}$$

(127) $$\Omega = \ln 2 - \sum_{n=1}^{\infty} c_n \frac{(2\ln 2 - 1)^n}{(1+\ln 2)^{2n-1}}$$

donde

(128) $$c_0 = -\frac{\ln 2}{1+\ln 2}\,, c_1 = \frac{1}{2}$$

(129) $$c_{n+2} = \frac{1}{2\ln 2\,(n+2)}\left((1+\ln 2)nc_{n+1} - \sum_{k=0}^{n}(k+1)c_{k+1}\left((1+\ln 2)^2 c_{n-k} - 2\ln 2\, c_{n-k+1}\right)\right)\,, n = 0,1,2,\ldots$$

(130) $$\{c_n : n \geq 0\} = \left\{-\frac{\ln 2}{1+\ln 2}, \frac{1}{2}, \frac{2+\ln 2}{8}, \frac{9 + 8\ln 2 + 2(\ln 2)^2}{48}, \frac{64 + 79\ln 2 + 36(\ln 2)^2 + 6(\ln 2)^3}{384}, \ldots\right\}$$

(131) $$\Omega = 1 - \sum_{n=1}^{\infty} c_n\,(1 - e^{-1})^n$$

donde

(132) $$c_0 = 0, c_1 = \frac{1}{2}$$

(133) $$c_{n+1} = \frac{1}{2(n+1)}\left((2n-1)c_n + \sum_{k=0}^{n-1}(k+1)c_{k+1}(c_{n-k} - c_{n-1-k})\right)\,, n = 1,2,3,\ldots$$

(134) $$\{c_n : n \in \mathbb{N}\} = \left\{\frac{1}{2}, \frac{3}{16}, \frac{19}{192}, \frac{185}{3072}, \frac{2437}{61440}, \frac{40523}{14745601}, \ldots\right\}$$

De la fórmula (79) , se obtiene:

$$(135)\ \Omega = \frac{1}{2} + \frac{1}{3}\left(\ln 2 - \frac{1}{2}\right) + \frac{2}{3^3}\left(\ln 2 - \frac{1}{2}\right)^2 - \frac{4}{3^7}\left(\ln 2 - \frac{1}{2}\right)^4 + \frac{16}{5 \cdot 3^9}\left(\ln 2 - \frac{1}{2}\right)^5 - \cdots$$

$$(136)\ \frac{1}{1+\Omega} = \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\pi^{2n-2k}(-2)^m \frac{\Gamma(2k-m+1,(2n-m+2)b)}{(2n-m+2)^{2k-m+1}}$$
$$+\frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{a}\right)$$
$$+\sum_{n=1}^{\infty}(-1)^n \sum_{k=1}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\pi^{2n-2k}2^m \frac{\Gamma(-2n+m-1,(2k-m)a)}{(2k-m)^{2n-m+1}}$$
$$+\int_{-a}^{b}\frac{1}{(e^x-x)^2+\pi^2}dx\ , a > 3.18.., b > 0.91 \ldots$$

donde

$$(137)\qquad \Gamma(x,y) = \int_{y}^{\infty} e^{-t}\,t^{x-1}\,dt\ , x > 0$$

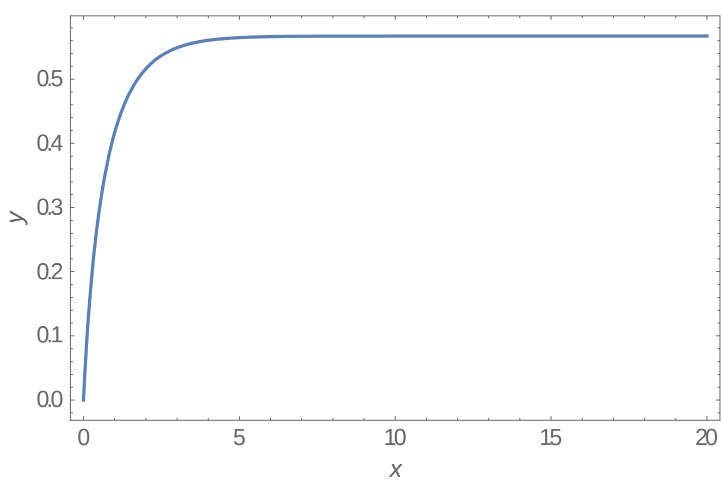
10. Ecuaciónes diferenciales para omega

$$(138)\qquad \frac{dy}{dx} = \frac{e^{-x}}{1-e^{-x}+e^{y}}\ , y(0) = 0$$

La función $y(x)$ satisface la propiedad:

$$(139)\qquad \lim_{x\to\infty} y(x) = \Omega$$

La gráfica de la función $y(x)$ es:

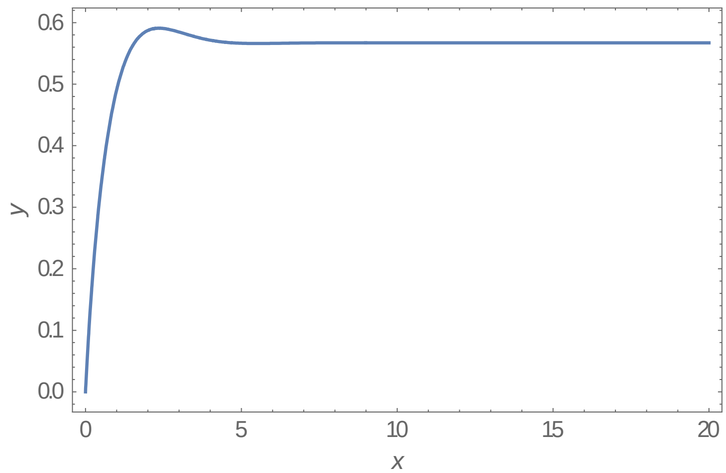


(140) $$\frac{dy}{dx} = \frac{e^{-x}(\sin x + \cos x)}{1 - e^{-x}\cos x + e^{y}} \quad , y(0) = 0$$

La función $y(x)$ satisface la propiedad:

(141) $$\lim_{x\to\infty} y(x) = \Omega$$

La gráfica de la función $y(x)$ es:



11. Cotas racionales para omega

Sea $f(x) = xe^{x}$, $x > 0$, sea X_n , $n \in \mathbb{N}$ la sucesión definida como sigue:

(142) $$X_1 = (x_{11}, x_{12}) \ , \ X_{n+1} = \begin{cases} \left(x_{n1}, \frac{1}{2}(x_{n1} + x_{n2})\right) & si \quad f\left(\frac{1}{2}(x_{n1} + x_{n2})\right) > 1 \\ \left(x_{n1}, \frac{1}{2}(x_{n1} + x_{n2})\right) & si \quad f\left(\frac{1}{2}(x_{n1} + x_{n2})\right) < 1 \end{cases}$$

donde

(143) $$x_{11} < \Omega < x_{12} \ \land \ x_{11}, x_{12} \in \mathbb{Q}$$

Entonces es válida la desigualdad:

(144) $$x_{n1} < \Omega < x_{n2} \ \land \ x_{n1}, x_{n2} \in \mathbb{Q} \, , n \in \mathbb{N}$$

además

(145) $$\lim_{n\to\infty} x_{n1} = \lim_{n\to\infty} x_{n2} = \Omega$$

Ejemplo:

(146) $$X_1 = \left(\frac{1}{2}, \frac{3}{5}\right)$$

(147) $$X_n = \left\{\left(\frac{1}{2}, \frac{3}{5}\right), \left(\frac{11}{20}, \frac{3}{5}\right), \left(\frac{11}{20}, \frac{23}{40}\right), \left(\frac{9}{16}, \frac{23}{40}\right), \left(\frac{9}{16}, \frac{91}{160}\right), \left(\frac{181}{320}, \frac{91}{160}\right), \dots\right\}$$

12. Recurrencias en dos variables

(148) $(x_{n+1}, y_{n+1}) = (e^{-x_n}\cos y_n, -e^{-x_n}\sin y_n)$, $(x_1, y_1) = (1,1), (x_n, y_n) \to (\Omega, 0)$

(149) $$\begin{cases} x_{n+1} = e^{x_n/(x_n^2+y_n^2)}\cos\left(\dfrac{y_n}{x_n^2+y_n^2}\right) \\ x_{n+1} = -e^{x_n/(x_n^2+y_n^2)}\sin\left(\dfrac{y_n}{x_n^2+y_n^2}\right) \\ (x_1, y_1) = (1,1)\,, (x_n, y_n) \to (\Omega^{-1}, 0) \end{cases}$$

