Double Conformal Space-Time Algebra

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Abstract

This paper introduces the $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA). $\mathcal{G}_{4,8}$ DCSTA is a straightforward extension of the $\mathcal{G}_{2,8}$ Double Conformal Space Algebra (DCSA), which is a different form of the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA). $\mathcal{G}_{4,8}$ DCSTA extends $\mathcal{G}_{2,8}$ DCSA with spacetime boost operations and differential operators for differentiation with respect to the pseudospatial time $w=ct$ direction and time t. The spacetime boost operation can implement anisotropic dilation (directed non-uniform scaling) of quadric surface entities. DCSTA is a high-dimensional 12D embedding of the $\mathcal{G}_{1,3}$ Space-Time Algebra (STA) and is a doubling of the $\mathcal{G}_{2,4}$ Conformal Space-Time Algebra (CSTA). The 2-vector quadric surface entities of the DCSA subalgebra appear in DCSTA as quadric surfaces at zero velocity that can be boosted into moving surfaces with constant velocities that display the length contraction effect of special relativity. DCSTA inherits doubled forms of all CSTA entities and versors. The doubled CSTA entities (standard DCSTA entities) include points, hypercones, hyperplanes, hyperpseudospheres, and other entities formed as their intersections, such as planes, lines, spatial spheres and circles, and spacetime hyperboloids (pseudospheres) and hyperbolas (pseudocircles). The doubled CSTA versors (DCSTA versors) include rotor, hyperbolic rotor (boost), translator, dilator, and their compositions such as the translated-rotor, translated-boost, and translated-dilator. The DCSTA versors provide a complete set of spacetime transformation operators on all DCSTA entities. DCSTA inherits the DCSA 2-vector spatial entities for Darboux cyclides (incl. parabolic and Dupin cyclides, general quadrics, and ring torus) and gains Darboux pseudocyclides formed in spacetime with the pseudospatial time dimension. All DCSTA entities can be reflected in, and intersected with, the standard DCSTA entities. To demonstrate $\mathcal{G}_{4,8}$ DCSTA as concrete mathematics with possible applications, this paper includes sample code and example calculations using the symbolic computer algebra system *SymPy*.

Keywords: conformal geometric algebra; space-time algebra; Clifford algebra

A.M.S. subject classification: 15A66, 83A05, 53A30, 14J70, 51K05

1 Introduction

This original research monograph[1.1] introduces the $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA), which is a straightforward extension of the $\mathcal{G}_{2,8}$ *Double Conformal Space Algebra* (DCSA) into spacetime. $\mathcal{G}_{2,8}$ DCSA is a different form of the $\mathcal{G}_{8,2}$ *Double Conformal / Darboux Cyclide Geometric Algebra* (DCGA). $\mathcal{G}_{8,2}$ DCGA is introduced in the original research monograph [7] and in the published short paper [8], and is discussed further in the papers [5] and [6]. All of the results of $\mathcal{G}_{8,2}$ DCGA have a similar form in $\mathcal{G}_{2,8}$ DCSA, and also in $\mathcal{G}_{4,8}$ DCSTA at time $w=ct=0$. $\mathcal{G}_{4,8}$ DCSTA is a high-dimensional 12D embedding of the $\mathcal{G}_{1,3}$ *Space-Time Algebra* (STA). $\mathcal{G}_{1,3}$ STA is introduced by Hestenes in [14]. $\mathcal{G}_{4,8}$ DCSTA is an application of the $\mathcal{G}_{4,8}$ Geometric Algebra. Geometric Algebra is introduced by Hestenes and Sobczyk in [15]. Familiarity with Geometric Algebra and $\mathcal{G}_{8,2}$ DCGA is assumed.

[1.1] Revised version v3, *July 7, 2016*, submitted to `http://vixra.org/author/robert_b_easter`. This version may be superseded at the above link by newer revised versions. Email: dirname@yahoo.com.

$\mathcal{G}_{4,8}$ DCSTA may offer new mathematical methods for some applications. However, the 12D high-dimensionality of DCSTA incurs high computational cost and applications may require an efficient implementation [10] using optimized hardware and software [11] for DCSTA. Other works on algebras similar to $\mathcal{G}_{4,8}$ DCSTA may exist in the mathematical physics literature, but no specific works essentially the same as $\mathcal{G}_{4,8}$ DCSTA were known by this author at the time of researching and writing this paper.

$\mathcal{G}_{4,8}$ DCSTA is a doubling extension of the $\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra* (CSTA). $\mathcal{G}_{2,4}$ CSTA is introduced by C.J.L. Doran and A.N. Lasenby in [3] as the *spacetime conformal group*. $\mathcal{G}_{2,4}$ CSTA embeds $\mathcal{G}_{1,3}$ STA using stereographic embedding and homogenization as discussed by Perwass [16] in the context of $\mathcal{G}_{4,1}$ *Conformal Geometric Algebra* (CGA) (chapters 1-3 in [20]) and as discussed (in French) by Anglès [1].

$\mathcal{G}_{4,8}$ DCSTA $\mathcal{D}$ has twelve unit vector elements $\mathbf{e}_i\colon 1\leq i\leq 12$ with metric matrix

$$m_{\mathcal{D}} = \operatorname{diag}(1,-1,-1,-1,1,-1,1,-1,-1,-1,1,-1) = [(m_{\mathcal{D}})_{ij}] = [\mathbf{e}_i\cdot\mathbf{e}_j]. \tag{1.1}$$

The first six elements are in the $\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra 1* (CSTA1) $\mathcal{C}^1$

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i\in\{1,5\} \\ -1 & : i\in\{2,3,4,6\}. \end{cases} \tag{1.2}$$

The next six elements are in the $\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra 2* (CSTA2) $\mathcal{C}^2$

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i\in\{7,11\} \\ -1 & : i\in\{8,9,10,12\}. \end{cases} \tag{1.3}$$

$\mathcal{G}_{2,4}$ CSTA1 $\mathcal{C}^1$ embeds the four elements of the $\mathcal{G}_{1,3}$ *Space-Time Algebra 1* (STA1) $\mathcal{M}^1$

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i\in\{1\} \\ -1 & : i\in\{2,3,4\}. \end{cases} \tag{1.4}$$

$\mathcal{G}_{2,4}$ CSTA2 $\mathcal{C}^2$ embeds the four elements of the $\mathcal{G}_{1,3}$ *Space-Time Algebra 2* (STA2) $\mathcal{M}^2$

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i\in\{7\} \\ -1 & : i\in\{8,9,10\}. \end{cases} \tag{1.5}$$

$\mathcal{G}_{1,3}$ STA1 $\mathcal{M}^1$ contains the three elements of the $\mathcal{G}_{0,3}$ *Space Algebra 1* (SA1) $\mathcal{S}^1$

$$\mathbf{e}_i^2\colon i\in\{2,3,4\} = -1. \tag{1.6}$$

$\mathcal{G}_{1,3}$ STA2 $\mathcal{M}^2$ contains the three elements of the $\mathcal{G}_{0,3}$ *Space Algebra 2* (SA2) $\mathcal{S}^2$

$$\mathbf{e}_i^2\colon i\in\{8,9,10\} = -1. \tag{1.7}$$

The $\mathcal{G}_{1,3}$ STA $\mathcal{M}$ elements, which are the Dirac gammas $\boldsymbol{\gamma}_i$ and Pauli sigmas $\boldsymbol{\sigma}_i$, can be defined in STA1 $\mathcal{M}^1$ as

$$\boldsymbol{\gamma}_i = \begin{cases} \mathbf{e}_{i+1} & : i\in\{0,1,2,3\} \\ \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 & : i=5 \end{cases} \tag{1.8}$$

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_x = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_0 \tag{1.9}$$

$$\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_y = \boldsymbol{\gamma}_2\boldsymbol{\gamma}_0 \tag{1.10}$$

$$\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_z = \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0. \tag{1.11}$$

The $\mathcal{G}_{1,3}$ STA $\mathcal{M}$ elements $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$ that are introduced in [14] are used in all general discussions of STA. The STA $\mathcal{M}$ elements can be identified isomorphically with either the STA1 $\mathcal{M}^1$ or STA2 $\mathcal{M}^2$ elements. The elements $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, and $\boldsymbol{\gamma}_3$ can also be denoted $\boldsymbol{\gamma}_x$, $\boldsymbol{\gamma}_y$, and $\boldsymbol{\gamma}_z$ when emphasizing their usage as the conventional x, y, and z spatial directions. The element $\boldsymbol{\gamma}_0$ is the pseudospatial time $w=ct$ direction.

In addition to the above subalgebras of DCSTA, there also exists the subalgebras for $\mathcal{G}_{1,4}$ *Conformal Space Algebra 1* (CSA1) $\mathcal{CS}^1$ with the five elements

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{5\} \\ -1 & : i \in \{2,3,4,6\} \end{cases} \tag{1.12}$$

and $\mathcal{G}_{1,4}$ *Conformal Space Algebra 2* (CSA2) $\mathcal{CS}^2$ with the five elements

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{11\} \\ -1 & : i \in \{8,9,10,12\} \end{cases} \tag{1.13}$$

and $\mathcal{G}_{2,8}$ *Double Conformal Space Algebra* (DCSA) $\mathcal{DS}$ that has the ten unit vector elements of $\mathcal{CS}^1$ and $\mathcal{CS}^2$.

$\mathcal{G}_{1,4}$ *Conformal Space Algebra* (CSA) $\mathcal{CS}$ is very similar to the well-known $\mathcal{G}_{4,1}$ *Conformal Geometric Algebra* (CGA) and will not be explicitly discussed at length in this paper, except to note that it is nearly identical to CGA in all respects, with differences only in the signs of some expressions. Likewise, $\mathcal{G}_{2,8}$ DCSA is very similar to $\mathcal{G}_{8,2}$ DCGA, except for differences in the signs of some expressions. $\mathcal{G}_{4,8}$ DCSTA becomes $\mathcal{G}_{2,8}$ DCSA when all times are zero, $w=ct=0$.

2 Space Algebra (SA)

$\mathcal{G}_{0,3}$ *Space Algebra* (SA) $\mathcal{S}$ is very similar to $\mathcal{G}_3$ *Algebra of Physical Space* (APS) [13]. Vectors in SA square negative $(\mathbf{v}^2=\mathbf{v}\cdot\mathbf{v})\leq 0$, which causes sign flips in many formulas adapted from $\mathcal{G}_3$ APS, $\mathcal{G}_{4,1}$ CGA, or $\mathcal{G}_{8,2}$ DCGA. While $\mathcal{G}_{8,2}$ DCGA embeds two Euclidean $\mathcal{G}_3$ APS spaces, $\mathcal{G}_{2,8}$ DCSA $\mathcal{DS}$ and $\mathcal{G}_{4,8}$ DCSTA $\mathcal{D}$ embed two anti-Euclidean $\mathcal{G}_{0,3}$ SA spaces $\mathcal{S}^1$ and $\mathcal{S}^2$.

The subscript $\mathcal{S}$ denotes an element or operation in generic SA. The subscript $\mathcal{S}^1$ denotes an element or operation in SA1. The subscript $\mathcal{S}^2$ denotes an element or operation in SA2. In most formulas, these subscripts can simply be substituted to write formulas in SA, SA1, and SA2. Such duplication of similar formulas in each representation is avoided unless it adds clarity to the discussion. In similar fashion, in STA (§3), STA1, and STA2, the subscripts $\mathcal{M}$, $\mathcal{M}^1$, and $\mathcal{M}^2$ indicate elments in STA, STA1, and STA2.

The 3-D spatial vectors in $\mathcal{G}_{0,3}$ SA are generally **bold** lowercase letters, such as $\mathbf{p}=\mathbf{p}_{\mathcal{S}}$. The 4-D spacetime vectors in $\mathcal{G}_{1,3}$ STA (§3) are generally ***bold italic*** lowercase letters, such as $\boldsymbol{p}=\boldsymbol{p}_{\mathcal{M}}$.

2.1 SA unit pseudoscalar

The SA 3-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{S}}$ with signature $(---)$ is

$$\mathbf{I}_{\mathcal{S}} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 \tag{2.1}$$

$$\mathbf{I}_{\mathcal{S}}^{\sim} = (-1)^{3(3-1)/2}\mathbf{I}_{\mathcal{S}} = -\mathbf{I}_{\mathcal{S}} \tag{2.2}$$

$$\mathbf{I}_{\mathcal{S}}^2 = -\mathbf{I}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{\sim} = 1 \tag{2.3}$$

$$\mathbf{I}_{\mathcal{S}}^{-1} = \mathbf{I}_{\mathcal{S}}. \tag{2.4}$$

The SA1 3-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{S}^1}$ with signature $(---)$ is

$$\mathbf{I}_{\mathcal{S}^1} \;=\; \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4. \tag{2.5}$$

The SA2 3-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{S}^2}$ with signature $(---)$ is

$$\mathbf{I}_{\mathcal{S}^2} \;=\; \mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}. \tag{2.6}$$

The notation $A^{\sim}$ is the *reverse* of A [4][16]. The $\mathcal{G}_{0,3}$ SA unit pseudoscalar $\mathbf{I}_{\mathcal{S}}$ is its own inverse and squares to 1 as a *hyperbolic unit*. In $\mathcal{G}_3$ APS, the unit pseudoscalar squares to -1 and is an *imaginary unit*. This difference affects how the SA dualization operation is defined.

2.2 SA dualization

The SA *dual* $A_{\mathcal{S}}^{*\mathcal{S}}$ of an SA multivector $A_{\mathcal{S}}$ is

$$A_{\mathcal{S}}^{*} = A_{\mathcal{S}}^{*\mathcal{S}} \;=\; A_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{\sim} = -A_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1}. \tag{2.7}$$

The SA *undual* $A_{\mathcal{S}}$ of a dual SA multivector $A_{\mathcal{S}}^{*\mathcal{S}}$ is

$$A_{\mathcal{S}} \;=\; A_{\mathcal{S}}^{*}\mathbf{I}_{\mathcal{S}}^{\sim} = -A_{\mathcal{S}}^{*}\mathbf{I}_{\mathcal{S}} \tag{2.8}$$

The SA dual and undual operations are the same, and the SA dualization is an *involution*.

The notation $A_{\mathcal{S}}$ denotes an element of the algebra denoted by $\mathcal{S}$, which is SA. In later sections, we will encounter the algebras denoted by $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{D}$, which are STA, CSTA, and DCSTA, respectively. For the subalgebras $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{C}$ of $\mathcal{D}$, there are two copies of them in $\mathcal{D}$, which are denoted by $\mathcal{S}^1$ and $\mathcal{S}^2$ and similarly for other subalgebras that have a double in DCSTA. For example, the PAULI subalgebra of STA is denoted $\mathcal{P}$, and there are $\mathcal{P}^1$ in $\mathcal{M}^1$ and $\mathcal{P}^2$ in $\mathcal{M}^2$.

The explicit dualization notation $A_{\mathcal{S}}^{*\mathcal{S}}$ denotes the dual of $A_{\mathcal{S}}$ in subspace $\mathcal{S}$ using the unit pseudoscalar $\mathbf{I}_{\mathcal{S}}$ of the subspace $\mathcal{S}$. The implicit dualization notation $A_{\mathcal{S}}^{*}$ denotes the same dualization as indicated by the subscript $\mathcal{S}$.

To introduce the notation further, the explicit dualizations are

$$A^{*} \;=\; \begin{cases} A_{\mathcal{S}}^{*\mathcal{S}} = A_{\mathcal{S}}^{*} = -A_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1} & : \text{ SA } \mathcal{S} \text{ dualization} \\ A_{\mathcal{M}}^{*\mathcal{M}} = A_{\mathcal{M}}^{*} = A_{\mathcal{M}}\mathbf{I}_{\mathcal{M}}^{-1} & : \text{ STA } \mathcal{M} \text{ dualization} \\ A_{\mathcal{C}}^{*\mathcal{C}} = A_{\mathcal{C}}^{*} = A_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} & : \text{ CSTA } \mathcal{C} \text{ dualization} \\ A_{\mathcal{D}}^{*\mathcal{D}} = A_{\mathcal{D}}^{*} = A_{\mathcal{D}}\mathbf{I}_{\mathcal{D}}^{-1} & : \text{ DCSTA } \mathcal{D} \text{ dualization.} \end{cases} \tag{2.9}$$

Duals are typically the result of division by the unit pseudoscalar. The SA dualization is defined as division by the negative unit pseudoscalar, and the reason is explained in (§2.6) on the SA rotor. These dualizations are discussed further in later sections.

2.3 SA test vector

The symbolic SA *test vector* $\mathbf{t}_{\mathcal{S}}$ is defined on the basis of the DIRAC gammas [14] as

$$\mathbf{t} = \mathbf{t}_{\mathcal{S}} \;=\; x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3. \tag{2.10}$$

The symbolic SA1 *test vector* $\mathbf{t}_{\mathcal{S}^1}$ is defined as

$$\mathbf{t}_{\mathcal{S}^1} \;=\; x\mathbf{e}_2 + y\mathbf{e}_3 + z\mathbf{e}_4. \tag{2.11}$$

The symbolic SA2 *test vector* $\mathbf{t}_{\mathcal{S}^2}$ is defined as

$$\mathbf{t}_{\mathcal{S}^2} \;=\; x\mathbf{e}_8 + y\mathbf{e}_9 + z\mathbf{e}_{10}. \tag{2.12}$$

The *symbolic* scalars x, y, and z are the conventional coordinates in space. Numerical scalars are denoted p_x, p_y, and p_z for a vector $\mathbf{p}$. This distinction between symbolic values and numerical values is helpful in symbolic computations. Symbolic computations using a symbolic computer algebra software, such as *SymPy* [19] with the $\mathcal{GA}$lgebra [2] module, can assist in the study of DCSTA and other high-dimensional Geometric Algebras.

A test vector, or other test entity, holds symbolic coordinates and parameters. A non-test vector, or other non-test entity, holds numeric coordinates and parameters. In symbolic calculations, a non-test entity, or simply an actual *entity*, can be evaluated against a symbolic test entity to obtain the symbolic algebraic expression, or implicit surface function, that is represented by the entity.

2.4 SA spatial velocity vector

An SA $\mathcal{S}$ *spatial velocity vector* $\mathbf{v}_{\mathcal{S}}$ has the form

$$\mathbf{v} = \mathbf{v}_{\mathcal{S}} \;=\; v_x\boldsymbol{\gamma}_1 + v_y\boldsymbol{\gamma}_2 + v_z\boldsymbol{\gamma}_3 = \beta c\hat{\mathbf{v}}. \tag{2.13}$$

An SA1 $\mathcal{S}^1$ *spatial velocity vector* $\mathbf{v}_{\mathcal{S}^1}$ has the form

$$\mathbf{v}_{\mathcal{S}^1} \;=\; v_x\mathbf{e}_2 + v_y\mathbf{e}_3 + v_z\mathbf{e}_4 = \beta c\hat{\mathbf{v}}_{\mathcal{S}^1}. \tag{2.14}$$

An SA2 $\mathcal{S}^2$ *spatial velocity vector* $\mathbf{v}_{\mathcal{S}^2}$ has the form

$$\mathbf{v}_{\mathcal{S}^2} \;=\; v_x\mathbf{e}_8 + v_y\mathbf{e}_9 + v_z\mathbf{e}_{10} = \beta c\hat{\mathbf{v}}_{\mathcal{S}^2}. \tag{2.15}$$

The scalars v_x, v_y, and v_z are coordinate speeds in the conventional x, y, and z spatial directions. Natural speed is $\beta = v/c = \|\mathbf{v}\|/c$, $|\beta| \leq 1$. Light speed is c.

The vector units $\mathbf{e}_1$ and $\mathbf{e}_7$ are in STA1 and STA2, respectively, where they serve as the unit directions for pseudospatial time $w = ct$. Pseudospatial time coordinate w is measured in distance that light travels in time t. Clock time (coordinate time) is $t = w\,/\,c$. In standard units of meters and seconds, $c = 299792458$ m/s, exactly. In natural units, $c = 1$, which is convenient for testing calculations and graphing implicit surfaces $F(x, y, z) = 0$.

In special relativity, the constant non-negative *norm* of an SA velocity

$$\|\mathbf{v}_{\mathcal{S}}\| \;=\; \sqrt{\mathbf{v}_{\mathcal{S}} \cdot \mathbf{v}_{\mathcal{S}}^{\dagger}} = \sqrt{-\mathbf{v}_{\mathcal{S}}^2} = \sqrt{v_x^2 + v_y^2 + v_z^2} = |\beta c| \tag{2.16}$$

must not exceed light speed c

$$0 \leq \|\mathbf{v}_{\mathcal{S}}\| \leq c. \tag{2.17}$$

The unit direction of velocity $\mathbf{v}$ is

$$\hat{\mathbf{v}} \;=\; \mathbf{v}/\|\mathbf{v}\|. \tag{2.18}$$

The *conjugate* $\mathbf{v}^{\dagger}$ is discussed by Perwass in [16] as anti-Euclidean subspace conjugation. The conjugate, or space conjugation, of any STA multivector $A_{\mathcal{M}}$, including any SA multivector, is

$$A_{\mathcal{M}}^{\dagger} \;=\; \boldsymbol{\gamma}_0\widetilde{A_{\mathcal{M}}}\boldsymbol{\gamma}_0. \tag{2.19}$$

This conjugation formula is valid for any $\mathcal{G}_{1,q}$ Geometric Algebra, where $\boldsymbol{\gamma}_0 = \mathbf{e}_1$ and $\boldsymbol{\gamma}_0^2 = 1$. The conjugate of an SA vector is simply its negative, $\mathbf{v}^\dagger = -\mathbf{v}$.

The notations as used by HESTENES in [14] are *not* adopted here and conflict with the notations as they are adopted here. In [14], the notation $\mathbf{v}^\dagger$ is called *hermitian conjugation* and is the *reverse* of an element in STA. The *reverse* $\mathbf{v}^\sim$ is the notation that is adopted here. In [14], the notation $\mathbf{v}^*$ is called *space conjugation* and is anti-Euclidean subspace conjugation in STA. The notation $\mathbf{v}^*$ is adopted here as the *dual* of $\mathbf{v}$. The *conjugate* $\mathbf{v}^\dagger$ is adopted here, following PERWASS in [16], and is discussed by this author in [9].

In general, a conjugation is an operation that selectively changes the signs on only certain elements and there are many kinds of conjugations and notations. It is thought that the notations that have been adopted here are the ones most commonly adopted in the current literature on Geometric Algebra. The notations of HESTENES in [14] may be found in physics literature.

2.5 SA spatial position vector

An SA *spatial position vector* $\mathbf{p}_{\mathcal{S}}(t)$, as a function of time t, has the form

$$\mathbf{p}(t) = \mathbf{p}_{\mathcal{S}}(t) \;=\; p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3 = \mathbf{p}_0 + \mathbf{v}t. \tag{2.20}$$

An SA1 *spatial position vector* $\mathbf{p}_{\mathcal{S}^1}$ has the form

$$\mathbf{p}_{\mathcal{S}^1}(t) \;=\; p_x\mathbf{e}_2 + p_y\mathbf{e}_3 + p_z\mathbf{e}_4. \tag{2.21}$$

An SA2 *spatial position vector* $\mathbf{p}_{\mathcal{S}^2}$ has the form

$$\mathbf{p}_{\mathcal{S}^2} \;=\; p_x\mathbf{e}_8 + p_y\mathbf{e}_9 + p_z\mathbf{e}_{10}. \tag{2.22}$$

The scalars p_x, p_y, and p_z are coordinate positions in the conventional x, y, and z spatial directions. The vector $\mathbf{p}_0$ is the *initial position* at time $t = 0$, and $\mathbf{v}$ is the velocity. A spatial vector $\mathbf{p}$ is the spatial position of a particle, observer, or other *observable* object. For all time $\forall t$, $\mathbf{p}(t)$ is the spatial *path* of the *worldline* of an observable moving at contant velocity $\mathbf{v}$. This paper only considers constant velocities in special relativity.

2.6 SA rotor

A rotation operator R, called a *rotor*, can be understood in terms of ratios, or products, of unit vectors, which are called *versors*. A rotor is isomorphic to a quaternion versor as discussed at length by this author in [9]. The concept of versors is generalized to k-versors in [15]. A k-versor is the product of k unit vectors. In DCSTA, we will encounter 4-versors for rotation, translation, dilation, boost, and planar reflection, and 2-versors for inversions in hyperpseudospheres.

In SA, the unit bivector rotor elements are the ratios

$$\mathbf{i} = \mathbf{k}/\mathbf{j} \;\cong\; \boldsymbol{\gamma}_3/\boldsymbol{\gamma}_2 = -\boldsymbol{\gamma}_3\boldsymbol{\gamma}_2 \tag{2.23}$$

$$\mathbf{j} = \mathbf{i}/\mathbf{k} \;\cong\; \boldsymbol{\gamma}_1/\boldsymbol{\gamma}_3 = -\boldsymbol{\gamma}_1\boldsymbol{\gamma}_3 \tag{2.24}$$

$$\mathbf{k} = \mathbf{j}/\mathbf{i} \;\cong\; \boldsymbol{\gamma}_2/\boldsymbol{\gamma}_1 = -\boldsymbol{\gamma}_2\boldsymbol{\gamma}_1. \tag{2.25}$$

The SA duals of the SA unit vector elements are

$$\begin{aligned}\gamma_1^* &= \gamma_1\mathbf{I}_{\widetilde{\mathcal{S}}} = -\gamma_1\mathbf{I}_{\mathcal{S}} = -\gamma_1\mathbf{I}_{\mathcal{S}}^{-1} = -\gamma_1(\gamma_1\gamma_2\gamma_3) = -\gamma_3\gamma_2 \cong \mathbf{i} && (2.26)\\ \gamma_2^* &= \gamma_2\mathbf{I}_{\widetilde{\mathcal{S}}} = -\gamma_2\mathbf{I}_{\mathcal{S}} = -\gamma_2\mathbf{I}_{\mathcal{S}}^{-1} = -\gamma_2(\gamma_1\gamma_2\gamma_3) = -\gamma_1\gamma_3 \cong \mathbf{j} && (2.27)\\ \gamma_3^* &= \gamma_3\mathbf{I}_{\widetilde{\mathcal{S}}} = -\gamma_3\mathbf{I}_{\mathcal{S}} = -\gamma_3\mathbf{I}_{\mathcal{S}}^{-1} = -\gamma_3(\gamma_1\gamma_2\gamma_3) = -\gamma_2\gamma_1 \cong \mathbf{k}. && (2.28)\end{aligned}$$

The SA dualization is defined such that the isomorphism to ($\cong$) quaternion units is via duals.

The dual of an SA vector $\mathbf{x}_{\mathcal{S}}$ is

$$\mathbf{x}^* = \mathbf{x}_{\mathcal{S}}^* = -\mathbf{x}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1}. \qquad (2.29)$$

The dual SA vector $\mathbf{x}^*$ is the rotor element or logarithm of a rotor $R = e^{\mathbf{x}^*}$, where

$$\begin{aligned} e^A &= \exp(A) && (2.30)\\ &= \sum_{n=0}^{\infty}\frac{A^n}{n!} && (2.31)\\ &= \sum_{n=0}^{\infty}\frac{A^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{A^{2n+1}}{(2n+1)!} && (2.32)\\ &= \cosh(A) + \sinh(A) && (2.33)\end{aligned}$$

for any multivector A. For a scalar x, imaginary unit $i = \sqrt{-1}$, hyperbolic unit $j^2 = 1$, and null (nilpotent) unit $\varepsilon^2 = 0$, we also have the standard formulas

$$\begin{aligned}\cosh(ix) &= \cos(x) && (2.34)\\ \sinh(ix) &= i\sin(x) && (2.35)\\ \cosh(jx) &= \cosh(x) && (2.36)\\ \sinh(jx) &= j\sinh(x) && (2.37)\\ \cosh(\varepsilon x) &= 1 && (2.38)\\ \sinh(\varepsilon x) &= \varepsilon x. && (2.39)\end{aligned}$$

Given the SA unit vector

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_{\mathcal{S}} = \frac{\mathbf{x}_{\mathcal{S}}}{\|\mathbf{x}_{\mathcal{S}}\|} = \frac{\mathbf{x}_{\mathcal{S}}}{\sqrt{-\mathbf{x}_{\mathcal{S}}^2}} \qquad (2.40)$$

as the *axis* of rotation, and $\|\mathbf{x}_{\mathcal{S}}\| = \frac{1}{2}\theta$ as *half* the non-negative *angle* θ of rotation, then $(\hat{\mathbf{x}}^*)^2 = -1$ and $\hat{\mathbf{x}}^* \cong \sqrt{-1}$, and the *rotor* $R_{\mathcal{S}}$ for the rotation is

$$\begin{aligned} R = R_{\mathcal{S}} &= e^{\mathbf{x}^*} = \exp(\mathbf{x}^*) = \cosh(\mathbf{x}^*) + \sinh(\mathbf{x}^*) && (2.41)\\ &= \cosh\left(\frac{1}{2}\theta\hat{\mathbf{x}}^*\right) + \sinh\left(\frac{1}{2}\theta\hat{\mathbf{x}}^*\right) && (2.42)\\ &= \cos\left(\frac{1}{2}\theta\right) + \hat{\mathbf{x}}^*\sin\left(\frac{1}{2}\theta\right) && (2.43)\\ &= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{x}}_{\mathcal{S}}\mathbf{I}_{\widetilde{\mathcal{S}}} && (2.44)\\ &= \exp\left(\frac{1}{2}\theta\hat{\mathbf{x}}_{\mathcal{S}}^*\right) = e^{\frac{1}{2}\theta\hat{\mathbf{x}}_{\mathcal{S}}^*}. && (2.45)\end{aligned}$$

The unit vector axis $\hat{\mathbf{x}}_{\mathcal{S}}$ and unit pseudoscalar $\mathbf{I}_{\mathcal{S}}$ may be in SA1 or SA2 by changing subscript $\mathcal{S}$ to $\mathcal{S}^1$ or $\mathcal{S}^2$ for rotors $R_{\mathcal{S}^1}$ and $R_{\mathcal{S}^2}$, respectively.

In $\mathcal{G}_{4,1}$ CGA, $\mathcal{G}_{1,4}$ CSA, and $\mathcal{G}_{2,4}$ CSTA, the (isotropic) dilator D has a logarithm unit that is isomorphic to ($\cong$) a hyperbolic unit j, and the translator T has a logarithm unit that is isomorphic to a null (nilpotent) unit ε. The CSTA boost B, like a dilator D, has a logarithm unit that is isomorphic to a hyperbolic unit j and can be used as a directed non-uniform (anisotropic) dilation operator.

The *rotor operation*, or *versor "sandwich" operation*, that rotates any SA multivector $A_{\mathcal{S}}$ around the axis $\hat{\mathbf{x}}_{\mathcal{S}}$ by angle θ is

$$A'_{\mathcal{S}} = R_{\mathcal{S}} A_{\mathcal{S}} R_{\mathcal{S}}^{-1} = R_{\mathcal{S}} A_{\mathcal{S}} R_{\widetilde{\mathcal{S}}}. \tag{2.46}$$

The SA multivector $A_{\mathcal{S}}$ is typically a vector $\mathbf{a}_{\mathcal{S}}$, but it can be any multivector in SA.

The sense of positive angle θ rotation around an axis $\hat{\mathbf{x}}$ usually follows the *right-hand rule* on a right-handed axes model. The sense of positive rotation around an axis follows the similar *left-hand rule* on a left-handed axes model. The choice of axes model does *not* affect the rotation mathematics, but it affects the orientation, or handedness, of the axes and the *interpretation* of rotation results on the chosen axes model.

In general, the rotor operation is an example of a *versor operation*, and each ith vector $\mathbf{a}_{ijk}$, of the jth k-blade $\mathbf{A}_{jk}$ of the k-vector $A_k = \langle A \rangle_k$ of a multivector A in a $\mathcal{G}_{p,q}$ Geometric Algebra $n = p + q$, is transformed by a $(1 \leq m \leq n)$-versor R as the versor operation

$$\begin{aligned}
A' &= RAR^{-1} && (2.47)\\
&= R\left(\sum_{k=0}^{n} \langle A \rangle_k\right)R^{-1} && (2.48)\\
&= R\left(\sum_{k=0}^{n}\left(\sum_{j=1}^{\binom{n}{k}} \mathbf{A}_{jk}\right)\right)R^{-1} && (2.49)\\
&= R\left(\sum_{k=0}^{n}\left(\sum_{j=1}^{\binom{n}{k}}\left(\bigwedge_{i=1}^{k} \mathbf{a}_{ijk}\right)\right)\right)R^{-1} && (2.50)\\
&= \left(\sum_{k=0}^{n}\left(\sum_{j=1}^{\binom{n}{k}}\left(\bigwedge_{i=1}^{k} R\mathbf{a}_{ijk}R^{-1}\right)\right)\right). && (2.51)
\end{aligned}$$

For grade $k = 0$, the single "vector" $\mathbf{a}_{\varnothing 10} = A_0$ is defined as the scalar part of A, which is not transformed by any versor operation since scalars commute with all multivectors. The general versor R is called an m-versor [15], which is the product of m vectors with inverses. This result is called *versor outermorphism*, which is discussed by PERWASS in [16] and by this author in [9].

If R is a rotor, then the whole multivector A is rotated, as a rigid body of blades or as a surface entity, vector by vector in the multivector and preserves the angles between all vectors as a conformal transformation. Other angle-preserving (i.e. conformal) transformations of surface points or geometric surface entities are uniform surface scaling (isotropic dilation) and surface translation, which preserve the angles between surface features. The conformal transformations, as versor operations, for dilation and translation of surface points or entities require the embedding of surface representations or surface points into the $\mathcal{G}_{p+1,q+1}$ Conformal Geometric Algebra (CGA) or conformal space model. The same conformal rotation versor R of the base space algebra $\mathcal{G}_{p,q}$ Geometric Algebra (GA) is also valid in the embedding of $\mathcal{G}_{p,q}$ GA into its higher-dimensional conformal space algebra $\mathcal{G}_{p+1,q+1}$ CGA. In a CGA, some types of surfaces have a full representation as a multivector-valued surface entity A, but general surfaces are represented point-wise as embedded surface points that can be transformed by versor operations. This should be familiar from books on CGA such as [4], where surface entities for flats and rounds are the types of surface entities available in $\mathcal{G}_{4,1}$ CGA. In $\mathcal{G}_{8,2}$ DCGA [7][5][6], we gain 2-vector surface entities for quadrics, Dupin cyclides, and Darboux cyclides. In $\mathcal{G}_{4,8}$ DCSTA, we gain 2-vector surface entities for quadrics that can be boosted and anisotropically dilated. In $\mathcal{G}_{2,4}$ CSTA, we have entities for spacetime flats and hyperbolics (pseudorounds). Still, other general surfaces that do not have any multivector-valued entity A representation in the algebra must be represented point-wise as meshes or clouds of surface points. When they are available in a CGA, multivector-valued surface entities are powerful representations of complete surfaces that have advantages over point-wise representations of surfaces, and they can be transformed by versor operations (e.g., $A' = RAR^{-1}$) as versor outermorphism, which preserves blades as transformed blades of the same grade that are composed of transformed vectors.

3 Space-Time Algebra (STA)

Space-Time Algebra (STA) is introduced in the book *Space-Time Algebra* by David Hestenes [14]. STA is also called Dirac Algebra (DA). As explained in [14], the *space-time split* generates a Pauli Algebra (PA) on a unit bivector basis. DCSTA contains two STA $\mathcal{M}$ subalgebras, STA1 $\mathcal{M}^1$ and STA2 $\mathcal{M}^2$.

The $\mathcal{M}$ is for Minkowski spacetime $(1,3)$ and is the subscript that denotes an element or operation in STA. The subscript $\mathcal{M}^1$ denotes an element or operation in STA1. The subscript $\mathcal{M}^2$ denotes an element or operation in STA2.

3.1 STA elements

3.1.1 Dirac gammas and Pauli sigmas in STA

The Dirac gammas and Pauli sigmas can be defined in STA1 as

$$\gamma_i = \begin{cases} \mathbf{e}_{i+1} & : \; i \in \{0,1,2,3\} \\ \gamma_0\gamma_1\gamma_2\gamma_3 & : \; i = 5 \end{cases} \tag{3.1}$$

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_x = \gamma_1\gamma_0 \tag{3.2}$$

$$\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_y = \gamma_2\gamma_0 \tag{3.3}$$

$$\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_z = \gamma_3\gamma_0. \tag{3.4}$$

The STA elements can also be defined similarly in STA2. The DIRAC gammas and PAULI sigmas are represented as matrices in other literature, but they have multivector representations in STA. See reference [14] for more information about these representations.

The gammas are used to denote elements in STA $\mathcal{M}$, but it should be understood that all discussions of STA $\mathcal{M}$ apply similarly in STA1 $\mathcal{M}^1$ and STA2 $\mathcal{M}^2$ by changing subscripting and elements

$$\begin{array}{lllll} \text{STA} & \cong & \text{STA1} & \cong & \text{STA2} \\ \mathcal{M} & \cong & \mathcal{M}^1 & \cong & \mathcal{M}^2 \\ \boldsymbol{\gamma}_0 & \cong & \mathbf{e}_1 & \cong & \mathbf{e}_7 \\ \boldsymbol{\gamma}_1 & \cong & \mathbf{e}_2 & \cong & \mathbf{e}_8 \\ \boldsymbol{\gamma}_2 & \cong & \mathbf{e}_3 & \cong & \mathbf{e}_9 \\ \boldsymbol{\gamma}_3 & \cong & \mathbf{e}_4 & \cong & \mathbf{e}_{10}. \end{array} \tag{3.5}$$

3.1.2 STA unit pseudoscalar

The $\mathcal{G}_{1,3}$ STA 4-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{M}}$ with signature $(+---)$ is

$$\mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_5 \tag{3.6}$$

$$\mathbf{I}_{\mathcal{M}}^{\sim} = (-1)^{4(4-1)/2}\mathbf{I}_{\mathcal{M}} = \mathbf{I}_{\mathcal{M}} \tag{3.7}$$

$$\mathbf{I}_{\mathcal{M}}^2 = -1 \tag{3.8}$$

$$\mathbf{I}_{\mathcal{M}}^{-1} = -\mathbf{I}_{\mathcal{M}} = -\mathbf{I}_{\mathcal{M}}^{\sim}. \tag{3.9}$$

The $\mathcal{G}_{1,3}$ STA1 4-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{M}^1}$ with signature $(+---)$ is

$$\mathbf{I}_{\mathcal{M}^1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4. \tag{3.10}$$

The $\mathcal{G}_{1,3}$ STA2 4-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{M}^2}$ with signature $(+---)$ is

$$\mathbf{I}_{\mathcal{M}^2} = \mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}. \tag{3.11}$$

3.1.3 STA test vector

The symbolic STA *test vector* $\boldsymbol{t}_{\mathcal{M}}$ is defined on the basis of the DIRAC gammas [14] as

$$\boldsymbol{t} = \boldsymbol{t}_{\mathcal{M}} = w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 = ct\boldsymbol{\gamma}_0 + \mathbf{t}_{\mathcal{S}} = \boldsymbol{o}_{\mathcal{M}}t + \mathbf{t}_{\mathcal{S}}. \tag{3.12}$$

The symbolic STA1 *test vector* $\boldsymbol{t}_{\mathcal{M}^1}$ is defined as

$$\boldsymbol{t}_{\mathcal{M}^1} = w\mathbf{e}_1 + x\mathbf{e}_2 + y\mathbf{e}_3 + z\mathbf{e}_4 = ct\mathbf{e}_1 + \mathbf{t}_{\mathcal{S}^1} = \boldsymbol{o}_{\mathcal{M}^1}t + \mathbf{t}_{\mathcal{S}^1}. \tag{3.13}$$

The symbolic STA2 *test vector* $\boldsymbol{t}_{\mathcal{M}^2}$ is defined as

$$\boldsymbol{t}_{\mathcal{M}^2} = w\mathbf{e}_7 + x\mathbf{e}_8 + y\mathbf{e}_9 + z\mathbf{e}_{10} = ct\mathbf{e}_7 + \mathbf{t}_{\mathcal{S}^2} = \boldsymbol{o}_{\mathcal{M}^2}t + \mathbf{t}_{\mathcal{S}^2}. \tag{3.14}$$

The symbolic scalars w, x, y, and z are the conventional coordinates in spacetime. The *observer* with coordinate time t is identified with the worldline $\boldsymbol{o}_{\mathcal{M}}t = ct\boldsymbol{\gamma}_0$.

The symbolic test vector $\boldsymbol{t}_{\mathcal{M}}$ is useful in symbolic computations and will be embedded as the $\mathcal{G}_{2,4}$ CSTA *test point* $\boldsymbol{T}_{\mathcal{C}}$. $\mathcal{G}_{2,4}$ CSTA1 and CSTA2 test points $\boldsymbol{T}_{\mathcal{C}^1}$ and $\boldsymbol{T}_{\mathcal{C}^2}$, respectively, are multiplied to form the $\mathcal{G}_{4,8}$ DCSTA *test point* $\boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{C}^1}\boldsymbol{T}_{\mathcal{C}^2} = \boldsymbol{T}_{\mathcal{C}^1} \wedge \boldsymbol{T}_{\mathcal{C}^2}$. The DCSTA point 2-vector value-*extraction elements* T_s are defined as inner product operators that extract values s from $\boldsymbol{T}_{\mathcal{D}}$ as $s = T_s = \boldsymbol{T}_{\mathcal{D}}$. Linear combinations of the elements T_s form the 2-vector DCSTA entities for quadric surfaces and cyclides.

3.1.4 STA observer

An STA $\mathcal{M}$ *observer position* (or *worldline*) $\boldsymbol{o}_{\mathcal{M}}t$ at the observer's proper (coordinate) time t is

$$\boldsymbol{o}t = \boldsymbol{o}_{\mathcal{M}}t \;=\; ct\boldsymbol{\gamma}_0. \tag{3.15}$$

For all times t, $\boldsymbol{o}t$ symbolically represents the worldline of the observer that passes through the origin of spacetime. An observable worldline can be represented non-symbolically in $\mathcal{G}_{2,4}$ CSTA as a CSTA GIPNS 3-vector line entity $\boldsymbol{L}_{\mathcal{C}}$ (§4.4.11).

An STA1 $\mathcal{M}^1$ observer position $\boldsymbol{o}_{\mathcal{M}^1}t$ has the form

$$\boldsymbol{o}_{\mathcal{M}^1}t \;=\; ct\mathbf{e}_1. \tag{3.16}$$

An STA2 $\mathcal{M}^2$ observer position $\boldsymbol{o}_{\mathcal{M}^2}t$ has the form

$$\boldsymbol{o}_{\mathcal{M}^2}t \;=\; ct\mathbf{e}_7. \tag{3.17}$$

An STA *translated-observer position* $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$, translated by spatial position $\mathbf{p}_0$, has the form

$$\boldsymbol{o}^{\mathbf{p}_0}(t) = \boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) \;=\; \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0 = ct\boldsymbol{\gamma}_0 + \mathbf{p}_0. \tag{3.18}$$

The observer worldline $\boldsymbol{o}_{\mathcal{M}}t$ of a translated-observer worldline $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ is

$$\boldsymbol{o}_{\mathcal{M}}t \;=\; \dot{\boldsymbol{o}}_{\mathcal{M}}^{\mathbf{p}_0}(t) = \frac{\mathrm{d}\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)}{\mathrm{d}t}t = ct\boldsymbol{\gamma}_0. \tag{3.19}$$

For a translated-observer position $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$, its spacetime velocity is $\boldsymbol{o}_{\mathcal{M}}$. The translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ represents the observer $\boldsymbol{o}_{\mathcal{M}}t$ at $\mathbf{p}_0$ and is the proper observer of any observable with initial position $\mathbf{p}_0$ at time $t = 0$. Boosts, spacetime contractions (dilations), and other spacetime transformations can be translated relative to $\mathbf{p}_0$ (e.g., translated-boost, translated-dilator) when transforming an observable with initial position $\mathbf{p}_0$.

In special relativity, the observer $\boldsymbol{o}_{\mathcal{M}}t$ and an observable particle $\boldsymbol{p}_{\mathcal{M}}$ that is being observed must coincide at the observer's proper time $t = 0$ (i.e., their worldlines should intersect in $\mathbf{p}_0$ at time $t = 0$). If the particle and observer do not coincide at $t = 0$, then the particle has an initial spatial position $\mathbf{p}_0$ at $t = 0$ and

$$\boldsymbol{p}_{\mathcal{M}}(t) \;=\; \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0 + \mathbf{v}_{\mathcal{S}}t. \tag{3.20}$$

The translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) = \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0$ and particle then coincide at $t = 0$, and $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ can be called the *proper observer* of $\boldsymbol{p}_{\mathcal{M}}(t)$. Hyperbolic rotations (boosts) that transform a particle must be relative to a proper observer that is coincident with the particle at $t = 0$, which is the translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$. The *translated-boost* (§4.6.9) relative to a translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ is similar to a rotation around a general line through a shifted or translated origin. A translated-boost is achieved as a translation of $\mathbf{p}_0$ to the origin (translaton by $-\mathbf{p}_0$), then a boost relative to $\boldsymbol{o}_{\mathcal{M}}t$, and then a translation back by $\mathbf{p}_0$. The translated-boost can be followed by a *translated-dilation* (§4.6.7) around $\mathbf{p}_0$ with dilation factor $1/\gamma$ for a *spacetime contraction* that exits the old observer frame of coordinate time t and enters the new observer frame of proper time τ ($\gamma = \mathrm{d}t/\mathrm{d}\tau$), where τ becomes the new coordinate time. The value of γ depends on the particular boosts and velocities involved, according to velocity additions or subtractions, where $\boldsymbol{o} = \gamma c\boldsymbol{\gamma}_0$ such that γ can always be extracted (after any number of boosts) from a spacetime velocity $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ as $\boldsymbol{v} \cdot \boldsymbol{\gamma}_0/c = \gamma$. For one simple boost, γ is the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$, but γ is different after successive boosts.