13. Fórmula de inversión ,fórmula de recurrencia

Sea $\Omega^* \cong \Omega$, una aproximación de Ω , entonces:

(150) $$\Omega = \Omega^* + c\,y + \frac{1}{2}c^3y^2 + \frac{1}{6}c^4(3-c)y^3 + \frac{1}{24}c^5(1-10c-15c^2)y^4 + \cdots$$

donde

(151) $$c = \left(1 + e^{\Omega^*}\right)^{-1}$$

(152) $$y = 1 - \Omega^* e^{\Omega^*}$$

Sea $\Omega_1 = \Omega^* \cong \Omega$,$n \in \mathbb{N}$, entonces:

(153) $$\Omega_{n+1} = \Omega_n + c_n y_n + \frac{1}{2}c_n^3y_n^2 + \frac{1}{6}c_n^4(3-c_n)y_n^3 + \frac{1}{24}c_n^5(1-10c_n-15c_n^2)y_n^4 + \cdots$$

donde

(154) $$c_n = (1 + e^{\Omega_n})^{-1}$$

(155) $$y_n = 1 - \Omega_n e^{\Omega_n}$$

(156) $$\Omega_n \to \Omega$$

En la fórmula (153) solo se considera un número finito de términos.

14. Función W de Lambert

Se define por la ecuación (for details see [2]):

(157) $$W(z)e^{W(z)} = z \ , z \in \mathbb{C}$$

Algunas fórmulas que relacionan la función $W(x)$ y constante Omega Ω .

(158) $$\Omega = W(1)$$

(159) $$\Omega = \frac{1}{n} W\left(\frac{n}{\Omega^{n-1}}\right) \quad ,n > 0$$

(160) $$\Omega = 1 + \int_0^1 W(x)\,dx - \int_1^e \frac{W(x)}{1 + W(x)}\,dx$$

(161) $$\Omega = e - 1 - \int_1^e W(x)\,dx + \int_0^1 \frac{W(x)}{1 + W(x)}\,dx$$

(162) $$\int_0^1 W(x)\,dx = \Omega + \Omega^{-1} - 1$$

(163) $$\Omega^{-1} = \int_1^e \frac{W(x)}{1 + W(x)}\,dx$$

(164) $$\Omega + \sqrt{2\pi} = \int_0^1 W(1/x^2)\,dx + \int_1^\infty \frac{2\,W(1/x^2)}{1 + W(1/x^2)}\,dx$$

(165) $$\Omega^{-1} = \int_1^\infty \frac{1}{x\,W(x)\big(1 + W(x)\big)}\,dx$$

(166) $$\Omega = W(y) - \int_1^y \frac{W(x)}{x\big(1 + W(x)\big)}\,dx \quad ,y \geq 0$$

(167) $$\Omega = W(y) - \int_1^y \frac{e^{-W(x)}}{1 + W(x)}\,dx \quad ,y \geq 0$$

(168) $$\Omega = W(y) - \int_1^y \frac{1}{x + e^{W(x)}}\,dx \quad ,y \geq 0$$

Sea $f(x)$ definida como:

(169) $$f(x) = \frac{(x^2 + (1 - x\cot x)^2)(1 - e^{-x\cot x} x\csc x)}{(1 + e^{-2x\cot x} x^2 (\csc x)^2)(1 + e^{-x\cot x} x\csc x)}$$

Se tiene:

(170) $$\Omega = a - \frac{1}{\pi}\int_0^{\pi} f(x)\, x e^{-x \cot x} \csc x \, dx$$

(171) $$\Omega = b + \frac{1}{\pi}\int_0^{\pi} f(x)\, dx$$

(172) $$\Omega^{-1} = c - \frac{1}{\pi}\int_0^{\pi} f(x)\, e^{-2x \cot x}\, dx$$

donde

(173) $$a = Re\big(W(i)\big) = 0.374699 \ldots$$

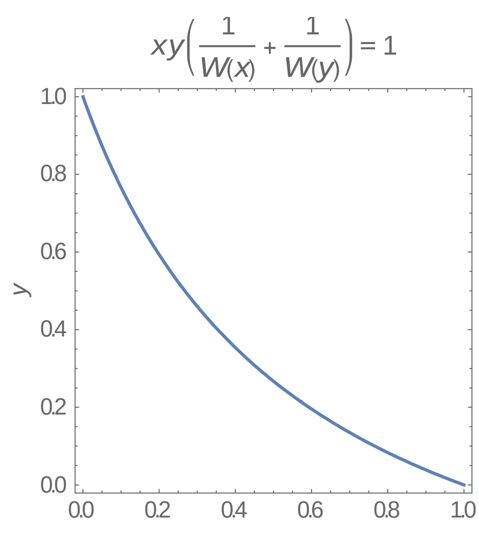
(174) $$b = Im\big(W(i)\big) = 0.576412 \ldots$$

(175) $$c = Re\left(\frac{i}{W(i)}\right) = \frac{b}{a^2 + b^2} = 1.21953 \ldots$$

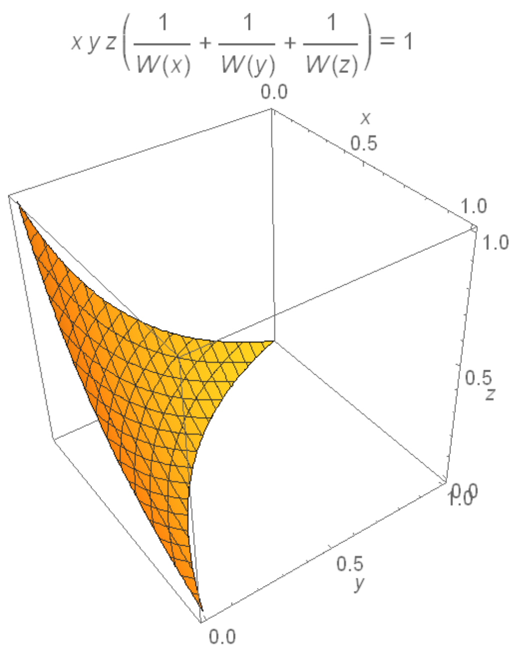
Una recurrencia para $a + bi = w = W(i)$ es:

(176) $$w_1 = \frac{1}{2} + \frac{1}{2} i \;, w_{n+1} = \frac{(1 + w_n)i}{i + e^{w_n}} \;, w_n \to a + bi$$

(177) $$0 < x < 1, 0 < y < 1, xy\left(\frac{1}{W(x)} + \frac{1}{W(y)}\right) = 1 \Rightarrow \Omega = W(x) + W(y)$$



$$(178) \qquad x, y, z \in (0,1), xyz\left(\frac{1}{W(x)W(y)} + \frac{1}{W(x)W(z)} + \frac{1}{W(y)W(z)}\right) = 1 \Rightarrow \Omega = W(x) + W(y) + W(z)$$



15. Serie para omega,números de Euler de segunda clase

$$(179) \qquad \Omega = 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\ 2^{2n-1}} c_n$$

donde

$$(180) \qquad c_n = \sum_{k=0}^{n-1} (-1)^k\, a_{n-1,k} \quad , n \in \mathbb{N}$$

$$(181) \qquad a_{n,n} = \begin{cases} 1 & n = 0 \\ 0 & en\ otro\ caso \end{cases}$$

$$(182) \qquad a_{n,0} = 1$$

$$(183) \qquad a_{n,k} = (k+1)a_{n-1,k} + (2n-1-k)a_{n-1,k-1}$$

$$(184) \qquad a_{n,k} = \{\{1\}, \{1,0\}, \{1,2,0\}, \{1,8,6,0\}, \{1,22,58,24,0\}, \{1,52,328,444,120,0\}, \dots\}$$

$$(185) \qquad c_n = \{1,1,-1,-1,13,-47,-73,2447,-16811, \dots\}$$

Los números $a_{n,k}$ se conocen como números de Euler de segunda clase.

16. Fracción continua para omega

$$\Omega = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{10+\cdots}}}}}}$$

$$\Omega = [0; 1,1,3,4,2,10,4,1,1,1,1,2,7,306,1,5,1, \dots]$$

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