3.1.5 STA spatial velocity

An STA *spatial velocity* $\mathbf{v}_{\mathcal{S}}$ has the form

$$\mathbf{v} = \mathbf{v}_{\mathcal{S}} = v_x\boldsymbol{\gamma}_1 + v_y\boldsymbol{\gamma}_2 + v_z\boldsymbol{\gamma}_3 = \beta c\hat{\mathbf{v}}. \tag{3.21}$$

An STA1 *spatial velocity* $\mathbf{v}_{\mathcal{S}^1}$ has the form

$$\mathbf{v}_{\mathcal{S}^1} = v_x\mathbf{e}_2 + v_y\mathbf{e}_3 + v_z\mathbf{e}_4. \tag{3.22}$$

An STA2 *spatial velocity* $\mathbf{v}_{\mathcal{S}^2}$ has the form

$$\mathbf{v}_{\mathcal{S}^2} = v_x\mathbf{e}_8 + v_y\mathbf{e}_9 + v_z\mathbf{e}_{10}. \tag{3.23}$$

STA spatial velocities are the same as SA spatial velocities. The v_x, v_y, and v_z are coordinate speeds in the conventional x, y, and z directions.

The non-negative *norm* of an SA spatial velocity $\mathbf{v}_{\mathcal{S}}$ is the speed

$$\|\mathbf{v}_{\mathcal{S}}\| = \sqrt{-\mathbf{v}_{\mathcal{S}}^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{3.24}$$

In special relativity, speed cannot exceed light speed c,

$$0 \leq \|\mathbf{v}_{\mathcal{S}}\| \leq c. \tag{3.25}$$

The *unit* direction of an STA or SA *spatial velocity* $\mathbf{v}_{\mathcal{S}}$ is

$$\hat{\mathbf{v}}_{\mathcal{S}} = \frac{\mathbf{v}_{\mathcal{S}}}{\|\mathbf{v}_{\mathcal{S}}\|}. \tag{3.26}$$

3.1.6 STA spatial position

An STA *spatial position* $\mathbf{p}_{\mathcal{S}}$ has the form

$$\mathbf{p}(t) = \mathbf{p}_{\mathcal{S}}(t) = \mathbf{p}_0 + \mathbf{v}_{\mathcal{S}}t = \mathbf{p}_0 + \beta c\hat{\mathbf{v}}_{\mathcal{S}}t = p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3. \tag{3.27}$$

An STA1 *spatial position* $\mathbf{p}_{\mathcal{S}^1}$ has the form

$$\mathbf{p}_{\mathcal{S}^1}(t) = \mathbf{p}_{0\mathcal{S}^1} + \mathbf{v}_{\mathcal{S}^1}t = \mathbf{p}_{0\mathcal{S}^1} + \beta c\hat{\mathbf{v}}_{\mathcal{S}^1}t = p_x\mathbf{e}_2 + p_y\mathbf{e}_3 + p_z\mathbf{e}_3. \tag{3.28}$$

An STA2 *spatial position* $\mathbf{p}_{\mathcal{S}^2}$ has the form

$$\mathbf{p}_{\mathcal{S}^2}(t) \;=\; \mathbf{p}_{0\mathcal{S}^2}+\mathbf{v}_{\mathcal{S}^2}t=\mathbf{p}_{0\mathcal{S}^2}+\beta c\hat{\mathbf{v}}_{\mathcal{S}^2}t=p_x\mathbf{e}_8+p_y\mathbf{e}_9+p_z\mathbf{e}_{10}. \tag{3.29}$$

In special relativity, the time t is called the *coordinate time* and is the *proper time* of the *observer* $\boldsymbol{o}_{\mathcal{M}}$. The spatial position $\mathbf{p}_{\mathcal{S}}$ is relative to the observer $\boldsymbol{o}_{\mathcal{M}}$ (§3.1.4) as the spacetime position (§3.1.8)

$$\boldsymbol{p}_{\mathcal{M}}(t) \;=\; \boldsymbol{o}_{\mathcal{M}}t+\mathbf{p}_{\mathcal{S}}(t)=\boldsymbol{o}_{\mathcal{M}}t+(\mathbf{p}_0+\mathbf{v}_{\mathcal{S}}t) \tag{3.30}$$

$$\;=\; (\boldsymbol{o}_{\mathcal{M}}t+\mathbf{p}_0)+\mathbf{v}_{\mathcal{S}}t=\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)+\mathbf{v}_{\mathcal{S}}t \tag{3.31}$$

$$\;=\; \mathbf{p}_0+(\boldsymbol{o}_{\mathcal{M}}+\mathbf{v}_{\mathcal{S}})t=\mathbf{p}_0+\boldsymbol{v}_{\mathcal{M}}t. \tag{3.32}$$

The spacetime velocity is $\boldsymbol{v}_{\mathcal{M}}$ (§3.1.7). The spacetime position $\boldsymbol{p}_{\mathcal{M}}$ (§3.1.8) is relative to the translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ (§3.1.4) such that $\boldsymbol{p}_{\mathcal{M}}=\mathbf{p}_0$ at time $t=0$.

3.1.7 STA spacetime velocity

An STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ has the form

$$\boldsymbol{v}=\boldsymbol{v}_{\mathcal{M}} \;=\; \boldsymbol{o}_{\mathcal{M}}+\mathbf{v}_{\mathcal{S}}=c\boldsymbol{\gamma}_0+\beta c\hat{\mathbf{v}}_{\mathcal{S}}=c\boldsymbol{\gamma}_0+(v_x\boldsymbol{\gamma}_1+v_y\boldsymbol{\gamma}_2+v_z\boldsymbol{\gamma}_3). \tag{3.33}$$

An STA1 *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}^1}$ has the normalized form

$$\boldsymbol{v}_{\mathcal{M}^1} \;=\; \boldsymbol{o}_{\mathcal{M}^1}+\mathbf{v}_{\mathcal{S}^1}=c\mathbf{e}_1+\beta c\hat{\mathbf{v}}_{\mathcal{S}^1}=c\mathbf{e}_1+(v_x\mathbf{e}_2+v_y\mathbf{e}_3+v_z\mathbf{e}_4). \tag{3.34}$$

An STA2 *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}^2}$ has the normalized form

$$\boldsymbol{v}_{\mathcal{M}^2} \;=\; \boldsymbol{o}_{\mathcal{M}^2}+\mathbf{v}_{\mathcal{S}^2}=c\mathbf{e}_7+\beta c\hat{\mathbf{v}}_{\mathcal{S}^2}=c\mathbf{e}_7+(v_x\mathbf{e}_8+v_y\mathbf{e}_9+v_z\mathbf{e}_{10}). \tag{3.35}$$

In special relativity, a spacetime velocity $\boldsymbol{v}_{\mathcal{M}}$ is the sum of an observer spacetime velocity $\boldsymbol{o}_{\mathcal{M}}$ and a spatial velocity $\mathbf{v}_{\mathcal{S}}$ relative to the observer, where $0\leq\|\mathbf{v}_{\mathcal{S}}\|\leq c$.

The *modulus* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$|\boldsymbol{v}_{\mathcal{M}}| \;=\; \sqrt{\boldsymbol{v}_{\mathcal{M}}^2}=\sqrt{c^2+\mathbf{v}_{\mathcal{S}}^2}=\sqrt{c^2-\|\mathbf{v}_{\mathcal{S}}\|^2}. \tag{3.36}$$

The square of a spacetime velocity $\boldsymbol{v}^2=c^2-\|\mathbf{v}\|^2$ may be positive, negative, or zero and represents a relative comparison of light speed to spatial speed. A spacetime velocity with positive signature $0<\boldsymbol{v}^2$ is *timelike*, with negative signature $\boldsymbol{v}^2<0$ is *spacelike*, and with null signature $\boldsymbol{v}^2=0$ is *lightlike*.

The *conjugate* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\boldsymbol{v}_{\mathcal{M}}^{\dagger} \;=\; \boldsymbol{\gamma}_0\boldsymbol{v}_{\mathcal{M}}\boldsymbol{\gamma}_0=\boldsymbol{o}_{\mathcal{M}}-\mathbf{v}_{\mathcal{S}}. \tag{3.37}$$

The *norm* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\|\boldsymbol{v}_{\mathcal{M}}\| \;=\; \sqrt{\boldsymbol{v}_{\mathcal{M}}\cdot\boldsymbol{v}_{\mathcal{M}}^{\dagger}}=\sqrt{\boldsymbol{v}_{\mathcal{M}}\cdot(\boldsymbol{\gamma}_0\boldsymbol{v}_{\mathcal{M}}\boldsymbol{\gamma}_0)}=\sqrt{c^2-\mathbf{v}_{\mathcal{S}}^2}=\sqrt{c^2+\|\mathbf{v}_{\mathcal{S}}\|^2}. \tag{3.38}$$

The *unit*, or *modulus-unit*, of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\hat{\boldsymbol{v}}_{\mathcal{M}}=\frac{\boldsymbol{v}_{\mathcal{M}}}{|\boldsymbol{v}_{\mathcal{M}}|} \;=\; \frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{\boldsymbol{o}_{\mathcal{M}}^2+\mathbf{v}_{\mathcal{S}}^2}}=\frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{c^2-v_x^2-v_y^2-v_z^2}}. \tag{3.39}$$

The *norm-unit* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\frac{\boldsymbol{v}_{\mathcal{M}}}{\|\boldsymbol{v}_{\mathcal{M}}\|} = \frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{\boldsymbol{o}_{\mathcal{M}}^2 - \mathbf{v}_{\mathcal{S}}^2}} = \frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{c^2 + v_x^2 + v_y^2 + v_z^2}}. \tag{3.40}$$

The overhat is on the modulus-unit of an STA spacetime vector $\boldsymbol{a}$ with $a_w \neq 0$ as $\hat{\boldsymbol{a}}$, but the overhat is on the norm-unit of an SA spatial vector $\mathbf{a}$ with $a_w = 0$ as $\hat{\mathbf{a}}$. In some contexts, it is explicitly noted when the overhat notation on spacetime vectors is taking the norm-unit.

There are two times associated with a spacetime velocity $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$. The *coordinate time of* $\boldsymbol{v}$, denoted $t_{c\boldsymbol{v}}$,

$$t_{c\boldsymbol{v}} = t_{p\boldsymbol{o}} = t \tag{3.41}$$

is the proper time $t_{p\boldsymbol{o}}$ of the observer $\boldsymbol{o} = c\boldsymbol{\gamma}_0$ of $\boldsymbol{v}$. The time t is the conventional notation for coordinate time. The observable $\boldsymbol{v}$ has spacetime position $\boldsymbol{v}t$ (assuming $\mathbf{p}_0 = 0$, §3.1.8) in the frame of $\boldsymbol{o}$. The *proper time of* $\boldsymbol{v}$, denoted $t_{p\boldsymbol{v}}$,

$$t_{p\boldsymbol{v}} = t_{c\boldsymbol{o}_{\boldsymbol{v}}} = \tau \tag{3.42}$$

is the coordinate time $t_{c\boldsymbol{o}_{\boldsymbol{v}}}$ of the observer $\boldsymbol{o}_{\boldsymbol{v}} \cong \boldsymbol{o}$, which is $\boldsymbol{v}$ actively transformed relative to itself in its own rest frame as (see Fig. 3.1 in §3.2.3)

$$\begin{aligned} \boldsymbol{o}_{\boldsymbol{v}} &= (\boldsymbol{v} \ominus \boldsymbol{v}) / \gamma_{\mathbf{v}}^{-1} & (3.43)\\ &= (\gamma_{\mathbf{v}} \boldsymbol{v} \ominus \boldsymbol{v}). & (3.44) \end{aligned}$$

The time τ is the conventional notation for proper time. The transformations of time by boosts can be cause for confusion. The notations $t_{c\boldsymbol{v}}$ and $t_{p\boldsymbol{v}}$ may help avoid some confusion. The (additive) "active" boosts $B_{\mathbf{v}}$ passively transform a *new* proper *time parameter* $t_{p\boldsymbol{v}} = \tau$ back into the old coordinate time $t_{p\boldsymbol{o}} = t$. The (subtractive or relative) "passive" boosts $B_{\mathbf{v}}^{\sim}$ passively transform the *same* old coordinate *time parameter* $t_{p\boldsymbol{o}} = t$ into a new proper time $t_{p\boldsymbol{v}} = \tau$. The "active" and "passive" boosts transform times in reverse of each other.

3.1.8 STA spacetime position

The STA *spacetime position* $\boldsymbol{p}_{\mathcal{M}}$ of SA spatial position $\mathbf{p}_{\mathcal{S}}$ relative to observer $\boldsymbol{o}_{\mathcal{M}}t$ is

$$\begin{aligned} \boldsymbol{p}(t) = \boldsymbol{p}_{\mathcal{M}}(t) &= \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_{\mathcal{S}}(t) = ct\boldsymbol{\gamma}_0 + (\mathbf{p}_0 + \mathbf{v}_{\mathcal{S}}t) & (3.45)\\ &= (\boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0) + \mathbf{v}_{\mathcal{S}}t = \boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) + \mathbf{v}_{\mathcal{S}}t & (3.46)\\ &= \mathbf{p}_0 + (\boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}})t = \mathbf{p}_0 + \boldsymbol{v}_{\mathcal{M}}t. & (3.47) \end{aligned}$$

In special relativity, if the initial position $\mathbf{p}_0 \neq 0$, then the proper observer that coincides with $\boldsymbol{p}_{\mathcal{M}}$ at $t = 0$ is the translated observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$. A spacetime boost of $\boldsymbol{p}_{\mathcal{M}}$ should be relative to the proper observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ by using a translated-boost around $\mathbf{p}_0$. CSTA has a versor for translated-boosts (§4.6.9).

The time t derivative of $\boldsymbol{p}_{\mathcal{M}}$ is

$$\dot{\boldsymbol{p}}_{\mathcal{M}} = \partial_t \boldsymbol{p}_{\mathcal{M}} = \frac{\partial \boldsymbol{p}_{\mathcal{M}}}{\partial t} = \boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}} = \boldsymbol{v}_{\mathcal{M}}. \tag{3.48}$$

The *modulus-unit*

$$\hat{\boldsymbol{p}}_{\mathcal{M}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{|\boldsymbol{p}_{\mathcal{M}}|} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{\boldsymbol{p}_{\mathcal{M}}^2}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{(ct)^2 - \|\mathbf{p}_{\mathcal{S}}\|^2}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{p_w^2 - p_x^2 - p_y^2 - p_z^2}} \tag{3.49}$$

and the *norm-unit*

$$\frac{\boldsymbol{p}_{\mathcal{M}}}{\|\boldsymbol{p}_{\mathcal{M}}\|} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{p}_{\mathcal{M}}^{\dagger}}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{(ct)^2 + \|\mathbf{p}_{\mathcal{S}}\|^2}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{p_w^2 + p_x^2 + p_y^2 + p_z^2}} \tag{3.50}$$

of an STA spacetime position $\boldsymbol{p}_{\mathcal{M}}$ are similar to those of an STA spacetime velocity $\boldsymbol{v}_{\mathcal{M}}$.

The square of a spacetime position $\boldsymbol{p}_{\mathcal{M}}^2$ is the *spacetime interval* between the origin of spacetime and $\boldsymbol{p}_{\mathcal{M}}$. Likewise, $(\boldsymbol{p}_{\mathcal{M}_2} - \boldsymbol{p}_{\mathcal{M}_1})^2$ is the spacetime interval between $\boldsymbol{p}_{\mathcal{M}_1}$ and $p_{\mathcal{M}_2}$. The passive boost of a spacetime position $\boldsymbol{p}_{\mathcal{M}}$ to become relative to the frame of a new observer preserves the spacetime interval $\boldsymbol{p}_{\mathcal{M}}^2$.

3.2 STA operations

3.2.1 STA dualization

The STA *dual* $A_{\mathcal{M}}^{*\mathcal{M}}$ of an STA multivector $A_{\mathcal{M}}$ is

$$A_{\mathcal{M}}^{*} = A_{\mathcal{M}}^{*\mathcal{M}} = A_{\mathcal{M}}\mathbf{I}_{\mathcal{M}}^{-1} = -A_{\mathcal{M}}\mathbf{I}_{\mathcal{M}}. \tag{3.51}$$

The STA *undual* $A_{\mathcal{M}}$ of an STA multivector $A_{\mathcal{M}}^{*\mathcal{M}}$ is

$$A_{\mathcal{M}} = A_{\mathcal{M}}^{*}\mathbf{I}_{\mathcal{M}} = A_{\mathcal{M}}\mathbf{I}_{\mathcal{M}}^{-1}\mathbf{I}_{\mathcal{M}}. \tag{3.52}$$

3.2.2 STA rotor

The STA spatial rotation operator, or *rotor*, $R_{\mathcal{M}} = R_{\mathcal{S}}$ is the SA rotor $R_{\mathcal{S}}$ (§2.6).

The STA 2-versor spatial *rotor* $R_{\mathcal{S}}$ for rotation in SA space around the SA unit vector *axis* $\hat{\mathbf{x}}_{\mathcal{S}}$ by *angle* θ is

$$\begin{aligned} R_{\mathcal{S}} &= e^{\frac{1}{2}\theta\hat{\mathbf{x}}_{\mathcal{S}}^{*\mathcal{S}}} = e^{\frac{1}{2}\theta\hat{\mathbf{x}}_{\mathcal{S}}\widetilde{\mathbf{I}_{\mathcal{S}}}} & (3.53)\\ &= \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right)\hat{\mathbf{x}}_{\mathcal{S}}\widetilde{\mathbf{I}_{\mathcal{S}}} & (3.54)\\ &= \cos\left(\frac{1}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)\hat{\mathbf{x}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1}. & (3.55)\end{aligned}$$

The rotor operation

$$A'_{\mathcal{M}} = R_{\mathcal{S}}A_{\mathcal{M}}\widetilde{R_{\mathcal{S}}} \tag{3.56}$$

rotates any multivector $A_{\mathcal{M}}$ in STA as expected in the spatial SA components, but leaves the STA timelike components unchanged.

3.2.3 STA spacetime boost

Introduction

Although it would be sufficient to just define the *boost operator*

$$B_{\mathcal{M}} = B_{\mathbf{v}} = e^{\frac{1}{2}\varphi\hat{\mathbf{v}}\gamma_0} \tag{3.57}$$

for a boost by spacetime velocity

$$\boldsymbol{v} \;=\; \boldsymbol{o}+\mathbf{v}=c\gamma_0+\beta c\hat{\mathbf{v}} \tag{3.58}$$

with natural speed $\beta=\beta_{\mathbf{v}}$ and rapidity

$$\varphi=\varphi_{\mathbf{v}} \;=\; \operatorname{atanh}(\beta_{\mathbf{v}})=\operatorname{atanh}(\|\mathbf{v}\|/c), \tag{3.59}$$

this section also attempts to discuss some of the basics of active and passive boosts, and velocity addition and subtraction.

The exponential function

The following functions and identities are frequently used to define versors.

$$\exp(A)=e^A \;=\; \sum_{n=0}^{\infty}\frac{A^n}{n!}=\cosh(A)+\sinh(A), \quad \text{where } A \text{ is any multivector [13]} \tag{3.60}$$

$$\cosh(A) \;=\; \sum_{n=0}^{\infty}\frac{A^{2n}}{(2n)!}=\frac{e^A+e^{-A}}{2} \tag{3.61}$$

$$\sinh(A) \;=\; \sum_{n=0}^{\infty}\frac{A^{2n+1}}{(2n+1)!}=\frac{e^A-e^{-A}}{2} \tag{3.62}$$

$$\tanh(A) \;=\; \frac{\sinh(A)}{\cosh(A)}=\frac{e^A-e^{-A}}{e^A+e^{-A}} \tag{3.63}$$

$$\cosh(iA) \;=\; \cos(A), \quad \text{where } i^2=-1 \text{ and } iA=Ai \tag{3.64}$$

$$\sinh(iA) \;=\; i\sin(A) \tag{3.65}$$

$$\cosh(jA) \;=\; \cosh(A), \quad \text{where } j^2=1 \text{ and } jA=Aj \tag{3.66}$$

$$\sinh(jA) \;=\; j\sinh(A) \tag{3.67}$$

$$\cosh(\varepsilon A) \;=\; 1, \quad \text{where } \varepsilon^2=0 \text{ and } \varepsilon A=A\varepsilon \tag{3.68}$$

$$\sinh(\varepsilon A) \;=\; \varepsilon A \tag{3.69}$$

$$\cosh\left(\frac{1}{2}\operatorname{atanh}(\beta_{\mathbf{v}})\right) \;=\; \frac{(1+\beta_{\mathbf{v}})+\sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{3.70}$$

$$\sinh\left(\frac{1}{2}\operatorname{atanh}(\beta_{\mathbf{v}})\right) \;=\; \frac{(1+\beta_{\mathbf{v}})-\sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{3.71}$$

$$\cosh(\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{3.72}$$

$$\sinh(\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{\beta_{\mathbf{v}}}{\sqrt{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{3.73}$$

$$\cosh(2\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{1+\beta_{\mathbf{v}}^2}{1-\beta_{\mathbf{v}}^2}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{3.74}$$

$$\sinh(2\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{2\beta_{\mathbf{v}}}{1-\beta_{\mathbf{v}}^2}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{3.75}$$

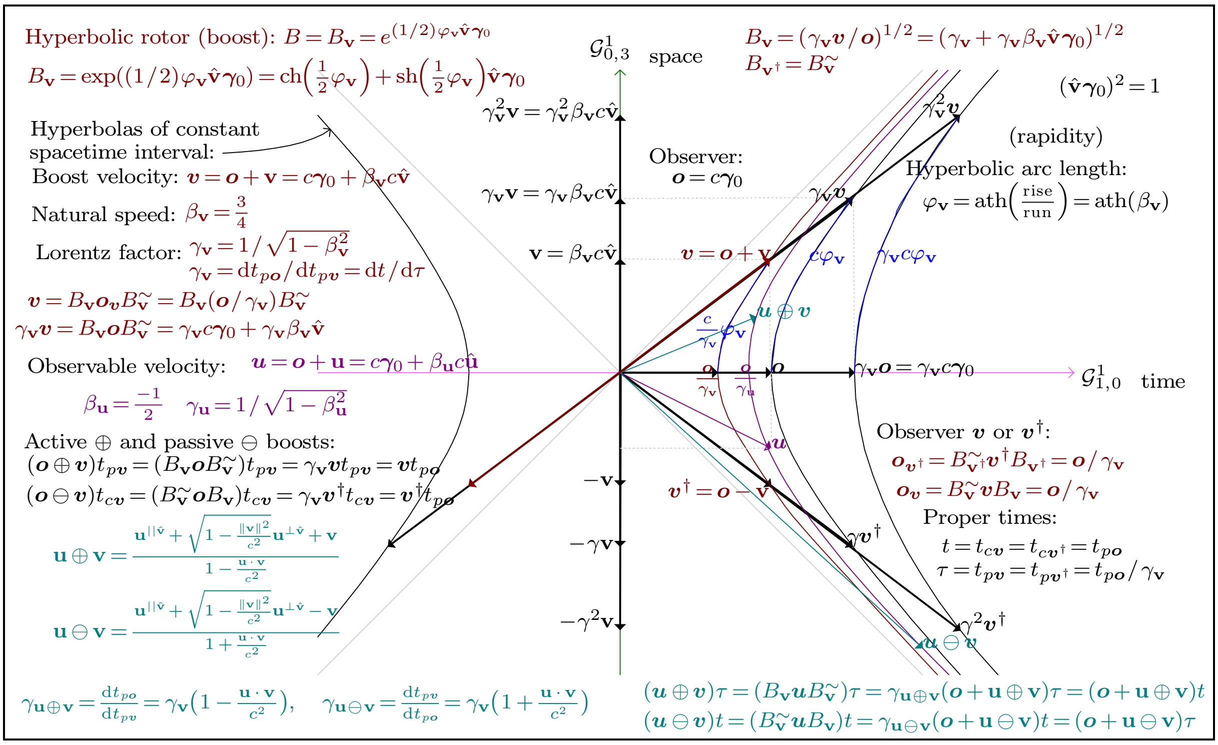
Hyperbolic rotor (boost): $B = B_{\mathbf{v}} = e^{(1/2)\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}$
$B_{\mathbf{v}} = \exp((1/2)\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0) = \mathrm{ch}\left(\frac{1}{2}\varphi_{\mathbf{v}}\right) + \mathrm{sh}\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\hat{\mathbf{v}}\boldsymbol{\gamma}_0$
Hyperbolas of constant spacetime interval:
Boost velocity: $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}}c\hat{\mathbf{v}}$
Natural speed: $\beta_{\mathbf{v}} = \frac{3}{4}$
Lorentz factor: $\gamma_{\mathbf{v}} = 1/\sqrt{1-\beta_{\mathbf{v}}^2}$
$\gamma_{\mathbf{v}} = \mathrm{d}t_{p\boldsymbol{o}}/\mathrm{d}t_{p\boldsymbol{v}} = \mathrm{d}t/\mathrm{d}\tau$
$\boldsymbol{v} = B_{\mathbf{v}}\boldsymbol{o}_{\boldsymbol{v}}B^{\sim}_{\mathbf{v}} = B_{\mathbf{v}}(\boldsymbol{o}/\gamma_{\mathbf{v}})B^{\sim}_{\mathbf{v}}$
$\gamma_{\mathbf{v}}\boldsymbol{v} = B_{\mathbf{v}}\boldsymbol{o}B^{\sim}_{\mathbf{v}} = \gamma_{\mathbf{v}}c\boldsymbol{\gamma}_0 + \gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}$
Observable velocity: $\boldsymbol{u} = \boldsymbol{o} + \mathbf{u} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{u}}c\hat{\mathbf{u}}$
$\beta_{\mathbf{u}} = \frac{-1}{2}$ $\gamma_{\mathbf{u}} = 1/\sqrt{1-\beta_{\mathbf{u}}^2}$
Active $\oplus$ and passive $\ominus$ boosts:
$(\boldsymbol{o}\oplus\boldsymbol{v})t_{p\boldsymbol{v}} = (B_{\mathbf{v}}\boldsymbol{o}B^{\sim}_{\mathbf{v}})t_{p\boldsymbol{v}} = \gamma_{\mathbf{v}}\boldsymbol{v}t_{p\boldsymbol{v}} = \boldsymbol{v}t_{p\boldsymbol{o}}$
$(\boldsymbol{o}\ominus\boldsymbol{v})t_{c\boldsymbol{v}} = (B^{\sim}_{\mathbf{v}}\boldsymbol{o}B_{\mathbf{v}})t_{c\boldsymbol{v}} = \gamma_{\mathbf{v}}\boldsymbol{v}^{\dagger}t_{c\boldsymbol{v}} = \boldsymbol{v}^{\dagger}t_{p\boldsymbol{o}}$
$\mathbf{u}\oplus\mathbf{v} = \dfrac{\mathbf{u}^{||\hat{\mathbf{v}}} + \sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}} + \mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}$
$\mathbf{u}\ominus\mathbf{v} = \dfrac{\mathbf{u}^{||\hat{\mathbf{v}}} + \sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}} - \mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}$
$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \frac{\mathrm{d}t_{p\boldsymbol{o}}}{\mathrm{d}t_{p\boldsymbol{v}}} = \gamma_{\mathbf{v}}\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right),$ $\gamma_{\mathbf{u}\ominus\mathbf{v}} = \frac{\mathrm{d}t_{p\boldsymbol{v}}}{\mathrm{d}t_{p\boldsymbol{o}}} = \gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)$
$\mathcal{G}^1_{0,3}$ space
$\gamma_{\mathbf{v}}^2\mathbf{v} = \gamma_{\mathbf{v}}^2\beta_{\mathbf{v}}c\hat{\mathbf{v}}$
$\gamma_{\mathbf{v}}\mathbf{v} = \gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}}$
$\mathbf{v} = \beta_{\mathbf{v}}c\hat{\mathbf{v}}$
$-\mathbf{v}$
$-\gamma\mathbf{v}$
$-\gamma^2\mathbf{v}$
Observer: $\boldsymbol{o} = c\boldsymbol{\gamma}_0$
$B_{\mathbf{v}} = (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{1/2} = (\gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0)^{1/2}$
$B_{\mathbf{v}^{\dagger}} = B^{\sim}_{\mathbf{v}}$
$(\hat{\mathbf{v}}\boldsymbol{\gamma}_0)^2 = 1$
$\gamma_{\mathbf{v}}^2\boldsymbol{v}$
(rapidity)
Hyperbolic arc length:
$\varphi_{\mathbf{v}} = \mathrm{ath}\left(\frac{\mathrm{rise}}{\mathrm{run}}\right) = \mathrm{ath}(\beta_{\mathbf{v}})$
$\gamma_{\mathbf{v}}\boldsymbol{v}$
$\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$
$c\varphi_{\mathbf{v}}$
$\gamma_{\mathbf{v}}c\varphi_{\mathbf{v}}$
$\boldsymbol{u}\oplus\boldsymbol{v}$
$\frac{c}{\gamma_{\mathbf{v}}}\varphi_{\mathbf{v}}$
$\frac{\boldsymbol{o}}{\gamma_{\mathbf{v}}}$
$\frac{\boldsymbol{o}}{\gamma_{\mathbf{u}}}$
$\boldsymbol{o}$
$\gamma_{\mathbf{v}}\boldsymbol{o} = \gamma_{\mathbf{v}}c\boldsymbol{\gamma}_0$
$\mathcal{G}^1_{1,0}$ time
$\boldsymbol{u}$
$\boldsymbol{v}^{\dagger} = \boldsymbol{o} - \mathbf{v}$
Observer $\boldsymbol{v}$ or $\boldsymbol{v}^{\dagger}$:
$\boldsymbol{o}_{\boldsymbol{v}^{\dagger}} = B^{\sim}_{\mathbf{v}^{\dagger}}\boldsymbol{v}^{\dagger}B_{\mathbf{v}^{\dagger}} = \boldsymbol{o}/\gamma_{\mathbf{v}}$
$\boldsymbol{o}_{\boldsymbol{v}} = B^{\sim}_{\mathbf{v}}\boldsymbol{v}B_{\mathbf{v}} = \boldsymbol{o}/\gamma_{\mathbf{v}}$
Proper times:
$t = t_{c\boldsymbol{v}} = t_{c\boldsymbol{v}^{\dagger}} = t_{p\boldsymbol{o}}$
$\tau = t_{p\boldsymbol{v}} = t_{p\boldsymbol{v}^{\dagger}} = t_{p\boldsymbol{o}}/\gamma_{\mathbf{v}}$
$\gamma\boldsymbol{v}^{\dagger}$
$\gamma^2\boldsymbol{v}^{\dagger}$
$\boldsymbol{u}\ominus\boldsymbol{v}$
$(\boldsymbol{u}\oplus\boldsymbol{v})\tau = (B_{\mathbf{v}}\boldsymbol{u}B^{\sim}_{\mathbf{v}})\tau = \gamma_{\mathbf{u}\oplus\mathbf{v}}(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})\tau = (\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})t$
$(\boldsymbol{u}\ominus\boldsymbol{v})t = (B^{\sim}_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}})t = \gamma_{\mathbf{u}\ominus\mathbf{v}}(\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})t = (\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\tau$

Figure 3.1. Spacetime diagram of observables $\boldsymbol{o}$, $\boldsymbol{v}$, and $\boldsymbol{u}$

Figure 3.1 shows the spacetime diagram of spacetime velocities for observer $\boldsymbol{o}$, boost observable $\boldsymbol{v}$, and observable $\boldsymbol{u}$. The time axis is horizontal and the space axis is vertical. The hyperbolic angle (rapidity) φ is positive anticlockwise. This orientation of the spacetime diagram of hyperbolic rotations by φ is analogous to circular rotations by an angle θ. By circular rotations, points are translated along a circle through a circular arc $r\theta$. By hyperbolic rotations, points are translated along a hyperbola through a hyperbolic arc $r\varphi$. For both circular and hyperbolic rotations, the radius r is an invariant distance (interval) from the origin to a point. A circular radius is a positive real scalar r, and a real hyperbolic radius (pseudoradius) is $r \in \left\{\frac{c}{\gamma}, c, \gamma c\right\}$ for the hyperbolas of constant spacetime interval that are shown in the figure. The boost operator $B_{\mathbf{v}}$ rotates from $\boldsymbol{o}$ toward $\boldsymbol{v}$ by $\varphi_{\mathbf{v}}$, which corresponds to a change of speed by $\beta_{\mathbf{v}}c$ in the direction $\hat{\mathbf{v}}$. The speed $\beta_{\mathbf{v}}c$ of a worldline is its slope in the diagram, $\gamma_{\mathbf{v}}$ is the Lorentz (time dilation) factor, and c is the speed of light. In the rest frame of an observable, the observable is the observer having proper time τ, zero spatial speed $\beta c = 0$, and a worldline $\boldsymbol{o}\tau = c\tau\boldsymbol{\gamma}_0$. An observable worldline $\boldsymbol{v}t$ that is hyperbolically rotated into the time axis by a passive boost

$$B^{\sim}_{\mathbf{v}}(\boldsymbol{v}t)B_{\mathbf{v}} \;=\; B^{\sim}_{\mathbf{v}}\boldsymbol{v}B_{\mathbf{v}}t = \boldsymbol{o}t/\gamma_{\mathbf{v}} = \boldsymbol{o}\tau \tag{3.76}$$

gives the proper time τ rest frame worldline $\boldsymbol{o}\tau$ of the observable $\boldsymbol{v}$ relative to the observer $\boldsymbol{o}t/\gamma_{\mathbf{v}}$. A hyperbolic rotation (boost) of an observable velocity

$$\boldsymbol{u} \;=\; \boldsymbol{o} + \mathbf{u} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{u}}c\hat{\mathbf{u}}, \qquad \text{for } 0 \le \beta_{\mathbf{u}} < 1, \tag{3.77}$$

preserves its spacetime interval

$$r_{\boldsymbol{u}} \;=\; |\boldsymbol{u}| = \sqrt{\boldsymbol{u}^2} = \sqrt{c^2 - \beta_{\mathbf{u}}^2 c^2} = c\sqrt{1-\beta_{\mathbf{u}}} = c/\gamma_{\mathbf{u}} \tag{3.78}$$

$$\;=\; |B_{\mathbf{v}}\boldsymbol{u}B^{\sim}_{\mathbf{v}}| = |B^{\sim}_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}| = |\boldsymbol{u}||B_{\mathbf{v}}\hat{\boldsymbol{u}}B^{\sim}_{\mathbf{v}}| = |\boldsymbol{u}||B^{\sim}_{\mathbf{v}}\hat{\boldsymbol{u}}B_{\mathbf{v}}|, \qquad \text{for } \boldsymbol{u}^2 \neq 0, \tag{3.79}$$

such that all boosted vectors remain on their hyperbola of constant spacetime interval. A null vector (at light speed) $\boldsymbol{u}^2 = 0$ is not on any hyperbola and cannot be boosted. A hyperbolic rotation by a negative hyperbolic angle is a passive transformation from current reference frame with time t into a new frame with time τ at the positive angle of rotation, representing a relativistic velocity subtraction $\mathbf{u} \ominus \mathbf{v}$. A hyperbolic rotation by a positive hyperbolic angle is an active transformation of a velocity vector into the boosted frame of the boost observable $\boldsymbol{v}$ with new coordinate time τ, representing a relativistic velocity addition $\mathbf{u} \oplus \mathbf{v}$. After a passive boost, the time is t, which passively transforms into a relative τ. After an active boost, the time is τ, which passively transforms into a relative t. The passive boost of a position $\boldsymbol{p}$ that has an initial position $\mathbf{p}_0$

$$\boldsymbol{p}(t) \;=\; \mathbf{p}_0 + \dot{\boldsymbol{p}} t = \mathbf{p}_0 + (\boldsymbol{o} + \dot{\mathbf{p}})t \tag{3.80}$$

is valid, but it should be performed using a translated-boost (§4.6.9) as

$$\begin{aligned}
B_{\mathbf{v}}^{\mathbf{p}_0 \sim} \boldsymbol{p} B_{\mathbf{v}}^{\mathbf{p}_0} &= \mathbf{p}_0 + B_{\mathbf{v}}^{\sim} \dot{\boldsymbol{p}} B_{\mathbf{v}} t && (3.81)\\
&= \mathbf{p}_0 + B_{\mathbf{v}}^{\sim} (\boldsymbol{o} + \dot{\mathbf{p}}) B_{\mathbf{v}} t && (3.82)\\
&= \mathbf{p}_0 + \gamma_{\dot{\mathbf{p}} \ominus \mathbf{v}} (\boldsymbol{o} + \dot{\mathbf{p}} \ominus \mathbf{v}) t && (3.83)\\
&= \mathbf{p}_0 + (\boldsymbol{o} + \dot{\mathbf{p}} \ominus \mathbf{v}) \tau, && (3.84)
\end{aligned}$$

which preserves the initial position $\mathbf{p}_0$ at time $t = \tau = 0$.

Derivation of boost operator

An "active" boost operation is a hyperbolic rotation operation in spacetime (Fig. 3.1) that passively turns (transforms) an observer spacetime velocity

$$\boldsymbol{o} \;=\; c\boldsymbol{\gamma}_0, \tag{3.85}$$

which has zero spatial velocity in its own frame with coordinate time t, into a relative spacetime velocity

$$\begin{aligned}
\boldsymbol{o}' &= B_{\mathbf{v}} \boldsymbol{o} B_{\mathbf{v}}^{\sim} = B_{\mathbf{v}^\dagger}^{\sim} \boldsymbol{o} B_{\mathbf{v}^\dagger} && (3.86)\\
&= \boldsymbol{o} \oplus \boldsymbol{v} = \boldsymbol{o} \ominus \boldsymbol{v}^\dagger && (3.87)\\
&= \gamma_{\mathbf{v}} \boldsymbol{v} = \gamma_{\mathbf{v}} (\boldsymbol{o} + \mathbf{v}) && (3.88)\\
&= \gamma_{\mathbf{v}} c \boldsymbol{\gamma}_0 + \gamma_{\mathbf{v}} \beta_{\mathbf{v}} c \hat{\mathbf{v}} && (3.89)
\end{aligned}$$

that is relative to ($\ominus$) the new observer $\boldsymbol{v}^\dagger = \boldsymbol{o} - \mathbf{v}$ with proper time $\tau = t_{\boldsymbol{v}^\dagger}$, consistent with special relativity. The "active" boost operator is $B_{\mathbf{v}}$, which is a "passive" boost operator $B_{\mathbf{v}^\dagger}^{\sim}$. Boosts operate on spacetime velocities, not on positions. Following a velocity boost, *the new time parameter*, which is to be multiplied into a transformed velocity $\boldsymbol{o}'$ as a spacetime displacement $\boldsymbol{o}'\tau$, *is the proper time τ of $\boldsymbol{v}^\dagger$*, not the coordinate time t of $\boldsymbol{o}$. The time and spatial displacement of $\boldsymbol{o}'\tau$ is passively transformed into coordinate time $t = \gamma_{\mathbf{v}} \tau$ and spatial displacement $\mathbf{d} = \gamma_{\mathbf{v}} \tau \mathbf{v} = \mathbf{v} t$ as seen by the coordinate time t observer $\boldsymbol{o}$ and corresponds to (but is not) a time τ and displacement seen by $\boldsymbol{v}^\dagger$. That is,

$$\begin{aligned}
\boldsymbol{o}'\tau &= (\gamma_{\mathbf{v}} c \boldsymbol{\gamma}_0 + \gamma_{\mathbf{v}} \beta_{\mathbf{v}} c \hat{\mathbf{v}}) \tau && (3.90)\\
&= ct\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}} ct \hat{\mathbf{v}} && (3.91)\\
&= \boldsymbol{o} t + \mathbf{v} t && (3.92)\\
&= \boldsymbol{o} t + \mathbf{d}. && (3.93)
\end{aligned}$$

A boost can be applied to any spacetime velocity, but its boosted speed can never exceed light speed c relative to any observer.

To derive the boost operator, we can start by defining the *ratio* of spacetime velocities of an observable (particle) $\boldsymbol{v}$ to its coordinate time t observer $\boldsymbol{o}$ as the *hyperbolic biradial* $\boldsymbol{v}/\boldsymbol{o}=\boldsymbol{v}\boldsymbol{o}^{-1}$ ("$\boldsymbol{v}$ by $\boldsymbol{o}$"). The term *biradial* was coined by HAMILTON in his original work on Quaternions [12].

The *hyperbolic biradial*

$$H = \boldsymbol{v}\boldsymbol{o}^{-1}=\frac{|\boldsymbol{v}|}{|\boldsymbol{o}|}\widehat{\boldsymbol{v}/\boldsymbol{o}}=\frac{\sqrt{c^2+\mathbf{v}^2}}{c}\hat{H} \tag{3.94}$$

$$= \sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\hat{H}=\sqrt{1-\beta_{\mathbf{v}}^2}\hat{H}=\frac{1}{\gamma_{\mathbf{v}}}\hat{H} \tag{3.95}$$

is an operator that *turns* the spacetime velocity of the observer $\boldsymbol{o}$ into the spacetime velocity of the boost particle

$$\boldsymbol{v} = \boldsymbol{o}+\mathbf{v} \tag{3.96}$$

as the one-sided versor operation

$$\boldsymbol{v} = H\boldsymbol{o}. \tag{3.97}$$

The *natural speed* $\beta_{\mathbf{v}}$ of the velocity $\mathbf{v}$ is

$$\beta_{\mathbf{v}} = \frac{\|\mathbf{v}\|}{c}=\frac{\sqrt{\mathbf{v}\cdot\mathbf{v}^{\dagger}}}{c}=\frac{\sqrt{-\mathbf{v}^2}}{c}=\frac{v}{c}. \tag{3.98}$$

The *Lorentz factor* (*spacetime dilation factor*) $\gamma_{\mathbf{v}}$ of the velocity $\mathbf{v}$ is

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}=\frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{3.99}$$

where $t=t_{c\boldsymbol{v}}=t_{p\boldsymbol{o}}$ is the coordinate time of $\boldsymbol{v}$ and $\tau=t_{p\boldsymbol{v}}$ is the proper time of $\boldsymbol{v}$.

The interval (pseudodistance) $c\tau$ is the hyperbolic arc length along the worldline of $\boldsymbol{v}$

$$c\mathrm{d}\tau = |\boldsymbol{v}|\mathrm{d}t=\sqrt{\boldsymbol{v}^2}\mathrm{d}t=\sqrt{c^2-\beta_{\mathbf{v}}^2c^2}\mathrm{d}t=\frac{c}{\gamma_{\mathbf{v}}}\mathrm{d}t \tag{3.100}$$

$$\tau = \int_0^t\frac{1}{\gamma_{\mathbf{v}}}\mathrm{d}t=\frac{t}{\gamma_{\mathbf{v}}}, \qquad \text{where } \tau=0 \text{ when } t=0. \tag{3.101}$$

Maximum $c\tau$ is when $\beta_{\mathbf{v}}=0$, such that inertial observers experience maximum time.

The *length contraction*, to length L from an initial length L_0 in the direction of boost velocity $\mathbf{v}$, is given by

$$L = \frac{L_0}{\gamma_{\mathbf{v}}}=L_0\sqrt{1-\beta_{\mathbf{v}}^2}. \tag{3.102}$$

The *dilation factor* of the velocity $\mathbf{v}$ is

$$d = \frac{1}{\gamma_{\mathbf{v}}} = \sqrt{1-\beta_{\mathbf{v}}^2}. \tag{3.103}$$

For a dilation factor d, the required natural speed is $\beta_{\mathbf{v}} = \sqrt{1-d^2}$. For $d \leq 1$, the dilation factor d can be called the *spacetime contraction factor*, which is the usual case. For $d > 1$, then $\beta_{\mathbf{v}}$ is an *imaginary* natural speed and it is possible to dilate lengths instead of contract lengths, but dilated lengths are only geometrical effects, not physics effects.

The *hyperbolic versor* $\hat{H}$ is the *unit hyperbolic biradial*

$$\begin{aligned}
\hat{H} &= \gamma_{\mathbf{v}} H = \gamma_{\mathbf{v}} \boldsymbol{v}\boldsymbol{o}^{-1} = \frac{\gamma_{\mathbf{v}}}{c}\boldsymbol{v}\boldsymbol{\gamma}_0 & (3.104)\\
&= \frac{\gamma_{\mathbf{v}}}{c}(\boldsymbol{v}\cdot\boldsymbol{\gamma}_0 + \boldsymbol{v}\wedge\boldsymbol{\gamma}_0) = \gamma_{\mathbf{v}} + \frac{\gamma_{\mathbf{v}}}{c}\mathbf{v}\boldsymbol{\gamma}_0 & (3.105)\\
&= \gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\frac{\|\mathbf{v}\|}{c}\hat{\mathbf{v}}\boldsymbol{\gamma}_0 = \gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0 & (3.106)\\
&= \cosh(\varphi_{\mathbf{v}}) + \sinh(\varphi_{\mathbf{v}})\hat{\mathbf{v}}\boldsymbol{\gamma}_0 = \exp(\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0) = e^{\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}, & (3.107)
\end{aligned}$$

where

$$\begin{aligned}
j^2 = (\hat{\mathbf{v}}\boldsymbol{\gamma}_0)^2 = (\hat{\mathbf{v}}\wedge\boldsymbol{\gamma}_0)^2 &= 1 & (3.108)\\
\gamma_{\mathbf{v}} &= \cosh(\varphi_{\mathbf{v}}) & (3.109)\\
\gamma_{\mathbf{v}}\beta_{\mathbf{v}} &= \sinh(\varphi_{\mathbf{v}}) & (3.110)\\
\beta_{\mathbf{v}} &= \tanh(\varphi_{\mathbf{v}}) = \frac{\sinh(\varphi_{\mathbf{v}})}{\cosh(\varphi_{\mathbf{v}})} & (3.111)\\
\varphi_{\mathbf{v}} &= \operatorname{atanh}(\beta_{\mathbf{v}}). & (3.112)
\end{aligned}$$

Using half of the *hyperbolic angle (rapidity)* $\varphi_{\mathbf{v}}$, the *hyperbolic rotation operator (hyperbolic rotor or boost operator)* $B_{\mathbf{v}}$ is the *square root of the hyperbolic versor*

$$\begin{aligned}
B_{\mathbf{v}} = \hat{H}^{\frac{1}{2}} &= \exp\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right) = e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} & (3.113)\\
&= \cosh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right) + \sinh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\hat{\mathbf{v}}\boldsymbol{\gamma}_0 & (3.114)\\
&= \frac{(1+\beta_{\mathbf{v}}) + \sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}} + \frac{(1+\beta_{\mathbf{v}}) - \sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0 \qquad \text{for } -1 < \beta_{\mathbf{v}} < 1. & (3.115)
\end{aligned}$$

The hyperbolic rotation by a natural speed $|\beta_{\mathbf{v}}| = 1$, corresponding to rapidity $|\varphi_{\mathbf{v}}| = \infty$, is invalid since it represents reaching the light speed asymptote $\boldsymbol{l} = c\boldsymbol{\gamma}_0 + c\hat{\mathbf{v}}$ (a null vector), which can never be reached on the hyperbola of constant (invariant) spacetime interval.

While the *hyperbolic versor* $\hat{H}$ is a one-sided versor, the *hyperbolic rotor (boost)* $B_{\mathbf{v}}$ is a two-sided versor or "sandwiching" versor with its reverse $B_{\mathbf{v}}^{\sim} = B_{\mathbf{v}}^{-1}$, such that

$$\begin{aligned}
\boldsymbol{v} &= \hat{H}\boldsymbol{o}/\gamma_{\mathbf{v}} = B_{\mathbf{v}}^2\boldsymbol{o}/\gamma_{\mathbf{v}} & (3.116)\\
&= B_{\mathbf{v}}\boldsymbol{o}B_{\mathbf{v}}^{\sim}/\gamma_{\mathbf{v}}. & (3.117)
\end{aligned}$$

The one-sided versor operation is valid only for collinear (coplanar in spacetime) velocity boosts, while the two-sided versor operation is valid for general boosts of any velocity that need not be collinear with $\mathbf{v}$. This is similar to the difference in quaternion rotations between planar rotation using the one-sided versor operation $e^{\theta\hat{\mathbf{n}}}\mathbf{r}^{\perp\hat{\mathbf{n}}}$ and conical rotation using the two-sided versor "sandwich" operation $e^{\frac{1}{2}\theta\hat{\mathbf{n}}}\mathbf{r}e^{-\frac{1}{2}\theta\hat{\mathbf{n}}} = \mathbf{r}^{||\hat{\mathbf{n}}} + e^{\theta\hat{\mathbf{n}}}\mathbf{r}^{\perp\hat{\mathbf{n}}}$.

Although boosts are generally valid on spacetime velocities and not generally valid on spacetime positions since the time after a velocity boost is subject to interpretation, *it is valid* to passively boost a spacetime position of the form $\boldsymbol{u}t = (\boldsymbol{o} + \mathbf{u})t$, which is the product of the spacetime velocity $\boldsymbol{u}$ and its coordinate time $t = t_{p\boldsymbol{o}} = t_{c\boldsymbol{u}}$. After a passive boost, the time to be applied to the boosted velocity is still t, which can be correctly factored out of the passive boost operation as

$$B_{\mathbf{v}}^{\sim}(\boldsymbol{u}t)B_{\mathbf{v}} \;=\; B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}t. \tag{3.118}$$

The passive boost is interpreted as transforming coordinate time t into the proper time $\tau = t_{p\boldsymbol{v}}$ of $\boldsymbol{v}$, and transforming distance into the distance relative to observable $\boldsymbol{v}$. Evaluating this passive position boost gives some useful results, as follows.

$$\begin{aligned}
B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}t &= e^{-\frac{1}{2}\hat{\mathbf{v}}\gamma_0}(\boldsymbol{o}+\mathbf{u})e^{\frac{1}{2}\hat{\mathbf{v}}\gamma_0}t = (\boldsymbol{u}\ominus\boldsymbol{v})t & (3.119)\\
&= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}(c\boldsymbol{\gamma}_0 + \mathbf{u}^{||\hat{\mathbf{v}}} + \mathbf{u}^{\perp\hat{\mathbf{v}}})(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}t & (3.120)\\
&= (c\boldsymbol{\gamma}_0 + \mathbf{u}^{||\hat{\mathbf{v}}})(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})t + \mathbf{u}^{\perp\hat{\mathbf{v}}}t & (3.121)\\
&= (c\boldsymbol{\gamma}_0 + \mathbf{u}^{||\hat{\mathbf{v}}})(\gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0)t + \mathbf{u}^{\perp\hat{\mathbf{v}}}t & (3.122)\\
&= (\gamma_{\mathbf{v}}c\boldsymbol{\gamma}_0 - \gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}} + \gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}} + \gamma_{\mathbf{v}}\beta_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0)t + \mathbf{u}^{\perp\hat{\mathbf{v}}}t & (3.123)\\
&= \left(\left(\gamma_{\mathbf{v}} + \frac{1}{c}\gamma_{\mathbf{v}}\beta_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}\hat{\mathbf{v}}\right)c\boldsymbol{\gamma}_0 + \gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}} + \mathbf{u}^{\perp\hat{\mathbf{v}}} - \gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}}\right)t & (3.124)\\
&= (\boldsymbol{o}' + \mathbf{u}')t = (\gamma_{\mathbf{u}\ominus\mathbf{v}}\boldsymbol{o} + \gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v})t. & (3.125)
\end{aligned}$$

The time transformation is

$$\begin{aligned}
\tau &= \gamma_{\mathbf{u}\ominus\mathbf{v}}t & (3.126)\\
&= \left(\gamma_{\mathbf{v}} + \frac{1}{c}\gamma_{\mathbf{v}}\beta_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}\hat{\mathbf{v}}\right)t & (3.127)\\
&= \gamma_{\mathbf{v}}\left(1 + \frac{1}{c}\beta_{\mathbf{v}}(\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}^{-1}\hat{\mathbf{v}}\right)t & (3.128)\\
&= \gamma_{\mathbf{v}}\left(1 + \frac{1}{c}\frac{\|\mathbf{v}\|}{c}\frac{(\mathbf{u}\cdot\mathbf{v})}{\|\mathbf{v}\|}\right)t & (3.129)\\
&= \gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)t. & (3.130)
\end{aligned}$$

The distance transformation is

$$\begin{aligned}
\mathbf{u}'t &= (\gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v})t & (3.131)\\
&= (\mathbf{u}\ominus\mathbf{v})\tau & (3.132)\\
&= \frac{\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}} + \mathbf{u}^{\perp\hat{\mathbf{v}}} - \gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}}}{\gamma_{\mathbf{u}\ominus\mathbf{v}}}\tau & (3.133)\\
&= \frac{\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}} + \mathbf{u}^{\perp\hat{\mathbf{v}}} - \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}\ominus\mathbf{v}}}\tau & (3.134)\\
&= \frac{\mathbf{u}^{||\hat{\mathbf{v}}} + \sqrt{1-\beta_{\mathbf{v}}^2}\,\mathbf{u}^{\perp\hat{\mathbf{v}}} - \mathbf{v}}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\tau. & (3.135)
\end{aligned}$$

Summary of useful results:

$$\begin{aligned}
\boldsymbol{u}t &= (\boldsymbol{o}+\mathbf{u})t=(c\boldsymbol{\gamma}_0+\beta_{\mathbf{u}}c\hat{\mathbf{u}})t && (3.136)\\
\boldsymbol{v}t &= (\boldsymbol{o}+\mathbf{v})t=(c\boldsymbol{\gamma}_0+\beta_{\mathbf{v}}c\hat{\mathbf{v}})t && (3.137)\\
t &= t_{p\boldsymbol{o}}=t_{c\boldsymbol{v}}=t_{c\boldsymbol{u}}=t_{c\boldsymbol{u}\ominus\boldsymbol{v}}=t_{c\mathbf{u}\oplus\mathbf{v}} && (3.138)\\
\tau &= t_{p\boldsymbol{v}}=t_{c\boldsymbol{u}\oplus\boldsymbol{v}}=t_{c\mathbf{u}\ominus\mathbf{v}} && (3.139)\\
B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}t &= (\boldsymbol{u}\ominus\boldsymbol{v})t=(\gamma_{\mathbf{u}\ominus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v})t=(\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\tau && (3.140)\\
B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{\sim}\tau &= (\boldsymbol{u}\oplus\boldsymbol{v})\tau=(\gamma_{\mathbf{u}\oplus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}}\mathbf{u}\oplus\mathbf{v})\tau=(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})t && (3.141)\\
\gamma_{\mathbf{u}\ominus\mathbf{v}} &= \gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) && (3.142)\\
\gamma_{\mathbf{u}\oplus\mathbf{v}} &= \gamma_{\mathbf{v}}\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) && (3.143)\\
\mathbf{u}\ominus\mathbf{v}\tau &= \frac{\mathbf{u}^{||\hat{\mathbf{v}}}+\sqrt{1-\beta_{\mathbf{v}}^2}\mathbf{u}^{\perp\hat{\mathbf{v}}}-\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\tau=\frac{(\mathbf{u}\cdot\mathbf{v})\mathbf{v}^{-1}+\sqrt{1-\beta_{\mathbf{v}}^2}(\mathbf{u}\wedge\mathbf{v})\mathbf{v}^{-1}-\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\tau && (3.144)\\
\mathbf{u}\oplus\mathbf{v}t &= \frac{\mathbf{u}^{||\hat{\mathbf{v}}}+\sqrt{1-\beta_{\mathbf{v}}^2}\mathbf{u}^{\perp\hat{\mathbf{v}}}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}t=\frac{(\mathbf{u}\cdot\mathbf{v})\mathbf{v}^{-1}+\sqrt{1-\beta_{\mathbf{v}}^2}(\mathbf{u}\wedge\mathbf{v})\mathbf{v}^{-1}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}t. && (3.145)
\end{aligned}$$

Active boost

The "active" boost $\boldsymbol{u}\oplus\boldsymbol{v}$ of spacetime velocity

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{o}+\mathbf{u}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{u}}c\hat{\mathbf{u}} && (3.146)\\
t_{c\boldsymbol{u}} &= t=t_{p\boldsymbol{o}}, && (3.147)
\end{aligned}$$

by spacetime velocity

$$\begin{aligned}
\boldsymbol{v} &= \boldsymbol{o}+\mathbf{v}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{v}}c\hat{\mathbf{v}} && (3.148)\\
t_{c\boldsymbol{v}} &= t=t_{p\boldsymbol{o}} && (3.149)\\
t_{p\boldsymbol{v}} &= \tau, && (3.150)
\end{aligned}$$

is the "active" boost operation

$$\begin{aligned}
\boldsymbol{u}\oplus\boldsymbol{v} &= B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{\sim}=B_{\mathbf{v}^\dagger}^{\sim}\boldsymbol{u}B_{\mathbf{v}^\dagger} && (3.151)\\
&= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}=(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{-\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{\frac{1}{2}} && (3.152)\\
&= \exp\left(\frac{1}{2}\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right)\boldsymbol{u}\exp\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}\right)=\exp\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}^\dagger\right)\boldsymbol{u}\exp\left(\frac{1}{2}\varphi\hat{\mathbf{v}}^\dagger\boldsymbol{\gamma}_0\right) && (3.153)\\
&= \boldsymbol{u}\oplus\boldsymbol{v} \quad = \quad \boldsymbol{u}\ominus\boldsymbol{v}^\dagger && (3.154)\\
&= \boldsymbol{u}\oplus(\boldsymbol{o}+\mathbf{v}) \quad = \quad \boldsymbol{u}\ominus(\boldsymbol{o}-\mathbf{v}) && (3.155)\\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}}\mathbf{u}\oplus\mathbf{v} \quad = \quad \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}\boldsymbol{o}+\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}\mathbf{u}\ominus\mathbf{v}^\dagger && (3.156)\\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\oplus\mathbf{v}}\beta_{\mathbf{u}\oplus\mathbf{v}}c\widehat{\mathbf{u}\oplus\mathbf{v}} \quad = \quad \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}\beta_{\mathbf{u}\ominus\mathbf{v}^\dagger}c\widehat{\mathbf{u}\ominus\mathbf{v}^\dagger}. && (3.157)
\end{aligned}$$

The hyperbolic angle (rapidity) is

$$\begin{aligned}
\varphi=\varphi_{\mathbf{v}}=\varphi_{\mathbf{v}^\dagger} &= \operatorname{atanh}(\beta_{\mathbf{v}})=\operatorname{atanh}\left(\frac{|\mathbf{v}|}{|\boldsymbol{o}|}\right)=\operatorname{atanh}\left(\frac{v}{c}\right)=\operatorname{atanh}\left(\frac{\text{rise}}{\text{run}}\right) && (3.158)\\
\varphi_{\mathbf{v}^\dagger}\mathbf{v}^\dagger &= -\varphi_{\mathbf{v}}\mathbf{v}. && (3.159)
\end{aligned}$$

The Lorentz factor for $\mathbf{v}$ is

$$\gamma_{\mathbf{v}}=\gamma_{\mathbf{v}^\dagger} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}}=\frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}. \qquad (3.160)$$

The spacetime dilation factor is

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger} \;=\; \gamma_{\mathbf{v}}\Big(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\Big). \tag{3.161}$$

The spatial relativistic velocity addition is

$$\mathbf{u}\oplus\mathbf{v} = \mathbf{u}\ominus\mathbf{v}^\dagger \;=\; \frac{\mathbf{u}^{||\hat{\mathbf{v}}} + \sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\,\mathbf{u}^{\perp\hat{\mathbf{v}}} + \mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}. \tag{3.162}$$

The natural speed of the spatial velocity addition $\mathbf{u}\oplus\mathbf{v}$ is

$$\beta_{\mathbf{u}\oplus\mathbf{v}} = \beta_{\mathbf{u}\ominus\mathbf{v}^\dagger} \;=\; \frac{\|\mathbf{u}\oplus\mathbf{v}\|}{c} = \frac{\sqrt{(\mathbf{u}\oplus\mathbf{v})\cdot(\mathbf{u}\oplus\mathbf{v})^\dagger}}{c}. \tag{3.163}$$

For parallel velocities $\mathbf{u}||\mathbf{v}$, then

$$\beta_{\mathbf{u}\oplus\mathbf{v}} \;=\; \frac{\beta_{\mathbf{u}}+\beta_{\mathbf{v}}}{1+\beta_{\mathbf{u}}\beta_{\mathbf{v}}}, \qquad \text{for } \mathbf{u}||\mathbf{v}; \mathbf{u} = \beta_{\mathbf{u}} c\hat{\mathbf{v}}. \tag{3.164}$$

For perpendicular velocities $\mathbf{u}\perp\mathbf{v}$, then

$$\beta_{\mathbf{u}\oplus\mathbf{v}} \;=\; \sqrt{(1-\beta_{\mathbf{v}}^2)\beta_{\mathbf{u}}^2+\beta_{\mathbf{v}}^2} \qquad \text{for } \mathbf{u}\perp\mathbf{v}. \tag{3.165}$$

Time transformations for active boost

Although $\boldsymbol{u}\oplus\boldsymbol{v}$ and $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$ are the same spacetime velocities $\boldsymbol{u}\oplus\boldsymbol{v} = \boldsymbol{u}\ominus\boldsymbol{v}^\dagger$, their associated time transformations are different. For $\boldsymbol{u}\oplus\boldsymbol{v}$, the boost operator is $(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}$ and $\boldsymbol{v}/\boldsymbol{o}$ transforms time t into τ internal to the boost operator, and then $\gamma_{\mathbf{v}}$ transforms time τ back into t external to the boost operator. Therefore, the time is τ for $\boldsymbol{u}\oplus\boldsymbol{v}$ and the displacement is

$$(\boldsymbol{u}\oplus\boldsymbol{v})\tau \;=\; (\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})\gamma_{\mathbf{u}\oplus\mathbf{v}}\tau \tag{3.166}$$

$$\;=\; (\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})t \tag{3.167}$$

that passively transforms τ into t. For $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$, the boost operator is $(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{-\frac{1}{2}}$ and time transformations are reciprocal such that time τ transforms into t internal to the boost operator, and then time t transforms back into τ external to the boost operator. Therefore, the time is t for $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$ and the displacement is

$$(\boldsymbol{u}\ominus\boldsymbol{v}^\dagger)t \;=\; (\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v}^\dagger)\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t \tag{3.168}$$

$$\;=\; (\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v}^\dagger)\tau \tag{3.169}$$

that passively transforms t into τ.

An "active" boost $(\boldsymbol{u}\oplus\boldsymbol{v})\tau$ passively transforms time τ into $t = \gamma_{\mathbf{u}\oplus\mathbf{v}}\tau$, from the frame of $\boldsymbol{v}$ into the frame of $\boldsymbol{o}$. The boost $\boldsymbol{u}\oplus\boldsymbol{v}$ adds/moves $\boldsymbol{u}$ into the frame of $\boldsymbol{v}$ with time $\tau = t_{\boldsymbol{pv}}$ that passively transforms to coordinate time $t = t_{\boldsymbol{po}}$ such that observer $\boldsymbol{o}$ sees a dilated time $t = \gamma_{\mathbf{u}\oplus\mathbf{v}}\tau$ and a velocity addition $\gamma_{\mathbf{u}\oplus\mathbf{v}}\tau\mathbf{u}\oplus\mathbf{v} = t\mathbf{u}\oplus\mathbf{v}$. The spacetime contraction $(\boldsymbol{u}\oplus\boldsymbol{v}/\gamma_{\mathbf{u}\oplus\mathbf{v}})t$ is an active velocity addition in the frame of $\boldsymbol{o}$.

A "passive" boost $(\boldsymbol{u}\ominus\boldsymbol{v}^\dagger)t$ passively transforms time t into $\tau = \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t$, from the frame of $\boldsymbol{o}$ into the frame of $\boldsymbol{v}^\dagger$. The boost $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$ subtracts/moves $\boldsymbol{u}$ from the frame of $\boldsymbol{v}^\dagger$ into the frame of $\boldsymbol{o}$ with time $t = t_{\boldsymbol{po}}$ that passively transforms to time $\tau = t_{\boldsymbol{pv}^\dagger}$ such that observer $\boldsymbol{v}^\dagger$ sees a relative dilated time $\tau = \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t$ and a relative velocity subtraction $\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t\mathbf{u}\ominus\mathbf{v}^\dagger = \tau\mathbf{u}\ominus\mathbf{v}^\dagger$. The spacetime contraction $(\boldsymbol{u}\ominus\boldsymbol{v}^\dagger/\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger})t$ is an active velocity subtraction in the frame of $\boldsymbol{o}$.

Passive boost

The "passive" boost is the reverse of the "active" boost. Therefore, the following is very similar to the "active" boost, but with everything going in reverse.

The "passive" boost $\boldsymbol{u}\ominus\boldsymbol{v}$ of spacetime velocity

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{o}+\mathbf{u}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{u}}c\hat{\mathbf{u}} && (3.170)\\ t_{c\boldsymbol{u}} &= t=t_{p\boldsymbol{o}}, && (3.171)\end{aligned}$$

by spacetime velocity

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{o}+\mathbf{v}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{v}}c\hat{\mathbf{v}} && (3.172)\\ t_{c\boldsymbol{v}} &= t=t_{p\boldsymbol{o}}, && (3.173)\end{aligned}$$

is the "passive" boost operation

$$\begin{aligned}\boldsymbol{u}\ominus\boldsymbol{v} &= B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}=B_{\mathbf{v}^\dagger}\boldsymbol{u}B_{\mathbf{v}^\dagger}^{\sim} && (3.174)\\ &= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}=(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{-\frac{1}{2}} && (3.175)\\ &= \exp\!\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}\right)\boldsymbol{u}\exp\!\left(\frac{1}{2}\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right)=\exp\!\left(\frac{1}{2}\varphi\hat{\mathbf{v}}^\dagger\boldsymbol{\gamma}_0\right)\boldsymbol{u}\exp\!\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}^\dagger\right) && (3.176)\\ &= \boldsymbol{u}\ominus\boldsymbol{v} \quad = \quad \boldsymbol{u}\oplus\boldsymbol{v}^\dagger && (3.177)\\ &= \boldsymbol{u}\ominus(\boldsymbol{o}+\mathbf{v}) \quad = \quad \boldsymbol{u}\oplus(\boldsymbol{o}-\mathbf{v}) && (3.178)\\ &= \gamma_{\mathbf{u}\ominus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v} \quad = \quad \gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\mathbf{u}\oplus\mathbf{v}^\dagger && (3.179)\\ &= \gamma_{\mathbf{u}\ominus\mathbf{v}}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\ominus\mathbf{v}}\beta_{\mathbf{u}\ominus\mathbf{v}}c\widehat{\mathbf{u}\ominus\mathbf{v}} \quad = \quad \gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\beta_{\mathbf{u}\oplus\mathbf{v}^\dagger}c\widehat{\mathbf{u}\oplus\mathbf{v}^\dagger}. && (3.180)\end{aligned}$$

The hyperbolic angle (rapidity) is

$$\begin{aligned}\varphi=\varphi_{\mathbf{v}}=\varphi_{\mathbf{v}^\dagger} &= \operatorname{atanh}(\beta_{\mathbf{v}})=\operatorname{atanh}\!\left(\frac{|\mathbf{v}|}{|\boldsymbol{o}|}\right)=\operatorname{atanh}\!\left(\frac{v}{c}\right)=\operatorname{atanh}\!\left(\frac{\text{rise}}{\text{run}}\right) && (3.181)\\ \varphi_{\mathbf{v}^\dagger}\mathbf{v}^\dagger &= -\varphi_{\mathbf{v}}\mathbf{v}. && (3.182)\end{aligned}$$

The Lorentz factor for $\mathbf{v}$ is

$$\gamma_{\mathbf{v}}=\gamma_{\mathbf{v}^\dagger} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}}=\frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}. \qquad (3.183)$$

The spacetime dilation factor is

$$\gamma_{\mathbf{u}\oplus\mathbf{v}}=\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger} = \gamma_{\mathbf{v}}\!\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right). \qquad (3.184)$$

The spatial relativistic velocity subtraction is

$$\mathbf{u}\ominus\mathbf{v}=\mathbf{u}\oplus\mathbf{v}^\dagger = \frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}}-\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}. \qquad (3.185)$$

The natural speed of the spatial velocity subtraction $\mathbf{u}\ominus\mathbf{v}$ is

$$\beta_{\mathbf{u}\ominus\mathbf{v}}=\beta_{\mathbf{u}\oplus\mathbf{v}^\dagger} = \frac{\|\mathbf{u}\ominus\mathbf{v}\|}{c}=\frac{\sqrt{(\mathbf{u}\ominus\mathbf{v})\cdot(\mathbf{u}\ominus\mathbf{v})^\dagger}}{c}. \qquad (3.186)$$

For parallel velocities $\mathbf{u}||\mathbf{v}$, then

$$\beta_{\mathbf{u}\ominus\mathbf{v}} = \frac{\beta_{\mathbf{u}}-\beta_{\mathbf{v}}}{1-\beta_{\mathbf{u}}\beta_{\mathbf{v}}} \qquad \text{for } \mathbf{u}||\mathbf{v}; \mathbf{u}=\beta_{\mathbf{u}}c\hat{\mathbf{v}}. \tag{3.187}$$

For perpendicular velocities $\mathbf{u}\perp\mathbf{v}$, then

$$\beta_{\mathbf{u}\ominus\mathbf{v}} = \sqrt{(1-\beta_{\mathbf{v}}^2)\beta_{\mathbf{u}}^2+\beta_{\mathbf{v}}^2} \qquad \text{for } \mathbf{u}\perp\mathbf{v}. \tag{3.188}$$

Time transformations for passive boost

Although $\boldsymbol{u}\ominus\boldsymbol{v}$ and $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$ are the same spacetime velocities $\boldsymbol{u}\ominus\boldsymbol{v}=\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$, their associated time transformations are different. For $\boldsymbol{u}\ominus\boldsymbol{v}$, the boost operator is $(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}$ and the reciprocal of $\boldsymbol{v}/\boldsymbol{o}$ transforms time τ into t internal to the boost operator, and then the reciprocal of $\gamma_{\mathbf{v}}$ transforms time t back into τ external to the boost operator. Therefore, the time is t for $\boldsymbol{u}\ominus\boldsymbol{v}$ and the displacement is

$$(\boldsymbol{u}\ominus\boldsymbol{v})t = (\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\gamma_{\mathbf{u}\ominus\mathbf{v}}t \tag{3.189}$$
$$= (\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\tau \tag{3.190}$$

that passively transforms t into τ. For $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$, the boost operator is $(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{\frac{1}{2}}$ and time t transforms into τ internal to the boost operator, and then time τ transforms back into t external to the boost operator. Therefore, the time is τ for $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$ and the displacement is

$$(\boldsymbol{u}\oplus\boldsymbol{v}^\dagger)\tau = (\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v}^\dagger)\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau \tag{3.191}$$
$$= (\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v}^\dagger)t \tag{3.192}$$

that passively transforms τ into t.

A "passive" boost $(\boldsymbol{u}\ominus\boldsymbol{v})t$ passively transforms time t into $\tau=\gamma_{\mathbf{u}\ominus\mathbf{v}}t$, from the frame of $\boldsymbol{o}$ into the frame of $\boldsymbol{v}$. The boost $\boldsymbol{u}\ominus\boldsymbol{v}$ subtracts/moves $\boldsymbol{u}$ from the frame of $\boldsymbol{v}$ into the frame of $\boldsymbol{o}$ with time $t=t_{p\boldsymbol{o}}$ that passively transforms to time $\tau=t_{p\boldsymbol{v}}$ such that observer $\boldsymbol{v}$ sees a relative dilated time $\tau=\gamma_{\mathbf{u}\ominus\mathbf{v}}t$ and a relative velocity subtraction $\gamma_{\mathbf{u}\ominus\mathbf{v}}t\mathbf{u}\ominus\mathbf{v}=\tau\mathbf{u}\ominus\mathbf{v}$. The spacetime contraction $(\boldsymbol{u}\ominus\boldsymbol{v}/\gamma_{\mathbf{u}\ominus\mathbf{v}})t$ is an active velocity subtraction in the frame of $\boldsymbol{o}$.

An "active" boost $(\boldsymbol{u}\oplus\boldsymbol{v}^\dagger)\tau$ passively transforms time τ into $t=\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau$, from the frame of $\boldsymbol{v}^\dagger$ into the frame of $\boldsymbol{o}$. The boost $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$ adds/moves $\boldsymbol{u}$ into the frame of $\boldsymbol{v}^\dagger$ with time $\tau=t_{p\boldsymbol{v}^\dagger}$ that passively transforms to coordinate time $t=t_{p\boldsymbol{o}}$ such that observer $\boldsymbol{o}$ sees a dilated time $t=\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau$ and a velocity addition $\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau\mathbf{u}\oplus\mathbf{v}^\dagger=t\mathbf{u}\oplus\mathbf{v}^\dagger$. The spacetime contraction $(\boldsymbol{u}\oplus\boldsymbol{v}^\dagger/\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger})t$ is an active velocity addition in the frame of $\boldsymbol{o}$.

Generalization of spacetime contraction operation

A spacetime velocity $\boldsymbol{u}$ that has been boosted successively has the general form

$$\boldsymbol{u}' = \gamma c\boldsymbol{\gamma}_0+\gamma\beta c\hat{\mathbf{u}}'. \tag{3.193}$$

In general, the spacetime contraction is

$$\boldsymbol{u}'' = \boldsymbol{u}'/\gamma = c\frac{\boldsymbol{u}'}{\boldsymbol{u}'\cdot\boldsymbol{\gamma}_0}. \tag{3.194}$$

The spacetime velocity $\boldsymbol{u}''$ can be interpreted as some combination of active velocity additions and subtractions in the frame of observer $\boldsymbol{o}$. Or, $\boldsymbol{u}''$ can be interpreted as a spacetime velocity transformed into the frame of a new observer $\boldsymbol{v}\to\boldsymbol{o}_{\boldsymbol{v}}$ with time $\tau=t_{p\boldsymbol{v}}=t_{c\boldsymbol{u}''}$.

If $\boldsymbol{u}=\boldsymbol{o}+\mathbf{u}$ is in the contracted frame of observable $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ with proper time τ, where $\boldsymbol{u}$ sees $\boldsymbol{v}$ as its observer $\boldsymbol{o}$, but $\boldsymbol{v}$ sees its observer $\boldsymbol{o}$ as the coordinate time t observer, then $\boldsymbol{u}$ can be transformed into the contracted frame of the coordinate time t observer as the active velocity addition with spacetime contraction

$$\begin{align}
\boldsymbol{u}\oplus\boldsymbol{v}/\gamma_{\mathbf{u}\oplus\mathbf{v}} &= B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{\sim}/\gamma_{\mathbf{u}\oplus\mathbf{v}} \tag{3.195}\\
&= \left(\gamma_{\mathbf{u}\oplus\mathbf{v}}c\gamma_0+\gamma_{\mathbf{u}\oplus\mathbf{v}}\beta_{\mathbf{u}\oplus\mathbf{v}}c\widehat{\mathbf{u}\oplus\mathbf{v}}\right)/\gamma_{\mathbf{u}\oplus\mathbf{v}} \tag{3.196}\\
&= \boldsymbol{o}+\mathbf{u}\oplus\mathbf{v} \tag{3.197}\\
&= \boldsymbol{o}+\frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}. \tag{3.198}
\end{align}$$

The transformation of this back into the contracted frame of $\boldsymbol{v}$ is

$$\begin{align}
B_{\mathbf{v}}^{\sim}(\boldsymbol{u}\oplus\boldsymbol{v}/\gamma_{\mathbf{u}\oplus\mathbf{v}})B_{\mathbf{v}}/\gamma &= B_{\mathbf{v}}^{\sim}(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})B_{\mathbf{v}}/\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}} \tag{3.199}\\
&= (\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}}\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v})/\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}} \tag{3.200}\\
&= \boldsymbol{o}+\mathbf{u}=\boldsymbol{u}. \tag{3.201}
\end{align}$$

where

$$\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v} = \mathbf{u} \tag{3.202}$$

but

$$\begin{align}
\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}} &\neq \gamma_{\mathbf{u}} \tag{3.203}\\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}}^{-1}. \tag{3.204}
\end{align}$$

The generalization, for transformation into the new contracted frame after passive boosts of $\boldsymbol{u}$, is to divide by the general spacetime contraction factor

$$\gamma = \frac{\boldsymbol{u}'\cdot\gamma_0}{c} \tag{3.205}$$

as

$$\boldsymbol{u}'' = \boldsymbol{u}'/\gamma=c\frac{\boldsymbol{u}'}{\boldsymbol{u}'\cdot\gamma_0}=\boldsymbol{o}+\mathbf{u}' \tag{3.206}$$

where $\boldsymbol{o}$ represents the new observer with new coordinate time $t_{p\boldsymbol{o}}=t_{c\boldsymbol{u}''}$.

Approximations

For boost speed $\|\mathbf{v}\|\ll c$ and initial speed $\|\mathbf{u}\|\ll c$, an active boost is approximately an addition of velocities

$$\mathbf{u}\oplus\mathbf{v} \approx \mathbf{u}+\mathbf{v}, \qquad \text{for } \|\mathbf{u}\|,\|\mathbf{v}\|\ll c. \tag{3.207}$$

and a passive boost is approximately the subtraction of velocities

$$\mathbf{u}\ominus\mathbf{v} \approx \mathbf{u}-\mathbf{v}, \qquad \text{for } \|\mathbf{u}\|,\|\mathbf{v}\|\ll c. \tag{3.208}$$

For a very small speed $(v=\beta_{\mathbf{v}}c)\ll c$, then $\beta_{\mathbf{v}}\ll 1$ and the rapidity $\varphi_{\mathbf{v}}$ is approximately equal to $\beta_{\mathbf{v}}$

$$\begin{align}
\varphi_{\mathbf{v}} &= \operatorname{atanh}(\beta_{\mathbf{v}}) \tag{3.209}\\
&\approx \beta_{\mathbf{v}}, \qquad \text{for } \beta_{\mathbf{v}}\ll 1. \tag{3.210}
\end{align}$$

For $\beta_{\mathbf{v}} \ll 1$, the proper velocity (celerity) $\varphi_{\mathbf{v}} c\hat{\mathbf{v}}$ is approximately equal to velocity

$$\varphi_{\mathbf{v}} c\hat{\mathbf{v}} \;\approx\; (\beta_{\mathbf{v}} c\hat{\mathbf{v}} = \mathbf{v}), \qquad \text{for } \beta_{\mathbf{v}} \ll 1. \tag{3.211}$$

For a small speed $v \ll c$, the Lorentz factor $\gamma_{\mathbf{v}} = 1/\sqrt{1 - v^2/c^2} \approx 1$.

Using the hyperbolic function composition identities, it can be shown that the boost operator that applies the boost $\beta_{\mathbf{v}}$ *twice* successively, and adds the rapidity $2\varphi_{\mathbf{v}}$, is

$$B_{\mathbf{v}} B_{\mathbf{v}} = B_{\beta_{\mathbf{v}}} B_{\beta_{\mathbf{v}}} \;=\; \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{1-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0 \tag{3.212}$$

and the boost operator that applies the boost $\frac{1}{2}\beta_{\mathbf{v}}$ *twice* successively is

$$B_{\frac{1}{2}\beta_{\mathbf{v}}} B_{\frac{1}{2}\beta_{\mathbf{v}}} \;=\; \frac{2}{\sqrt{4-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{4-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0. \tag{3.213}$$

The double boost operator $B_{\beta_{\mathbf{v}}} B_{\beta_{\mathbf{v}}}$ can be defined as successive reflections in two space-time planes through the origin, where the first plane contains the observer $\boldsymbol{o}$ and the second plane contains the boost observable (particle) $\boldsymbol{v}$. The two planes bound the hyperbolic angle $\varphi_{\mathbf{v}}$ that turns from the first plane at $\beta_{\boldsymbol{o}} = 0$ into the second plane at $\beta_{\mathbf{v}}$, toward the direction in space of the boost velocity $\mathbf{v}$.

For very small $\beta_{\mathbf{v}} \ll c$, then $\varphi_{\mathbf{v}} = \operatorname{atanh}(\beta_{\mathbf{v}}) \approx \beta_{\mathbf{v}}$ and then

$$\begin{aligned} B_{\mathbf{v}} \;&=\; e^{\frac{1}{2}\varphi_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} = e^{\frac{1}{4}\varphi_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} e^{\frac{1}{4}\varphi_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} = B_{\frac{1}{2}\varphi_{\mathbf{v}}} B_{\frac{1}{2}\varphi_{\mathbf{v}}} & (3.214) \\ &\approx\; \frac{2}{\sqrt{4-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{4-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0 = B_{\frac{1}{2}\beta_{\mathbf{v}}} B_{\frac{1}{2}\beta_{\mathbf{v}}} & (3.215) \\ &\approx\; e^{\frac{1}{2}\beta_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} = \cosh\!\left(\frac{1}{2}\beta_{\mathbf{v}}\right) + \sinh\!\left(\frac{1}{2}\beta_{\mathbf{v}}\right) \hat{\mathbf{v}} \boldsymbol{\gamma}_0 & (3.216) \end{aligned}$$

The good approximation of $B_{\mathbf{v}}$ for very small $\beta_{\mathbf{v}} \ll c$ is

$$B_{\mathbf{v}} \;\approx\; \frac{2}{\sqrt{4-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{4-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0. \tag{3.217}$$

4 Conformal Space-Time Algebra (CSTA)

$\mathcal{G}_{2,4}$ Conformal Space-Time Algebra (CSTA) is introduced in [3] as the *spacetime conformal group*.

$\mathcal{G}_{2,4}$ CSTA is a straightforward extension and adaptation of the $\mathcal{G}_{4,1}$ Conformal Geometric Algebra (CGA). CGA is introduced by Hestenes, Li, and Rockwood in [20]. CGA is also discussed by Perwass in [16], and by Dorst, Fontijne, and Mann in [4].

$\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) $\mathcal{D}$ contains two copies of $\mathcal{G}_{2,4}$ CSTA $\mathcal{C}$, which are called CSTA1 $\mathcal{C}^1$ and CSTA2 $\mathcal{C}^2$. Elements and operations in CSTA1 are subscripted with $\mathcal{C}^1$. Elements and operations in CSTA2 are subscripted with $\mathcal{C}^2$. Elements and operations in generic CSTA are subscripted with $\mathcal{C}$.

Most formulas are expressed in CSTA $\mathcal{C}$ and written explicitly in CSTA1 $\mathcal{C}^1$ and CSTA2 $\mathcal{C}^2$ only when helpful to see how the particular CSTA1 and CSTA2 elements are used in formulas. Most formulas in CSTA can be written in CSTA1 and CSTA2 by just changing subscripts. CSTA uses the origin $\mathbf{e}_{o\gamma}$ and infinity $\mathbf{e}_{\infty\gamma}$ points and the Dirac gammas $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$ for the timelike $w = ct$ and spatial x, y, z axes, respectively. The generic CSTA point embedding is $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$.

4.1 CSTA unit pseudoscalar

The $\mathcal{G}_{2,4}$ CSTA 6-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{C}}$ with signature $(+---+-)$ is

$$\begin{aligned}\mathbf{I}_{\mathcal{C}} &= \mathbf{I}_{\mathcal{M}}(\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}) = \boldsymbol{\gamma}_0\mathbf{I}_{\mathcal{S}}(\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}) = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3(\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}) = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\mathbf{e}_+\mathbf{e}_- && (4.1)\\ \mathbf{I}_{\widetilde{\mathcal{C}}} &= (-1)^{6(6-1)/2}\mathbf{I}_{\mathcal{C}} = -\mathbf{I}_{\mathcal{C}} && (4.2)\\ \mathbf{I}_{\mathcal{C}}^2 &= -1 && (4.3)\\ \mathbf{I}_{\mathcal{C}}^{-1} &= -\mathbf{I}_{\mathcal{C}} = \mathbf{I}_{\widetilde{\mathcal{C}}}. && (4.4)\end{aligned}$$

The $\mathcal{G}_{2,4}$ CSTA1 6-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{C}^1}$ with signature $(+---+-)$ is

$$\mathbf{I}_{\mathcal{C}^1} = \mathbf{I}_{\mathcal{M}^1}(\mathbf{e}_{\infty1}\wedge\mathbf{e}_{o1}) = \mathbf{e}_1\mathbf{I}_{\mathcal{S}^1}(\mathbf{e}_{\infty1}\wedge\mathbf{e}_{o1}) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4(\mathbf{e}_{\infty1}\wedge\mathbf{e}_{o1}) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6. \quad (4.5)$$

The $\mathcal{G}_{2,4}$ CSTA2 6-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{C}^2}$ with signature $(+---+-)$ is

$$\mathbf{I}_{\mathcal{C}^2} = \mathbf{I}_{\mathcal{M}^2}(\mathbf{e}_{\infty2}\wedge\mathbf{e}_{o2}) = \mathbf{e}_7\mathbf{I}_{\mathcal{S}^2}(\mathbf{e}_{\infty2}\wedge\mathbf{e}_{o2}) = \mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}(\mathbf{e}_{\infty2}\wedge\mathbf{e}_{o2}) = \mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}\mathbf{e}_{11}\mathbf{e}_{12}. \quad (4.6)$$

4.2 CSTA point

The CSTA null 1-vector *point* entity is very similar to the CGA null 1-vector *point* entity. The following subsections define the CSTA points at the origin and at infinity, and the CSTA point embedding.

4.2.1 Stereographic embedding and homogenization

The embedding of an $\mathcal{G}_{1,3}$ STA position *vector* $\boldsymbol{p}_{\mathcal{M}}$ into a $\mathcal{G}_{2,4}$ CSTA null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}}$ is done in exactly the same way a $\mathcal{G}_3$ APS point $\mathbf{p}$ is embedded into a $\mathcal{G}_{4,1}$ CGA point $\mathbf{P}_{\mathcal{C}}$. There are many references that explain the stereographic embedding and homogenization, such as PERWASS [16], ROSENHAHN [18], and the paper on $\mathcal{G}_{8,2}$ DCGA [7].

4.2.2 CSTA point at the origin

The CSTA null 1-vector *point at the origin* is defined as

$$\mathbf{e}_{o\gamma} = \frac{1}{2}(-\mathbf{e}_+ + \mathbf{e}_-) \quad (4.7)$$

where $\mathbf{e}_+$ is the stereographic unit and $\mathbf{e}_-$ is the homogeneous unit, and

$$\mathbf{e}_+ = \begin{cases}\mathbf{e}_5 & : \text{ in CSTA1}\\ \mathbf{e}_{11} & : \text{ in CSTA2}\end{cases} \quad (4.8)$$

$$\mathbf{e}_- = \begin{cases}\mathbf{e}_6 & : \text{ in CSTA1}\\ \mathbf{e}_{12} & : \text{ in CSTA2.}\end{cases} \quad (4.9)$$

The CSTA1 null 1-vector *point at the origin* is defined as

$$\mathbf{e}_{o1} = \frac{1}{2}(-\mathbf{e}_5 + \mathbf{e}_6). \quad (4.10)$$

The CSTA2 null 1-vector *point at the origin* is defined as

$$\mathbf{e}_{o2} = \frac{1}{2}(-\mathbf{e}_{11} + \mathbf{e}_{12}). \quad (4.11)$$

The CSTA null 1-vector *point at the origin* $\mathbf{e}_{o\gamma}$ represents either $\mathbf{e}_{o1}$ or $\mathbf{e}_{o2}$.

4.2.3 CSTA point at infinity

The CSTA null 1-vector *point at infinity* is defined as

$$\mathbf{e}_{\infty\gamma} \;=\; \mathbf{e}_{+}+\mathbf{e}_{-}. \tag{4.12}$$

The CSTA1 null 1-vector *point at infinity* is defined as

$$\mathbf{e}_{\infty 1} \;=\; \mathbf{e}_{5}+\mathbf{e}_{6}. \tag{4.13}$$

The CSTA2 null 1-vector *point at infinity* is defined as

$$\mathbf{e}_{\infty 2} \;=\; \mathbf{e}_{11}+\mathbf{e}_{12}. \tag{4.14}$$

The CSTA null 1-vector *point at infinity* $\mathbf{e}_{\infty\gamma}$ represents either $\mathbf{e}_{\infty 1}$ or $\mathbf{e}_{\infty 2}$.

4.2.4 CSTA point embedding

The generic CSTA null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}}$ entity is the embedding of an STA *position* $\boldsymbol{p}_{\mathcal{M}}$ as

$$\boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) \;=\; \boldsymbol{p}_{\mathcal{M}}+\frac{1}{2}\boldsymbol{p}_{\mathcal{M}}^{2}\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma}. \tag{4.15}$$

The CSTA1 null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}^1}$ entity is the embedding of an STA1 *position* $\boldsymbol{p}_{\mathcal{M}^1}$ as

$$\boldsymbol{P}_{\mathcal{C}^1}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}^1}) \;=\; \boldsymbol{p}_{\mathcal{M}^1}+\frac{1}{2}\boldsymbol{p}_{\mathcal{M}^1}^{2}\mathbf{e}_{\infty 1}+\mathbf{e}_{o1}. \tag{4.16}$$

The CSTA2 null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}^2}$ entity is the embedding of an STA2 *position* $\boldsymbol{p}_{\mathcal{M}^2}$ as

$$\boldsymbol{P}_{\mathcal{C}^2}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}^2}) \;=\; \boldsymbol{p}_{\mathcal{M}^2}+\frac{1}{2}\boldsymbol{p}_{\mathcal{M}^2}^{2}\mathbf{e}_{\infty 2}+\mathbf{e}_{o2}. \tag{4.17}$$

The embedding function $\mathcal{C}$ is implemented as a piecewise embedding function that embeds an STA, STA1, or STA2 vector into the corresponding CSTA, CSTA1, or CSTA2 point. The generic CSTA embedding will used to avoid duplication in generic discussions that can apply just as well in either CSTA1 or CSTA2 by only changing the subscripts accordingly.

The CSTA point $\boldsymbol{P}_{\mathcal{C}}$ is similar to a CGA point $\mathbf{P}_{\mathcal{C}}$ as in [7] *when* $\boldsymbol{P}_{\mathcal{C}}$ is the embedding of a spatial point $\boldsymbol{p}_{\mathcal{M}}=\mathbf{p}_{\mathcal{S}}$ and we hold $w=ct=0$, which gives the $\mathcal{G}_{1,4}$ CSA null 1-vector point $\mathbf{P}_{\mathcal{CS}}$.

As a GOPNS entity, a CSTA point $\boldsymbol{P}_{\mathcal{C}}$ simply represents the *point* (§4.5.2), as expected.

As a GIPNS entity, a finite CSTA point $\boldsymbol{P}_{\mathcal{C}}$, excluding $\mathbf{e}_{\infty\gamma}$, actually represents a *hypercone* (§4.4.2) in spacetime of the form

$$(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_z)^2 \;=\; 0 \tag{4.18}$$

where

$$\boldsymbol{p}_{\mathcal{M}} \;=\; p_w\boldsymbol{\gamma}_0+p_x\boldsymbol{\gamma}_1+p_y\boldsymbol{\gamma}_2+p_z\boldsymbol{\gamma}_3=p_w\boldsymbol{\gamma}_0+\mathbf{p}_{\mathcal{S}}. \tag{4.19}$$

In general, a *hyper*surface in an n-D space has an $(n-1)$-D surface. A cone or other surface in 3-D space has a 2-D surface, but a hypercone or other hypersurface in 4-D spacetime has a 3-D surface. A hypersurface is treated and conceptualized in most respects the same as a 2-D surface, but it embeds extended dimensions and its mathematical forms contain an additional term per extended dimension.

The hypercone is a result of the MINKOWSKI spacetime metric (1, 3), which can be seen in the hypercone equation. For comparison to $\mathcal{G}_{4,1}$ CGA, a CGA point embeds a 3-D Euclidean vector with metric (3, 0) and represents an implicit surface equation of a sphere with zero radius

$$(x-p_x)^2+(y-p_y)^2+(z-p_z)^2 = 0. \tag{4.20}$$

In 3-D spacetime with only two spatial dimensions by holding $z-p_z=0$, the hypercone reduces to the circular cone

$$(x-p_x)^2+(y-p_y)^2-(w-p_w)^2 = 0 \tag{4.21}$$

which is an expanding circle in the xy-plane as the time-like coordinate $w=ct$ increases past p_w. The hypercone is an expanding sphere in space that is expanding with time t in radius

$$r = w-p_w=ct-p_w \tag{4.22}$$

at the speed of light c. The point begins expanding after time $t=p_w/c$ and is contracting before that time.

A CSTA point, as an expanding sphere, represents a *light-cone* in spacetime that is centered at the vertex point $\boldsymbol{p}_{\mathcal{M}}$. In spacetime, the light-cone is a spherical hypercone, which is a cone with a 3-D hypersurface. A surface is usually 2-D, but a hypersurface is imagined as a surface while it is actually a higher-dimensional space. The light-cone is often depicted as a *cone* in a 3-D spacetime of two spatial dimensions and a time-like dimension, wherein the cone is a *circular wave front* of light that expands in space as time t increases. The expanding radius $r=ct-p_w$ of the wave front is centered at a point light source $\boldsymbol{p}_{\mathcal{M}}$. A CSTA point represents a *spherical wave front* of light in space, or light-cone in spacetime, centered at a point light source $\boldsymbol{p}_{\mathcal{M}}$ that flashes at time $t=p_w/c$.

4.2.5 CSTA point normalization

A homogeneous CSTA point embedding with scalar weight s is

$$s\boldsymbol{P}_{\mathcal{C}}=s\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = s\boldsymbol{p}_{\mathcal{M}}+s\frac{1}{2}\boldsymbol{p}_{\mathcal{M}}^2\mathbf{e}_{\infty\gamma}+s\mathbf{e}_{o\gamma}. \tag{4.23}$$

A *normalized* point is scaled to weight $s=1$.

The *normalization* of a weighted CSTA point $s\boldsymbol{P}_{\mathcal{C}}$ is

$$\boldsymbol{P}_{\mathcal{C}} = \frac{(s\boldsymbol{P}_{\mathcal{C}})}{-(s\boldsymbol{P}_{\mathcal{C}})\cdot\mathbf{e}_{\infty\gamma}}=\frac{s\boldsymbol{P}_{\mathcal{C}}}{s}. \tag{4.24}$$

Many formulas require points and other entities to be unit weight. The normalization of an entity can be particular to the type of the entity.

A normalized point can be denoted

$$\hat{\boldsymbol{P}}_{\mathcal{C}} = \frac{\boldsymbol{P}_{\mathcal{C}}}{-\boldsymbol{P}_{\mathcal{C}}\cdot\mathbf{e}_{\infty\gamma}}. \tag{4.25}$$

4.2.6 CSTA point projection (inverse embedding)

The projection of CSTA point $\boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ to STA position $\boldsymbol{p}_{\mathcal{M}}$ is

$$\begin{aligned}\boldsymbol{p}_{\mathcal{M}}=\mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{C}}) &= \left(\frac{\boldsymbol{P}_{\mathcal{C}}}{-\boldsymbol{P}_{\mathcal{C}}\cdot\mathbf{e}_{\infty\gamma}}\cdot\mathbf{I}_{\mathcal{M}}\right)\mathbf{I}_{\mathcal{M}}^{-1} &(4.26)\\ &= (\hat{\boldsymbol{P}}_{\mathcal{C}}\cdot\mathbf{I}_{\mathcal{M}})\mathbf{I}_{\mathcal{M}}^{-1}. &(4.27)\end{aligned}$$

4.2.7 CSTA test point

The symbolic CSTA *test point* $\boldsymbol{T}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ is the embedding of the symbolic STA *test vector*

$$\boldsymbol{t}_{\mathcal{M}} \;=\; w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 = ct\boldsymbol{\gamma}_0 + \mathbf{t}_{\mathcal{S}} = \boldsymbol{o}_{\mathcal{M}}t + \mathbf{t}_{\mathcal{S}}. \tag{4.28}$$

The symbolic CSTA1 *test point* $\boldsymbol{T}_{\mathcal{C}^1} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}^1})$ is the embedding of the symbolic STA1 *test vector*

$$\boldsymbol{t}_{\mathcal{M}^1} \;=\; w\mathbf{e}_1 + x\mathbf{e}_2 + y\mathbf{e}_3 + z\mathbf{e}_4 = ct\mathbf{e}_1 + \mathbf{t}_{\mathcal{S}^1} = \boldsymbol{o}_{\mathcal{M}^1}t + \mathbf{t}_{\mathcal{S}^1}. \tag{4.29}$$

The symbolic CSTA2 *test point* $\boldsymbol{T}_{\mathcal{C}^2} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}^2})$ is the embedding of the symbolic STA2 *test vector*

$$\boldsymbol{t}_{\mathcal{M}^2} \;=\; w\mathbf{e}_7 + x\mathbf{e}_8 + y\mathbf{e}_9 + z\mathbf{e}_{10} = ct\mathbf{e}_7 + \mathbf{t}_{\mathcal{S}^2} = \boldsymbol{o}_{\mathcal{M}^2}t + \mathbf{t}_{\mathcal{S}^2}. \tag{4.30}$$

The symbolic scalars w, x, y, and z are the conventional coordinates in spacetime. The time-like coordinate $w = ct$ represents the distance traveled by light in time t. The *observer*, as defined for special relativity, is identified as the symbolic time-like velocity $\boldsymbol{o}_{\mathcal{M}}$.

CSTA1 and CSTA2 test points $\boldsymbol{T}_{\mathcal{C}^1}$ and $\boldsymbol{T}_{\mathcal{C}^2}$, respectively, are wedged to form the $\mathcal{G}_{4,8}$ DCSTA *test point* $\boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{C}^1} \wedge \boldsymbol{T}_{\mathcal{C}^2}$. The DCSTA point value-*extraction elements* T_s are defined as elements that extract values from the DCSTA test point $\boldsymbol{T}_{\mathcal{D}}$ as $s = \boldsymbol{T}_{\mathcal{D}} \cdot T_s$.

4.3 CSTA point value-extraction elements

The CSTA 1-vector point value-*extraction elements* C_s extract the value s from a test point $\boldsymbol{T}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ as $s = \boldsymbol{T}_{\mathcal{C}} \cdot C_s$. The CSTA value-extraction elements are

$$C_1 \;=\; -\mathbf{e}_{\infty\gamma} \tag{4.31}$$
$$C_w \;=\; \boldsymbol{\gamma}_0 \tag{4.32}$$
$$C_t \;=\; \frac{1}{c}C_w \tag{4.33}$$
$$C_x \;=\; -\boldsymbol{\gamma}_1 \tag{4.34}$$
$$C_y \;=\; -\boldsymbol{\gamma}_2 \tag{4.35}$$
$$C_z \;=\; -\boldsymbol{\gamma}_3 \tag{4.36}$$
$$C_{\boldsymbol{t}^2} \;=\; -2\mathbf{e}_{o\gamma}. \tag{4.37}$$

These elements are straightforward to verify. When $w = ct$, the extraction C_t gives t. The extraction

$$\boldsymbol{T}_{\mathcal{C}} \cdot C_{\boldsymbol{t}^2} \;=\; \boldsymbol{t}_{\mathcal{M}}^2 = |\boldsymbol{t}_{\mathcal{M}}|^2 = w^2 - r^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{4.38}$$

is the squared modulus, or interval from the origin, of the STA test vector $\boldsymbol{t}_{\mathcal{M}}$.

The CSTA geometric inner product null space (GIPNS) 1-vector surface entities can be defined in terms of these extraction elements by writing their implicit surface functions. Two of these entities are the CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ and the CSTA GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* $\boldsymbol{\Sigma}_{\mathcal{C}}$. A hyperhyperboloid can degenerate into a hypercone, which is a CSTA GIPNS null 1-vector point entity $\boldsymbol{P}_{\mathcal{C}}$. The CSTA GIPNS 1-vector entities $\boldsymbol{\Sigma}_{\mathcal{C}}$ and $\boldsymbol{E}_{\mathcal{C}}$ are similar in form to the CGA sphere $\mathbf{S}$ and plane $\mathbf{\Pi}$. The other CSTA GIPNS entities are of grades 2 to 5 and are formed as intersections (wedges) of hyperpseudospheres and hyperplanes or by specific formulas.

4.4 CSTA GIPNS entities

The $\mathcal{G}_{2,4}$ CSTA GIPNS entities are similar to $\mathcal{G}_{4,1}$ CGA GIPNS entities, but with some changes to account for the anti-Euclidean signature of $\mathcal{G}_{0,3}$ SA and the pseudo-Euclidean spacetime signature of $\mathcal{G}_{1,3}$ STA in a 4-D spacetime. The CSTA GIPNS entities of forms similar to CGA GIPNS entities are representing hypersurfaces in 4-D spacetime.

4.4.1 Geometric inner product null space (GIPNS)

Geometric inner product null space (GIPNS) entities are introduced by PERWASS in [16], and are reviewed by this author in [7] and [9].

4.4.2 CSTA GIPNS 1-vector hypercone

The implicit quadric surface equation for a circular *hypercone* is

$$(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_z)^2 \;=\; 0. \tag{4.39}$$

The CSTA GIPNS null 1-vector *hypercone* $\boldsymbol{K}_{\mathcal{C}}$ is the point embedding

$$\boldsymbol{K}_{\mathcal{C}} \;=\; \boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) \tag{4.40}$$

with center vertex point $\boldsymbol{p}_{\mathcal{M}}$. The hypercone is a sphere in space that expands from a point at $\boldsymbol{p}_{\mathcal{M}}$ with squared radius

$$r^2 \;=\; (w-p_w)^2=(ct-p_w)^2. \tag{4.41}$$

4.4.3 CSTA GIPNS 1-vector hyperplane

A hyperplane is a linear subspace of dimension $(n-1)$ in a space of dimension n. In 4D spacetime, a hyperplane is a 3D subspace. The hyperplane space can be a MINKOWSKI spacetime $(1,2)$ or an anti-Euclidean space $(0,3)$.

An implicit surface equation for a hyperplane in spacetime through the origin can be written

$$\begin{aligned} \boldsymbol{t}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}} \;&=\; \\ n_w w-n_x x-n_y y-n_z z \;&=\; 0. \end{aligned} \tag{4.42, 4.43}$$

The STA vector

$$\boldsymbol{n}_{\mathcal{M}} \;=\; n_w\gamma_0+n_x\gamma_1+n_y\gamma_2+n_z\gamma_3 \tag{4.44}$$

is the *normal vector* to the hyperplane. Only the direction of $\boldsymbol{n}_{\mathcal{M}}$ is significant, and its magnitude can be arbitrary. However, as a normalization of the scale, normal vectors with unit magnitudes (norms) $\boldsymbol{n}/\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}^\dagger}$ are sometimes required. The STA test vector $\boldsymbol{t}_{\mathcal{M}}$ is

$$\boldsymbol{t}_{\mathcal{M}} \;=\; w\gamma_0+x\gamma_1+y\gamma_2+z\gamma_3. \tag{4.45}$$

The equation holds good for any point $\boldsymbol{t}_{\mathcal{M}}$ on the hyperplane through the origin orthogonal to $\boldsymbol{n}_{\mathcal{M}}$. Using the CSTA point value-extraction elements (§4.3), the hyperplane implicit surface function can be written as the CSTA GIPNS entity

$$\begin{aligned} n_w w-n_x x-n_y y-n_z z \;&\rightarrow & (4.46)\\ n_w C_w-n_x C_x-n_y C_y-n_z C_z \;&= & (4.47)\\ n_w\gamma_0+n_x\gamma_1+n_y\gamma_2+n_z\gamma_3 \;&= & (4.48)\\ \boldsymbol{n}_{\mathcal{M}} \;&. & (4.49) \end{aligned}$$

The CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ through the origin with normal vector $\boldsymbol{n}_{\mathcal{M}}$ is defined as

$$\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{n}_{\mathcal{M}}. \tag{4.50}$$

The hyperplane through the origin $\boldsymbol{n}_{\mathcal{M}}$ can be translated from the origin to a point $\boldsymbol{d}_{\mathcal{M}}$ using the translator (§4.6.4) operation

$$\begin{aligned} T_{\mathcal{C}}\boldsymbol{n}_{\mathcal{M}}\tilde{T_{\mathcal{C}}} &= \qquad (4.51)\\ \left(1-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}\right)\boldsymbol{n}_{\mathcal{M}}\left(1-\frac{1}{2}\mathbf{e}_{\infty\gamma}\boldsymbol{d}_{\mathcal{M}}\right) &= \qquad (4.52)\\ \boldsymbol{n}_{\mathcal{M}}+(\boldsymbol{d}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma} &\quad . \qquad (4.53)\end{aligned}$$

The CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ through the point $\boldsymbol{p}_{\mathcal{M}}$ with normal vector $\boldsymbol{n}_{\mathcal{M}}$ is defined as

$$\begin{aligned} \boldsymbol{E}_{\mathcal{C}} &= \boldsymbol{n}_{\mathcal{M}}+(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma} && (4.54)\\ &\simeq \boldsymbol{E}^{*}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} && (4.55)\end{aligned}$$

and is equal to the CSTA undual of the dual CSTA GOPNS 5-vector *hyperplane* $\boldsymbol{E}^{*}_{\mathcal{C}}$ (§4.5.13) up to a homogeneous scalar factor. The normal vector $\boldsymbol{n}_{\mathcal{M}}$ can have any magnitude, and $\boldsymbol{p}_{\mathcal{M}}$ can be any point on the hyperplane. The hyperplane $\boldsymbol{E}_{\mathcal{C}}$ has a form that is similar to a $\mathcal{G}_{4,1}$ CGA plane $\mathbf{\Pi}$, and when we hold $w=ct=0$, then the form gives the $\mathcal{G}_{1,4}$ CSA GIPNS 1-vector plane $\mathbf{\Pi}_{\mathcal{CS}}=\mathbf{n}_{\mathcal{S}}+(\mathbf{p}_{\mathcal{S}}\cdot\mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty\gamma}$.

If $\boldsymbol{n}_{\mathcal{M}}$ is normal (perpendicular) to the hyperplane and also a point on the hyperplane, then the hyperplane can be defined as

$$\begin{aligned} \boldsymbol{E}_{\mathcal{C}} &= \boldsymbol{n}_{\mathcal{M}}+\boldsymbol{n}^{2}_{\mathcal{M}}\mathbf{e}_{\infty\gamma} && (4.56)\\ &= \boldsymbol{n}_{\mathcal{M}}+d^{2}\mathbf{e}_{\infty\gamma} && (4.57)\end{aligned}$$

where $d=\sqrt{\boldsymbol{n}^{2}_{\mathcal{M}}}$ is the hyperbolic distance (modulus) of $\boldsymbol{n}_{\mathcal{M}}$ from the origin. The modulus d may be a real or imaginary number, but the *spacetime interval* d^2 of $\boldsymbol{n}_{\mathcal{M}}$ from the origin is a real scalar. For $\boldsymbol{n}_{\mathcal{M}}$ both normal to and on the hyperplane, the squared modulus $d^2=\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}}$ from the origin, as well as d itself, is constant for all points $\boldsymbol{p}_{\mathcal{M}}$ on the hyperplane. By using the squared modulus d^2, it is possible to avoid imaginary numbers. The actual magnitude of $\boldsymbol{n}_{\mathcal{M}}$ does not affect the representation of the hyperplane surface since the hyperplane entity is a homogeneous entity that may be arbitrarily scaled by any non-zero scalar without affecting the surface that is represented. The scaling of entities affects metrical distance calculations between entities, and the formulas for distances between entities must include methods for normalizing the scale of entities.

The hyperplane $\boldsymbol{E}_{\mathcal{C}}$ is the set of STA points $\boldsymbol{t}_{\mathcal{M}}$

$$\mathbb{NI}_{G}(\boldsymbol{E}_{\mathcal{C}}) = \{\, \boldsymbol{t}_{\mathcal{M}} \;:\; \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})\cdot\boldsymbol{E}_{\mathcal{C}}=0 \,\} \tag{4.58}$$

of the geometric inner product null space of $\boldsymbol{E}_{\mathcal{C}}$, denoted $\mathbb{NI}_{G}(\boldsymbol{E}_{\mathcal{C}})$ [16]. A similar set holds for all other GIPNS entities.

If $\boldsymbol{n}_{\mathcal{M}}$ is a null vector, then the hyperplane $\boldsymbol{E}_{\mathcal{C}}$ degenerates into the representation of a CSTA GIPNS null 1-vector *light-line (null line)* entity $\mathbb{L}_{\mathcal{C}}$ (§4.4.4) parallel to $\boldsymbol{n}_{\mathcal{M}}$ and through the point $\boldsymbol{p}_{\mathcal{M}}$. The null line entity $\mathbb{L}_{\mathcal{C}}$ includes the point at infinity $\mathbf{e}_{\infty\gamma}$ on the line.

The intersections of two, three, four, or five hyperplanes $\boldsymbol{E}_{\mathcal{C}_i}$ can define entities for planes, lines, flat points, and the point at infinity, respectively as follows:

Two hyperplanes intersect as a CSTA GIPNS 2-vector *plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ (§4.4.10)

$$\boldsymbol{E}_{\mathcal{C}_1}\wedge\boldsymbol{E}_{\mathcal{C}_2} = \tag{4.59}$$

$$(\boldsymbol{n}_1+d_1^2\mathbf{e}_{\infty\gamma})\wedge(\boldsymbol{n}_2+d_2^2\mathbf{e}_{\infty\gamma}) = \tag{4.60}$$

$$\boldsymbol{n}_1\wedge\boldsymbol{n}_2+(d_2^2\boldsymbol{n}_1-d_1^2\boldsymbol{n}_2)\mathbf{e}_{\infty\gamma} = \tag{4.61}$$

$$\boldsymbol{D}^{*\mathcal{M}}-(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} = \boldsymbol{\Pi}_{\mathcal{C}} \tag{4.62}$$

where $\boldsymbol{D}^{*\mathcal{M}}=\boldsymbol{n}_1\wedge\boldsymbol{n}_2$ is the STA dual of the plane $\boldsymbol{\Pi}_{\mathcal{C}}$ direction bivector $\boldsymbol{D}$.

Three hyperplanes intersect as a CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ (§4.4.11)

$$\boldsymbol{E}_{\mathcal{C}_1}\wedge\boldsymbol{E}_{\mathcal{C}_2}\wedge\boldsymbol{E}_{\mathcal{C}_3} = \tag{4.63}$$

$$(\boldsymbol{n}_1\wedge\boldsymbol{n}_2+(d_2^2\boldsymbol{n}_1-d_1^2\boldsymbol{n}_2)\mathbf{e}_{\infty\gamma})\wedge(\boldsymbol{n}_3+d_3^2\mathbf{e}_{\infty\gamma}) = \tag{4.64}$$

$$\boldsymbol{n}_1\wedge\boldsymbol{n}_2\wedge\boldsymbol{n}_3+(d_2^2\boldsymbol{n}_1-d_1^2\boldsymbol{n}_2)\wedge\mathbf{e}_{\infty\gamma}\wedge\boldsymbol{n}_3+d_3^2\boldsymbol{n}_1\wedge\boldsymbol{n}_2\wedge\mathbf{e}_{\infty\gamma} = \tag{4.65}$$

$$\boldsymbol{n}_1\wedge\boldsymbol{n}_2\wedge\boldsymbol{n}_3+(d_1^2\boldsymbol{n}_2\wedge\boldsymbol{n}_3-d_2^2\boldsymbol{n}_1\wedge\boldsymbol{n}_3+d_3^2\boldsymbol{n}_1\wedge\boldsymbol{n}_2)\mathbf{e}_{\infty\gamma} = \tag{4.66}$$

$$\boldsymbol{d}^{*\mathcal{M}}+(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{d}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} = \boldsymbol{L}_{\mathcal{C}} \tag{4.67}$$

where $\boldsymbol{d}^{*\mathcal{M}}=\boldsymbol{n}_1\wedge\boldsymbol{n}_2\wedge\boldsymbol{n}_3$ is the STA dual of the line $\boldsymbol{L}_{\mathcal{C}}$ direction vector $\boldsymbol{d}$.

Four hyperplanes intersect as a CST GIPNS 4-vector *flat point* (§4.4.16)

$$\boldsymbol{E}_{\mathcal{C}_1}\wedge\boldsymbol{E}_{\mathcal{C}_2}\wedge\boldsymbol{E}_{\mathcal{C}_3}\wedge\boldsymbol{E}_{\mathcal{C}_4} = \tag{4.68}$$

$$(\boldsymbol{n}_1\wedge\boldsymbol{n}_2\wedge\boldsymbol{n}_3+(d_1^2\boldsymbol{n}_2\wedge\boldsymbol{n}_3-d_2^2\boldsymbol{n}_1\wedge\boldsymbol{n}_3+d_3^2\boldsymbol{n}_1\wedge\boldsymbol{n}_2)\mathbf{e}_{\infty\gamma})\wedge(\boldsymbol{n}_4+d_4^2\mathbf{e}_{\infty\gamma}) = \tag{4.69}$$

$$\alpha\mathbf{I}_{\mathcal{M}}-\alpha(\boldsymbol{p}_{\mathcal{M}}\cdot\mathbf{I}_{\mathcal{M}})\mathbf{e}_{\infty\gamma} \simeq \tag{4.70}$$

$$\mathbf{I}_{\mathcal{M}}+\boldsymbol{p}_{\mathcal{M}}^{*}\mathbf{e}_{\infty\gamma} = \mathbb{P}_{\mathcal{C}}. \tag{4.71}$$

The CSTA dual of the flat point is

$$(\mathbf{I}_{\mathcal{M}}+\boldsymbol{p}_{\mathcal{M}}^{*}\mathbf{e}_{\infty\gamma})\mathbf{I}_{\mathcal{C}}^{-1} = \tag{4.72}$$

$$\mathbf{e}_{\infty\gamma}\wedge\boldsymbol{P}_{\mathcal{C}} = \mathbb{P}_{\mathcal{C}}^{*}. \tag{4.73}$$

Five hyperplanes intersect as the CSTA GIPNS 5-vector *point* at infinity $\mathbf{e}_{\infty\gamma}^{\star}$ (§4.4.17)

$$\boldsymbol{E}_{\mathcal{C}_1}\wedge\boldsymbol{E}_{\mathcal{C}_2}\wedge\boldsymbol{E}_{\mathcal{C}_3}\wedge\boldsymbol{E}_{\mathcal{C}_4}\wedge\boldsymbol{E}_{\mathcal{C}_5} = \tag{4.74}$$

$$(\mathbf{I}_{\mathcal{M}}+\boldsymbol{p}_{\mathcal{M}}^{*}\mathbf{e}_{\infty\gamma})\wedge(\boldsymbol{n}_5+d_5^2\mathbf{e}_{\infty\gamma}) = \tag{4.75}$$

$$\boldsymbol{p}_{\mathcal{M}}^{*}\wedge\mathbf{e}_{\infty\gamma}\wedge\boldsymbol{n}_5+d_5^2\mathbf{I}_{\mathcal{M}}\wedge\mathbf{e}_{\infty\gamma} \simeq \tag{4.76}$$

$$\mathbf{I}_{\mathcal{M}}\wedge\mathbf{e}_{\infty\gamma} = \tag{4.77}$$

$$\mathbf{I}_{\mathcal{M}}\wedge\mathbf{e}_{+}+\mathbf{I}_{\mathcal{M}}\wedge\mathbf{e}_{-} = \tag{4.78}$$

$$-\mathbf{I}_{\mathcal{C}}\cdot\mathbf{e}_{-}-\mathbf{I}_{\mathcal{C}}\cdot\mathbf{e}_{+} = \tag{4.79}$$

$$-\mathbf{I}_{\mathcal{C}}\cdot\mathbf{e}_{\infty\gamma} = \tag{4.80}$$

$$-(-1)^{1(6-1)}\mathbf{e}_{\infty\gamma}\cdot\mathbf{I}_{\mathcal{C}} = \tag{4.81}$$

$$\mathbf{e}_{\infty\gamma}\mathbf{I}_{\mathcal{C}} = \mathbf{e}_{\infty\gamma}^{\star}. \tag{4.82}$$

4.4.4 CSTA GIPNS null 1-vector light-line (null line)

As a CSTA GIPNS entity, a null vector $\boldsymbol{n}_{\mathcal{M}}=\alpha(\boldsymbol{\gamma}_0+\hat{\mathbf{n}})$ represents the line through the origin in the direction of $\boldsymbol{n}_{\mathcal{M}}$. This is a degenerate case of the hyperplane $\boldsymbol{E}_{\mathcal{C}}$ (§4.4.3) with null vector $\boldsymbol{n}_{\mathcal{M}}$ as the hyperplane normal vector.

The CSTA GIPNS null 1-vector *light-line (null line)* $\mathbb{L}_{\mathcal{C}}$ through the origin in the direction of the null vector $\boldsymbol{n}_{\mathcal{M}}$ is defined as

$$\mathbb{L}_{\mathcal{C}} = \boldsymbol{n}_{\mathcal{M}}. \tag{4.83}$$

The null line $\mathbb{L}_{\mathcal{C}} = \boldsymbol{n}_{\mathcal{M}}$ can be translated in spacetime from the origin to an arbitrary point $\boldsymbol{p}_{\mathcal{M}}$ using a translation operation (§4.6.4). The CSTA GIPNS null 1-vector *null line* $\mathbb{L}_{\mathcal{C}}$ through the point $\boldsymbol{p}_{\mathcal{M}}$ in the direction of null vector $\boldsymbol{n}_{\mathcal{M}}$ is defined, via translation, as

$$\mathbb{L}_{\mathcal{C}} = \boldsymbol{n}_{\mathcal{M}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma}. \tag{4.84}$$

The null line $\mathbb{L}_{\mathcal{C}}$ includes the point at infinity $\mathbf{e}_{\infty\gamma}$ on the line.

4.4.5 CSTA GIPNS 1-vector hyperhyperboloid of one sheet

The implicit quadric surface equation for a circular *hyperhyperboloid of one sheet* is

$$r_0^2 + (w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2 = 0 \tag{4.85}$$

where r_0 is the initial radius of the expanding sphere in space with time-varying radius

$$r = \sqrt{r_0^2 + (w - p_w)^2} = \sqrt{r_0^2 + (ct - p_w)^2} \tag{4.86}$$

and center position

$$\boldsymbol{p}_{\mathcal{M}} = p_w\boldsymbol{\gamma}_0 + p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3 = p_w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}} \tag{4.87}$$

in 4-D spacetime.

When $w - p_w = 0$, the surface is a sphere with radius r_0. The circular hyperhyperboloid of one sheet can also be called a ***hyperpseudosphere***. Like a sphere, a hyperpseudosphere does not include the point at infinity.

In 3-D spacetime with only two spatial dimensions by holding $(z - p_z) = 0$, the circular hyperhyperboloid of one sheet reduces to the circular *hyperboloid of one sheet*

$$\frac{(x - p_x)^2}{r_0^2} + \frac{(y - p_y)^2}{r_0^2} - \frac{(w - p_w)^2}{r_0^2} = 1. \tag{4.88}$$

When $(w - p_w) = 0$, the hyperboloid of one sheet is a circle in the xy-plane with initial radius r_0 at initial time $t = p_w/c$, or $w = p_w$. The circle radius $r = \sqrt{r_0^2 + (w - p_w)^2}$ is expanding after time $t = p_w/c$ and is contracting before that time. The radius is expanding with time t at the rate

$$\dot{r} = \frac{\partial r}{\partial t} = \frac{1}{2}r^{-1}2(ct - p_w)c = \frac{ct - p_w}{r}c = \frac{ct - p_w}{\sqrt{(ct - p_w)^2 + r_0^2}}c. \tag{4.89}$$

The initial rate at time $t = p_w/c$ is $\dot{r}(p_w/c) = 0$ and increases to $\dot{r}(\infty) = c$ as $t \rightarrow \infty$. In natural units, $c = 1$ and the hyperhyperboloid of one sheet is asymptotically the hypercone of a spherically expanding point $\boldsymbol{P}_{\mathcal{C}}$. The acceleration of the radius r is

$$\ddot{r} = \frac{\partial \dot{r}}{\partial t} = \partial_t \frac{ct - p_w}{r}c = \frac{cr - \dot{r}(ct - p_w)}{r^2}c = \frac{c^2 - \dot{r}^2}{r}. \tag{4.90}$$

The initial acceleration at $t = p_w/c$ is $\ddot{r}(p_w/c) = c^2/r_0$ and decreases to $\ddot{r}(\infty) = 0$ as $t \rightarrow \infty$. In natural units, $c = 1$ and $\ddot{r}$ is a measure of circular, or spherical, curvature at time t.

Using the CSTA point value-extraction elements (§4.3) the hyperhyperboloid of one sheet implicit surface function entity can be written

$$\begin{aligned}
r_0^2 + (w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_x)^2 &\rightarrow && (4.91)\\
(r_0^2 + \boldsymbol{p}_{\mathcal{M}}^2)C_1 + C_{\boldsymbol{t}^2} - 2p_wC_w + 2p_xC_x + 2p_yC_y + 2p_zC_z &= && (4.92)\\
-(r_0^2 + \boldsymbol{p}_{\mathcal{M}}^2)\mathbf{e}_{\infty\gamma} - 2\mathbf{e}_{o\gamma} - 2p_w\boldsymbol{\gamma}_0 - 2p_x\boldsymbol{\gamma}_1 - 2p_y\boldsymbol{\gamma}_2 - 2p_z\boldsymbol{\gamma}_3 &= && (4.93)\\
-2\boldsymbol{p}_{\mathcal{M}} - (r_0^2 + \boldsymbol{p}_{\mathcal{M}}^2)\mathbf{e}_{\infty\gamma} - 2\mathbf{e}_{o\gamma} &. && (4.94)
\end{aligned}$$

Normalizing $\mathbf{e}_{o\gamma}$ by scaling $-1/2$ gives

$$\boldsymbol{p}_{\mathcal{M}}+\frac{1}{2}(r_0^2+\boldsymbol{p}_{\mathcal{M}}^2)\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} \;=\; \tag{4.95}$$

$$\boldsymbol{P}_{\mathcal{C}}+\frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma} \quad . \tag{4.96}$$

The CSTA GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* $\boldsymbol{\Sigma}_{\mathcal{C}}$ in spacetime with initial radius r_0 centered at CSTA point $\boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ is defined as

$$\boldsymbol{\Sigma}_{\mathcal{C}} \;=\; \boldsymbol{P}_{\mathcal{C}}+\frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma} \tag{4.97}$$

$$\simeq\; \boldsymbol{\Sigma}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} \tag{4.98}$$

and equals the CSTA undual of the dual CSTA GOPNS 5-vector *hyperpseudosphere* $\boldsymbol{\Sigma}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor.

The CSTA hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{C}}$ is similar to a CGA sphere $\mathbf{S}$ discussed in [7] *when* $\boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\mathbf{p}_{\mathcal{S}})$ is the embedding of a spatial point $\boldsymbol{p}_{\mathcal{M}}=\mathbf{p}_{\mathcal{S}}$ with $w=ct=0$, which gives the $\mathcal{G}_{1,4}$ CSA GIPNS 1-vector sphere $\mathbf{S}_{\mathcal{CS}}=\mathbf{P}_{\mathcal{CS}}+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma}$. When $r_0=0$, $\boldsymbol{\Sigma}_{\mathcal{C}}=\boldsymbol{K}_{\mathcal{C}}$ is a *hypercone*, which is the CSTA point embedding $\boldsymbol{P}_{\mathcal{C}}=\boldsymbol{K}_{\mathcal{C}}$ as a GIPNS entity.

Two hyperpseudospheres can intersect in a growing spatial circle, which is a CSTA GIPNS 2-vector spacetime hyperboloid or pseudosphere $\boldsymbol{S}_{\mathcal{C}}$. Three hyperpseudospheres can intersect in a growing spatial point pair, which is a CSTA GIPNS 3-vector spacetime hyperbola or pseudocircle $\boldsymbol{C}_{\mathcal{C}}$. Four hyperpseudospheres can intersect in a CSTA GIPNS 4-vector spacetime point pair $\mathbb{P}_{\mathcal{C}}$.

4.4.6 CSTA GIPNS 1-vector hyperhyperboloid of two sheets

The implicit quadric surface equation for a circular *hyperhyperboloid of two sheets* is

$$-r_0^2+(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_z)^2 \;=\; 0. \tag{4.99}$$

The CSTA GIPNS 1-vector *hyperhyperboloid of two sheets (imaginary hyperpseudosphere)* is

$$\boldsymbol{\Xi}_{\mathcal{C}} \;=\; \boldsymbol{P}_{\mathcal{C}}-\frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma}. \tag{4.100}$$

The imaginary radius is $\sqrt{-1}r_0$.

The intersection of $\boldsymbol{\Xi}_{\mathcal{C}}$ and hyperplane $\boldsymbol{E}_{\mathcal{C}}=\boldsymbol{\gamma}_0+p_w\mathbf{e}_{\infty\gamma}$ holds $w=p_w$ and produces an imaginary sphere. The intersection of $\boldsymbol{\Xi}_{\mathcal{C}}$ and hyperplane $\boldsymbol{E}_{\mathcal{C}}=\boldsymbol{\gamma}_3-p_z\mathbf{e}_{\infty\gamma}$ holds $z=p_z$ and produces a hyperboloid of two sheets in wxy-spacetime opening up and down the w-axis. The intersection of $\boldsymbol{\Xi}_{\mathcal{C}}$ and spacetime plane $\boldsymbol{\Pi}_{\mathcal{C}}$ produces a hyperbola in the spacetime plane that opens up and down the time axis.

4.4.7 CSTA GIPNS 2-vector spatial sphere

The implicit quadric surface equation for a *hyperpseudosphere* is

$$r_0^2+(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_z)^2 \;=\; 0. \tag{4.101}$$

If we set the time coordinate w, then we get the implicit surface equation for a *sphere* in xyz-space

$$(x-p_x)^2+(y-p_y)^2+(z-p_z)^2-r^2 \;=\; 0 \tag{4.102}$$

with radius

$$r \;=\; \sqrt{r_0^2+(w-p_w)^2}. \tag{4.103}$$

To set w, we can intersect a hyperpseudosphere

$$\boldsymbol{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma} \tag{4.104}$$

with radius r_0 centered at

$$\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3) = \mathcal{C}(p_w\gamma_0 + \mathbf{p}_{\mathcal{S}}) \tag{4.105}$$

with the hyperplane

$$\boldsymbol{E}_{\mathcal{C}} = \gamma_0 + w\mathbf{e}_{\infty\gamma} \tag{4.106}$$

of xyz-space at w. The sphere of radius r_0 centered at $\mathbf{p}_{\mathcal{S}}$ is at the time $w = p_w$.

The CSTA GIPNS 2-vector *sphere* $\boldsymbol{S}_{\mathcal{C}}$ centered at $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with radius r_0 is the intersection

$$\begin{aligned}
\boldsymbol{S}_{\mathcal{C}} &= \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}} && (4.107)\\
&= \left(\boldsymbol{P}_{\mathcal{C}} + \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma}\right) \wedge (\gamma_0 + (\boldsymbol{p}_{\mathcal{M}} \cdot \gamma_0)\mathbf{e}_{\infty\gamma}) && (4.108)\\
&= \left(\boldsymbol{P}_{\mathcal{C}} + \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma}\right) \wedge (\gamma_0 + p_w\mathbf{e}_{\infty\gamma}) && (4.109)\\
&\simeq \boldsymbol{S}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} && (4.110)
\end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *sphere* $\boldsymbol{S}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor.

4.4.8 CSTA GIPNS 2-vector spacetime hyperboloid of one sheet

The implicit quadric surface equation for a *hyperpseudosphere* is

$$r_0^2 + (w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2 = 0. \tag{4.111}$$

If we set one spatial coordinate, for instance z, then we get the implicit surface equation for a circular *hyperboloid of one sheet* in wxy-spacetime

$$\begin{aligned}
(x - p_x)^2 + (y - p_y)^2 - (w - p_w)^2 - (r_0^2 - (z - p_z)^2) &= && (4.112)\\
\frac{(x - p_x)^2}{r^2} + \frac{(y - p_y)^2}{r^2} - \frac{(w - p_w)^2}{r^2} - 1 &= 0 && (4.113)
\end{aligned}$$

with central radius

$$r = \sqrt{r_0^2 - (z - p_z)^2} \tag{4.114}$$

and circular conic section radius at w, z

$$r_c = \sqrt{(w - p_w)^2 + (r_0^2 - (z - p_z)^2)} = \sqrt{(w - p_w)^2 + r^2}. \tag{4.115}$$

The spacetime hyperboloid of one sheet is also called a *pseudosphere*. Like a sphere, a pseudosphere does not include the point at infinity. The pseudosphere in wxy-spacetime is a *circle* in xy-space that changes in radius with w, z. To set z, we can intersect a hyperpseudosphere

$$\boldsymbol{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma} \tag{4.116}$$

with radius r_0 centered at

$$\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3) = \mathcal{C}(p_w\gamma_0 + \mathbf{p}_{\mathcal{S}}) \tag{4.117}$$

with the hyperplane

$$\begin{aligned} \boldsymbol{E}_{\mathcal{C}} &= z\gamma_3 - z^2\mathbf{e}_{\infty\gamma} && (4.118)\\ &\simeq \gamma_3 - z\mathbf{e}_{\infty\gamma} && (4.119)\end{aligned}$$

of wxy-space at $z\gamma_3$. To have pseudosphere central radius r_0, set $z = p_z$. The xy-plane circle of radius r_0 centered at $\mathbf{p}_{\mathcal{S}}$ is at the time $w = p_w$. More generally, the hyperplane can be through point $\boldsymbol{p}_{\mathcal{M}}$ with spatial normal vector $\mathbf{n}_{\mathcal{S}}$ as

$$\boldsymbol{E}_{\mathcal{C}} = \mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty}. \qquad (4.120)$$

The CSTA GIPNS 2-vector *spacetime hyperboloid of one sheet (pseudosphere)* $\boldsymbol{S}_{\mathcal{C}}$ centered at $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with central radius r_0 in the spatial plane orthogonal to normal vector $\mathbf{n}_{\mathcal{S}}$ is the intersection

$$\begin{aligned} \boldsymbol{S}_{\mathcal{C}} &= \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}} && (4.121)\\ &= \left(\boldsymbol{P}_{\mathcal{C}} + \frac{1}{2}r_0^2\mathbf{e}_{\infty}\right) \wedge (\mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty}) && (4.122)\\ &\simeq \boldsymbol{S}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} && (4.123)\end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *pseudosphere* $\boldsymbol{S}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor.

The CSTA GIPNS null 2-vector *spacetime cone (null cone)* is the pseudosphere $\boldsymbol{S}_{\mathcal{C}}$ with central radius $r = 0$.

The spacetime hyperboloid of one sheet is always in a 3D spacetime and is not a purely spatial surface. The spatial part is circles in planes parallel to the $\mathbf{n}_{\mathcal{S}}^{*}$-plane centered at $\mathbf{p}_{\mathcal{S}}$ with radius $r_c = \sqrt{(w - p_w)^2 + r_0^2}$. The CSTA GIPNS 3-vector *circle* entity $\boldsymbol{C}_{\mathcal{C}}$ with radius r_0 is obtained by another intersection with the hyperplane at $p_w\gamma_0$.

4.4.9 CSTA GIPNS 2-vector spacetime hyperboloid of two sheets

The CSTA GIPNS 2-vector *spacetime hyperboloid of two sheets (imaginary pseudosphere)* $\boldsymbol{S}_{\mathcal{C}}$ centered at $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with central radius r_0 in the spatial plane orthogonal to normal vector $\mathbf{n}_{\mathcal{S}}$ is the intersection

$$\begin{aligned} \boldsymbol{S}_{\mathcal{C}} &= \boldsymbol{\Xi}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}} && (4.124)\\ &= \left(\boldsymbol{P}_{\mathcal{C}} - \frac{1}{2}r_0^2\mathbf{e}_{\infty}\right) \wedge (\mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty}) && (4.125)\\ &\simeq \boldsymbol{S}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} && (4.126)\end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *imaginary pseudosphere* $\boldsymbol{S}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor. The two sheets open up and down the w-axis and have circular sections in the $\mathbf{n}_{\mathcal{S}}^{*}$-plane.

4.4.10 CSTA GIPNS 2-vector plane

A plane in spacetime can be defined by two orthogonal unit-norm direction vectors

$$\begin{aligned} \boldsymbol{d}_{\mathcal{M}_1} &= \frac{\boldsymbol{d}_{\mathcal{M}_1}}{\|\boldsymbol{d}_{\mathcal{M}_1}\|} = d_{w_1}\gamma_0 + d_{x_1}\gamma_1 + d_{y_1}\gamma_2 + d_{z_1}\gamma_3 && (4.127)\\ \boldsymbol{d}_{\mathcal{M}_2} &= \frac{\boldsymbol{d}_{\mathcal{M}_2}}{\|\boldsymbol{d}_{\mathcal{M}_2}\|} = d_{w_2}\gamma_0 + d_{x_2}\gamma_1 + d_{y_2}\gamma_2 + d_{z_2}\gamma_3 && (4.128)\\ \boldsymbol{d}_{\mathcal{M}_1} \cdot \boldsymbol{d}_{\mathcal{M}_2}^{\dagger} &= 0 && (4.129)\end{aligned}$$

and a point

$$\boldsymbol{p}_{\mathcal{M}} = p_w\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 \tag{4.130}$$

on the plane. The direction of the plane is represented by the normalized unit bivector

$$\begin{aligned}\boldsymbol{D} &= \boldsymbol{d}_{\mathcal{M}_1} \wedge \boldsymbol{d}_{\mathcal{M}_2} &(4.131)\\ &= \frac{\boldsymbol{D}}{\sqrt{\boldsymbol{D}\cdot\boldsymbol{D}^{\dagger}}} = \frac{\boldsymbol{D}}{\sqrt{\boldsymbol{D}\cdot(\gamma_0\boldsymbol{D}^{\sim}\gamma_0)}}. &(4.132)\end{aligned}$$

The notation $\mathbf{A}^{\dagger}_{\mathcal{M}} = \gamma_0\widetilde{\mathbf{A}}_{\mathcal{M}}\gamma_0$ is the anti-Euclidean space conjugation, or SA space conjugation, which is necessary for the case where $\boldsymbol{D}$ is a null bivector. For blade $\mathbf{A}_{\mathcal{M}}$ in spacetime, the *conjugate* [16] has the property

$$\mathbf{A}_{\mathcal{M}}\cdot\mathbf{A}^{\dagger}_{\mathcal{M}} = \mathbf{A}_{\mathcal{M}}\cdot\big(\gamma_0\widetilde{\mathbf{A}_{\mathcal{M}}}\gamma_0\big) = \|\mathbf{A}_{\mathcal{M}}\|^2. \tag{4.133}$$

Any test point

$$\boldsymbol{t}_{\mathcal{M}} = w\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \tag{4.134}$$

on the plane must satisfy the plane equation

$$(\boldsymbol{t}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}}) \wedge \boldsymbol{D} = 0 \tag{4.135}$$

which can also be written in the dual form

$$(\boldsymbol{t}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}}) \cdot \boldsymbol{D}^{*\mathcal{M}} = \boldsymbol{t}_{\mathcal{M}}\cdot\boldsymbol{D}^{*\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{D}^{*\mathcal{M}} = 0. \tag{4.136}$$

The dual form plane equation is vector-valued and the components represent a system of implicit surface equations for an intersection of hyperplanes that gives the plane.

The CSTA GIPNS 2-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}$ through point $\boldsymbol{p}_{\mathcal{M}}$ in the planar direction of the unit bivector $\boldsymbol{D}$ in spacetime can be defined as

$$\begin{aligned}\mathbf{\Pi}_{\mathcal{C}} &= \boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{D}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma} &(4.137)\\ &\simeq \mathbf{\Pi}^{*}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} &(4.138)\end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *plane* $\mathbf{\Pi}^{*}_{\mathcal{C}}$ up to a homogeneous scalar factor.

The CSTA translation operation on any CSTA entity can be defined as its successive reflections in two parallel CSTA planes. The CSTA 2-versor *translator* (translation operator) $T_{\mathcal{C}}$ can be defined by two parallel planes $\mathbf{\Pi}_{\mathcal{C}_1}$ and $\mathbf{\Pi}_{\mathcal{C}_2}$ that are separated by a spacetime displacement vector $\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}$ from $\mathbf{\Pi}_{\mathcal{C}_1}$ to $\mathbf{\Pi}_{\mathcal{C}_2}$ as

$$T_{\mathcal{C}} = \mathbf{\Pi}_{\mathcal{C}_2}\mathbf{\Pi}_{\mathcal{C}_1}. \tag{4.139}$$

The translator versor operation on a CSTA point $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$, for example, is

$$\boldsymbol{P}'_{\mathcal{C}} = T_{\mathcal{C}}\boldsymbol{P}_{\mathcal{C}}\widetilde{T_{\mathcal{C}}} = \mathbf{\Pi}_{\mathcal{C}_2}\mathbf{\Pi}_{\mathcal{C}_1}\boldsymbol{P}_{\mathcal{C}}\widetilde{\mathbf{\Pi}_{\mathcal{C}_1}}\widetilde{\mathbf{\Pi}_{\mathcal{C}_2}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}} + \boldsymbol{d}_{\mathcal{M}}). \tag{4.140}$$

The successive reflections in two parallel planes translates by *twice* the spacetime displacement between the parallel planes.

The *rotor* (spatial rotation operator) $R_{\mathcal{S}}$ for a rotation by *twice* the angle between two non-parallel spatial planes $\mathbf{\Pi}_{\mathcal{C}_1}$ and $\mathbf{\Pi}_{\mathcal{C}_2}$ can be defined as

$$R_{\mathcal{S}} = \mathbf{\Pi}_{\mathcal{C}_2}\mathbf{\Pi}_{\mathcal{C}_1}. \tag{4.141}$$

The spatial rotation operator $R_{\mathcal{S}}$ is equivalent to the SA rotor and is the same spatial rotor that is used in STA and CSTA. The spatial rotor $R_{\mathcal{S}}$ can spatially rotate any multivector by *versor outermorphism* [16] that rotates all vectors within outer products.

The double *boost* operator $B_{\mathbf{v}}B_{\mathbf{v}}$, where $B_{\mathbf{v}} = (\gamma_{\mathbf{v}}\boldsymbol{v} \,/\, \boldsymbol{o})^{\frac{1}{2}}$ and $\boldsymbol{v} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}}c\hat{\mathbf{v}}$, that adds the double rapidity $2\varphi_{\mathbf{v}} = 2\text{atanh}(\beta_{\mathbf{v}})$ in the direction of $\boldsymbol{v}$ can be defined as the successive reflections in two non-parallel spacetime planes. The first plane $\boldsymbol{\Pi}_{\mathcal{C}_1}$ should represent the observer as a plane through the origin and observer $\boldsymbol{o}$ that spans the time axis and another spatial axis perpendicular to $\mathbf{v}$. The second plane $\boldsymbol{\Pi}_{\mathcal{C}_2}$ should represent the boost velocity $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ by passing through the origin and $\boldsymbol{v}$ and spanning the same direction perpendicular to $\mathbf{v}$ as the first plane. The planes should be unit scale by using unit bivectors to define the plane directions. The two planes contain the hyperbolic angle $\varphi_{\mathbf{v}}$ that turns positive from $\boldsymbol{\Pi}_{\mathcal{C}_1}$ toward $\boldsymbol{\Pi}_{\mathcal{C}_2}$ into the direction of $\mathbf{v}$. The double boost $B_{\mathbf{v}}B_{\mathbf{v}}$ of a spacetime velocity $\boldsymbol{u}$ is obtained by the successive reflections

$$\boldsymbol{u}' \;=\; \boldsymbol{\Pi}_{\mathcal{C}_2}\boldsymbol{\Pi}_{\mathcal{C}_1}\boldsymbol{u}\boldsymbol{\Pi}^{\sim}_{\mathcal{C}_1}\boldsymbol{\Pi}^{\sim}_{\mathcal{C}_2} = B_{\mathbf{v}}B_{\mathbf{v}}\boldsymbol{u}B^{\sim}_{\mathbf{v}}B^{\sim}_{\mathbf{v}}. \tag{4.142}$$

4.4.11 CSTA GIPNS 3-vector line

Implicit surface equation of line

An implicit equation for a line in spacetime through two points can be written as

$$(\boldsymbol{t} - \boldsymbol{p}_1) \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_1)^{*\mathcal{M}} \;=\; 0 \tag{4.143}$$

where $\boldsymbol{t}$ is the CSTA test point. The equation holds good for any $\boldsymbol{t}$ on the line of the two points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. The unit norm direction $\boldsymbol{d}$ of the line can be written as

$$\begin{aligned}
\boldsymbol{d} \;&=\; \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{\sqrt{(\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot (\boldsymbol{p}_2 - \boldsymbol{p}_1)^{\dagger}}} && (4.144)\\
&=\; \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{\sqrt{(\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot (\boldsymbol{\gamma}_0(\boldsymbol{p}_2 - \boldsymbol{p}_1)\boldsymbol{\gamma}_0)}} && (4.145)\\
&=\; \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{\|\boldsymbol{p}_2 - \boldsymbol{p}_1\|}. && (4.146)
\end{aligned}$$

The unit norm trivector dual to the line direction is

$$\boldsymbol{d}^{*\mathcal{M}} \;=\; \boldsymbol{d}\mathbf{I}_{\mathcal{M}}^{-1}. \tag{4.147}$$

The implicit equation can be rewritten as

$$\begin{aligned}
(\boldsymbol{t} - \boldsymbol{p}) \cdot \boldsymbol{d}^{*\mathcal{M}} \;&=\; && (4.148)\\
\boldsymbol{t} \cdot \boldsymbol{d}^{*\mathcal{M}} - \boldsymbol{p} \cdot \boldsymbol{d}^{*\mathcal{M}} \;&=\; 0 && (4.149)
\end{aligned}$$

where $\boldsymbol{p}$ is any point on the line.

Definition of line entity

The CSTA 3-vector *line* entity $\boldsymbol{L}_{\mathcal{C}}$ through point $\boldsymbol{p}_{\mathcal{M}}$ in the direction of unit norm vector $\boldsymbol{d}_{\mathcal{M}}$ can be defined as

$$\begin{aligned}
\boldsymbol{L}_{\mathcal{C}} \;&=\; \boldsymbol{d}^{*\mathcal{M}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{d}^{*\mathcal{M}}) \wedge \mathrm{e}_{\infty\gamma} && (4.150)\\
&\simeq\; \boldsymbol{L}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} && (4.151)
\end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}^{*}$ (§4.5.6) up to a homogeneous scalar factor. If the line direction $\boldsymbol{d}$ is a null vector, then the line entity $\boldsymbol{L}_{\mathcal{C}}$ is a null 3-vector representing a null line (light-line), otherwise it is a non-null 3-vector representing a timelike or spacelike line. The point at infinity $\mathbf{e}_{\infty\gamma}$ is on all 3-vector lines $\boldsymbol{L}_{\mathcal{C}}$.

Observable representation

The observable worldline

$$\begin{aligned}\boldsymbol{p}_{\mathcal{M}}^{\mathbf{p}_0}(t) &= \mathbf{p}_0+\boldsymbol{v}t && (4.152)\\ &= \mathbf{p}_0+\boldsymbol{o}t+\mathbf{v}t && (4.153)\\ &= \boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)+\mathbf{v}t, && (4.154)\end{aligned}$$

which intersects the worldline of the translated observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ at

$$\mathbf{p}_0 = o_x\gamma_1+o_y\gamma_2+o_z\gamma_3 \qquad (4.155)$$

when the coordinate time is $t=t_{\boldsymbol{po}}=t_{\boldsymbol{cv}}=0$, can be represented as the CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\boldsymbol{p}}$ that is constructed as

$$\boldsymbol{L}_{\boldsymbol{p}} = \boldsymbol{v}^{*\mathcal{M}}+(\mathbf{p}_0\cdot\boldsymbol{v}^{*\mathcal{M}})\wedge\mathbf{e}_{\infty\gamma}. \qquad (4.156)$$

The observable worldline $\boldsymbol{L}_{\boldsymbol{p}}$ can be operated on by all of the CSTA versors. For example, the translated-boost $B_{\mathcal{C}\mathbf{u}}^{\mathbf{p}_0}$ (§4.6.9) of $\boldsymbol{L}_{\boldsymbol{p}}$ can boost $\boldsymbol{L}_{\boldsymbol{p}}$ into the frame of $\boldsymbol{u}=\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)+\mathbf{u}$ with proper time $\tau=t_{\boldsymbol{pu}}=t_{\boldsymbol{cv}\oplus\boldsymbol{u}}$ relative to the translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ as a relativistic velocity addition $\boldsymbol{v}\oplus\boldsymbol{u}$ while the initial position remains $\mathbf{p}_0$. Other, even more complicated, spacetime transformations can be achieved by compositions of the CSTA versors applied to an observable line $\boldsymbol{L}_{\boldsymbol{p}}$.

The position *point* $\boldsymbol{P}_{\mathcal{C}}^{\mathbf{p}_0}(t)$ of the observable $\boldsymbol{L}_{\boldsymbol{p}}$ at time t is represented by the CSTA GIPNS 4-vector *flat point* (§4.4.16)

$$\begin{aligned}\mathbb{P}_{\mathcal{C}} &= \boldsymbol{L}_{\boldsymbol{p}}\wedge\boldsymbol{E}_{\mathcal{C}} && (4.157)\\ &\simeq (\boldsymbol{P}_{\mathcal{C}}^{\mathbf{p}_0}(t)\wedge\mathbf{e}_{\infty\gamma})\mathbf{I}_{\mathcal{C}} && (4.158)\end{aligned}$$

with time hyperplane

$$\boldsymbol{E}_{\mathcal{C}} = \gamma_0+t\mathbf{e}_{\infty}. \qquad (4.159)$$

Flat point decomposition (§4.5.5) of the CSTA GOPNS 2-vector *flat point* $\mathbb{P}_{\mathcal{C}}^{*}\simeq\mathbb{P}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1}$ can *project* the CSTA position *point* $\boldsymbol{P}_{\mathcal{C}}^{\mathbf{p}_0}(t)$ as the STA position *vector* $\boldsymbol{p}_{\mathcal{M}}^{\mathbf{p}_0}(t)$.

The rapidity of $\boldsymbol{L}_{\boldsymbol{p}}$ is given by

$$\varphi = \operatorname{acosh}\left(\frac{\boldsymbol{L}_{\boldsymbol{p}}\cdot(\gamma_1\gamma_2\gamma_3)}{\sqrt{\boldsymbol{L}_{\boldsymbol{p}}^2}}\right) \qquad (4.160)$$

and the natural speed β of $\boldsymbol{L}_{\boldsymbol{p}}$ is

$$\beta = \tanh(\varphi). \qquad (4.161)$$

The formula for rapidity φ is similar to the standard formula for the angle θ between two Euclidean plane vectors $\mathbf{v}$ and $\mathbf{u}$, $\theta=\operatorname{acos}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$, except that this plane is a Minkowski spacetime plane with hyperbolic angle φ.

The spacetime velocity $\boldsymbol{v}$ of $\boldsymbol{L_p}$ is the line direction

$$\boldsymbol{v} = \boldsymbol{d} \;=\; ((\boldsymbol{L_p} \cdot \mathbf{I}_{\mathcal{M}})\mathbf{I}_{\mathcal{M}}^{-1})\mathbf{I}_{\mathcal{M}} = \boldsymbol{L_p} \cdot \mathbf{I}_{\mathcal{M}}. \tag{4.162}$$

If the observable $\boldsymbol{L_p}$ is boosted by translated-boost operations, then $\boldsymbol{v}$ is boosted according to all the same results and interpretations as $\boldsymbol{v}$ boosted by STA boost operations centered on the observer $\boldsymbol{o}$.

One advantage of using the CSTA worldline representation $\boldsymbol{L_p}$ is the ability to easily incorporate initial positions $\mathbf{p}_0$ and use translated spacetime operations with the CSTA translator (§4.6.4). Other advantages may include the ability to compute various intersections of a worldline with other CSTA spacetime entities.

4.4.12 CSTA GIPNS 3-vector spatial circle

The CSTA GIPNS 1-vector *hyperpseudosphere* with radius r_0

$$\boldsymbol{\Sigma}_{\mathcal{C}} \;=\; \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma} \tag{4.163}$$

centered at

$$\boldsymbol{P}_{\mathcal{C}} \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w \gamma_0 + p_x \gamma_1 + p_y \gamma_2 + p_z \gamma_3) \tag{4.164}$$

can be intersected with two CSTA GIPNS 1-vector *hyperplanes*

$$\boldsymbol{E}_{\mathcal{C}_1} \;=\; \mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty\gamma} \tag{4.165}$$

$$\boldsymbol{E}_{\mathcal{C}_2} \;=\; \gamma_0 + (\boldsymbol{p}_{\mathcal{M}} \cdot \gamma_0)\mathbf{e}_{\infty\gamma} \tag{4.166}$$

to obtain a circle with radius r_0 centered at $\boldsymbol{p}_{\mathcal{M}}$ in the spatial plane through $\boldsymbol{p}_{\mathcal{M}}$ with direction bivector $\mathbf{N}_{\mathcal{S}} = \mathbf{n}_{\mathcal{S}}^* = -\mathbf{n}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}^{-1}$.

The CSTA GIPNS 3-vector *circle* entity $\boldsymbol{C}_{\mathcal{C}}$ centered at $\boldsymbol{p}_{\mathcal{M}}$ with radius r_0 at time p_w in the plane of bivector $\mathbf{N}_{\mathcal{S}} = \mathbf{n}_{\mathcal{S}}^* = -\mathbf{n}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}^{-1}$ dual to normal vector $\mathbf{n}_{\mathcal{S}}$ can be formed as

$$\boldsymbol{C}_{\mathcal{C}} \;=\; \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2} \tag{4.167}$$

$$\;=\; \boldsymbol{S}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_2} \tag{4.168}$$

Without setting the time $w = p_w$ by intersecting $\boldsymbol{E}_{\mathcal{C}_2}$, the circle changes radius with time as the CSTA GIPNS 2-vector *hyperboloid (pseudosphere)* $\boldsymbol{S}_{\mathcal{C}} = \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_1}$ (§4.4.8).

The CSTA GIPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}$ can also be represented as the intersection of the CSTA GIPNS 1-vector *hyperpseudosphere* $\boldsymbol{\Sigma}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ as

$$\boldsymbol{C}_{\mathcal{C}} \;=\; \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{\Pi}_{\mathcal{C}} \tag{4.169}$$

where the hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{C}}$ is the same as above and sets the center $\boldsymbol{p}_{\mathcal{M}}$ and radius r_0, and the plane $\boldsymbol{\Pi}_{\mathcal{C}}$ with spatial direction bivector $\mathbf{N}_{\mathcal{S}}$ through point $\boldsymbol{p}_{\mathcal{M}}$ is

$$\boldsymbol{\Pi}_{\mathcal{C}} \;=\; \boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma} \tag{4.170}$$

$$\;=\; \mathbf{N}_{\mathcal{S}}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{N}_{\mathcal{S}}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma} \tag{4.171}$$

$$\;=\; \boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2}. \tag{4.172}$$

The CSTA GIPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}$ is equal to the CSTA undual of the dual CSTA GOPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}^*$

$$\boldsymbol{C}_{\mathcal{C}} \;\simeq\; \boldsymbol{C}_{\mathcal{C}}^* \mathbf{I}_{\mathcal{C}} \tag{4.173}$$

up to a homogeneous scalar factor.

A CSTA GIPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}$ can also be formed as the intersection of a CSTA GIPNS 1-vector *imaginary hyperpseudosphere* $\Xi_{\mathcal{C}}$ or CSTA GIPNS 1-vector *hypercone* $\boldsymbol{K}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *spatial plane* $\mathbf{\Pi}_{\mathcal{C}}$ as

$$\begin{aligned}\boldsymbol{C}_{\mathcal{C}} &= \Xi_{\mathcal{C}}\wedge\mathbf{\Pi}_{\mathcal{C}} &(4.174)\\ &= \boldsymbol{K}_{\mathcal{C}}\wedge\mathbf{\Pi}_{\mathcal{C}}. &(4.175)\end{aligned}$$

4.4.13 CSTA GIPNS 3-vector spacetime hyperbola (pseudocircle)

The *circle* $\boldsymbol{C}_{\mathcal{C}}$ with radius r_0 centered at $\boldsymbol{p}_{\mathcal{M}}=p_w\gamma_0+\mathbf{p}_{\mathcal{S}}$ in spatial plane $\mathbf{N}_{\mathcal{S}}=\mathbf{n}_{\mathcal{S}}^*$ is formed by intersecting the plane $\mathbf{\Pi}_{\mathcal{C}}$ of $\mathbf{N}_{\mathcal{S}}$ through $\boldsymbol{p}_{\mathcal{M}}$ with the hyperpseudosphere $\mathbf{\Sigma}_{\mathcal{C}}$ of radius r_0 at $\boldsymbol{p}_{\mathcal{M}}$. Similarly, the *pseudocircle* $\boldsymbol{C}_{\mathcal{C}}$ with central radius r_0 centered at $\boldsymbol{p}_{\mathcal{M}}=p_w\gamma_0+\mathbf{p}_{\mathcal{S}}$ in Minkowski spacetime plane $\boldsymbol{D}_{\mathcal{M}}=\boldsymbol{\gamma}_0\mathbf{d}_{\mathcal{S}}$ is formed by intersecting the plane $\mathbf{\Pi}_{\mathcal{C}}$ of $\boldsymbol{D}_{\mathcal{M}}$ through $\boldsymbol{p}_{\mathcal{M}}$ with the hyperpseudosphere $\mathbf{\Sigma}_{\mathcal{C}}$ of radius r_0 at $\boldsymbol{p}_{\mathcal{M}}$. The hyperbola opens up and down the spatial vector axis $\mathbf{d}_{\mathcal{S}}$ for a hyperpseudosphere $\mathbf{\Sigma}_{\mathcal{C}}$, and it opens up and down the time axis $\boldsymbol{\gamma}_0$ for an imaginary hyperpseudosphere $\Xi_{\mathcal{C}}$.

The CSTA GIPNS 3-vector *spacetime hyperbola (pseudocircle)* $\boldsymbol{C}_{\mathcal{C}}$ can be defined as

$$\boldsymbol{C}_{\mathcal{C}} = \mathbf{\Sigma}_{\mathcal{C}}\wedge\mathbf{\Pi}_{\mathcal{C}} \tag{4.176}$$

where the hyperpseudosphere $\mathbf{\Sigma}_{\mathcal{C}}$ sets the central position $\boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ and initial radius r_0 as

$$\mathbf{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}}+\frac{1}{2}r_0\mathbf{e}_{\infty\gamma} \tag{4.177}$$

and the plane $\mathbf{\Pi}_{\mathcal{C}}$ sets the Minkowski spacetime plane $\boldsymbol{D}=\boldsymbol{\gamma}_0\mathbf{d}_{\mathcal{S}}$ of spatial unit direction vector $\mathbf{d}_{\mathcal{S}}$ and time direction $\boldsymbol{\gamma}_0$ as

$$\mathbf{\Pi}_{\mathcal{C}} = \boldsymbol{D}^{*\mathcal{M}}-(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{D}^{*\mathcal{M}})\wedge\mathbf{e}_{\infty\gamma}. \tag{4.178}$$

The hyperbola can be visualized as a point pair on the spatial line $\mathbf{d}_{\mathcal{S}}$, centered on $\mathbf{p}_{\mathcal{S}}$, and separated by an initial distance $2r=2r_0$ at time $w=p_w$. As time w changes away from the initial time p_w, the radius r increases to $r=\sqrt{r_0^2+(w-p_w)^2}$. The CSTA GIPNS 4-vector spacetime point pair can be obtained as

$$\mathbf{2}_{\mathcal{C}} = \boldsymbol{S}_{\mathcal{C}}\wedge\boldsymbol{E}_{\mathcal{C}} \tag{4.179}$$

where $\boldsymbol{E}_{\mathcal{C}}$ is the xyz-space hyperplane

$$\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0+w\mathbf{e}_{\infty\gamma} \tag{4.180}$$

at the time w for the point pair with radius r around $\mathbf{p}_{\mathcal{S}}$ on the line direction $\mathbf{d}_{\mathcal{S}}$. The hyperplane sets the time w component of the points in the spacetime point pair. The points appear to move apart spatially with time away from p_w.

A CSTA GIPNS 3-vector *spacetime hyperbola (pseudocircle)* $\boldsymbol{C}_{\mathcal{C}}$ can also be formed as the intersection of a CSTA GIPNS 1-vector *imaginary hyperpseudosphere* $\Xi_{\mathcal{C}}$ or CSTA GIPNS 1-vector *hypercone* $\boldsymbol{K}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *spacetime plane* $\mathbf{\Pi}_{\mathcal{C}}$ as

$$\begin{aligned}\boldsymbol{C}_{\mathcal{C}} &= \Xi_{\mathcal{C}}\wedge\mathbf{\Pi}_{\mathcal{C}} &(4.181)\\ &= \boldsymbol{K}_{\mathcal{C}}\wedge\mathbf{\Pi}_{\mathcal{C}} &(4.182)\end{aligned}$$

which open up and down the time w-axis.

4.4.14 CSTA GIPNS 4-vector point pair

The CSTA GIPNS 4-vector *point pair* $\mathbf{2}_{\mathcal{C}}$ is

$$\begin{aligned}\mathbf{2}_{\mathcal{C}} &= -\boldsymbol{P}_{\mathcal{C}_1}\cdot\boldsymbol{P}^*_{\mathcal{C}_2} &(4.183)\\ &= -\boldsymbol{P}_{\mathcal{C}_1}\cdot(\boldsymbol{P}_{\mathcal{C}_2}\mathbf{I}_{\mathcal{C}}^{-1}) = \boldsymbol{P}_{\mathcal{C}_1}\cdot(\boldsymbol{P}_{\mathcal{C}_2}\mathbf{I}_{\mathcal{C}}) = (\boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2})\mathbf{I}_{\mathcal{C}} &(4.184)\\ &= \mathbf{2}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} &(4.185)\end{aligned}$$

which is exactly the CSTA undual of the dual CSTA GOPNS 2-vector *point pair* $\mathbf{2}^*_{\mathcal{C}}$.

If the two points are *relatively lightlike*, then the point pair is actually the CSTA GIPNS null 4-vector *light-line (null line)* $\mathcal{L}_{\mathcal{C}}$ that is exactly undual to the dual CSTA GOPNS null 2-vector *light-line (null line)* $\mathcal{L}^*_{\mathcal{C}}$ (§4.5.4). The point pair $\mathbf{2}_{\mathcal{C}}$ of two not relatively lightlike points is *non-null*.

If one of the two points is $\mathbf{e}_{\infty\gamma}$, then the point pair is actually the CSTA GIPNS 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ that is exactly undual to the dual CSTA GOPNS 2-vector *flat point* $\mathbb{P}^*_{\mathcal{C}}$ (§4.5.5). A flat point is non-null.

4.4.15 CSTA GIPNS null 4-vector light-line (null line)

The CSTA GIPNS null 4-vector *null line (light-line)* $\mathcal{L}_{\mathcal{C}}$ is exactly the undual $\mathcal{L}_{\mathcal{C}} = \mathcal{L}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}$ of the dual CSTA GOPNS null 2-vector *null line (light-line)* $\mathcal{L}^*_{\mathcal{C}}$ (§4.5.4).

The CSTA GIPNS null 4-vector *light-line (null line)* $\mathcal{L}_{\mathcal{C}}$ is

$$\begin{aligned}\mathcal{L}_{\mathcal{C}} &= -\boldsymbol{P}_{\mathcal{L}_1}\cdot\boldsymbol{P}^*_{\mathcal{L}_2} &(4.186)\\ &= -\boldsymbol{P}_{\mathcal{L}_1}\cdot(\boldsymbol{P}_{\mathcal{L}_2}\mathbf{I}_{\mathcal{C}}^{-1}) = \boldsymbol{P}_{\mathcal{L}_1}\cdot(\boldsymbol{P}_{\mathcal{L}_2}\mathbf{I}_{\mathcal{C}}) = (\boldsymbol{P}_{\mathcal{L}_1}\wedge\boldsymbol{P}_{\mathcal{L}_2})\mathbf{I}_{\mathcal{C}} &(4.187)\\ &= \mathcal{L}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} &(4.188)\end{aligned}$$

where $\boldsymbol{P}_{\mathcal{L}_i} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i}) = \mathcal{C}(p_{w_i}\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}_i})$ denotes points that are *relatively lightlike* in spacetime positions. The two relatively lightlike points $\boldsymbol{P}_{\mathcal{L}_1}$ and $\boldsymbol{P}_{\mathcal{L}_2}$ are on a light-line in spacetime having equal changes in time components $|p_{w_1} - p_{w_2}|$ to space components $\|\mathbf{p}_{\mathcal{S}_1} - \mathbf{p}_{\mathcal{S}_2}\|$,

$$\begin{aligned}\frac{|p_{w_1} - p_{w_2}|}{\|\mathbf{p}_{\mathcal{S}_1} - \mathbf{p}_{\mathcal{S}_2}\|} &= 1 &(4.189)\\ \partial_t\|\mathbf{p}_{\mathcal{S}_1} - \mathbf{p}_{\mathcal{S}_2}\| = \partial_t|p_{w_1} - p_{w_2}| &= c. &(4.190)\end{aligned}$$

Light speed c is required to travel between the two points, or any two points on a light-line, in spacetime. The vector $\boldsymbol{p}_{\mathcal{M}_1} - \boldsymbol{p}_{\mathcal{M}_2}$ is a null vector in spacetime, and any two points in spacetime with a null difference vector are relatively lightlike.

A null 4-vector *light-line* $\mathcal{L}_{\mathcal{C}}$ can be converted into a non-null 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ as

$$\boldsymbol{L}_{\mathcal{C}} = (\mathcal{L}^*_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma})\mathbf{I}_{\mathcal{C}} = \boldsymbol{L}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}. \qquad (4.191)$$

4.4.16 CSTA GIPNS 4-vector flat point

The CSTA GIPNS 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ is

$$\begin{aligned}\mathbb{P}_{\mathcal{C}} &= -\boldsymbol{P}_{\mathcal{C}}\cdot\mathbf{e}^*_{\infty\gamma} &(4.192)\\ &= -\boldsymbol{P}_{\mathcal{C}}\cdot(\mathbf{e}_{\infty\gamma}\mathbf{I}_{\mathcal{C}}^{-1}) = \boldsymbol{P}_{\mathcal{C}}\cdot(\mathbf{e}_{\infty\gamma}\mathbf{I}_{\mathcal{C}}) = (\boldsymbol{P}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma})\mathbf{I}_{\mathcal{C}} &(4.193)\\ &= \mathbb{P}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} &(4.194)\end{aligned}$$

which is exactly the undual of the dual CSTA GOPNS 2-vector *flat point* $\mathbb{P}^*_{\mathcal{C}}$.

A flat spatial point $\mathbb{P}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}}) \wedge \mathbf{e}_{\infty\gamma}$ at $w = 0$ can be represented as the intersection of a CSTA GIPNS 2-vector plane $\boldsymbol{\Pi}_{\mathcal{C}}$ and CSTA GIPNS 3-vector line $\boldsymbol{L}_{\mathcal{C}}$ that are in the common xyz-space hyperplane at any times in spacetime as

$$\begin{aligned} \mathbb{P}_{\mathcal{C}} &= \boldsymbol{\gamma}_0 \wedge (\boldsymbol{\gamma}_0 \cdot \boldsymbol{\Pi}_{\mathcal{C}}) \wedge (\boldsymbol{\gamma}_0 \cdot \boldsymbol{L}_{\mathcal{C}}) && (4.195) \\ &= \mathbb{P}_{\mathcal{C}}^{*} \mathbf{I}_{\mathcal{C}} && (4.196) \end{aligned}$$

where the common hyperplane of xyz-space $\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0$ is contracted out of the plane and line before they are intersected, and then $\boldsymbol{\gamma}_0$ is intersected back into the result. The time components of the plane and line do not affect the result, which is spatial intersection at $w = 0$. The flat spatial point $\mathbb{P}_{\mathcal{C}}$ can exactly match the undual $\mathbb{P}_{\mathcal{C}}^{*} \mathbf{I}_{\mathcal{C}}$. The flat spatial point represents the point $\boldsymbol{P}_{\mathcal{C}}$ of intersection on the plane where the line passes through, and it also represents $\mathbf{e}_{\infty}$ where the plane and line also intersect. A flat spacetime point as intersections may also be possible but is not considered here.

4.4.17 CSTA GIPNS 5-vector point

The CSTA null 1-vector *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ is the

- CSTA *GIPNS* null 1-vector *hypercone* $\boldsymbol{P}_{\mathcal{C}}$ centered at $\boldsymbol{p}_{\mathcal{M}}$
- CSTA *GOPNS* null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}}$ representing the point $\boldsymbol{p}_{\mathcal{M}}$.

Therefore, the CSTA GIPNS null 5-vector *point* $\boldsymbol{P}_{\mathcal{C}}^{\star}$ is the undual

$$\boldsymbol{P}_{\mathcal{C}}^{\star} = \boldsymbol{P}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}} \qquad (4.197)$$

which also introduces a notation for the undual operation. The undual notation has been omitted on other undual entities. For the 5-vector point, the undual notation avoids a notational conflict since $\boldsymbol{P}_{\mathcal{C}}$ is the dual, not $\boldsymbol{P}_{\mathcal{C}}^{*}$.

For STA vectors, the undual is

$$\begin{aligned} \boldsymbol{\gamma}_0^{\star} &= \boldsymbol{\gamma}_0 \mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_1 \boldsymbol{\gamma}_2 \boldsymbol{\gamma}_3 && (4.198) \\ \boldsymbol{\gamma}_1^{\star} &= \boldsymbol{\gamma}_1 \mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0 \boldsymbol{\gamma}_2 \boldsymbol{\gamma}_3 && (4.199) \\ \boldsymbol{\gamma}_2^{\star} &= \boldsymbol{\gamma}_2 \mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0 \boldsymbol{\gamma}_3 \boldsymbol{\gamma}_1 && (4.200) \\ \boldsymbol{\gamma}_3^{\star} &= \boldsymbol{\gamma}_3 \mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0 \boldsymbol{\gamma}_1 \boldsymbol{\gamma}_2 && (4.201) \end{aligned}$$

which is consistent, in this case, with *Hodge dual* that is denoted $\star A$ in other literature.

The CSTA GIPNS null 5-vector *point* $\boldsymbol{P}_{\mathcal{C}}^{\star}$ can also be represented as the intersection of a hypercone $\boldsymbol{P}_{\mathcal{C}}$ with the four hyperplanes $\boldsymbol{E}_{\mathcal{C}_i}$ through the hypercone vertex $\boldsymbol{p}_{\mathcal{M}}$ as

$$\boldsymbol{P}_{\mathcal{C}}^{\star} = \boldsymbol{P}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2} \wedge \boldsymbol{E}_{\mathcal{C}_3} \wedge \boldsymbol{E}_{\mathcal{C}_4} \qquad (4.202)$$

where

$$\begin{aligned} \boldsymbol{P}_{\mathcal{C}} &= \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w \boldsymbol{\gamma}_0 + p_x \boldsymbol{\gamma}_1 + p_y \boldsymbol{\gamma}_2 + p_z \boldsymbol{\gamma}_3) && (4.203) \\ \boldsymbol{E}_{\mathcal{C}_1} &= \boldsymbol{\gamma}_0 + p_w \mathbf{e}_{\infty\gamma} && (4.204) \\ \boldsymbol{E}_{\mathcal{C}_2} &= -\boldsymbol{\gamma}_1 + p_x \mathbf{e}_{\infty\gamma} && (4.205) \\ \boldsymbol{E}_{\mathcal{C}_3} &= -\boldsymbol{\gamma}_2 + p_y \mathbf{e}_{\infty\gamma} && (4.206) \\ \boldsymbol{E}_{\mathcal{C}_4} &= -\boldsymbol{\gamma}_3 + p_z \mathbf{e}_{\infty\gamma}. && (4.207) \end{aligned}$$

Each hyperplane fixes one coordinate to hold a value.

4.5 CSTA GOPNS entities

In $\mathcal{G}_{2,4}$ CSTA, *five or less points* can be wedged into CSTA GOPNS entities, allowing a greater variety of entities than in $\mathcal{G}_{4,1}$ CGA, which uses four or less points. The CSTA 6-vector unit pseudoscalar and the CGA 5-vector unit pseudoscalar are dualization operators that can also be interpreted as GOPNS entities that represent the whole 4-D spacetime in CSTA or the whole 3-D space CGA.

The familiar flat and round GOPNS entities of $\mathcal{G}_{4,1}$ CGA have a similar representation in $\mathcal{G}_{2,4}$ CSTA as the wedge four or less points that are the embeddings of spatial points $\mathbf{P}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}})$. These flat and round CGA-like entities are at $w = ct = 0$ in spacetime unless translated to time $w \neq 0$, and can be constructed as the intersections of CSTA GIPNS flat and hyperbolic entities with the CSTA GIPNS hyperplane $\boldsymbol{\gamma}_0 + w\mathbf{e}_\infty$ for time w. These CGA-like entities are the entities of the $\mathcal{G}_{1,4}$ Conformal Space Algebra (CSA) that is without time components. $\mathcal{G}_{1,4}$ CSA is similar to $\mathcal{G}_{4,1}$ CGA, except that there are differences in the signs of some similar CGA expressions. For example, the distance between two $\mathcal{G}_{1,4}$ CSA points is $d = \sqrt{2\mathbf{P}_{\mathcal{CS}_1} \cdot \mathbf{P}_{\mathcal{CS}_2}}$, while in $\mathcal{G}_{4,1}$ CGA it is $d = \sqrt{-2\mathbf{P}_{\mathcal{C}_1} \cdot \mathbf{P}_{\mathcal{C}_2}}$. As a subalgebra of CSTA, the CSA versors are the CSTA versors for spatial operations, which excludes the spacetime boost versors. The dilator operation, or successive inversions in two concentric spheres for a dilation by factor r_2^2 / r_1^2, can *isotropically dilate* CSA entities in space *and time*. The translator operation, or successive reflections in parallel spacetime planes, can translate CSA entities in space and time. The rotor operation, or successive reflections in non-parallel spatial planes, can rotate CSA entities in space, leaving the time unaffected.

The GOPNS entities are called *dual* to the *undual* GIPNS entities, but this naming is quite often reversed in other literature. This naming is chosen to be consistent with DCSTA entities, where the DCSTA GIPNS entities are unduals and the DCSTA GOPNS entities are duals.

4.5.1 Geometric outer product null space (GOPNS)

Geometric outer product null space (GOPNS) entities are introduced by Perwass in [16], and are reviewed by this author in [7] and [9].

The $\mathcal{G}_{2,4}$ CSTA unit pseudoscalar $\mathbf{I}_{\mathcal{C}}$ is grade 6, and it can be interpreted to be a GOPNS 6-vector entity that represents the entire 4D spacetime. Otherwise, the CSTA GOPNS surface entities are formed as the wedge of *five or less* CSTA GOPNS null 1-vector points $\bigwedge \boldsymbol{P}_{\mathcal{C}_i}$ (§4.5.2) on the surface that span the surface. In $\mathcal{G}_{4,1}$ CGA, the CGA GOPNS entities are formed as the wedge of four or less points. Compared to CGA, CSTA has a larger set of GOPNS entities.

The subset of $\mathcal{G}_{2,4}$ CSTA GOPNS entities that are similar to $\mathcal{G}_{4,1}$ CGA GOPNS entities are the $\mathcal{G}_{1,4}$ CSA GOPNS entities, which are defined as the wedge of four or less CSTA spatial points $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$, or $\mathcal{G}_{1,4}$ CSA null 1-vector points $\mathbf{P}_{\mathcal{CS}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$, that are on the surface and that also *span the surface* of the entity. The CSA GOPNS entities are constructed as wedges of spatial points by the same forms as in $\mathcal{G}_{4,1}$ CGA. The spatial CSA entities are located at time $w = ct = 0$ in the CSTA spacetime, but can be translated (§4.6.4) to exist at any time $w = ct = p_w$. The $\mathcal{G}_{1,4}$ CSA entities and $\mathcal{G}_{4,1}$ CGA entities represent the same surfaces, but there are some sign changes. For example, the distance d between two spatial points $\mathbf{P}_{\mathcal{CS}_1}$ and $\mathbf{P}_{\mathcal{CS}_2}$ is now

$$d = \sqrt{2\mathbf{P}_{\mathcal{CS}_1} \cdot \mathbf{P}_{\mathcal{CS}_2}} \qquad (4.208)$$

while in ordinary $\mathcal{G}_{4,1}$ CGA, $d = \sqrt{-2\mathbf{P}_{\mathcal{C}_1}\cdot\mathbf{P}_{\mathcal{C}_2}}$.

4.5.2 CSTA GOPNS 1-vector point

As a GOPNS entity, the CSTA null 1-vector *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ represents the *point* of the embedded STA position $\boldsymbol{p}_{\mathcal{M}}$. The GOPNS test

$$\boldsymbol{T}_{\mathcal{C}} \wedge \boldsymbol{P}_{\mathcal{C}} \;=\; 0 \tag{4.209}$$

holds good *if and only if* (iff)

$$\boldsymbol{T}_{\mathcal{C}} \;\equiv\; \boldsymbol{P}_{\mathcal{C}}. \tag{4.210}$$

As a GIPNS entity, a point $\boldsymbol{P}_{\mathcal{C}}$ represents a null *hypercone* (§4.4.2) in spacetime. The GIPNS test

$$\boldsymbol{T}_{\mathcal{C}} \cdot \boldsymbol{P}_{\mathcal{C}} \;=\; 0 \tag{4.211}$$

holds good for any point $\boldsymbol{T}_{\mathcal{C}}$ on the hypercone with vertex $\boldsymbol{P}_{\mathcal{C}}$. A point $\boldsymbol{T}_{\mathcal{C}}$ on the hypercone is a point that is located at a *lightlike* (null vector) displacement from the vertex $\boldsymbol{P}_{\mathcal{C}}$. The hypercone is a sphere in space, centered at $\boldsymbol{P}_{\mathcal{C}}$, with time-varying radius $r = w - p_w = ct - p_w$.

A CSTA null 1-vector *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ represents

$$\boldsymbol{T}_{\mathcal{C}} \circ \boldsymbol{P}_{\mathcal{C}} \;=\; \begin{cases} \text{null } \textit{hypercone} \text{ centered at vertex } \boldsymbol{p}_{\mathcal{M}} & : \;\circ \text{ is } \cdot \\ \text{null } \textit{point} \text{ at } \boldsymbol{p}_{\mathcal{M}}. & : \;\circ \text{ is } \wedge \end{cases} \tag{4.212}$$

4.5.3 CSTA GOPNS 2-vector point pair

The CSTA GOPNS 2-vector *point pair* $\mathbf{2}^*_{\mathcal{C}}$ is the wedge of two finite CSTA points that are *not relatively lightlike* (i.e., $(\boldsymbol{p}_{\mathcal{M}_2} - \boldsymbol{p}_{\mathcal{M}_1})^2 \neq 0$)

$$\begin{aligned} \mathbf{2}^*_{\mathcal{C}} &= \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} && (4.213)\\ &= \mathbf{2}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} && (4.214)\end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 4-vector *point pair* $\mathbf{2}_{\mathcal{C}}$. Two points are relatively lightlike if they are separated by a null vector displacement. The GOPNS test

$$\boldsymbol{T}_{\mathcal{C}} \cdot \mathbf{2}^*_{\mathcal{C}} \;=\; 0 \tag{4.215}$$

holds good for the point pair $\mathbf{2}^*_{\mathcal{C}}$ of two finite points that are not relatively lightlike *if and only if* (iff)

$$\boldsymbol{T}_{\mathcal{C}} \;\in\; \{\boldsymbol{P}_{\mathcal{C}_1}, \boldsymbol{P}_{\mathcal{C}_2}\}. \tag{4.216}$$

A valid *point pair* $\mathbf{2}^*_{\mathcal{C}}$ represents the two distinct points as a single entity.

The *point pair decomposition* [4]

$$\hat{\boldsymbol{P}}_{\mathcal{C}_\pm} \;=\; \frac{\mathbf{2}^*_{\mathcal{C}} \mp \sqrt{(\mathbf{2}^*_{\mathcal{C}})^2}}{-\mathbf{e}_{\infty\gamma}\cdot\mathbf{2}^*_{\mathcal{C}}} = (\mathbf{2}^*_{\mathcal{C}} \mp \sqrt{\mathbf{2}^*_{\mathcal{C}}\cdot\mathbf{2}^*_{\mathcal{C}}})(-\mathbf{e}_{\infty\gamma}\cdot\mathbf{2}^*_{\mathcal{C}})^{-1} \tag{4.217}$$

gives the two normalized (unit scale) finite points $\hat{\boldsymbol{P}}_{\mathcal{C}_+}$ and $\hat{\boldsymbol{P}}_{\mathcal{C}_-}$ of the point pair $\mathbf{2}^*_{\mathcal{C}}$.

A *light-line (null line)* $\mathcal{L}^*_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{L}_1} \wedge \boldsymbol{P}_{\mathcal{L}_2}$ (§4.5.4) is the wedge of two *relatively lightlike points* $\boldsymbol{P}_{\mathcal{L}_i}$ and represents the line of the two points, excluding $\mathbf{e}_{\infty\gamma}$. A *flat point* $\mathbb{P}^*_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} \wedge \mathbf{e}_{\infty\gamma}$ (§4.5.5) is the wedge of one finite point $\boldsymbol{P}_{\mathcal{C}}$ and the point at infinity $\mathbf{e}_{\infty\gamma}$.

4.5.4 CSTA GOPNS 2-vector light-line (null line)

An STA null *lightlike position* vector $\boldsymbol{l}_{\mathcal{M}}$ relative to the origin has the form

$$\boldsymbol{l}_{\mathcal{M}} = ct(\boldsymbol{\gamma}_0+\hat{\mathbf{n}}_{\mathcal{S}})=w(\boldsymbol{\gamma}_0+\hat{\mathbf{n}}_{\mathcal{S}})=ct\boldsymbol{n}_{\mathcal{M}}. \tag{4.218}$$

It can be verified that a vector of the form of $\boldsymbol{n}_{\mathcal{M}}=\boldsymbol{\gamma}_0+\hat{\mathbf{n}}_{\mathcal{S}}$ is a null vector $\boldsymbol{n}_{\mathcal{M}}^2=0$, where $\hat{\mathbf{n}}_{\mathcal{S}}$ is any spatial unit direction vector. A lightlike position relative to an STA position vector $\boldsymbol{p}_{\mathcal{M}}$ is

$$\boldsymbol{p}_{\mathcal{L}} = \boldsymbol{p}_{\mathcal{M}}+\boldsymbol{l}_{\mathcal{M}}. \tag{4.219}$$

Let any three collinear positions $\boldsymbol{p}_{\mathcal{L}_i}$ and their CSTA point embeddings $\boldsymbol{P}_{\mathcal{L}_i}$ be

$$\boldsymbol{P}_{\mathcal{L}_1}=\mathcal{C}(\boldsymbol{p}_{\mathcal{L}_1}) = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+w_1\boldsymbol{n}_{\mathcal{M}})=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+ct_1\boldsymbol{n}_{\mathcal{M}}) \tag{4.220}$$
$$\boldsymbol{P}_{\mathcal{L}_2}=\mathcal{C}(\boldsymbol{p}_{\mathcal{L}_2}) = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+w_2\boldsymbol{n}_{\mathcal{M}})=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+ct_2\boldsymbol{n}_{\mathcal{M}}) \tag{4.221}$$
$$\boldsymbol{P}_{\mathcal{L}_3}=\mathcal{C}(\boldsymbol{p}_{\mathcal{L}_3}) = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+w_3\boldsymbol{n}_{\mathcal{M}})=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+ct_3\boldsymbol{n}_{\mathcal{M}}). \tag{4.222}$$

These three points, called *relatively lightlike* points, are along a light-line in the null direction $\boldsymbol{n}_{\mathcal{M}}$ on a light-cone with vertex $\boldsymbol{p}_{\mathcal{M}}$. Two points, $\boldsymbol{p}_{\mathcal{L}_1}$ and $\boldsymbol{p}_{\mathcal{L}_2}$, are relatively lightlike if their difference vector $\boldsymbol{l}_{\mathcal{M}}=c(t_2-t_1)\boldsymbol{n}_{\mathcal{M}}=\boldsymbol{p}_{\mathcal{L}_2}-\boldsymbol{p}_{\mathcal{L}_1}$ is a null vector $\boldsymbol{n}_{\mathcal{M}}^2=0$. It can be verified that for any three collinear relatively lightlike points

$$\boldsymbol{P}_{\mathcal{L}_1}\wedge\boldsymbol{P}_{\mathcal{L}_2}\wedge\boldsymbol{P}_{\mathcal{L}_3} = 0. \tag{4.223}$$

Therefore, the *light-line* $\mathcal{L}_{\mathcal{C}}$ in the direction of $\boldsymbol{n}_{\mathcal{M}}$ through the point $\boldsymbol{p}_{\mathcal{M}}$ is characterized by the wedge of any two points on the light-line. The point at infinity $\mathbf{e}_{\infty\gamma}$ is *not* a point on a light-line that is represented like a point pair.

The CSTA GOPNS null 2-vector *light-line* $\mathcal{L}_{\mathcal{C}}$ is the wedge of any two *relatively lightlike* points on the light-line

$$\mathcal{L}_{\mathcal{C}}^* = \boldsymbol{P}_{\mathcal{L}_1}\wedge\boldsymbol{P}_{\mathcal{L}_2} \tag{4.224}$$
$$\simeq \mathcal{L}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} \tag{4.225}$$

and is the CSTA dual of the CSTA null 4-vector *light-line* $\mathcal{L}_{\mathcal{C}}$ up to a homogeneous scalar factor.

The light-line $\mathcal{L}_{\mathcal{C}}^*$ does *not* include the point at infinity $\mathbf{e}_{\infty\gamma}$. A light-line exists only in spacetime. In general, the two points of $\mathcal{L}_{\mathcal{C}}^*$ are along a light-line, which is a line through spacetime with slope $m=\pm1$ of time to space distance on a light-cone. A light-line is also called a *null line* since

$$(\mathcal{L}_{\mathcal{C}}^*)^2 = 0. \tag{4.226}$$

For any two *coplanar light-lines* $\mathcal{L}_{\mathcal{C}_1}^*$ and $\mathcal{L}_{\mathcal{C}_2}^*$, the lines share a light-cone vertex $\boldsymbol{p}_{\mathcal{M}}$ and

$$\mathcal{L}_{\mathcal{C}_1}^*\wedge\mathcal{L}_{\mathcal{C}_2}^* = 0 \tag{4.227}$$

which is a result that holds in general for all coplanar lines.

A *light-line* $\mathcal{L}_{\mathcal{C}}^*$ is a special type of line that requires only two points to define the line. A light-line is also called a *lightlike line*. Other lines in spacetime are *timelike* or *spacelike* lines and require the wedge of three collinear points to define them as GOPNS entities.

The CSTA GOPNS 3-vector line $\boldsymbol{L}_{\mathcal{C}}^*$ (§4.5.6) always includes $\mathbf{e}_{\infty\gamma}$ on the line. A light-line $\mathcal{L}_{\mathcal{C}}^*$ can be extended to include $\mathbf{e}_{\infty\gamma}$ as a null 3-vector line $\boldsymbol{L}_{\mathcal{C}}^*$. A *lightlike line* $\mathcal{L}_{\mathcal{C}}^*$ is converted into a CSTA GOPNS null 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}^*$ as

$$\boldsymbol{L}_{\mathcal{C}}^* = \mathcal{L}_{\mathcal{C}}^*\wedge\mathbf{e}_{\infty\gamma}. \tag{4.228}$$

See also, the CSTA GIPNS null 1-vector *light-line (null line)* (§4.4.4), which also includes $\mathbf{e}_{\infty\gamma}$ on the null line.

The two points of a lightlike line $\mathcal{L}_{\mathcal{C}}^{*}$ cannot be decomposed from the line entity.

4.5.5 CSTA GOPNS 2-vector flat point

A CSTA *flat point* $\mathbb{P}_{\mathcal{C}}^{*}$ is the wedge of a *finite* CSTA point $\boldsymbol{P}_{\mathcal{C}}$ and the CSTA *point at infinity* $\mathbf{e}_{\infty\gamma}$

$$\begin{aligned}\mathbb{P}_{\mathcal{C}}^{*} &= \boldsymbol{P}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}})\wedge\mathbf{e}_{\infty\gamma} && (4.229)\\ &\simeq \mathbb{P}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} && (4.230)\end{aligned}$$

and equals the CSTA dual of the CSTA 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ up to a homogeneous scalar factor.

As introduced in [4] in the context of $\mathcal{G}_{4,1}$ CGA, a flat point is the intersection point of a plane and line in space. However, a plane and line both also include the point at infinity. Therefore, a flat point represents the *two points* where a line and plane intersect in space. In $\mathcal{G}_{2,4}$ CSTA, a line and plane are in spacetime and may intersect at a spacetime flat point. The CSTA GIPNS 2-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}$ is the intersection of two hyperplanes

$$\mathbf{\Pi}_{\mathcal{C}} = \boldsymbol{E}_{\mathcal{C}_1}\wedge\boldsymbol{E}_{\mathcal{C}_2} \qquad (4.231)$$

and the CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ is the intersection of three hyperplanes

$$\boldsymbol{L}_{\mathcal{C}} = \boldsymbol{E}_{\mathcal{C}_3}\wedge\boldsymbol{E}_{\mathcal{C}_4}\wedge\boldsymbol{E}_{\mathcal{C}_5}. \qquad (4.232)$$

In CGA, the intersection of a line and plane is simply $\mathbf{L}\wedge\mathbf{\Pi}$, but this form cannot work as simply in CSTA. There can be *zero, one, or two* hyperplanes that are the same in the line and plane. If *zero* are the same, then $\mathbf{\Pi}_{\mathcal{C}}\wedge\boldsymbol{L}_{\mathcal{C}}\neq 0$ and the intersection is $\mathbf{\Pi}_{\mathcal{C}}\wedge\boldsymbol{L}_{\mathcal{C}}\simeq\mathbf{e}_{\infty\gamma}^{\star}$ (§4.4.17). If *two* are the same, then $\boldsymbol{L}_{\mathcal{C}}=\mathbf{\Pi}_{\mathcal{C}}\wedge\boldsymbol{E}_{\mathcal{C}}$ and the intersection is $\boldsymbol{L}_{\mathcal{C}}$. If *one* hyperplane is the same, then the intersection is a finite spacetime point $\boldsymbol{P}_{\mathcal{C}}$ and $\mathbf{e}_{\infty\gamma}$, which are represented as a CSTA GOPNS 2-vector *flat point* $\mathbb{P}_{\mathcal{C}}^{*}=\boldsymbol{P}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}$. In all three cases, a line and plane intersect at $\mathbf{e}_{\infty\gamma}$.

Assume, for now, that there is only *one* common hyperplane

$$\boldsymbol{E}_{\mathcal{C}}=\boldsymbol{E}_{\mathcal{C}_1}=\boldsymbol{E}_{\mathcal{C}_3} = \boldsymbol{n}_{\mathcal{M}}+(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma}. \qquad (4.233)$$

We expect to obtain a CSTA GIPNS 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ as the intersection of the CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}$. If we contract $\boldsymbol{E}_{\mathcal{C}}$ into the line or the plane and then wedge them, then we get the 4-vector flat point. The pseudoscalar of the spacetime projections of $\mathbf{\Pi}_{\mathcal{C}}$ and $\boldsymbol{L}_{\mathcal{C}}$ is $\mathbf{I}_{\mathcal{M}}$, which can be used to project their directional blades. The conjugate normal vector $\boldsymbol{n}_{\mathcal{M}}^{\dagger}$ of the common hyperplane $\boldsymbol{E}_{\mathcal{C}}$ is given by the spacetime *meet* product $\vee_{\mathcal{M}}$ of the plane and line as

$$\begin{aligned}\boldsymbol{n}_{\mathcal{M}}^{\dagger} &= (\mathbf{\Pi}_{\mathcal{C}}\vee_{\mathcal{M}}\boldsymbol{L}_{\mathcal{C}})^{\dagger} && (4.234)\\ &= \boldsymbol{\gamma}_0(((\mathbf{\Pi}_{\mathcal{C}}\cdot\mathbf{I}_{\mathcal{M}}^{-1})\wedge(\boldsymbol{L}_{\mathcal{C}}\cdot\mathbf{I}_{\mathcal{M}}^{-1}))\cdot\mathbf{I}_{\mathcal{M}})\boldsymbol{\gamma}_0. && (4.235)\end{aligned}$$

The conjugate normal vector $\boldsymbol{n}_{\mathcal{M}}^{\dagger}$ can be used to contract $\boldsymbol{E}_{\mathcal{C}}$ in either the plane or line, which then allows intersections of the plane and line to be formed as the two flat points

$$\begin{aligned}\mathbb{P}_{\mathcal{C}_1} &= (\boldsymbol{n}_{\mathcal{M}}^{\dagger}\cdot\mathbf{\Pi}_{\mathcal{C}})\wedge\boldsymbol{L}_{\mathcal{C}} && (4.236)\\ \mathbb{P}_{\mathcal{C}_2} &= \mathbf{\Pi}_{\mathcal{C}}\wedge(\boldsymbol{n}_{\mathcal{M}}^{\dagger}\cdot\boldsymbol{L}_{\mathcal{C}}). && (4.237)\end{aligned}$$

For *one* common hyperplane $\boldsymbol{E}_{\mathcal{C}}$, as assumed, then $\mathbb{P}_{\mathcal{C}_1} = \pm\mathbb{P}_{\mathcal{C}_2}$. Now, if *two* hyperplanes are common in the plane and line, then the spacetime meet produces zero and the flat points are zero. These results allow the following definition for the intersection $\boldsymbol{\Pi}_{\mathcal{C}} \cap \boldsymbol{L}_{\mathcal{C}}$ of a line and plane.

The CSTA GIPNS intersection $\boldsymbol{\Pi}_{\mathcal{C}} \cap \boldsymbol{L}_{\mathcal{C}}$ of a CSTA GIPNS 2-vector *plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ and CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ can be defined as

$$\boldsymbol{\Pi}_{\mathcal{C}} \cap \boldsymbol{L}_{\mathcal{C}} = \begin{cases} \boldsymbol{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} \simeq \mathbf{e}_{\infty\gamma}^{\star} & : \ \boldsymbol{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} \neq 0 \\ \mathbb{P}_{\mathcal{C}_1} = \pm\mathbb{P}_{\mathcal{C}_2} & : \ \boldsymbol{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} = 0, \boldsymbol{\Pi}_{\mathcal{C}} \vee_{\mathcal{M}} \boldsymbol{L}_{\mathcal{C}} \neq 0 \\ \boldsymbol{L}_{\mathcal{C}} & : \ \boldsymbol{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} = 0, \boldsymbol{\Pi}_{\mathcal{C}} \vee_{\mathcal{M}} \boldsymbol{L}_{\mathcal{C}} = 0. \end{cases} \tag{4.238}$$

The intersection is valid for any null or non-null 3-vector line $\boldsymbol{L}_{\mathcal{C}}$ and any spacetime plane $\boldsymbol{\Pi}_{\mathcal{C}}$.

The point $\boldsymbol{P}_{\mathcal{C}}$ of a flat point $\mathbb{P}_{\mathcal{C}}^{*} = \mathbb{P}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1}$ is projected [4] as

$$\boldsymbol{p}_{\mathcal{M}} = \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{C}}) \ = \ \frac{(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot (\mathbf{e}_{o\gamma} \wedge \mathbb{P}_{\mathcal{C}}^{*})}{-(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot \mathbb{P}_{\mathcal{C}}^{*}} = \frac{-\mathbb{P}_{\mathcal{C}}^{*}}{(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot \mathbb{P}_{\mathcal{C}}^{*}} \cdot \mathbf{e}_{o\gamma} - \mathbf{e}_{o\gamma}. \tag{4.239}$$

4.5.6 CSTA GOPNS 3-vector line

The CSTA GOPNS line $\boldsymbol{L}_{\mathcal{C}}^{*}$ is similar to the CGA GOPNS line $\mathbf{L}^{*}$ discussed in [7]. In general, any line in spacetime can be represented as the wedge of three well-chosen points on the line. A CSTA GOPNS *2-vector lightlike line* $\mathcal{L}_{\mathcal{C}}^{*}$ (§4.5.4) is represented by the wedge of just two points but it does not include the point at infinity $\mathbf{e}_{\infty\gamma}$ on the line.

A CSTA GOPNS null 3-vector **lightlike** *line* $\boldsymbol{L}_{\mathcal{C}}^{*}$ is the wedge of *any two relatively lightlike points* $\boldsymbol{P}_{\mathcal{L}_i}$ on the line and the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$

$$\boldsymbol{L}_{\mathcal{C}}^{*} \ = \ \boldsymbol{P}_{\mathcal{L}_1} \wedge \boldsymbol{P}_{\mathcal{L}_2} \wedge \mathbf{e}_{\infty\gamma} = \mathcal{L}_{\mathcal{C}}^{*} \wedge \mathbf{e}_{\infty\gamma}. \tag{4.240}$$

A CSTA GOPNS non-null 3-vector **timelike or spacelike** *line* $\boldsymbol{L}_{\mathcal{C}}^{*}$ can be the wedge of *any two points* $\boldsymbol{P}_{\mathcal{C}_i}$ on the line and the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$ *or* the wedge of *any three collinear points* $\boldsymbol{P}_{\mathcal{C}_i}$ on the line

$$\boldsymbol{L}_{\mathcal{C}}^{*} \ = \ \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \mathbf{e}_{\infty\gamma} \tag{4.241}$$

$$\simeq \ \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \tag{4.242}$$

$$\simeq \ \boldsymbol{L}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} \tag{4.243}$$

and is equal to the CSTA dual of the CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ (§4.4.11) up to a homogeneous scalar factor.

The 3-vector line entity can be used to represent an observable (§4.4.11).

4.5.7 CSTA GOPNS 3-vector spatial circle

The system of implicit surface equations for a *spatial circle* with radius r_0 centered at (p_x, p_y, p_z) in the xy-plane at $z = p_z$ is

$$(x - p_x)^2 + (y - p_y)^2 - r_0^2 \ = \ 0 \tag{4.244}$$

$$z - p_z \ = \ 0 \tag{4.245}$$

$$w - p_w \ = \ 0. \tag{4.246}$$

The CSTA spatial circle entity $\boldsymbol{C}_{\mathcal{C}}$ represents a system of implicit surface equations of this form for the intersection of a circular cylinder and plane. The center position of the circle

$$\boldsymbol{p}_{\mathcal{M}} \ = \ p_w\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 \tag{4.247}$$

includes a time component $p_w\gamma_0$ that indicates *when* the circle exists.

The CSTA GOPNS spatial circle $\boldsymbol{C}^*_{\mathcal{C}}$ is similar to the CGA GOPNS circle $\mathbf{C}^*$ discussed in [7] and is the wedge of *any three points* on the circle in space at the same time. Three points are always *coplanar cocircular* points. Three *collinear* points are on a circle of infinite radius, which is a line.

The CSTA GOPNS 3-vector *spatial circle* $\boldsymbol{C}^*_{\mathcal{C}}$ is the wedge of *any three* CSTA *points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(p_w\gamma_0 + \mathbf{p}_{\mathcal{S}_i})$ *at the same time* p_w on the circle

$$\begin{aligned} \boldsymbol{C}^*_{\mathcal{C}} &= \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} && (4.248)\\ &\simeq \boldsymbol{C}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} && (4.249)\end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 3-vector *spatial circle* $\boldsymbol{C}_{\mathcal{C}}$ up to a homogeneous scalar factor.

The wedge of three points that are not all at the same time may produce a *spacetime hyperbola* (§4.5.8). The circle is produced for three points at the same time.

4.5.8 CSTA GOPNS 3-vector spacetime hyperbola (pseudocircle)

The system of implicit surface equations for a *spacetime circular hyperbola* in the xw-plane centered at (p_w, p_x, p_y, p_z), with central radius r, opening up and down the w-axis is

$$\begin{aligned} (x-p_x)^2 + r^2 - (w-p_w)^2 &= 0 && (4.250)\\ y - p_y &= 0 && (4.251)\\ z - p_z &= 0. && (4.252)\end{aligned}$$

The spacetime circular hyperbola can also be called a *pseudocircle*. The CSTA pseudocircle entity represents a system of implicit surface equations of this form. This hyperbola is not general, but circular. To get the expected shape, the points have to be chosen carefully. At $x = p_x$, $w = p_w \pm r$. At $w = p_w + \sqrt{2}r$, $x = p_x \pm r$. The axes may be transposed. Spatial rotations, spacetime translations, and spacetime isotropic dilations permit the pseudocircle to be in any Minkowski space-time plane, at any spacetime center point, and with any central radius. The hyperbola is generally a conic section of a related circular hyperboloid (§4.5.10) cut through a spacetime plane and has lightlike asymptotes. By cutting the related hyperboloid in different spacetime planes, it is possible to get hyperbolas that open up and down the time or space axis. The hyperboloids are w-axis (time-axis) aligned with circles in the xy-planes.

The CSTA GOPNS 3-vector *spacetime circular hyperbola* $\boldsymbol{C}^*_{\mathcal{C}}$ is the wedge of *any three non-collinear* CSTA *spacetime points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ on the spacetime circular hyperbola

$$\begin{aligned} \boldsymbol{C}^*_{\mathcal{C}} &= \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} && (4.253)\\ &\simeq \boldsymbol{C}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} && (4.254)\end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 3-vector *spacetime hyperbola* $\boldsymbol{C}_{\mathcal{C}}$ up to a homogeneous scalar factor. Similar to a circle, the point at infinity $\mathbf{e}_{\infty\gamma}$ is *not* a point on the pseudocircle.

The spacetime hyperbola $\boldsymbol{C}^*_{\mathcal{C}}$ becomes a *light-line pair* of a light-cone when the three non-collinear points are *relatively lightlike* points $\boldsymbol{P}_{\mathcal{L}_i}$ (§4.5.4). The points are relatively lightlike when any two points are relatively lightlike, forming one of the light-lines. The perpendicular line through the third point is the other light-line. The light-cone vertex is the point of intersection of the two light-lines, which could be one of the points.

In general, the wedge of three non-collinear CSTA *spatial points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ produces a *spatial circle* (§4.5.7) at $w = ct = 0$.

4.5.9 CSTA GOPNS 4-vector spatial sphere

The CSTA GOPNS spatial sphere $\boldsymbol{S}_{\mathcal{C}}^*$ is similar to the CGA GOPNS sphere $\mathbf{S}^*$ discussed in [7].

The CSTA GOPNS 4-vector *spatial sphere* $\boldsymbol{S}_{\mathcal{C}}^*$ is the wedge of four CSTA *spatial points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ on the sphere surface that span the sphere

$$\begin{aligned} \boldsymbol{S}_{\mathcal{C}}^* &= \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \wedge \boldsymbol{P}_{\mathcal{C}_4} & (4.255) \\ &\simeq \boldsymbol{S}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} & (4.256) \end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 2-vector *spatial sphere* $\boldsymbol{S}_{\mathcal{C}}$ up to a homogeneous scalar factor. To span the sphere, the points cannot be all coplanar. The spatial sphere $\boldsymbol{S}_{\mathcal{C}}^*$ holds $w = ct = 0$ and is a sphere in space that exists at time $t = 0$, but it can be translated to any time $w = p_w$ (or to any spacetime position) using the CSTA translator (§4.6.4).

4.5.10 CSTA GOPNS 4-vector spacetime hyperboloid (pseudosphere)

The implicit quadric surface equation of a *spacetime circular hyperboloid* of one sheet with circular sections in the xy-plane and central radius r is

$$(w - p_w)^2 + r^2 - (x - p_x)^2 - (y - p_y) \;=\; 0. \qquad (4.257)$$

The spacetime circular hyperboloid can also be called a *pseudosphere*. Spatial rotations, spacetime translations, and spacetime isotropic dilations permit the pseudosphere to be in any spatial plane, at any spacetime center point, and with any central radius.

The CSTA GOPNS 4-vector *spacetime circular hyperboloid of one sheet (pseudosphere)* $\boldsymbol{S}_{\mathcal{C}}^*$ is the wedge of four CSTA *spacetime points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ on the surface that span the surface

$$\begin{aligned} \boldsymbol{S}_{\mathcal{C}}^* &= \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \wedge \boldsymbol{P}_{\mathcal{C}_4} & (4.258) \\ &\simeq \mathcal{L}_{\mathcal{C}_1}^* \wedge \mathcal{L}_{\mathcal{C}_2}^* & (4.259) \\ &\simeq \boldsymbol{S}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} & (4.260) \end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 2-vector *spacetime circular hyperboloid of one sheet* $\boldsymbol{S}_{\mathcal{C}}$ up to a homogeneous scalar factor. Similar to a sphere, the point at infinity $\mathbf{e}_{\infty\gamma}$ is *not* a point on the pseudosphere. Two *non-coplanar* light-lines $\mathcal{L}_{\mathcal{C}_1}^*$ and $\mathcal{L}_{\mathcal{C}_2}^*$ can span a pseudosphere as asymptotes that are tangent to the hyperboloid.

The pseudosphere $\boldsymbol{S}_{\mathcal{C}}^*$ becomes a *light-cone*, also called a null cone, when the four points are *relatively lightlike* points $\boldsymbol{P}_{\mathcal{L}_i}$. The points are relatively lightlike when any three points are relatively lightlike to the fourth point, which is the vertex center point of the light-cone. The wedge of the light-cone vertex and another point is a light-line $\mathcal{L}_{\mathcal{C}}^*$, and the light-cone is spanned by three light-lines sharing the vertex.

In general, the wedge of four non-coplanar CSTA *spatial points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ produces a *spatial sphere* that holds $w = ct = 0$.

It is also possible to produce the CSTA GOPNS 4-vector *spacetime hyperboloid of two sheets (imaginary pseudosphere)* as the wedge of four well-chosen points that span the surface.

4.5.11 CSTA GOPNS 4-vector plane

The CSTA GOPNS plane $\mathbf{\Pi}^*_{\mathcal{C}}$ is similar to the CGA GOPNS plane $\mathbf{\Pi}^*$ discussed in [7]. In CGA, a plane $\mathbf{\Pi}^*$ is the wedge of *any four coplanar non-collinear non-cocircular points* on the plane. The four well-chosen points that define a plane in CGA are nearly the same for $\mathbf{\Pi}^*_{\mathcal{C}}$, but light-lines $\boldsymbol{\mathcal{L}}^*_{\mathcal{C}}$ (§4.5.4) introduce an additional constraint on the choice of the four coplanar points in spacetime.

Three non-collinear finite points $\boldsymbol{P}_{\mathcal{C}_i}$ are co(pseudo)circular and define a finite (pseudo)circle $\boldsymbol{C}^*_{\mathcal{C}}$. The fourth point can be the point at infinity $\mathbf{e}_{\infty\gamma}$ or some other coplanar non-co(pseudo)circular finite point $\boldsymbol{P}_{\mathcal{C}_4}$.

Three collinear, not relatively lightlike, finite points $\boldsymbol{P}_{\mathcal{C}_i}$ define a line $\boldsymbol{L}^*_{\mathcal{C}}$. The fourth point *cannot* be the point at infinity $\mathbf{e}_{\infty\gamma}$ since it is collinear. The fourth point can be some other non-collinear finite point $\boldsymbol{P}_{\mathcal{C}_4}$.

Two relatively lightlike finite points $\boldsymbol{P}_{\mathcal{L}_i}$ define a light-line $\boldsymbol{\mathcal{L}}^*_{\mathcal{C}}$. The other two points can be $\mathbf{e}_{\infty\gamma}$ and a coplanar non-collinear finite point $\boldsymbol{P}_{\mathcal{C}_4}$. The other two points can also be not relatively lightlike finite points $\boldsymbol{P}_{\mathcal{C}_3}$ and $\boldsymbol{P}_{\mathcal{C}_4}$ that are coplanar non-collinear points to $\boldsymbol{\mathcal{L}}^*_{\mathcal{C}}$.

The CSTA GOPNS 4-vector *plane* $\mathbf{\Pi}^*_{\mathcal{C}}$ is the wedge of *four well-chosen points* $\boldsymbol{P}_{\mathcal{C}_i}$ on the plane in space or spacetime

$$\begin{aligned}
\mathbf{\Pi}^*_{\mathcal{C}} &= \boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\boldsymbol{P}_{\mathcal{C}_4}=\boldsymbol{C}_{\mathcal{C}}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (4.261)\\
&\simeq \boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\mathbf{e}_{\infty\gamma}=\boldsymbol{C}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma} && (4.262)\\
&\simeq \boldsymbol{L}^*_{\mathcal{C}}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (4.263)\\
&\simeq \boldsymbol{\mathcal{L}}^*_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (4.264)\\
&\simeq \boldsymbol{\mathcal{L}}^*_{\mathcal{C}}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (4.265)\\
&\simeq \mathbf{\Pi}_{\mathcal{C}}\mathbf{I}^{-1}_{\mathcal{C}} && (4.266)
\end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 2-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}$ (§4.4.10) up to a homogeneous scalar factor. The four points must be well-chosen as explained above.

The entity $\mathbf{\Pi}^*_{\mathcal{C}}$ is a *plane in space* that holds $w=ct=0$ when its points $\boldsymbol{P}_{\mathcal{C}_i}=\mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ are the embeddings of spatial points $\mathbf{p}_{\mathcal{S}_i}$ in 3D SA space. In the general case of points $\boldsymbol{P}_{\mathcal{C}_i}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ in spacetime, the entity $\mathbf{\Pi}^*_{\mathcal{C}}$ is a *plane in spacetime*. The plane entity is generally valid in both space and spacetime.

As explained in §4.4.10, the rotor $R_{\mathcal{C}}$, translator $T_{\mathcal{C}}$, and boost $B_{\mathcal{C}}$ can be defined as reflections in planes. Reflections in either the GIPNS plane $\mathbf{\Pi}_{\mathcal{C}}$ or GOPNS plane $\mathbf{\Pi}^*_{\mathcal{C}}$ are both valid on all entities. The dilator $D_{\mathcal{C}}$ (§4.6.6) can be defined as inversions in hyperpseudospheres (§4.4.5).

4.5.12 CSTA GOPNS 5-vector hyperhyperboloid

The implicit quadric surface equation for a circular *hyperhyperboloid of one sheet (hyperpseudosphere)* is

$$r_0^2+(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_x)^2 \;=\; 0 \qquad (4.267)$$

where r_0 is the initial radius of the expanding sphere in space with time-varying radius

$$r \;=\; \sqrt{r_0^2+(w-p_w)^2}=\sqrt{r_0^2+(ct-p_w)^2} \qquad (4.268)$$

and center position

$$\boldsymbol{p}_{\mathcal{M}} \;=\; p_w\gamma_0+p_x\gamma_1+p_y\gamma_2+p_z\gamma_3 \qquad (4.269)$$

in spacetime.

The hyperhyperboloid can be spanned by five surface points that do not form entities for any (pseudo)sphere, plane, line, or (pseudo)circle. Planes and lines are avoided by excluding the point at infinity. Spheres and circles are avoided by using only one or two points in any circle on the surface. The choice of points is otherwise arbitrary. For example, using an arbitrary scalar $l \neq 0$, three values of time

$$w \;\in\; \{p_w + l, p_w + 2l, p_w - 3l\} \tag{4.270}$$

and corresponding values of radius

$$r \;\in\; \left\{\sqrt{r_0^2 + l^2}, \sqrt{r_0^2 + 4l^2}, \sqrt{r_0^2 + 9l^2}\right\} \tag{4.271}$$

can be chosen. Then, use at most two surface points per value of w. The hyperhyperboloid, a sphere that expands with time, has the five surface points that span the surface

$$\boldsymbol{P}_{\mathcal{C}_1} \;=\; \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}} + l\gamma_0 + \sqrt{r_0^2 + l^2}\gamma_1\Big) \tag{4.272}$$

$$\boldsymbol{P}_{\mathcal{C}_2} \;=\; \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}} + 2l\gamma_0 - \sqrt{r_0^2 + 4l^2}\gamma_2\Big) \tag{4.273}$$

$$\boldsymbol{P}_{\mathcal{C}_3} \;=\; \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}} + 2l\gamma_0 - \sqrt{r_0^2 + 4l^2}\gamma_3\Big) \tag{4.274}$$

$$\boldsymbol{P}_{\mathcal{C}_4} \;=\; \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}} - 3l\gamma_0 + \sqrt{r_0^2 + 9l^2}\gamma_1\Big) \tag{4.275}$$

$$\boldsymbol{P}_{\mathcal{C}_5} \;=\; \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}} - 3l\gamma_0 + \sqrt{r_0^2 + 9l^2}\gamma_2\Big). \tag{4.276}$$

These points are just an example of five well-chosen points on the surface that span the surface, and other points could be chosen.

The CSTA GOPNS 5-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* $\boldsymbol{\Sigma}_{\mathcal{C}}^*$ is the wedge of five CSTA points $\boldsymbol{P}_{\mathcal{C}_i}$ on the surface that span the surface

$$\boldsymbol{\Sigma}_{\mathcal{C}}^* \;=\; \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \wedge \boldsymbol{P}_{\mathcal{C}_4} \wedge \boldsymbol{P}_{\mathcal{C}_5} \tag{4.277}$$

$$\simeq\; \boldsymbol{\Sigma}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} \tag{4.278}$$

and is the CSTA dual of the CSTA GIPNS 1-vector *hyperhyperboloid* $\boldsymbol{\Sigma}_{\mathcal{C}}$ up to a homogeneous scalar factor.

The hyperhyperboloid with $r_0 = 0$ degenerates into the CSTA GOPNS null 5-vector *hypercone* $\boldsymbol{P}_{\mathcal{C}}^* = \boldsymbol{\Sigma}_{\mathcal{C}}^*(\boldsymbol{p}_{\mathcal{M}}, r_0 = 0)$. The CSTA GIPNS null 1-vector hypercone $\boldsymbol{P}_{\mathcal{C}} = \boldsymbol{K}_{\mathcal{C}}$ at $\boldsymbol{p}_{\mathcal{M}}$ is the *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$. The undual $\boldsymbol{P}_{\mathcal{C}}^{\star} = \boldsymbol{P}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}}^* \mathbf{I}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}} = -\boldsymbol{P}_{\mathcal{C}}^*$ is the CSTA GIPNS null 5-vector *point* $\boldsymbol{P}_{\mathcal{C}}^{\star}$.

It is also possible to produce the CSTA GOPNS 5-vector *hyperhyperboloid of two sheets (imaginary hyperpseudosphere)* $\boldsymbol{\Xi}_{\mathcal{C}}^*$ as the wedge of five well-chosen points that span the surface.

4.5.13 CSTA GOPNS 5-vector hyperplane

A hyperplane is a subspace of dimension $(n-1)$ in a space of dimension n. In 4D spacetime, a hyperplane is a 3D subspace at a fixed coordinate along a fourth perpendicular axis. Intersecting with a hyperplane serves to set or fix one coordinate. The signature of the hyperplane space can be $(2,1)$ or $(3,0)$.

The wedge of three non-collinear CSTA points $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ spans a spatial circle or spacetime pseudocircle $\boldsymbol{C}_{\mathcal{C}}^*$. Adding the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$, then the four CSTA points span a spatial plane or spacetime plane $\boldsymbol{\Pi}_{\mathcal{C}}^*$. Adding a fifth CSTA point that is non-coplanar to the other four points, then the five points span a 3D space. The wedge of five well-chosen CSTA points is the CSTA GOPNS 5-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}^*$ that represents a 3D subspace of the 4D spacetime.

The CSTA GOPNS 5-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}^*$ is the wedge the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$ and four CSTA points $\boldsymbol{P}_{\mathcal{C}_i}$ on the surface that span the hyperplane

$$\begin{aligned}\boldsymbol{E}_{\mathcal{C}}^* &= \boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\boldsymbol{P}_{\mathcal{C}_4}\wedge\mathbf{e}_{\infty\gamma} &(4.279)\\ &= \boldsymbol{E}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} &(4.280)\end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ (§4.4.3) up to a homogeneous scalar factor.

4.6 CSTA operations

4.6.1 CSTA dualization

The CSTA *dual* $A_{\mathcal{C}}^{*\mathcal{C}}$ of a CSTA multivector $A_{\mathcal{C}}$ is

$$A_{\mathcal{C}}^{*\mathcal{C}} = A_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} = A_{\mathcal{C}}\widetilde{\mathbf{I}_{\mathcal{C}}}. \qquad (4.281)$$

The CSTA *undual* $A_{\mathcal{C}}$ of a CSTA multivector $A_{\mathcal{C}}^{*\mathcal{C}}$ is

$$A_{\mathcal{C}} = A_{\mathcal{C}}^{*\mathcal{C}}\mathbf{I}_{\mathcal{C}} = A_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1}\mathbf{I}_{\mathcal{C}}. \qquad (4.282)$$

4.6.2 CSTA spatial projection

The $\mathcal{G}_{1,4}$ CSA1 *spatial projection* $A_{\mathcal{CS}^1}$ of a $\mathcal{G}_{2,4}$ CSTA1 multivector $A_{\mathcal{C}^1}$ is

$$A_{\mathcal{CS}^1} = (A_{\mathcal{C}^1}\cdot\mathbf{I}_{\mathcal{CS}^1})\mathbf{I}_{\mathcal{CS}^1}^{-1} \qquad (4.283)$$

where the $\mathcal{G}_{1,4}$ Conformal Space Algebra 1 (CSA1) *unit pseudoscalar* $\mathbf{I}_{\mathcal{CS}^1}$ is

$$\mathbf{I}_{\mathcal{CS}^1} = \mathbf{e}_1\cdot\mathbf{I}_{\mathcal{C}^1} = \mathbf{I}_{\mathcal{S}^1}\mathbf{e}_5\mathbf{e}_6. \qquad (4.284)$$

The $\mathcal{G}_{1,4}$ CSA2 *spatial projection* $A_{\mathcal{CS}^2}$ of a $\mathcal{G}_{2,4}$ CSTA2 multivector $A_{\mathcal{C}^2}$ is

$$A_{\mathcal{CS}^2} = (A_{\mathcal{C}^2}\cdot\mathbf{I}_{\mathcal{CS}^2})\mathbf{I}_{\mathcal{CS}^2}^{-1} \qquad (4.285)$$

where the $\mathcal{G}_{1,4}$ Conformal Space Algebra 2 (CSA2) *unit pseudoscalar* $\mathbf{I}_{\mathcal{CS}^2}$ is

$$\mathbf{I}_{\mathcal{CS}^2} = \mathbf{e}_7\cdot\mathbf{I}_{\mathcal{C}^2} = \mathbf{I}_{\mathcal{S}^2}\mathbf{e}_{11}\mathbf{e}_{12}. \qquad (4.286)$$

The spatial projections $A_{\mathcal{CS}}$ drop the time components of $A_{\mathcal{C}}$, and may be useful for extracting geometrical results in space.

4.6.3 CSTA spatial rotor

The spatial rotor R is the same in $\mathcal{G}_{0,3}$ SA $\mathcal{S}$, $\mathcal{G}_{1,3}$ STA $\mathcal{M}$, $\mathcal{G}_{1,4}$ CSA $\mathcal{CS}$, and $\mathcal{G}_{2,4}$ CSTA $\mathcal{C}$.

The CSTA 2-versor *spatial rotor* $R_{\mathcal{C}}$ is equal to the SA *rotor* $R_{\mathcal{S}}$

$$\begin{aligned}R = R_{\mathcal{S}} = R_{\mathcal{M}} = R_{\mathcal{CS}} = R_{\mathcal{C}} &= e^{\frac{1}{2}\theta\hat{\mathbf{n}}_{\mathcal{S}}^{*\mathcal{S}}} &(4.287)\\ &= \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right)\hat{\mathbf{n}}_{\mathcal{S}}\widetilde{\mathbf{I}_{\mathcal{S}}} &(4.288)\end{aligned}$$

where the SA unit vector $\hat{\mathbf{n}}_{\mathcal{S}}$ is the axis of rotation, and θ is the angle of rotation around the axis by the right-hand rule on a system of right-handed axes. The *rotor operation* $R_{\mathcal{C}}A_{\mathcal{C}}\widetilde{R_{\mathcal{C}}}$ on any CSTA entity $A_{\mathcal{C}}$ spatially rotates the entity in the usual way in space, leaving any time component unchanged.

The CSTA *rotor operation* that spatially rotates any CSTA entity $A_{\mathcal{C}}$ by angle θ around SA axis $\hat{\mathbf{n}}_{\mathcal{S}}$ is defined as

$$A'_{\mathcal{C}} = R_{\mathcal{C}} A_{\mathcal{C}} \widetilde{R_{\mathcal{C}}}. \tag{4.289}$$

The rotor R can be defined as

$$R = \mathbf{\Pi}_{\mathcal{C}_2} \mathbf{\Pi}_{\mathcal{C}_1}, \tag{4.290}$$

which is the successive reflections in two non-parallel *spatial* CSTA GIPNS planes (§4.4.10) that intersect in the rotation axis $\hat{\mathbf{n}}_{\mathcal{S}}$ through the origin. The angle of rotation θ is *twice* the angle subtended by the two planes. More generally, the two *spatial* planes can intersect in an arbitrary *spatial* CSA GIPNS 2-vector line

$$\begin{aligned}\mathbf{L}_{\mathcal{CS}} &= -\boldsymbol{\gamma}_0 \cdot \boldsymbol{L}_{\mathcal{C}} &(4.291)\\ &= (-\boldsymbol{\gamma}_0 \cdot \mathbf{\Pi}_{\mathcal{C}_2}) \wedge (-\boldsymbol{\gamma}_0 \cdot \mathbf{\Pi}_{\mathcal{C}_1}) &(4.292)\\ &= \mathbf{\Pi}_{\mathcal{CS}_2} \wedge \mathbf{\Pi}_{\mathcal{CS}_1} &(4.293)\end{aligned}$$

as the rotation axis (§4.6.5).

As a 2-versor, the rotor R can also be defined as

$$R = \boldsymbol{E}_{\mathcal{C}_2} \boldsymbol{E}_{\mathcal{C}_1}, \tag{4.294}$$

which is the successive reflections in two non-parallel CSTA GIPNS 1-vector hyperplanes (§4.4.3). The rotation is by *twice* the angle subtended by the two *spatial* hyperplane normal vectors, from $\mathbf{n}_{\mathcal{S}_1}$ toward $\mathbf{n}_{\mathcal{S}_2}$. The right-handed rotation axis is the SA undual $\mathbf{n} = -(\mathbf{n}_{\mathcal{S}_2} \wedge \mathbf{n}_{\mathcal{S}_1})\mathbf{I}_{\mathcal{S}}$. If the two hyperplanes are both centered on $\mathbf{p}_{\mathcal{S}}$, then the rotation axis is the line through $\mathbf{p}_{\mathcal{S}}$ in the direction of $\mathbf{n}$.

4.6.4 CSTA translator

The CSTA 2-versor *translator* $T_{\mathcal{C}}$, adapted from the CGA translator, is defined as

$$\begin{aligned}T_{\mathcal{C}} &= e^{-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} &(4.295)\\ &= 1 - \frac{1}{2}\boldsymbol{d}_{\mathcal{M}} \wedge \mathbf{e}_{\infty\gamma}. &(4.296)\end{aligned}$$

The translation vector $\boldsymbol{d}_{\mathcal{M}}$ is an STA *spacetime displacement* vector. Translations through space and of time are possible.

The CSTA *translator operation* that translates CSTA entity $A_{\mathcal{C}}$ in spacetime by STA spacetime displacement $\boldsymbol{d}_{\mathcal{M}}$ is the two-sided versor "sandwich" operation

$$A'_{\mathcal{C}} = T_{\mathcal{C}} A_{\mathcal{C}} \widetilde{T_{\mathcal{C}}}. \tag{4.297}$$

The translator $T_{\mathcal{C}}$ can be defined as

$$T_{\mathcal{C}} = \mathbf{\Pi}_{\mathcal{C}_2} \mathbf{\Pi}_{\mathcal{C}_1}, \tag{4.298}$$

which is the successive reflections in two *parallel* CSTA GIPNS planes (§4.4.10) that are separated by $\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}$, half the spacetime displacement $\boldsymbol{d}_{\mathcal{M}}$ of the translator. The translation is by *twice* the spacetime displacement between the planes. The orientation of the displacement $\boldsymbol{d}_{\mathcal{M}}$ is from $\mathbf{\Pi}_{\mathcal{C}_1}$ toward $\mathbf{\Pi}_{\mathcal{C}_2}$.

As a 2-versor, the translator $T_{\mathcal{C}}$ can also be defined as

$$T_{\mathcal{C}} = \boldsymbol{E}_{\mathcal{C}_2} \boldsymbol{E}_{\mathcal{C}_1}, \tag{4.299}$$

which is the successive reflections in two parallel CSTA GIPNS 1-vector hyperplanes (§4.4.3). The translation is by *twice* the spacetime displacement between the two hyperplanes. For the spacetime translation vector is $\boldsymbol{d}_{\mathcal{M}} = \boldsymbol{p}_{\mathcal{M}_2} - \boldsymbol{p}_{\mathcal{M}_1}$, the hyperplanes can be

$$\begin{aligned}\boldsymbol{E}_{\mathcal{C}_1} &= \frac{1}{2}\boldsymbol{p}_{\mathcal{M}_1} + \frac{1}{4}\boldsymbol{p}_{\mathcal{M}_1}^2 \mathbf{e}_{\infty\gamma} && (4.300)\\ \boldsymbol{E}_{\mathcal{C}_2} &= \frac{1}{2}\boldsymbol{p}_{\mathcal{M}_2} + \frac{1}{4}\boldsymbol{p}_{\mathcal{M}_2}^2 \mathbf{e}_{\infty\gamma}. && (4.301)\end{aligned}$$

4.6.5 CSTA spatial rotor around a line

The CSTA *spatial line rotor* $L_{\mathcal{C}}$ is defined as

$$\begin{aligned}L_{\mathcal{C}} &= e^{-\frac{1}{2}\theta\gamma_0 \cdot \boldsymbol{L}_{\mathcal{C}}} && (4.302)\\ &= \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right)(-\gamma_0 \cdot \boldsymbol{L}_{\mathcal{C}}). && (4.303)\end{aligned}$$

The line rotor $L_{\mathcal{C}}$ rotates around the spatial line $\boldsymbol{L}_{\mathcal{C}}$ by the angle θ. The CSTA GIPNS 3-vector line (§4.4.11)

$$\boldsymbol{L}_{\mathcal{C}} = \boldsymbol{d}^{*\mathcal{M}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{d}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} \qquad (4.304)$$

should be a purely spatial line, with a spatial *unit vector* direction $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ through a spatial point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$. The direction of rotation follows the right-hand rule, which is anticlockwise θ radians around the spatial direction $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$.

To understand how the spatial line rotor $L_{\mathcal{C}}$ is derived, consider the following. The STA dual of the unit spatial direction $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ of the line $\boldsymbol{L}_{\mathcal{C}}$ is

$$\begin{aligned}\boldsymbol{d}^{*\mathcal{M}} &= && (4.305)\\ \hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{M}}^{-1} &= && (4.306)\\ -\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{M}} &= && (4.307)\\ \gamma_0\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}} &\;\; . && (4.308)\end{aligned}$$

Therefore, the line $\boldsymbol{L}_{\mathcal{C}}$ in the spatial direction of $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ through the spatial point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$ is

$$\begin{aligned}\boldsymbol{L}_{\mathcal{C}} &= \gamma_0\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}} + \big(\mathbf{p}_{\mathcal{S}} \cdot \big(\gamma_0 \wedge \big(\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}\big)\big)\big) \wedge \mathbf{e}_{\infty\gamma} && (4.309)\\ &= \gamma_0 \wedge \big(\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}\big) - \gamma_0 \wedge \big(\mathbf{p}_{\mathcal{S}} \cdot \big(\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}\big)\big) \wedge \mathbf{e}_{\infty\gamma}. && (4.310)\end{aligned}$$

The spatial line rotor $L_{\mathcal{C}}$ uses the generator

$$\begin{aligned}-\gamma_0 \cdot \boldsymbol{L}_{\mathcal{C}} &= && (4.311)\\ -\big(\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}} - \big(\mathbf{p}_{\mathcal{S}} \cdot \big(\hat{\mathbf{d}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}\big)\big) \wedge \mathbf{e}_{\infty\gamma}\big) &= && (4.312)\\ \hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}} - \big(\mathbf{p}_{\mathcal{S}} \cdot \hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}\big) \wedge \mathbf{e}_{\infty\gamma} &= \mathbf{L}_{\mathcal{CS}} = \mathbf{L} && (4.313)\end{aligned}$$

which has the same form as a spatial $\mathcal{G}_{4,1}$ CGA GIPNS 2-vector line $\mathbf{L}$, but is a line entity in the similar $\mathcal{G}_{1,4}$ Conformal Space Algebra (CSA) $\mathcal{CS}$. If the line $\mathbf{L}$ were at (or through) the origin, then it would be $\mathbf{L} = \hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}$, which should be a *unit bivector* as a rotor generator. Therefore, $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ should be a unit SA direction. But, the line $\mathbf{L}$ is translated from the origin to point $\mathbf{p}_{\mathcal{S}}$ as

$$\mathbf{L} = T_{\mathcal{C}}\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}\tilde{T}_{\mathcal{C}} = \hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}} - \big(\mathbf{p}_{\mathcal{S}} \cdot \hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}\big) \wedge \mathbf{e}_{\infty\gamma} \qquad (4.314)$$

by CSTA translator $T_{\mathcal{C}}$ (§4.6.4) for translation by $\mathbf{p}_{\mathcal{S}}$.

The SA dualization of an SA unit direction vector $\hat{\mathbf{d}}_{\mathcal{S}}$ is defined to create a rotor unit bivector generator $\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}$ that is isomorphic to a quaternion versor without a reversal in orientation or sign and obeying the usual right-hand rule for rotation orientation around an axis $\hat{\mathbf{d}}_{\mathcal{S}}$ through the origin.

Now, consider a rotor R and translators T and $T^{\sim}$ by $\mathbf{p}_{\mathcal{S}}$ and $-\mathbf{p}_{\mathcal{S}}$, respectively,

$$R = e^{\frac{1}{2}\theta\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}} \tag{4.315}$$

$$T = e^{-\frac{1}{2}\mathbf{p}_{\mathcal{S}}\wedge\mathbf{e}_{\infty\gamma}} \tag{4.316}$$

$$T^{\sim} = e^{\frac{1}{2}\mathbf{p}_{\mathcal{S}}\wedge\mathbf{e}_{\infty\gamma}} \tag{4.317}$$

and their composition that translates $\mathbf{p}_{\mathcal{S}}$ to the origin, then rotates around the line of the unit vector $\mathbf{d}_{\mathcal{S}}$ through the origin, and then translates the origin back to $\mathbf{p}_{\mathcal{S}}$, which is applied to an entity $\mathbf{E}$ as

$$TRT^{\sim}\mathbf{E}\mathrm{TR}^{\sim}T^{\sim}\ \ . \tag{4.318}$$

The versor $TRT^{\sim}$ is a spatially *translated rotor*

$$TRT^{\sim} = \tag{4.319}$$

$$T\left(\cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}\right)T^{\sim} = \tag{4.320}$$

$$\cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)T\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}T^{\sim} = \tag{4.321}$$

$$\cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\mathbf{L} = \tag{4.322}$$

$$e^{\frac{1}{2}\theta\mathbf{L}} = \tag{4.323}$$

$$e^{\frac{1}{2}\theta(-\gamma_0\cdot\boldsymbol{L}_{\mathcal{C}})} = L_{\mathcal{C}}. \tag{4.324}$$

This composition, the translated rotor $L_{\mathcal{C}}$, is a versor for rotation around the spatial line $\boldsymbol{L}_{\mathcal{C}}$ by the angle θ. The spatial line $\boldsymbol{L}_{\mathcal{C}}$ should be unit scale, with the spatial line direction given by a unit vector $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ that passes through the spatial point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$.

All of the CSTA GIPNS 1-vector entities can be transformed by the CSTA versors that are defined in this section, including $L_{\mathcal{C}}$ in this subsection. All of the CSTA GIPNS k-vector entities can be constructed as the wedge, or intersection, of five or less CSTA GIPNS 1-vector entities. By versor outermorphism, all of the CSTA GIPNS k-vector entities can be correctly transformed by the CSTA versors. By the CSTA dualization transformation of the CSTA GIPNS entities into CSTA GOPNS entities, all of the CSTA GOPNS k-vector entities can also be correctly transformed by the CSTA versors.

4.6.6 CSTA isotropic dilator

The CSTA 2-versor *isotropic dilator* $D_{\mathcal{C}}$, adapted from the CGA dilator, is defined as

$$D_{\mathcal{C}} = \frac{1}{2}(1+d)+\frac{1}{2}(1-d)\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}. \tag{4.325}$$

The scalar d is the *dilation factor*. The CSTA isotropic dilator $D_{\mathcal{C}}$ is a *spacetime dilator*, which includes the dilation of the time and space components of an entity by the factor d.

The CSTA *isotropic dilation operation* that isotropically dilates CSTA entity $A_{\mathcal{C}}$ by factor d in spacetime is the two-sided versor "sandwich" operation

$$A'_{\mathcal{C}} = D_{\mathcal{C}}A_{\mathcal{C}}\widetilde{D_{\mathcal{C}}}. \tag{4.326}$$

It can be verified algebraically that the dilator $D_{\mathcal{C}}$ correctly dilates by factor d any CSTA GIPNS 1-vector entity. By versor outermorphism, $D_{\mathcal{C}}$ also correctly dilates any CSTA GIPNS k-vector entity, which can always be formed as the wedge of k CSTA GIPNS 1-vector entities. By CSTA dualization of GIPNS entities to GOPNS entities, all CSTA GOPNS entities are also dilated correctly by the dilator.

The dilator $D_{\mathcal{C}}$ can be derived from successive inversions in two CSTA GIPNS 1-vector *hyperpseudospheres* $\boldsymbol{\Sigma}_{\mathcal{C}_1}$ and $\boldsymbol{\Sigma}_{\mathcal{C}_2}$ (§4.4.5) centered on the origin $\mathbf{e}_{\infty\gamma}$ with radius $r_1 = 1$ and $r_2 = \sqrt{d}$, respectively, as

$$D_{\mathcal{C}} = -\boldsymbol{\Sigma}_{\mathcal{C}_2}\boldsymbol{\Sigma}_{\mathcal{C}_1} \simeq \boldsymbol{\Sigma}_{\mathcal{C}_2}\boldsymbol{\Sigma}_{\mathcal{C}_1} \tag{4.327}$$

The minus sign can be dropped since it cancels in the versor operation. $D_{\mathcal{C}}$ dilates relative to (around) the origin $\mathbf{e}_{o\gamma}$, but it can be translated by $\boldsymbol{p}_{\mathcal{M}}$ using a translator $T_{\mathcal{C}}$ to make the CSTA *translated-dilator* $D_{\mathcal{C}}^{\boldsymbol{p}}$ (§4.6.7) around point $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$.

4.6.7 CSTA translated-dilator

The CSTA 2-versor *translated-dilator* $D_{\mathcal{C}}^{\boldsymbol{p}}$ that dilates by factor d around $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ is

$$\begin{aligned} D_{\mathcal{C}}^{\boldsymbol{p}} &= T_{\mathcal{C}} D_{\mathcal{C}} \widetilde{T_{\mathcal{C}}} && (4.328)\\ &= T_{\mathcal{C}}\left(\frac{1}{2}(1+d) + \frac{1}{2}(1-d)\mathbf{e}_{\infty\gamma} \wedge \mathbf{e}_{o\gamma}\right)\widetilde{T_{\mathcal{C}}} && (4.329)\\ &= \frac{1}{2}(1+d) + \frac{1}{2}(1-d) T_{\mathcal{C}}(\mathbf{e}_{\infty\gamma} \wedge \mathbf{e}_{o\gamma})\widetilde{T_{\mathcal{C}}} && (4.330)\\ &= \frac{1}{2}(1+d) + \frac{1}{2}(1-d)\big(\mathbf{e}_{\infty\gamma} \wedge \hat{\boldsymbol{P}}_{\mathcal{C}}\big). && (4.331) \end{aligned}$$

The flat point (§4.5.5) in reverse orientation

$$\widetilde{\hat{\mathbb{P}}_{\mathcal{C}}} = \mathbf{e}_{\infty\gamma} \wedge \hat{\boldsymbol{P}}_{\mathcal{C}} \tag{4.332}$$

should be unit scale

$$\begin{aligned} \big(\widetilde{\hat{\mathbb{P}}_{\mathcal{C}}}\big)^2 &= \hat{\mathbb{P}}_{\mathcal{C}}\hat{\mathbb{P}}_{\mathcal{C}} = 1 && (4.333)\\ \hat{\boldsymbol{P}}_{\mathcal{C}} &= \frac{\boldsymbol{P}_{\mathcal{C}}}{-\boldsymbol{P}_{\mathcal{C}} \cdot \mathbf{e}_{\infty\gamma}}. && (4.334) \end{aligned}$$

The orientation of $\widetilde{\hat{\mathbb{P}}_{\mathcal{C}}}$ is important since its reverse makes reverse operations. For $d > 0$, the translated dilator can also be formulated as

$$\begin{aligned} D_{\mathcal{C}}^{\boldsymbol{p}} &= e^{\operatorname{atanh}\left(\frac{1-d}{1+d}\right)\mathbf{e}_{\infty\gamma} \wedge \hat{\boldsymbol{P}}_{\mathcal{C}}} && (4.335)\\ &= e^{-\frac{1}{2}\ln(d)\mathbf{e}_{\infty\gamma} \wedge \hat{\boldsymbol{P}}_{\mathcal{C}}}. && (4.336) \end{aligned}$$

Using the unit scale flat point in standard orientation $\hat{\mathbb{P}}_{\mathcal{C}} = \hat{\boldsymbol{P}}_{\mathcal{C}} \wedge \mathbf{e}_{\infty\gamma}$ (per §4.5.5), the translated dilator can be written as

$$\begin{aligned} D_{\mathcal{C}}^{\boldsymbol{p}} &= e^{\frac{1}{2}\ln(d)\hat{\mathbb{P}}_{\mathcal{C}}} && (4.337)\\ &= \cosh\left(\frac{1}{2}\ln(d)\right) + \sinh\left(\frac{1}{2}\ln(d)\right)\hat{\mathbb{P}}_{\mathcal{C}}, \quad \text{for } d > 0. && (4.338) \end{aligned}$$

The translated-dilator $D_{\mathcal{C}}^{\boldsymbol{p}}$ can be derived from successive inversions in two CSTA GIPNS 1-vector *hyperpseudospheres* $\boldsymbol{\Sigma}_{\mathcal{C}_1}$ and $\boldsymbol{\Sigma}_{\mathcal{C}_2}$ (§4.4.5) centered on $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with radius $r_1 = 1$ and $r_2 = \sqrt{d}$, respectively, as

$$D_{\mathcal{C}}^{\boldsymbol{p}} = -\boldsymbol{\Sigma}_{\mathcal{C}_2}\boldsymbol{\Sigma}_{\mathcal{C}_1} \simeq \boldsymbol{\Sigma}_{\mathcal{C}_2}\boldsymbol{\Sigma}_{\mathcal{C}_1} \tag{4.339}$$

The minus sign can be dropped since it cancels in a versor operation. Any CSTA entity $\mathbf{E}$ with center position $\boldsymbol{p}_{\mathcal{M}}$ can be dilated in situ as

$$\mathbf{E}' \;=\; D_{\mathcal{C}}^{\boldsymbol{p}}\mathbf{E}D_{\mathcal{C}}^{\boldsymbol{p}\sim}. \tag{4.340}$$

The identity

$$\operatorname{atanh}\left(\frac{1-d}{1+d}\right) \;=\; -\frac{1}{2}\ln(d), \tag{4.341}$$

which can also be written as

$$\ln(d) \;=\; 2\operatorname{atanh}\left(\frac{d-1}{d+1}\right), \tag{4.342}$$

may not be familiar, but can be derived or verified as follows.

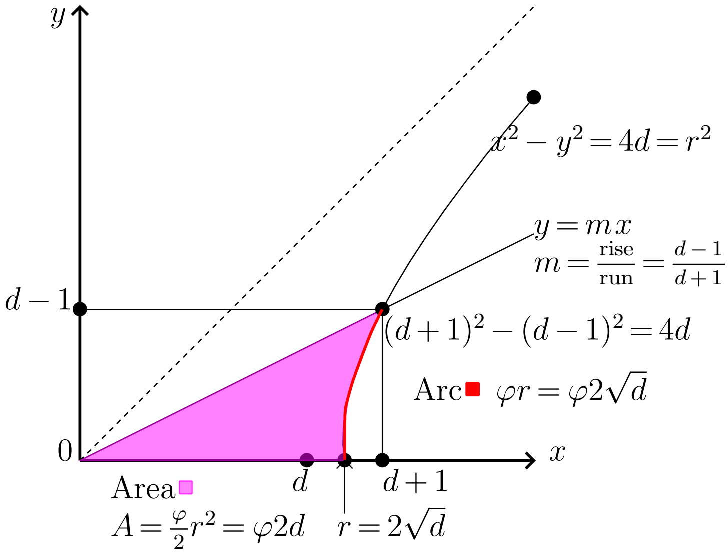
y
$x^2-y^2=4d=r^2$
$y=mx$
$m=\frac{\text{rise}}{\text{run}}=\frac{d-1}{d+1}$
$d-1$
$(d+1)^2-(d-1)^2=4d$
Arc 🟥 $\varphi r=\varphi 2\sqrt{d}$
0
x
Area 🟪
$A=\frac{\varphi}{2}r^2=\varphi 2d$
d
$d+1$
$r=2\sqrt{d}$

Figure 4.1. Area $A=\varphi r^2/2=\varphi 2d$ of hyperbolic angle φ

Referring to Figure 4.1, and noting the analogy between the trigonometry of circles and the trigonometry of hyperbolas (spacetime pseudocircles), then

$$2\operatorname{atanh}\left(\frac{d-1}{d+1}\right) \;=\; \tag{4.343}$$

$$2\operatorname{atanh}\left(\frac{r\sinh(\varphi)}{r\cosh(\varphi)}\right)=2\,\varphi=2\frac{1}{2d}A \;=\; \tag{4.344}$$

$$\frac{1}{d}\int_0^{d-1}\left(\sqrt{4d+y^2}-\frac{d+1}{d-1}y\right)\mathrm{d}y \;=\; \tag{4.345}$$

$$\frac{1}{d}\left(\left(\frac{y}{2}\sqrt{4d+y^2}+\frac{4d}{2}\ln\left(y+\sqrt{4d+y^2}\right)\right)^{[21]}-\left(\frac{d+1}{d-1}\right)\frac{y^2}{2}\right)_0^{d-1} \;=\; \tag{4.346}$$

$$\frac{1}{d}\left(\frac{4d}{2}\ln(d-1+d+1)\right)-\frac{1}{d}\left(\frac{4d}{2}\ln(2\sqrt{d})\right) \;=\; \tag{4.347}$$

$$2\ln(2d)-2\ln(2\sqrt{d})=2\ln\left(\frac{2d}{2\sqrt{d}}\right)=2\ln(\sqrt{d}) \;=\; \ln(d) \tag{4.348}$$

4.6.8 CSTA spacetime boost

CSTA inherits the STA boost operator $B_{\mathcal{M}}$ (§3.2.3) as the CSTA boost operator $B_{\mathcal{C}}=B_{\mathcal{M}}$. The boost $B_{\mathcal{C}}=B_{\mathbf{v}}$ by a spacetime velocity $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ (with proper time $\tau=t_{p\boldsymbol{v}}$) can be applied to any CSTA spacetime surface entity. An STA spacetime surface point $\boldsymbol{p}$ (with coordinate time $t=t_{c\boldsymbol{u}}=t_{p\boldsymbol{o}}$) of a CSTA spacetime entity $\boldsymbol{E}$ represents an observable spacetime position of the form

$$\begin{aligned}\boldsymbol{p}(t) &= \mathbf{p}_0+\dot{\boldsymbol{p}}t &(4.349)\\ &= \mathbf{p}_0+(\boldsymbol{o}+\dot{\mathbf{p}})t. &(4.350)\end{aligned}$$

The boost of the entity $B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim}$ is congruent to the set of all boosted surface points $B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim}$. For GIPNS entity $\boldsymbol{E}$, the set of spacetime surface points is

$$\mathbb{NI}_G(\boldsymbol{E}) = \{\, \boldsymbol{p} \;:\; \mathcal{C}(\boldsymbol{p})\cdot\boldsymbol{E}=0 \,\} \qquad (4.351)$$

and the set of the boosted entity is

$$\mathbb{NI}_G(B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim}) = \{\, B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim} \;:\; \mathcal{C}(B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim})\cdot(B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim})=0 \,\}. \qquad (4.352)$$

A point $\boldsymbol{p}$ is boosted as

$$\begin{aligned}B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim} &= B_{\mathbf{v}}(\mathbf{p}_0+\dot{\boldsymbol{p}}t)B_{\mathbf{v}}^{\sim}=\boldsymbol{p}\oplus\boldsymbol{v} &(4.353)\\ &= B_{\mathbf{v}^\dagger}^{\sim}(\mathbf{p}_0+\boldsymbol{o}t+\dot{\mathbf{p}}t)B_{\mathbf{v}^\dagger}=\boldsymbol{p}\ominus\boldsymbol{v}^\dagger &(4.354)\\ &= B_{\mathbf{v}^\dagger}^{\sim}(-\boldsymbol{o}+\boldsymbol{o}+\mathbf{p}_0)B_{\mathbf{v}^\dagger}+B_{\mathbf{v}^\dagger}^{\sim}(\boldsymbol{o}+\dot{\mathbf{p}})B_{\mathbf{v}^\dagger}t &(4.355)\\ &= -B_{\mathbf{v}^\dagger}^{\sim}\boldsymbol{o}B_{\mathbf{v}^\dagger}+B_{\mathbf{v}^\dagger}^{\sim}(\boldsymbol{o}+\mathbf{p}_0)B_{\mathbf{v}^\dagger}+B_{\mathbf{v}^\dagger}^{\sim}(\boldsymbol{o}+\dot{\mathbf{p}})B_{\mathbf{v}^\dagger}t &(4.356)\\ &= -\gamma_{\mathbf{v}}(\boldsymbol{o}+\mathbf{v})+\gamma_{\mathbf{p}_0\ominus\mathbf{v}^\dagger}(\boldsymbol{o}+\mathbf{p}_0\ominus\mathbf{v}^\dagger)+\gamma_{\dot{\mathbf{p}}\ominus\mathbf{v}^\dagger}(\boldsymbol{o}+\dot{\mathbf{p}}\ominus\mathbf{v}^\dagger)t &(4.357)\\ &= \mathbf{p}_0\ominus\boldsymbol{v}^\dagger+(\dot{\boldsymbol{p}}\ominus\boldsymbol{v}^\dagger)t &(4.358)\\ &= \mathbf{p}_0\ominus\boldsymbol{v}^\dagger+(\boldsymbol{o}+\dot{\mathbf{p}}\ominus\mathbf{v}^\dagger)\tau &(4.359)\\ &= \mathbf{p}_0\oplus\boldsymbol{v}+(\boldsymbol{o}+\dot{\mathbf{p}}\oplus\mathbf{v})t. &(4.360)\end{aligned}$$

This boost can be interpreted at least two ways: (Eq. 4.359) as $\boldsymbol{p}$ relative to $\boldsymbol{v}^\dagger=\boldsymbol{o}-\mathbf{v}$ and expressed in the frame of $\boldsymbol{v}^\dagger$ as a passive change of frame, or (Eq. 4.360) as $\boldsymbol{p}$ actively boosted up into the frame of $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ but passively expressed in the frame of $\boldsymbol{o}$ as an active relativistic velocity addition. The boost of a spacetime surface entity $B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim}$, representing the set of all boosted spacetime surface points $B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim}$, has a similar interpretation: that the entity is either (Eq. 4.359) relative to $\boldsymbol{v}^\dagger$ in its frame as a passive frame change, or (Eq. 4.360) boosted up into the frame of $\boldsymbol{v}$ but expressed (viewed) in the frame of $\boldsymbol{o}$ as an active relativistic velocity addition.

The CSTA *boost operator* $B_{\mathcal{C}}$ for a natural speed $\beta_{\mathbf{v}}$ in the SA *unit* direction $\hat{\mathbf{v}}_{\mathcal{S}}$ is defined as

$$\begin{aligned}B_{\mathcal{C}}=B_{\mathcal{M}}=B_{\mathbf{v}} &= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}=e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}}\boldsymbol{\gamma}_0} &(4.361)\\ &= \exp\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}}\boldsymbol{\gamma}_0\right) &(4.362)\\ &= \cosh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)+\sinh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\hat{\mathbf{v}}_{\mathcal{S}}\wedge\boldsymbol{\gamma}_0. &(4.363)\end{aligned}$$

For more information, see the STA boost operator (§3.2.3).

4.6.9 CSTA translated-boost

The CSTA 2-versor *translated-boost* $B_{\mathcal{C}}^{\boldsymbol{d}}$ by $\varphi = \operatorname{atanh}(\beta)$ in direction $\hat{\mathbf{v}}$ centered at $\boldsymbol{d}_{\mathcal{M}}$ is defined as

$$\begin{aligned} B_{\mathcal{C}}^{\boldsymbol{d}} = T_{\mathcal{C}} B_{\mathcal{C}} \widetilde{T_{\mathcal{C}}} &= e^{-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} e^{\frac{1}{2}\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0} e^{\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} && (4.364) \\ &= \cosh\!\left(\frac{1}{2}\varphi\right) + \sin\!\left(\frac{1}{2}\varphi\right) e^{-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} \hat{\mathbf{v}}\boldsymbol{\gamma}_0 e^{\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} && (4.365) \\ &= \cosh\!\left(\frac{1}{2}\varphi\right) + \sin\!\left(\frac{1}{2}\varphi\right) (\hat{\mathbf{v}}\boldsymbol{\gamma}_0 - (\boldsymbol{d}_{\mathcal{M}} \cdot (\hat{\mathbf{v}}\boldsymbol{\gamma}_0))\mathbf{e}_{\infty\gamma}) && (4.366) \\ &= e^{\frac{1}{2}\varphi(\hat{\mathbf{v}}\boldsymbol{\gamma}_0 - (\boldsymbol{d}_{\mathcal{M}} \cdot (\hat{\mathbf{v}}\boldsymbol{\gamma}_0))\mathbf{e}_{\infty\gamma})} && (4.367) \\ &= e^{\frac{1}{2}\varphi\boldsymbol{\Pi}_{\mathcal{C}}} && (4.368) \end{aligned}$$

where the plane

$$\boldsymbol{\Pi}_{\mathcal{C}} = \boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} \qquad (4.369)$$

has *unit bivector* direction $\boldsymbol{D} = \hat{\mathbf{v}}\boldsymbol{\gamma}_0 \mathbf{I}_{\mathcal{M}}$ through point $\boldsymbol{p}_{\mathcal{M}} = \boldsymbol{d}_{\mathcal{M}}$.

4.6.10 CSTA differential operators

Some of the CSTA point *value-extraction elements* C_s have inverses. These inverses allow the following CSTA 2-vector *differential elements* to be defined as

$$\begin{aligned} D_w^{\mathcal{C}} &= C_1 C_w^{-1} = \boldsymbol{\gamma}_0 \wedge \mathbf{e}_{\infty\gamma} && (4.370) \\ D_t^{\mathcal{C}} &= C_1 C_t^{-1} = c\boldsymbol{\gamma}_0 \wedge \mathbf{e}_{\infty\gamma} && (4.371) \\ D_x^{\mathcal{C}} &= C_1 C_x^{-1} = \boldsymbol{\gamma}_1 \wedge \mathbf{e}_{\infty\gamma} && (4.372) \\ D_y^{\mathcal{C}} &= C_1 C_y^{-1} = \boldsymbol{\gamma}_2 \wedge \mathbf{e}_{\infty\gamma} && (4.373) \\ D_z^{\mathcal{C}} &= C_1 C_z^{-1} = \boldsymbol{\gamma}_3 \wedge \mathbf{e}_{\infty\gamma}. && (4.374) \end{aligned}$$

The CSTA differential elements are *free vectors* [4], which are translation-invariant and represent directions without location.

Using the commutator product ×, the CSTA *differential operators* are defined as

$$\begin{aligned} \partial_w^{\mathcal{C}} &= \frac{\partial}{\partial w} = D_w^{\mathcal{C}} \times && (4.375) \\ \partial_t^{\mathcal{C}} &= \frac{\partial}{\partial t} = D_t^{\mathcal{C}} \times && (4.376) \\ \partial_x^{\mathcal{C}} &= \frac{\partial}{\partial x} = D_x^{\mathcal{C}} \times && (4.377) \\ \partial_y^{\mathcal{C}} &= \frac{\partial}{\partial y} = D_y^{\mathcal{C}} \times && (4.378) \\ \partial_z^{\mathcal{C}} &= \frac{\partial}{\partial z} = D_z^{\mathcal{C}} \times . && (4.379) \end{aligned}$$

The differential elements and operators in CSTA1 and CSTA2 can be denoted $D_s^{\mathcal{C}_1}$ and $\partial_s^{\mathcal{C}_1}$, and $D_s^{\mathcal{C}_2}$ and $\partial_s^{\mathcal{C}_2}$, respectively. The CSTA differential operators can be used for entity analysis. A different, but similar, set of differential elements and operators are defined in DCSTA (§5.8.1).

A CSTA directional derivative operator $D_{\boldsymbol{n}}^{\mathcal{C}} \times$, in the direction of a vector

$$\boldsymbol{n}_{\mathcal{M}} = n_w\boldsymbol{\gamma}_0 + n_x\boldsymbol{\gamma}_1 + n_y\boldsymbol{\gamma}_2 + n_z\boldsymbol{\gamma}_3, \qquad (4.380)$$

can be formed as the linear combination

$$D_{\boldsymbol{n}}^{\mathcal{C}} \times = (n_w D_w^{\mathcal{C}} + n_x D_x^{\mathcal{C}} + n_y D_y^{\mathcal{C}} + n_z D_z^{\mathcal{C}}) \times . \qquad (4.381)$$

The simplest example of using a CSTA differential operator $D^{\mathcal{C}}_{\boldsymbol{n}}\times$ is to take the derivative of any CSTA GIPNS 1-vector entity $\boldsymbol{\Sigma}_{\mathcal{C}}$ in the $\boldsymbol{n}$-direction as

$$\partial^{\mathcal{C}}_{\boldsymbol{n}}\boldsymbol{\Sigma}_{\mathcal{C}} \;=\; \frac{\partial\boldsymbol{\Sigma}_{\mathcal{C}}}{\partial\boldsymbol{n}} = D^{\mathcal{C}}_{\boldsymbol{n}}\times\boldsymbol{\Sigma}_{\mathcal{C}}. \tag{4.382}$$

5 Double Conformal Space-Time Algebra (DCSTA)

The $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) is a straightforward extension of the $\mathcal{G}_{2,8}$ Double Conformal Space Algebra (DCSA), which is similar to the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) that is introduced in the paper [7] and discussed further in the papers [5] and [6]. There are only some differences in signs between $\mathcal{G}_{2,8}$ DCSA and $\mathcal{G}_{8,2}$ DCGA, such that all the results of DCGA transfer to DCSA with only some sign changes.

The key idea of DCSTA is that any CSTA1 entity or versor $A_{\mathcal{C}^1}$ and its double $A_{\mathcal{C}^2}$ in CSTA2 can be multiplied to form the corresponding DCSTA entity or versor $A_{\mathcal{D}}$, where

$$A_{\mathcal{D}} \;=\; A_{\mathcal{C}^1}A_{\mathcal{C}^2} = A_{\mathcal{C}^1}\wedge A_{\mathcal{C}^2}. \tag{5.1}$$

According to the *outermorphism* property for transformation operators [16], or versors that operate as the two-sided versor “sandwich” operation, any doubled versor $V_{\mathcal{D}}$, which can be for rotation $R_{\mathcal{D}}$ (§5.7.6), translation $T_{\mathcal{D}}$ (§5.7.7), dilation $D_{\mathcal{D}}$ (§5.7.8), boost $B_{\mathcal{D}}$ (§5.7.3), or any of their compositions, operates on a doubled entity $E_{\mathcal{D}}$ as

$$\begin{aligned}
E'_{\mathcal{D}} \;&=\; V_{\mathcal{D}}E_{\mathcal{D}}\widetilde{V_{\mathcal{D}}} &(5.2)\\
&=\; V_{\mathcal{C}^1}V_{\mathcal{C}^2}(E_{\mathcal{C}^1}\wedge E_{\mathcal{C}^2})\widetilde{V_{\mathcal{C}^2}}\widetilde{V_{\mathcal{C}^1}} &(5.3)\\
&=\; (V_{\mathcal{C}^1}V_{\mathcal{C}^2}E_{\mathcal{C}^1}\widetilde{V_{\mathcal{C}^2}}\widetilde{V_{\mathcal{C}^1}})\wedge(V_{\mathcal{C}^1}V_{\mathcal{C}^2}E_{\mathcal{C}^2}\widetilde{V_{\mathcal{C}^2}}\widetilde{V_{\mathcal{C}^1}}) &(5.4)\\
&=\; (V_{\mathcal{C}^1}E_{\mathcal{C}^1}\widetilde{V_{\mathcal{C}^1}})\wedge(V_{\mathcal{C}^2}E_{\mathcal{C}^2}\widetilde{V_{\mathcal{C}^2}}) &(5.5)\\
&=\; E'_{\mathcal{C}^1}\wedge E'_{\mathcal{C}^2} = E'_{\mathcal{C}^1}E'_{\mathcal{C}^2}. &(5.6)
\end{aligned}$$

Therefore, the CSTA1 entity $E_{\mathcal{C}^1}$ is correctly transformed by the CSTA1 versor $V_{\mathcal{C}^1}$, and similarly for the CSTA2 entity $E_{\mathcal{C}^2}$. The product of the two correctly transformed CSTA entities is the correctly transformed DCSTA entity. For example, the DCSTA point entity $\boldsymbol{P}_{\mathcal{D}}$ (§5.2) is correctly transformed by all of the DCSTA versors. The DCSTA point value-extraction elements T_s (§5.2.3) extract correctly transformed values from a point $\boldsymbol{P}_{\mathcal{D}}$, leading to the ability to form entities in the general form of Darboux cyclides that can be correctly transformed by all of the DCSTA versors.

As a subalgebra of DCSTA, all the results of DCSA (or DCGA) are available in DCSTA. DCSTA extends DCSA with the pseudospatial time axis $(w = ct)\boldsymbol{\gamma}_0$, a variety of spacetime entities, and the spacetime boost (hyperbolic rotor) operator. In DCSTA, the DCSA GIPNS 2-vector quadric surfaces are surfaces in spacetime at zero velocity that can be boosted into any velocity. The boosted quadric surfaces display spacetime contraction effects.

DCSTA includes many operations on quadric surface entities, including

- Rotation in space
- Translation in spacetime
- Isotropic dilation in spacetime
- Anisotropic dilation (directed length dilation) in space

- Spacetime active boosts of velocity with length contraction effect
- Spacetime passive boosts relative to a new observer frame
- Intersection with standard entities that are doubled CSTA entities.

The general DCSTA GIPNS 2-vector surface entity $\boldsymbol{\Omega}$ has the general form of a Darboux cyclide in spacetime, which has degenerate forms that include Dupin cyclides, horned Dupin cyclides, parabolic cyclides, and the quadric surfaces. In DCSTA, the Darboux cyclide surface entities can be formed, similarly as in DCSA or DCGA, as linear combinations of the spatial DCSTA point value-extraction elements T_s (§5.2.3) that represent spatial cyclide surfaces in the 3D $\mathcal{G}_{0,3}$ SA space (at zero velocity). Darboux cyclide entities can also be formed from spacetime value-extraction elements to represent spacetime cyclides in a 3D $\mathcal{G}_{1,2}$ STA spacetime, where the spacetime cyclides are called pseudocyclides, pseudoquadrics, etc.

The DCSTA quadric surfaces support anisotropic length contraction and dilation (§5.7.9) since these forms can be written in terms of the DCSTA value-extraction elements. On the other hand, the higher-degree surfaces, which include cubic parabolic cyclides and quartic Darboux and Dupin cyclides, do not support anisotropic length contraction and dilation forms. Any DCSTA GIPNS 2-vector surface entity $\boldsymbol{\Omega}$ represents an implicit surface function in spacetime $F(w,x,y,z)$ and supports function differentiation $\partial_{\boldsymbol{n}}F$ (or $2F\partial_{\boldsymbol{n}}F$ for doubled entities $E_{\mathcal{D}}\widehat{=}F^2$) using the differential operations (§5.8) for

- Differentiation with respect to $w=ct$, t, x, y, or z
- Directional derivative with respect to a unit-norm direction $\boldsymbol{n}$ in spacetime.

The DCSTA forms of conic sections can also support the operations for

- Orthographic projection
- Perspective projection

as discussed in the paper [5].

5.1 DCSTA unit pseudoscalar

The DCSTA 12-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{D}}$ with signature $(+---+-+---+-)$ is

$$
\begin{aligned}
\mathbf{I}_{\mathcal{D}} &= \mathbf{I}_{\mathcal{C}^1}\wedge\mathbf{I}_{\mathcal{C}^2}=\bigwedge_{i=1}^{12}\mathbf{e}_i && (5.7)\\
&= \mathbf{I}_{\mathcal{D}}^{\sim}=\mathbf{I}_{\mathcal{D}}^{-1} && (5.8)\\
\mathbf{I}_{\mathcal{D}}^2 &= 1. && (5.9)
\end{aligned}
$$

5.2 DCSTA point

5.2.1 DCSTA point embedding

The DCSTA null 2-vector *point embedding* $\boldsymbol{P}_{\mathcal{D}}=\mathcal{D}(\boldsymbol{p})$ of STA position vector

$$
\begin{aligned}
\boldsymbol{p}=\boldsymbol{p}_{\mathcal{M}} &= p_w\boldsymbol{\gamma}_0+p_x\boldsymbol{\gamma}_1+p_y\boldsymbol{\gamma}_2+p_z\boldsymbol{\gamma}_3 && (5.10)\\
&= p_w\boldsymbol{\gamma}_0+\mathbf{p}_{\mathcal{S}} && (5.11)\\
&= \mathbf{p}_0+\dot{\boldsymbol{p}}t && (5.12)\\
&= \mathbf{p}_0+(\boldsymbol{o}+\dot{\mathbf{p}})t && (5.13)
\end{aligned}
$$

is defined as the doubling of the CSTA point $\boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p})$ as

$$\begin{aligned}\boldsymbol{P}_{\mathcal{D}} &= \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^1})\wedge\mathcal{C}(\boldsymbol{p}_{\mathcal{M}^2}) && (5.14)\\ &= \boldsymbol{P}_{\mathcal{C}^1}\wedge\boldsymbol{P}_{\mathcal{C}^2} && (5.15)\\ &= \mathcal{D}(\boldsymbol{p}) && (5.16)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{p}_{\mathcal{M}^1} &= p_w\mathbf{e}_1+p_x\mathbf{e}_2+p_y\mathbf{e}_3+p_z\mathbf{e}_4=p_w\mathbf{e}_1+\mathbf{p}_{\mathcal{S}^1} && (5.17)\\ \boldsymbol{p}_{\mathcal{M}^2} &= p_w\mathbf{e}_7+p_x\mathbf{e}_8+p_y\mathbf{e}_9+p_z\mathbf{e}_{10}=p_w\mathbf{e}_7+\mathbf{p}_{\mathcal{S}^2}. && (5.18)\end{aligned}$$

The DCSTA null 2-vector *point at the origin* is

$$\mathbf{e}_o = \mathbf{e}_{o1}\wedge\mathbf{e}_{o2}. \tag{5.19}$$

The DCSTA null 2-vector *point at infinity* is

$$\mathbf{e}_\infty = \mathbf{e}_{\infty1}\wedge\mathbf{e}_{\infty2}. \tag{5.20}$$

In general, the doubled CSTA entities, as DCSTA entities, represent the same entities as defined in CSTA (§4).

As a GIPNS entity, the DCSTA *point embedding* $\boldsymbol{P}_{\mathcal{D}}=\mathcal{D}(\boldsymbol{p})$ represents a space-time hypercone (lightcone) centered on vertex $\boldsymbol{p}$ and is the DCSTA GIPNS null 2-vector standard *hypercone* $\boldsymbol{P}_{\mathcal{D}}=\boldsymbol{K}_{\mathcal{D}}$.

As a GOPNS entity, the DCSTA *point embedding* $\boldsymbol{P}_{\mathcal{D}}=\mathcal{D}(\boldsymbol{p})$ represents the point $\boldsymbol{p}$ and is the DCSTA GOPNS null 2-vector standard *point* $\boldsymbol{P}_{\mathcal{D}}$.

A *point embedding* $\boldsymbol{P}_{\mathcal{D}}=\mathcal{D}(\boldsymbol{p})$ will often be called a *point*, but it should be understood that it is a GOPNS point and not a GIPNS point. For the purpose of *testing* any GIPNS or GOPNS surface entity for a surface point $\boldsymbol{p}$, the point embedding $\boldsymbol{P}_{\mathcal{D}}=\mathcal{D}(\boldsymbol{p})$ is the *test point entity*. Point $\boldsymbol{p}$ is on the surface of GIPNS entity $\boldsymbol{E}$ iff $\mathcal{D}(\boldsymbol{p})\cdot\boldsymbol{E}=0$. Point $\boldsymbol{p}$ is on the surface of GOPNS entity $\boldsymbol{E}^*$ iff $\mathcal{D}(\boldsymbol{p})\wedge\boldsymbol{E}^*=0$.

5.2.2 DCSTA point projection (inverse embedding)

The projection of DCSTA point $\boldsymbol{P}_{\mathcal{D}}$ back to STA1 vector $\boldsymbol{p}_{\mathcal{M}^1}$ is

$$\begin{aligned}\boldsymbol{p}_{\mathcal{M}^1} &= \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{D}}\cdot\mathbf{e}_{\infty2}) && (5.21)\\ &= \left(\frac{\boldsymbol{P}_{\mathcal{D}}\cdot\mathbf{e}_{\infty2}}{-(\boldsymbol{P}_{\mathcal{D}}\cdot\mathbf{e}_{\infty2})\cdot\mathbf{e}_{\infty1}}\cdot\mathbf{I}_{\mathcal{M}^1}\right)\cdot\mathbf{I}_{\mathcal{M}^1}^{-1}. && (5.22)\end{aligned}$$

The projection of DCSTA point $\boldsymbol{P}_{\mathcal{D}}$ back to STA2 vector $\boldsymbol{p}_{\mathcal{M}^2}$ is

$$\begin{aligned}\boldsymbol{p}_{\mathcal{M}^2} &= \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{D}}\cdot\mathbf{e}_{\infty1}) && (5.23)\\ &= \left(\frac{\boldsymbol{P}_{\mathcal{D}}\cdot\mathbf{e}_{\infty1}}{-(\boldsymbol{P}_{\mathcal{D}}\cdot\mathbf{e}_{\infty1})\cdot\mathbf{e}_{\infty2}}\cdot\mathbf{I}_{\mathcal{M}^2}\right)\cdot\mathbf{I}_{\mathcal{M}^2}^{-1}. && (5.24)\end{aligned}$$

5.2.3 DCSTA point value-extraction elements

The DCSTA *test point* $\boldsymbol{T}_{\mathcal{D}}=\mathcal{D}(\boldsymbol{t})$ is the point embedding of the STA test vector

$$\begin{aligned}\boldsymbol{t} &= w\boldsymbol{\gamma}_0+x\boldsymbol{\gamma}_1+y\boldsymbol{\gamma}_2+z\boldsymbol{\gamma}_3 && (5.25)\\ &= ct\boldsymbol{\gamma}_0+x\boldsymbol{\gamma}_1+y\boldsymbol{\gamma}_2+z\boldsymbol{\gamma}_3. && (5.26)\end{aligned}$$

The DCSTA 2-vector *extraction elements* T_s are defined as

$$T_w = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7) \tag{5.27}$$

$$T_t = \frac{1}{c} T_w \tag{5.28}$$

$$T_x = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{\infty 1}) \tag{5.29}$$

$$T_y = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{\infty 1}) \tag{5.30}$$

$$T_z = \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{\infty 1}) \tag{5.31}$$

$$T_{w^2} = \mathbf{e}_7 \wedge \mathbf{e}_1 \tag{5.32}$$

$$T_{t^2} = \frac{1}{c^2} T_{w^2} \tag{5.33}$$

$$T_{x^2} = \mathbf{e}_8 \wedge \mathbf{e}_2 \tag{5.34}$$

$$T_{y^2} = \mathbf{e}_9 \wedge \mathbf{e}_3 \tag{5.35}$$

$$T_{z^2} = \mathbf{e}_{10} \wedge \mathbf{e}_4 \tag{5.36}$$

$$T_{wx} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_8 + \mathbf{e}_2 \wedge \mathbf{e}_7) \tag{5.37}$$

$$T_{wy} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_9 + \mathbf{e}_3 \wedge \mathbf{e}_7) \tag{5.38}$$

$$T_{wz} = \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{10} + \mathbf{e}_4 \wedge \mathbf{e}_7) \tag{5.39}$$

$$T_{tx} = \frac{1}{c} T_{wx} \tag{5.40}$$

$$T_{ty} = \frac{1}{c} T_{wy} \tag{5.41}$$

$$T_{tz} = \frac{1}{c} T_{wz} \tag{5.42}$$

$$T_{xy} = \frac{1}{2}(\mathbf{e}_9 \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_3) \tag{5.43}$$

$$T_{yz} = \frac{1}{2}(\mathbf{e}_{10} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_4) \tag{5.44}$$

$$T_{zx} = \frac{1}{2}(\mathbf{e}_8 \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_2) \tag{5.45}$$

$$T_{w\boldsymbol{t}^2} = \mathbf{e}_1 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_7 \tag{5.46}$$

$$T_{t\boldsymbol{t}^2} = \frac{1}{c} T_{w\boldsymbol{t}^2} \tag{5.47}$$

$$T_{x\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{o1} \tag{5.48}$$

$$T_{y\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{o1} \tag{5.49}$$

$$T_{z\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{o1} \tag{5.50}$$

$$T_1 = -\mathbf{e}_\infty \tag{5.51}$$

$$T_{\boldsymbol{t}^2} = \mathbf{e}_{o2} \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} \wedge \mathbf{e}_{o1} \tag{5.52}$$

$$T_{\boldsymbol{t}^4} = -4\mathbf{e}_o. \tag{5.53}$$

The value s is extracted from a point $\boldsymbol{T}_{\mathcal{D}}$ as

$$s = T_s \cdot \boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{D}} \cdot T_s. \tag{5.54}$$

The DCSTA 2-vector extraction elements T_s are *inner product extraction operators*.

The DCSTA 2-vector extraction elements T_s are used to define the DCSTA GIPNS 2-vector entities that are similar to those that can be defined in DCSA or DCGA. The general DCSTA GIPNS 2-vector spacetime surface entity $\mathbf{\Omega}$ is a linear combination of the DCSTA extraction elements T_s. For example, an ellipsoid entity $\mathbf{E}$ can be defined as

$$\mathbf{E} \;=\; T_{x^2}/a^2 + T_{y^2}/b^2 + T_{z^2}/c^2 - T_1. \tag{5.55}$$

In general, any DCSTA GIPNS 2-vector *spatial quadric* entity $\mathbf{Q}$, formed similarly to the ellipsoid entity $\mathbf{E}$ as a linear combination of spatial extraction elements T_s, is independent of time $w = ct$ and exists for all time at its current spatial position, which can be translated using the DCSTA translator (§5.7.7). A spatial quadric surface entity $\mathbf{Q}$ is "pseudocylindrical" with the pseudospatial time w-axis as a type of hypercylinder entity. The interpretation in spacetime is that the spatial quadric $\mathbf{Q}$ is at zero velocity $\beta = 0$. The DCSTA 4-versor *boost* operator $B_{\mathbf{v}}$ (§5.7.3) can actively boost any spatial quadric $\mathbf{Q}$ into any velocity $\mathbf{v} = \beta_{\mathbf{v}} c\hat{\mathbf{v}}$ (into the rest frame of $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$) as a *spacetime quadric* $\boldsymbol{Q} = B_{\mathbf{v}}\mathbf{Q}B_{\mathbf{v}}^{\sim}$. Only a DCSTA GIPNS 2-vector spatial quadric surface entity $\mathbf{Q}$ can be formed independently of time $w = ct$ as a linear combination of spatial extraction elements T_s. The DCSTA GIPNS 2-vector cubic (parabolic cyclide) and quartic (Darboux and Dupin cyclide) entities use the extraction elements T_s that include $\boldsymbol{t}$ as its square $\boldsymbol{t}^2$ or square square $\boldsymbol{t}^4$ and are dependent on time $w = ct$, with the interpretation that the spatial cubic and quartic entities exist at time $w = ct = 0$ or are translated to exist at some time $w = p_w = ct_w$.

5.2.4 DCSTA point value-extraction pseudo-inverse elements

The pseudo-inverse of A is denoted A^+ and has the relation

$$A\cdot A^+ \;=\; 1. \tag{5.56}$$

If A^{-1} exists, it may be equal to A^+. The inverse or pseudo-inverse of an extraction element T_s can be useful for formulating certain other elements and operators, such as *pseudo-integral operators* (§5.9). The pseudo-inverses of some of the extraction elements are

$$T_{w^2}^{-1} = T_{w^2}^{+} \;=\; -T_{w^2} \tag{5.57}$$

$$T_{t^2}^{-1} = T_{t^2}^{+} \;=\; -c^2 T_{w^2} \tag{5.58}$$

$$T_{x^2}^{-1} = T_{x^2}^{+} \;=\; -T_{x^2} \tag{5.59}$$

$$T_{y^2}^{-1} = T_{y^2}^{+} \;=\; -T_{y^2} \tag{5.60}$$

$$T_{z^2}^{-1} = T_{z^2}^{+} \;=\; -T_{z^2} \tag{5.61}$$

$$T_{w}^{+} \;=\; T_{w\boldsymbol{t}^2} \tag{5.62}$$

$$T_{t}^{+} \;=\; c^2 T_{t\boldsymbol{t}^2} \tag{5.63}$$

$$T_{x}^{+} \;=\; -T_{x\boldsymbol{t}^2} \tag{5.64}$$

$$T_{y}^{+} \;=\; -T_{y\boldsymbol{t}^2} \tag{5.65}$$

$$T_{z}^{+} \;=\; -T_{z\boldsymbol{t}^2} \tag{5.66}$$

$$T_{wx}^{+} \;=\; 2T_{wx} \tag{5.67}$$

$$T_{wy}^{+} \;=\; 2T_{wy} \tag{5.68}$$

$$T_{wz}^{+} \;=\; 2T_{wz} \tag{5.69}$$

$$T_{tx}^{+} \;=\; 2c^2 T_{tx} \tag{5.70}$$

$$T_{ty}^{+} \;=\; 2c^2 T_{ty} \tag{5.71}$$

$$
\begin{aligned}
T^{+}_{tz} &= 2c^2 T_{tz} && (5.72)\\
T^{+}_{xy} &= -2T_{xy} && (5.73)\\
T^{+}_{yz} &= -2T_{yz} && (5.74)\\
T^{+}_{zx} &= -2T_{zx} && (5.75)\\
T^{+}_{1} &= -\frac{1}{4}T_{\boldsymbol{t}^4} && (5.76)\\
T^{+}_{\boldsymbol{t}^2} &= -\frac{1}{2}T_{\boldsymbol{t}^2} && (5.77)\\
T^{+}_{\boldsymbol{t}^4} &= -\frac{1}{4}T_{1}. && (5.78)
\end{aligned}
$$

5.3 DCSTA GIPNS standard entities

The DCSTA GIPNS *standard* surface entities are the doubling of the CSTA GIPNS entities. The wedge of corresponding CSTA1 and CSTA2 GIPNS entities, $\boldsymbol{X}_{\mathcal{C}^1}$ and $\boldsymbol{X}_{\mathcal{C}^2}$, forms the DCSTA GIPNS standard entity $\boldsymbol{X}_{\mathcal{D}} = \boldsymbol{X}_{\mathcal{C}^1} \wedge \boldsymbol{X}_{\mathcal{C}^2}$ representing the same surface. The following subsections provide some explicit examples of the doubling.

The DCSTA GIPNS standard entities have special properties and can act as operators for reflections and intersections. All DCSTA entities can be reflected in the standard entities. The reflection in a standard sphere is called inversion in a sphere. All DCSTA entities can be intersected with standard entities. A DCSTA GIPNS *intersection* entity is a wedge of GIPNS entities, similar to a DCGA GIPNS intersection entity and with similar limitations on what combinations of entities can be wedged to form a valid intersection entity. The basic examples of intersection entities are the DCSTA GIPNS 6-vector *standard line* $\boldsymbol{L}_{\mathcal{D}} = \boldsymbol{\Pi}_{\mathcal{D}} \wedge \boldsymbol{E}_{\mathcal{D}}$ (§5.3.7) and DCSTA GIPNS 6-vector *standard (pseudo)circle* $\boldsymbol{C}_{\mathcal{D}} = \boldsymbol{\Pi}_{\mathcal{D}} \wedge \boldsymbol{\Sigma}_{\mathcal{D}}$ (§5.3.8).

5.3.1 DCSTA GIPNS null 2-vector hypercone

The DCSTA GIPNS null 2-vector *standard hypercone* $\boldsymbol{K}_{\mathcal{D}}$ is defined as

$$\boldsymbol{K}_{\mathcal{D}} = \boldsymbol{K}_{\mathcal{C}^1} \wedge \boldsymbol{K}_{\mathcal{C}^2} = \boldsymbol{P}_{\mathcal{C}^1} \wedge \boldsymbol{P}_{\mathcal{C}^2} = \boldsymbol{P}_{\mathcal{D}} \qquad (5.79)$$

which is the wedge of the same *point embedding (hypercone)* (§4.2) in CSTA1 and CSTA2.

5.3.2 DCSTA GIPNS 2-vector standard hyperplane

The DCSTA GIPNS 2-vector *standard hyperplane* $\boldsymbol{E}_{\mathcal{D}}$ is defined as

$$\boldsymbol{E}_{\mathcal{D}} = \boldsymbol{E}_{\mathcal{C}^1} \wedge \boldsymbol{E}_{\mathcal{C}^2} \qquad (5.80)$$

which is the wedge of the same *hyperplane* (§4.4.3) embedded in CSTA1 and CSTA2.

5.3.3 DCSTA GIPNS 2-vector standard hyperhyperboloid of one sheet

The DCSTA GIPNS 2-vector *standard hyperhyperboloid of one sheet (hyperpseudosphere)* $\boldsymbol{\Sigma}_{\mathcal{D}}$ is defined as

$$\boldsymbol{\Sigma}_{\mathcal{D}} = \boldsymbol{\Sigma}_{\mathcal{C}^1} \wedge \boldsymbol{\Sigma}_{\mathcal{C}^2} \qquad (5.81)$$

which is the wedge of the same *hyperpseudosphere* (§4.4.5) embedded in CSTA1 and CSTA2.

5.3.4 DCSTA GIPNS 2-vector standard hyperhyperboloid of two sheets

The DCSTA GIPNS 2-vector *standard hyperhyperboloid of two sheets (imaginary hyperpseudosphere)* $\mathbf{\Xi}_{\mathcal{D}}$ is defined as

$$\mathbf{\Xi}_{\mathcal{D}} = \mathbf{\Xi}_{\mathcal{C}^1} \wedge \mathbf{\Xi}_{\mathcal{C}^2} \tag{5.82}$$

which is the wedge of the same *imaginary hyperpseudosphere* (§4.4.6) embedded in CSTA1 and CSTA2.

5.3.5 DCSTA GIPNS 4-vector standard sphere or pseudosphere

The DCSTA GIPNS 4-vector *standard sphere or pseudosphere* $\boldsymbol{S}_{\mathcal{D}}$ is defined as

$$\boldsymbol{S}_{\mathcal{D}} = \boldsymbol{S}_{\mathcal{C}^1} \wedge \boldsymbol{S}_{\mathcal{C}^2} \tag{5.83}$$

which is the wedge of the same *sphere* (§4.4.7) or *pseudosphere* (§4.4.8) embedded in CSTA1 and CSTA2.

5.3.6 DCSTA GIPNS 4-vector standard plane

The DCSTA GIPNS 4-vector *standard plane* $\mathbf{\Pi}_{\mathcal{D}}$ is defined as

$$\mathbf{\Pi}_{\mathcal{D}} = \mathbf{\Pi}_{\mathcal{C}^1} \wedge \mathbf{\Pi}_{\mathcal{C}^2} \tag{5.84}$$

which is the wedge of the same *plane* (§4.4.10) embedded in CSTA1 and CSTA2.

5.3.7 DCSTA GIPNS 6-vector standard line

The DCSTA GIPNS 6-vector *standard line* $\boldsymbol{L}_{\mathcal{D}}$ is defined as

$$\boldsymbol{L}_{\mathcal{D}} = \boldsymbol{L}_{\mathcal{C}^1} \wedge \boldsymbol{L}_{\mathcal{C}^2} \tag{5.85}$$

which is the wedge of the same *line* (§4.4.11) embedded in CSTA1 and CSTA2.

5.3.8 DCSTA GIPNS 6-vector standard circle or pseudocircle

The DCSTA GIPNS 6-vector *standard circle or pseudocircle* $\boldsymbol{C}_{\mathcal{D}}$ is defined as

$$\boldsymbol{C}_{\mathcal{D}} = \boldsymbol{C}_{\mathcal{C}^1} \wedge \boldsymbol{C}_{\mathcal{C}^2} \tag{5.86}$$

which is the wedge of the same *circle* (§4.4.12) or *pseudocircle* (§4.4.13) embedded in CSTA1 and CSTA2.

5.3.9 DCSTA GIPNS 8-vector standard point pair

The DCSTA GIPNS 8-vector *standard point pair* $\mathbf{2}_{\mathcal{D}}$ is defined as

$$\mathbf{2}_{\mathcal{D}} = \mathbf{2}_{\mathcal{C}^1} \wedge \mathbf{2}_{\mathcal{C}^2} \tag{5.87}$$

which is the wedge of the same *point pair* (§4.4.14) embedded in CSTA1 and CSTA2.

5.3.10 DCSTA GIPNS null 10-vector standard point

The DCSTA GIPNS null 10-vector *standard point* $\boldsymbol{P}^{\star}_{\mathcal{D}}$ is defined as

$$\boldsymbol{P}^{\star}_{\mathcal{D}} = \boldsymbol{P}^{\star}_{\mathcal{C}^1} \wedge \boldsymbol{P}^{\star}_{\mathcal{C}^2} \tag{5.88}$$

which is the wedge of the same *GIPNS point* (§4.4.17) embedded in CSTA1 and CSTA2.

5.4 DCSTA GOPNS standard entities

The DCSTA GOPNS *standard entities* are the DCSTA duals (§5.7.1) of the DCSTA GIPNS *standard entities*. The DCSTA GOPNS *standard entities* can also be formed as the wedges of DCSTA points by the same formulas as in CSTA. Only the DCSTA GOPNS *standard entities* can be formed as wedges of DCSTA null 2-vector *points*.

The DCSTA GIPNS 2-vector *non-standard entities* that are formed as linear combinations of the DCSTA 2-vector *value-extraction elements* T_s (§5.2.3) have DCSTA dual forms as DCSTA GOPNS 10-vector *non-standard entities*, but these GOPNS entities cannot be formed as wedges of DCSTA points. All DCSTA GIPNS entities have a DCSTA dual that is the DCSTA GOPNS entity that represents the same surface.

5.5 DCSTA GIPNS 2-vector non-standard surface entities

The DCSTA GIPNS 2-vector *non-standard surface entities* $\mathbf{\Omega}$ are defined as linear combinations

$$\mathbf{\Omega} = \sum \alpha_i T_i \tag{5.89}$$

of the DCSTA 2-vector *value-extraction elements* T_s (§5.2.3).

In a straightforward way, an entity $\mathbf{\Omega}$, which has the general form of a spacetime Darboux (pseudo)cyclide, is formulated in terms of the value-extraction elements T_s to represent an implicit surface function

$$F(w,x,y,z) = \boldsymbol{T}_{\mathcal{D}} \cdot \mathbf{\Omega}. \tag{5.90}$$

A pseudocyclide or pseudoquadric has the form of a spatial cyclide or quadric, but one of the spatial axes is replaced by the pseudospatial time w-axis.

In terms of the value-extraction elements T_s, these entities $\mathbf{\Omega}$ are defined exactly as they are in the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) that is introduced in [7]. The reader should refer to [7] for additional details that are not repeated in this paper.

The DCSTA GIPNS 2-vector, or dual GOPNS 10-vector, non-standard surface entities $\mathbf{\Omega}$ can be translated in spacetime using the DCSTA *translator* $T_{\mathcal{D}}$ (§5.7.7), spatially rotated in space using the DCSTA spatial *rotor* $R_{\mathcal{D}}$ (§5.7.6), and isotropically dilated in spacetime using the DCSTA *isotropic dilator* $D_{\mathcal{D}}$ (§5.7.8).

The DCSTA GIPNS 2-vector, or dual GOPNS 10-vector, non-standard *quadric* surface entities $\boldsymbol{Q}$ can also be boosted in spacetime using the DCSTA spacetime *boost* operator (§5.7.3) with the interpretation that the *quadric* surface relativistically gains the spacetime velocity $\mathbf{v}$ of the active boost $B_{\mathbf{v}}$. A *spatial quadric surface* (at zero velocity) $\mathbf{Q}$ at position $\mathbf{p}$ can also be *anisotropically dilated* (§5.7.9) in position $\mathbf{p}$ by a factor d in a specific direction $\hat{\mathbf{v}}$ in space using the DCSTA *translated-boost* $B^{\mathbf{p}}_{\mathcal{D}\mathbf{v}}$ (§5.7.4) with an imaginary natural boost speed $\beta_{\mathbf{v}}$ that is followed by a DCSTA *spatial projection* $\mathbf{Q}' = (\boldsymbol{Q} \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}^{-1}_{\mathcal{DS}}$ (§5.7.2). The spatial projection discards all time components to recover a spatial quadric surface $\mathbf{Q}'$ that is again at zero velocity.

5.6 DCSTA conic section entities

It should be straightforward to adapt the DCGA conic sections into DCSTA and its spatial subalgebra DCSA. The reader is referred to the paper [5] for details on conic sections in DCGA and possible applications that include orthographic and perspective projection of conic sections.

5.7 DCSTA operations

5.7.1 DCSTA dualization

The dual DCSTA GOPNS $(12-k)$-vector surface entity $\mathbf{Q}^{*\mathcal{D}}$ of any DCSTA GIPNS k-vector surface entity $\mathbf{Q}$ is obtained by the DCSTA dualization as

$$\mathbf{Q}^{*\mathcal{D}} = \mathbf{Q}\mathbf{I}_{\mathcal{D}} = \mathbf{Q}\cdot\mathbf{I}_{\mathcal{D}}. \tag{5.91}$$

The undual operation is

$$\mathbf{Q} = \mathbf{Q}^{*\mathcal{D}}\cdot\mathbf{I}_{\mathcal{D}}. \tag{5.92}$$

The dual and undual operations are the repeated application of the same dualization operation. Therefore, DCSTA dualization is an involution.

The DCSTA GIPNS k-vector surface entity $\mathbf{Q}$ and its dual DCSTA GOPNS $(12-k)$-vector surface entity $\mathbf{Q}^{*\mathcal{D}}$ represent the same surface.

5.7.2 DCSTA spatial projection

The DCSTA *spatial projection* of a DCSTA entity $\mathbf{\Omega}_{\mathcal{D}}$ is defined as

$$\mathbf{\Omega}_{\mathcal{DS}} = (\mathbf{\Omega}_{\mathcal{D}}\cdot\mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1}, \tag{5.93}$$

which is the projection into the $\mathcal{G}_{2,8}$ DCSA subalgebra, where

$$\begin{aligned}\mathbf{I}_{\mathcal{DS}} &= (\mathbf{e}_1\wedge\mathbf{e}_7)\cdot\mathbf{I}_{\mathcal{D}} = \mathbf{I}_{\mathcal{S}^1}\mathbf{e}_5\mathbf{e}_6\mathbf{I}_{\mathcal{S}^2}\mathbf{e}_{11}\mathbf{e}_{12} && (5.94)\\ &= -\tilde{\mathbf{I}}_{\mathcal{DS}} = -\mathbf{I}_{\mathcal{DS}}^{-1} && (5.95)\end{aligned}$$

is the DCSA unit pseudoscalar. The projection produces the $\mathcal{G}_{2,8}$ DCSA entity $\mathbf{\Omega}_{\mathcal{DS}}$ representing the $\mathcal{G}_{4,8}$ DCSTA entity $\mathbf{\Omega}_{\mathcal{D}}$ at time $w = ct = 0$.

The DCSA null 2-vector *point* is defined as

$$\begin{aligned}\mathbf{P}_{\mathcal{DS}} &= \mathcal{DS}(\mathbf{p}_{\mathcal{S}}) && (5.96)\\ &= \mathcal{CS}^1(\mathbf{p}_{\mathcal{S}^1})\wedge\mathcal{CS}^2(\mathbf{p}_{\mathcal{S}^2}) = \mathcal{CS}^1(\mathbf{p}_{\mathcal{S}^1})\mathcal{CS}^2(\mathbf{p}_{\mathcal{S}^2}) && (5.97)\\ &= \mathcal{C}^1(\mathbf{p}_{\mathcal{S}^1})\wedge\mathcal{C}^2(\mathbf{p}_{\mathcal{S}^2}) = \mathcal{C}^1(\mathbf{p}_{\mathcal{S}^1})\mathcal{C}^2(\mathbf{p}_{\mathcal{S}^2}) && (5.98)\\ &= \mathcal{D}(\mathbf{p}_{\mathcal{S}}) && (5.99)\\ &= \left(\mathbf{p}_{\mathcal{S}^1}+\frac{1}{2}\mathbf{p}_{\mathcal{S}^1}^2\mathbf{e}_{\infty1}+\mathbf{e}_{o1}\right)\wedge\left(\mathbf{p}_{\mathcal{S}^2}+\frac{1}{2}\mathbf{p}_{\mathcal{S}^2}^2\mathbf{e}_{\infty2}+\mathbf{e}_{o2}\right), && (5.100)\end{aligned}$$

which is just the doubled embedding of a $\mathcal{G}_{0,3}$ SA spatial point of the form

$$\mathbf{p}_{\mathcal{S}} = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 \tag{5.101}$$

without a time $w = p_w$ component.

The DCSA GIPNS entity $\mathbf{\Omega}_{\mathcal{DS}}$, or dual GOPNS entity $\mathbf{\Omega}_{\mathcal{DS}}^{*\mathcal{DS}}$, that is produced by the DCSTA spatial projection should be tested against only spatial DCSA points $\mathbf{P}_{\mathcal{DS}} = \mathcal{D}(\mathbf{p}_{\mathcal{S}})$ that represent spatial positions at time $w = ct = 0$.

The DCSTA *spatial projection* is specifically defined for the spatial projection of any DCSTA *quadric surface entity* $\boldsymbol{Q}$ as

$$\mathbf{Q}=\mathbf{Q}_{\mathcal{DS}} \;=\; (\boldsymbol{Q}\cdot\mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1}, \tag{5.102}$$

which is a DCSA quadric surface entity that represents the DCSTA quadric $\boldsymbol{Q}$ at time $w=ct=0$. The DCSTA anisotropic (directed or non-uniform) dilation operation (§5.7.9) is defined for any DCSA quadric entity $\mathbf{Q}$ formed as a linear combination of quadric extraction elements T_s without a time component. The anisotropic dilation is implemented as a boost by a natural speed $\beta=\sqrt{1-d^2}$ that may be imaginary for dilations $d>1$. Directed dilation using an imaginary β results in imaginary time components that are only artifacts of the directed dilation operation and should usually be discarded. The DCSTA spatial projection of a DCSTA quadric $\boldsymbol{Q}$, which may be a spatial DCSA quadric $\mathbf{Q}$ that has been boosted for directed dilation, discards any time components and produces the quadric at time $w=ct=0$ as a DCSA quadric $\mathbf{Q}$. The DCSA quadric $\mathbf{Q}$ is again a DCSTA quadric $\boldsymbol{Q}$ at zero velocity $\beta=0$.

5.7.3 DCSTA spacetime boost

The DCSTA 4-versor *boost* operator is defined as

$$B_{\mathcal{D}} \;=\; B_{\mathcal{C}^1}\wedge B_{\mathcal{C}^2}, \tag{5.103}$$

which is the doubling of the CSTA 2-versor *boost* $B_{\mathcal{C}}$ (§4.6.8) in CSTA1 and CSTA2.

For a DCSTA 4-versor boost, the notation $B_{\mathbf{v}}$ is defined as

$$\begin{aligned} B_{\mathbf{v}}=B_{\mathcal{D}\mathbf{v}} \;&=\; B_{\mathcal{C}^1\mathbf{v}}\wedge B_{\mathcal{C}^2\mathbf{v}} && (5.104)\\ &=\; \exp\!\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}^1}\mathbf{e}_1\right)\wedge\exp\!\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}^2}\mathbf{e}_7\right), && (5.105)\end{aligned}$$

which is an active boost into the frame of the observable spacetime velocity

$$\begin{aligned} \boldsymbol{v} \;&=\; \boldsymbol{o}+\mathbf{v} && (5.106)\\ &=\; c\gamma_0+\beta_{\mathbf{v}}c\hat{\mathbf{v}}. && (5.107)\end{aligned}$$

The observable worldline $\boldsymbol{v}t$, with initial position $\mathbf{p}_0=0$ (the spacetime origin) at time $t=0$, is also being called the *observable* $\boldsymbol{v}$ or the *frame* $\boldsymbol{v}$. The frame of $\boldsymbol{v}$ is carried at the position $\boldsymbol{v}t$ of the observable.

All DCSTA entities can be boosted. A boosted surface entity represents the set of all boosted surface points. For a full discussion of the boost operator, see the CSTA boost $B_{\mathcal{C}}$ (§4.6.8).

A DCSTA GIPNS 2-vector *quadric* surface entity $\boldsymbol{Q}$ is *boosted* as

$$\boldsymbol{Q}' \;=\; B_{\mathcal{D}}\boldsymbol{Q}B_{\mathcal{D}}^{\sim}. \tag{5.108}$$

The quadric surface $\boldsymbol{Q}$ should have initial position $\mathbf{p}_0=0$ at time $w=ct=0$. If $\mathbf{p}_0\neq 0$ at time $w=ct=0$, then the quadric $\boldsymbol{Q}$ with position $\mathbf{p}=\mathbf{p}_0$ can be boosted using the DCSTA 4-versor *translated-boost* operator (§5.7.4)

$$B_{\mathcal{D}}^{\mathbf{p}_0} \;=\; B_{\mathcal{C}^1}^{\mathbf{p}_0}B_{\mathcal{C}^2}^{\mathbf{p}_0} \tag{5.109}$$

that is centered on the initial position $\mathbf{p}_0$. The translated-boost of $\boldsymbol{Q}$ is

$$\boldsymbol{Q}' \;=\; B_{\mathcal{D}}^{\mathbf{p}_0}\boldsymbol{Q}B_{\mathcal{D}}^{\mathbf{p}_0\sim}, \tag{5.110}$$

where the position of $\boldsymbol{Q}'$ is preserved as $\mathbf{p}=\mathbf{p}_0$ at time $w=ct=0$.

Since entities are homogeneous and any general spacetime dilation factor γ of the boosts can always be divided out (as a normalization of the entity) without affecting the surface representation, the boost of a quadric surface has the interpretation that the quadric surface undergoes a relativistic velocity addition for an active boost and a relativistic velocity subtraction for a passive boost. A boosted quadric displays the spacetime (length) contraction effect of the boost, which is possible since general quadrics can be represented by the DCSTA extraction elements T_s (§5.2.3). For active boosts $B_{\mathbf{v}}$, the length contraction of displacements is as seen by the observable $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ in its contracted spacetime frame. For passive boosts $B_{\mathbf{v}}^{\sim}=B_{-\mathbf{v}}$, the length contraction of displacements is as seen by the observable $\boldsymbol{v}^{\dagger}=\boldsymbol{o}-\mathbf{v}$ in its contracted spacetime frame, as if the boost is the active boost $B_{-\mathbf{v}}$. The boost is viewed in the active orientation, and the quadric is boosted up into the frame of the active boost observable $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ for $B_{\mathbf{v}}$, or the frame of the active boost observable $\boldsymbol{v}^{\dagger}=\boldsymbol{o}-\mathbf{v}$ for $B_{\mathbf{v}}^{\sim}=B_{\mathbf{v}^{\dagger}}=B_{-\mathbf{v}}$. In either case, the length contraction factor is the same, $1/\gamma_{\mathbf{v}}=1/\gamma_{-\mathbf{v}}=\sqrt{1-\beta_{\mathbf{v}}^2}$, and the quadric surface relativistically gains the velocity $\mathbf{v}$ or $-\mathbf{v}$ of the frame it is boosted into.

For example, consider a spatial quadric $\mathbf{Q}=\mathbf{Q}_{\mathcal{DS}}$ that is initially at zero velocity $\beta_0=0$ and at position $\mathbf{p}_0=0$ (the origin) in the rest frame of the conventional coordinate time t observable $\boldsymbol{o}$. It is then boosted as

$$\boldsymbol{Q} \;=\; B_{\mathbf{v}}\mathbf{Q}B_{\mathbf{v}}^{\sim}=\mathbf{Q}\oplus\boldsymbol{v}. \tag{5.111}$$

$\boldsymbol{Q}$ is $\mathbf{Q}$ boosted or moved *into the rest frame* of $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$. Local to the frame of $\boldsymbol{v}$, $\boldsymbol{Q}$ is still at the origin and still at rest as it was in the frame of $\boldsymbol{o}$. $\boldsymbol{Q}$ is carried along in the frame of $\boldsymbol{v}$ at the frame velocity $\mathbf{v}=\beta_{\mathbf{v}}c\hat{\mathbf{v}}$ and with position displacement $\mathbf{v}t$ in the frame of coordinate time t observable $\boldsymbol{o}$. Since the entity $\boldsymbol{Q}$ is homogeneous, it displays as the contracted quadric, by contraction factor $\sqrt{1-\beta_{\mathbf{v}}^2}$ in direction $\hat{\mathbf{v}}$, as seen by the observable $\boldsymbol{v}$ in its own contracted frame, but $\boldsymbol{Q}$ moves with coordinate time t in the frame of observable $\boldsymbol{o}$ at the velocity $\mathbf{v}$ as seen by $\boldsymbol{o}$ while $\boldsymbol{Q}$ is carried at rest and at the origin *in the frame of* $\boldsymbol{v}$ that moves at velocity $\mathbf{v}$.

The boosted quadric surface $\boldsymbol{Q}$ can be symbolically evaluated [19] for its implicit surface function F as

$$\mathcal{D}(\boldsymbol{t}_{\mathcal{M}})\cdot\boldsymbol{Q} \;=\; F(w,x,y,z), \tag{5.112}$$

which can be graphed in xyz-space at any selected time $w=ct$. The graph of $\boldsymbol{Q}$ should show it to be centered in space at a position consistent with the elapsed coordinate time t in the frame of the observable $\boldsymbol{o}$, and the shape of the quadric surface should show a length contraction effect consistent with the speed of the boost that has been applied.

5.7.4 DCSTA translated-boost

The DCSTA 4-versor *translated-boost* operator $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}}$ is the doubling of the CSTA 2-versor translated-boost $B_{\mathcal{C}\mathbf{v}}^{\mathbf{p}}$ (§4.6.9) and defined as

$$\begin{aligned} B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}} &= B_{\mathcal{C}^1\mathbf{v}}^{\mathbf{p}}\wedge B_{\mathcal{C}^2\mathbf{v}}^{\mathbf{p}}=B_{\mathcal{C}^1\mathbf{v}}^{\mathbf{p}}B_{\mathcal{C}^2\mathbf{v}}^{\mathbf{p}} & (5.113)\\ &= \exp\!\left(\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\Pi}_{\mathcal{C}^1}\right)\wedge\exp\!\left(\frac{1}{2}\varphi_{\mathbf{v}}\boldsymbol{\Pi}_{\mathcal{C}^2}\right), & (5.114)\end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\Pi}_{\mathcal{C}^1} &= \boldsymbol{D}_{\mathcal{C}^1}^{*\mathcal{M}^1}-(\mathbf{p}_{0\mathcal{S}^1}\cdot\boldsymbol{D}_{\mathcal{C}^1}^{*\mathcal{M}^1})\mathbf{e}_{\infty 1} & (5.115)\\ \boldsymbol{\Pi}_{\mathcal{C}^2} &= \boldsymbol{D}_{\mathcal{C}^2}^{*\mathcal{M}^2}-(\mathbf{p}_{0\mathcal{S}^2}\cdot\boldsymbol{D}_{\mathcal{C}^2}^{*\mathcal{M}^2})\mathbf{e}_{\infty 2} & (5.116)\end{aligned}$$

and

$$
\begin{aligned}
\boldsymbol{D}_{\mathcal{C}^1} &= \hat{\mathbf{v}}_{\mathcal{S}^1}\mathbf{e}_1\mathbf{I}_{\mathcal{M}^1} && (5.117)\\
\boldsymbol{D}_{\mathcal{C}^2} &= \hat{\mathbf{v}}_{\mathcal{S}^2}\mathbf{e}_7\mathbf{I}_{\mathcal{M}^2} && (5.118)\\
\mathbf{p}_{\mathcal{S}^1} &= p_x\mathbf{e}_2 + p_y\mathbf{e}_3 + p_z\mathbf{e}_4 && (5.119)\\
\mathbf{p}_{\mathcal{S}^2} &= p_x\mathbf{e}_8 + p_y\mathbf{e}_9 + p_z\mathbf{e}_{10}. && (5.120)
\end{aligned}
$$

The translated-boost is centered on the spatial point

$$
\mathbf{p} = \mathbf{p}_0 = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 \qquad (5.121)
$$

and actively boosts into the rest frame of the translated observable with worldline

$$
\begin{aligned}
\boldsymbol{v}^{\mathbf{p}_0}(t) &= \mathbf{p}_0 + (\boldsymbol{o} + \mathbf{v})t && (5.122)\\
&= \mathbf{p}_0 + (c\gamma_0 + \beta_{\mathbf{v}}c\hat{\mathbf{v}})t. && (5.123)
\end{aligned}
$$

If $\mathbf{p}_0 = 0$, then $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}_0} = B_{\mathcal{D}\mathbf{v}}$ (§5.7.3).

For example, the point $\mathbf{p}_0 = \mathbf{p}$ can be the center *position* $\mathbf{p}$ of a spatial quadric surface $\mathbf{Q}$, or it can be the *initial point* $\mathbf{p}_0$ at time $w = ct = 0$ of a DCSTA GIPNS 6-vector standard *line* $\boldsymbol{L}_{\mathcal{D}}$ (§5.3.7) that represents an observable worldline.

The translated-boost operator $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}}$ is used in the definition of the DCSTA *anisotropic dilation operation* (§5.7.9), which is valid for the directed scaling of DCSTA quadric surface entities.

5.7.5 DCSTA spacetime reframe (reverse boost)

The DCSTA reframe operation is the reversed application of the active boost operation (§5.7.3) and is often interpreted as a passive transformation relative to a new frame of reference. For a full discussion of the boost operator, see the CSTA boost (§4.6.8).

A DCSTA GIPNS 2-vector quadric surface entity $\boldsymbol{Q}$ is reframed (*passively transformed*) into (*relative to*) the frame of observable $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ as

$$
\begin{aligned}
\boldsymbol{Q}' &= B_{\mathbf{v}}^{\sim}\boldsymbol{Q}B_{\mathbf{v}} = \boldsymbol{Q} \ominus \boldsymbol{v} && (5.124)\\
&= B_{-\mathbf{v}}\boldsymbol{Q}B_{-\mathbf{v}}^{\sim} = B_{\mathbf{v}^\dagger}\boldsymbol{Q}B_{\mathbf{v}^\dagger}^{\sim} = \boldsymbol{Q} \oplus \boldsymbol{v}^\dagger. && (5.125)
\end{aligned}
$$

One way to interpret the reframe is that $\boldsymbol{Q}$ is initially (being carried) in the frame of $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$, where it sees $\boldsymbol{v}$ as its conventional coordinate time $\tau = t_{p\boldsymbol{v}}$ observer $\boldsymbol{o}_{\boldsymbol{v}}$, and it is actively boosted down or moved into the frame of the actual coordinate time $t = t_{p\boldsymbol{o}}$ observer $\boldsymbol{o}$ as $\boldsymbol{Q}'$. From the perspective of observer $\boldsymbol{o}$, $\boldsymbol{Q}'$ relativistically loses the velocity $\mathbf{v}$ as a velocity subtraction.

Another way to interpret the reframe is that $\boldsymbol{Q}$ is initially in the frame of $\boldsymbol{o}$, and it is actively boosted up into the frame of $\boldsymbol{v}^\dagger = \boldsymbol{o} - \mathbf{v}$. The frame of $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ and the frame of $\boldsymbol{v}^\dagger$ are conjugate frames that carry boosted entities in opposite directions with the same proper time $\tau = t_{p\boldsymbol{v}} = t_{p\boldsymbol{v}^\dagger}$ relative to the coordinate time $t = t_{p\boldsymbol{o}}$ of observer $\boldsymbol{o}$. In many respects, conjugate frames are the same frame, and they also see the same length contraction factor $\sqrt{1 - \beta_{\mathbf{v}}^2}$. Entities that are actively boosted or moved into the frame of $\boldsymbol{v}^\dagger$ are how entities carried in the frame of $\boldsymbol{o}$ appear to $\boldsymbol{v}$ when they are passively transformed (not actively boosted or moved) into the frame of $\boldsymbol{v}$ by a relative or passive boost.

If an entity $\boldsymbol{Q}$ has an initial position $\mathbf{p}_0 \neq 0$ at time $w = ct = 0$, then a passive translated-boost $B_{\mathbf{v}}^{\mathbf{p}_0\sim}$ (§5.7.4) should be used.

5.7.6 DCSTA rotor

The DCSTA 4-versor *rotor* $R_{\mathcal{D}}$ is defined as

$$R_{\mathcal{D}} \;=\; R_{\mathcal{C}^1} \wedge R_{\mathcal{C}^2} \tag{5.126}$$

which is the wedge of the same *rotor* (§4.6.3) in CSTA1 and CSTA2.

Any DCSTA entity $\mathbf{Q}$ is rotated by the rotor operation

$$\mathbf{Q}' \;=\; R_{\mathcal{D}}\mathbf{Q}\widetilde{R_{\mathcal{D}}}. \tag{5.127}$$

The CSTA 2-versor *line rotor (translated rotor)* $R_{\mathcal{C}}^{\mathbf{p}} = L_{\mathcal{C}}$ (§4.6.5) can also be doubled into the DCSTA 4-versor *translated rotor* $R_{\mathcal{D}}^{\mathbf{p}} = L_{\mathcal{D}} = L_{\mathcal{C}^1} \wedge L_{\mathcal{C}^2}$.

5.7.7 DCSTA translator

The DCSTA 4-versor *translator* $T_{\mathcal{D}}$ is defined as

$$T_{\mathcal{D}} \;=\; T_{\mathcal{C}^1} \wedge T_{\mathcal{C}^2} \tag{5.128}$$

which is the wedge of the same *translator* (§4.6.4) in CSTA1 and CSTA2.

A DCSTA entity $\mathbf{Q}$ is translated by the translator operation

$$\mathbf{Q}' \;=\; T_{\mathcal{D}}\mathbf{Q}\widetilde{T_{\mathcal{D}}}. \tag{5.129}$$

5.7.8 DCSTA isotropic dilator

The DCSTA 4-versor *isotropic dilator* $D_{\mathcal{D}}$ is defined as

$$D_{\mathcal{D}} \;=\; D_{\mathcal{C}^1} \wedge D_{\mathcal{C}^2} \tag{5.130}$$

which is the wedge of the same *isotropic dilator* (§4.6.6) in CSTA1 and CSTA2.

A DCSTA entity $\mathbf{Q}$ is isotropically dilated by the dilator operation

$$\mathbf{Q}' \;=\; D_{\mathcal{D}}\mathbf{Q}\widetilde{D_{\mathcal{D}}}. \tag{5.131}$$

The DCSTA 4-versor *translated dilator* $D_{\mathcal{D}}^{\boldsymbol{p}}$ that dilates relative to the center point $\boldsymbol{p}_{\mathcal{M}}$ is the doubling of the CSTA 2-versor *translated dilator* $D_{\mathcal{C}}^{\boldsymbol{p}}$ (§4.6.7) as

$$D_{\mathcal{D}}^{\boldsymbol{p}} \;=\; D_{\mathcal{C}^1}^{\boldsymbol{p}} \wedge D_{\mathcal{C}^2}^{\boldsymbol{p}}. \tag{5.132}$$

5.7.9 DCSTA anisotropic dilator

Introduction

A *spatial* DCSTA GIPNS 2-vector *quadric surface* entity $\mathbf{Q}$ at center *position*

$$\mathbf{p} = \mathbf{p}_0 = \mathbf{p}_{\mathcal{S}} \;=\; p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3, \tag{5.133}$$

formed similarly to a DCGA GIPNS 2-vector quadric as a linear combination of *quadratic* DCSTA point value-*extraction elements* T_s (§5.2.3) without time components, can be anisotropically dilated (for non-uniform, directed scaling) in situ at $\mathbf{p} = \mathbf{p}_0$, by dilation factor d in a unit direction $\hat{\mathbf{v}} = \hat{\mathbf{v}}_{\mathcal{S}}$, as a *translated-boost* $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}}$ (§5.7.4) that is followed by a DCSTA *spatial projection* using $\mathbf{I}_{\mathcal{DS}}$ (§5.7.2). The natural speed $\beta_{\mathbf{v}}$ of the translated-boost, for the dilation factor d, is

$$\beta_{\mathbf{v}} \;=\; \sqrt{1-d^2}, \tag{5.134}$$

which is an imaginary number for $d>1$. The rapidity of the translated-boost is

$$\varphi_{\mathbf{v}} = \operatorname{atanh}(\beta_{\mathbf{v}}) = \operatorname{atanh}\left(\sqrt{1-d^2}\right), \tag{5.135}$$

which can also be imaginary when $\beta_{\mathbf{v}}$ is imaginary. The translated-boost spacetime velocity is

$$\boldsymbol{v} = \boldsymbol{o} + \mathbf{v} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}} c\hat{\mathbf{v}}, \tag{5.136}$$

which can be an imaginary spacetime velocity when $\beta_{\mathbf{v}}$ is imaginary. A spatial quadric $\mathbf{Q}$ (at zero velocity) is boosted by the translated-boost $B^{\mathbf{p}}_{\mathcal{D}\mathbf{v}}$ to be in the moving observable frame as $\boldsymbol{Q}$ and is carried along the observable worldline at velocity $\mathbf{v}$ with position

$$\begin{aligned} \boldsymbol{v}^{\mathbf{p}_0}(t) &= \mathbf{p}_0 + (\boldsymbol{o} + \mathbf{v})t & (5.137)\\ &= \mathbf{p}_0 + (c\boldsymbol{\gamma} + \beta_{\mathbf{v}} c\hat{\mathbf{v}})t. & (5.138) \end{aligned}$$

The observed dilation of quadric surface $\mathbf{Q}$ in the moving frame as $\boldsymbol{Q}$ is by factor

$$d = \sqrt{1-\beta_{\mathbf{v}}^2} \tag{5.139}$$

in the direction $\hat{\mathbf{v}}$, where $d>1$ for imaginary $\beta_{\mathbf{v}}$. The length dilation in the direction $\hat{\mathbf{v}}$ of the boost follows from the standard formula for special relativity length contraction

$$L = \frac{L_0}{\gamma_{\mathbf{v}}} = L_0\sqrt{1-\beta_{\mathbf{v}}^2} = L_0 d. \tag{5.140}$$

The moving, and anisotropically dilated, quadric $\boldsymbol{Q}$ can be evaluated at time $w = ct = 0$ to observe the anisotropic dilation at its position $\mathbf{p}$. By projecting $\boldsymbol{Q}$ into the spatial $\mathcal{G}_{2,8}$ CSTA subalgebra, an anisotropically dilated spatial quadric $\mathbf{Q}'$ at zero velocity is recovered.

Definition

The DCSTA *anisotropic dilator operation*, on a spatial DCSTA GIPNS 2-vector *quadric surface* $\mathbf{Q}$ with position $\mathbf{p}$, for dilation factor d in the direction $\hat{\mathbf{v}}$ is defined as

$$\begin{aligned} \mathbf{Q}' = \mathbf{Q}'_{\mathcal{DS}} &= ((B^{\mathbf{p}}_{\mathcal{D}\mathbf{v}} \mathbf{Q} B^{\mathbf{p}\sim}_{\mathcal{D}\mathbf{v}}) \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}^{-1}_{\mathcal{DS}} & (5.141)\\ &= (\boldsymbol{Q} \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}^{-1}_{\mathcal{DS}}, & (5.142) \end{aligned}$$

where the observable of the translated-boost (§5.7.4) is

$$\begin{aligned} \boldsymbol{v}^{\mathbf{p}}(t) &= \mathbf{p} + (\boldsymbol{o} + \mathbf{v})t & (5.143)\\ &= \mathbf{p} + (c\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}} c\hat{\mathbf{v}})t & (5.144) \end{aligned}$$

with

$$\begin{aligned} \beta_{\mathbf{v}} &= \sqrt{1-d^2} & (5.145)\\ \varphi_{\mathbf{v}} &= \operatorname{atanh}(\beta_{\mathbf{v}}) & (5.146)\\ \mathbf{p} &= p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3. & (5.147) \end{aligned}$$

Discussion

The dilated entity $\mathbf{Q}'_{\mathcal{DS}}$ can be symbolically evaluated as an implicit surface function

$$F(x,y,z) \;=\; \boldsymbol{T}_{\mathcal{D}}\cdot\mathbf{Q}_{\mathcal{DS}} \tag{5.148}$$

that graphs at position $\mathbf{p}$ as anisotropically dilated in the unit direction $\hat{\mathbf{v}}_{\mathcal{S}}$ by factor d.

The spatial quadric $\mathbf{Q}'$ has zero velocity and exists at all times at the same position $\mathbf{p}$. If $\mathbf{Q}'$ should exist only at a specific time $w=p_w$, then this can be represented as an intersection of $\mathbf{Q}'$ with a DCSTA 2-vector standard *hyperplane* $\boldsymbol{E}_{\mathcal{D}}$ (§5.3.2),

$$\begin{aligned}
\boldsymbol{E}_{\mathcal{D}} &= \boldsymbol{E}_{\mathcal{C}^1}\wedge\boldsymbol{E}_{\mathcal{C}^2} && (5.149)\\
\boldsymbol{E}_{\mathcal{C}^1} &= p_w\mathbf{e}_1+p_w^2\mathbf{e}_{\infty 1} && (5.150)\\
\boldsymbol{E}_{\mathcal{C}^2} &= p_w\mathbf{e}_7+p_w^2\mathbf{e}_{\infty 2}, && (5.151)
\end{aligned}$$

that fixes the time $w=ct=p_w$. The intersection is the GIPNS 4-vector intersection entity

$$\mathbf{Q}'^{\,w=p_w}_{\langle 4\rangle} \;=\; \mathbf{Q}'\wedge\boldsymbol{E}_{\mathcal{D}}. \tag{5.152}$$

The entity $\mathbf{Q}'^{\,w=p_w}_{\langle 4\rangle}$ can be translated in spacetime using the DCSTA translator (§5.7.7).

The entity $\mathbf{Q}'$ can subsequently be boosted from zero velocity into a real velocity $\mathbf{v}=\mathbf{v}_{\mathcal{S}}=\beta_{\mathbf{v}}c\hat{\mathbf{v}}_{\mathcal{S}}$ with natural speed $0\leq\beta_{\mathbf{v}}<1$ by the action of an active DCSTA translated-boost operation (§5.7.4)

$$\mathbf{Q}'_{\mathbf{v}} \;=\; B^{\mathbf{p}}_{\mathcal{D}\mathbf{v}}\mathbf{Q}'B^{\mathbf{p}\sim}_{\mathcal{D}\mathbf{v}}. \tag{5.153}$$

The boosted quadric entity $\mathbf{Q}'_{\mathbf{v}}$ exists at all times t, but its spacetime position

$$\begin{aligned}
\boldsymbol{p}^{\mathbf{p}}(t) &= \mathbf{p}+\dot{\boldsymbol{p}}t && (5.154)\\
&= \mathbf{p}+(\boldsymbol{o}+\dot{\mathbf{p}})t && (5.155)\\
&= \mathbf{p}+(\boldsymbol{o}+\mathbf{v})t && (5.156)
\end{aligned}$$

has velocity $\mathbf{v}$ relative to the observer $\boldsymbol{o}$, and its geometric surface shape undergoes a contraction by factor $d=\sqrt{1-\beta_{\mathbf{v}}^2}$ along the direction of its velocity $\mathbf{v}$. The contraction is consistent with special relativity.

5.8 DCSTA differential calculus

The DCSTA differential calculus is a straightforward extension of the DCGA differential calculus that is introduced in the paper [6].

5.8.1 DCSTA differential elements

Some of the DCSTA point *value-extraction elements* T_s have inverses. These inverses allow the following DCSTA 2-vector *differential elements* to be defined as

$$\begin{aligned}
D_w &= 2T_wT_{w^2}^{-1} && (5.157)\\
D_t &= 2T_tT_{t^2}^{-1} && (5.158)\\
D_x &= 2T_xT_{x^2}^{-1} && (5.159)\\
D_y &= 2T_yT_{y^2}^{-1} && (5.160)\\
D_z &= 2T_zT_{z^2}^{-1}. && (5.161)
\end{aligned}$$

5.8.2 DCSTA antisymmetric differential operators

The DCSTA antisymmetric *differential operators* are defined as

$$\partial_w = \frac{\partial}{\partial w} = D_w \times \tag{5.162}$$

$$\partial_t = \frac{\partial}{\partial t} = D_t \times \tag{5.163}$$

$$\partial_x = \frac{\partial}{\partial x} = D_x \times \tag{5.164}$$

$$\partial_y = \frac{\partial}{\partial y} = D_y \times \tag{5.165}$$

$$\partial_z = \frac{\partial}{\partial z} = D_z \times \tag{5.166}$$

where the symbol × is the antisymmetric *commutator product*. For any multivectors A and B, the commutator product is

$$A \times B = \frac{1}{2}(AB - BA). \tag{5.167}$$

Any DCSTA GIPNS 2-vector surface entity $\boldsymbol{\Omega}$ (§5.5), defined in terms of the extraction elements T_s (§5.2.3), can be differentiated as $D_{\boldsymbol{n}} \times \boldsymbol{\Omega}$, where $D_{\boldsymbol{n}}$ is one of the differential elements or is a linear combination of differential elements (§5.8.1).

Higher-order mixed partial derivatives can also be computed as successive differential operations. For example,

$$\frac{d^2\boldsymbol{\Omega}}{\partial x \partial y} = D_x \times (D_y \times \boldsymbol{\Omega}) = D_y \times (D_x \times \boldsymbol{\Omega}). \tag{5.168}$$

As required of partial differential operators, the sequence in which the derivatives are computed does not affect the result.

5.8.3 DCSTA directional derivative operator

The DCSTA $\boldsymbol{n}$-directional derivative operator is defined as

$$\partial_{\boldsymbol{n}} = \frac{\partial}{\partial \boldsymbol{n}} = (n_w D_w + n_x D_x + n_y D_y + n_z D_z) \times \tag{5.169}$$

where $\boldsymbol{n}$ is a unit norm spacetime direction

$$\boldsymbol{n} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{\boldsymbol{n}}{\sqrt{\boldsymbol{n} \cdot \boldsymbol{n}^\dagger}} = n_w \gamma_0 + n_x \gamma_1 + n_y \gamma_2 + n_z \gamma_3. \tag{5.170}$$

5.8.4 DCSTA time derivative operator

The DCSTA time t derivative operator is

$$\partial_t = \frac{\partial}{\partial t} = D_t \times . \tag{5.171}$$

The time t derivative of any DCSTA GIPNS 2-vector *spacetime entity* $\boldsymbol{\Omega}$ (§5.5) is

$$\dot{\boldsymbol{\Omega}} = \partial_t \boldsymbol{\Omega} = \frac{\partial \boldsymbol{\Omega}}{\partial t} = D_t \times \boldsymbol{\Omega}. \tag{5.172}$$

The DCSTA 2-vector *spacetime entity* $\boldsymbol{\Omega}$ (§5.5) is the most general DCSTA GIPNS 2-vector *non-standard surface entity* that is formed as a linear combination of the DCSTA 2-vector *extraction elements* T_s (§5.2.3).

5.9 DCSTA pseudo-integral calculus

In the paper [6], the DCGA pseudo-integral calculus is introduced. A straightforward adaptation and extension into DCSTA is possible by using the DCSTA *extraction pseudo-inverse elements* T_s^+ (§5.2.4).

6 DCSTA computing with SymPy

DCSTA computing with *SymPy* (http://sympy.org) [19] is possible by using the *Geometric Algebra Module for Sympy* ($\mathcal{GA}$lgebra) by Alan Bromborsky (https://github.com/brombo/galgebra) [2]. This section provides sample code listings and example computations in DCSTA using $\mathcal{GA}$lgebra. The *Anaconda* and *SciPy* python distributions both include SymPy and the *Mayavi* [17] data visualization package. The current version of the $\mathcal{GA}$lgebra module for SymPy can be downloaded and installed from **GitHub**. The *Jupyter Notebook* web application (http://jupyter.org) is recommended for running the sample code and example computations.

6.1 Sample code

The sample code that is listed in the following subsections can be inserted into cells of a Jupyter notebook file and executed in the order listed. The sample code initializes the $\mathcal{GA}$lgebra modules and defines functions and symbols for DCSTA computing. The example computations use the functions and symbols that are defined in the sample code. The sample code is provided as is for experimental testing and educational purposes only!

6.1.1 Imports

Import the SymPy and $\mathcal{GA}$lgebra modules:

```
from sympy import *
from sympy.printing import *
from galgebra.ga import *
from galgebra.mv import *
from galgebra.lt import *
from galgebra.metric import *
from galgebra.printer import *
init_printing()
```

6.1.2 Basis vectors

$\mathcal{G}_{4,8}$ DCSTA requires twelve unit vectors (§1), which can be setup as follows:

```
(e1,e2,e3,e4,e5,e6,e7,e8,e9,e10,e11,e12) = MV.setup(
    ’e*1|2|3|4|5|6|7|8|9|10|11|12’,
    metric=[1,-1,-1,-1,1,-1, 1,-1,-1,-1,1,-1]
)
```

6.1.3 Points at the origin and at infinity

The CSTA1, CSTA2 (§4.2), and DCSTA (§5.2) points at the origin and at infinity are defined as follows:

```
(eo1,ei1,eo2,ei2,eo,ei) = symbols(
    ’e_o1 e_i1 e_o2 e_i2 e_o e_i’
)
# CSTA1 points
eo1 = Pow(2,-1)*(-e5+e6)
ei1 = (e5+e6)
# CSTA2 points
eo2 = Pow(2,-1)*(-e11+e12)
ei2 = (e11+e12)
# DCSTA points
eo = eo1^eo2
ei = ei1^ei2
```

6.1.4 Unit pseudoscalars

The unit pseudoscalars in $\mathcal{G}_{0,3}$ SA1 (§2.1), $\mathcal{G}_{1,3}$ STA1 (§3.1.2), $\mathcal{G}_{2,4}$ CSTA1 (§4.1), $\mathcal{G}_{0,3}$ SA2, $\mathcal{G}_{1,3}$ STA2, $\mathcal{G}_{2,4}$ CSTA2, and $\mathcal{G}_{4,8}$ DCSTA (§5.1), respectively, are defined as follows:

```
(I31,I41,I61,I32,I42,I62,ID IDS) = symbols(
    ’I_31 I_41 I_61 I_32 I_42 I_62 I_D I_DS’
)
# SA1 unit pseudoscalar
I31 = e2^e3^e4
# STA1 unit pseudoscalar
I41 = e1^I31
# CSTA1 unit pseudoscalar
I61 = I41^ei1^eo1
# SA2 unit pseudoscalar
I32 = e8^e9^e10
# STA2 unit pseudoscalar
I42 = e7^I32
# CSTA2 unit pseudoscalar
I62 = I42^ei2^eo2
# DCSTA unit pseudoscalar
ID = I61^I62
# G2,8 anti-DCGA (spatial) unit pseudoscalar
IDS = (e1^e7)|ID
```

The last value, `IDS`, is the $\mathcal{G}_{2,8}$ DCSA unit pseudoscalar for an algebra that is very similar to $\mathcal{G}_{8,2}$ DCGA. The `IDS` unit pseudoscalar is used to project entities (§5.7.2) into a purely spatial algebra that drops the two time dimensions $\mathbf{e}_1$ and $\mathbf{e}_7$. When these time dimensions are dropped, or rejected, by a projection of an entity onto `IDS`, then the entity is effectively located at $w = ct = 0$ in spacetime as a spatial DCSA entity that should be tested against spatial DCSA points. The projection onto `IDS` is useful after a directed scaling (anisotropic dilation) of a quadric surface (§5.7.9).

6.1.5 Point embeddings

CSTA1 point embedding (§4.2):

```
def EV1(v):
    # Embed STA1 vector v as CSTA1 point.
    v1 = v
    return ( v1 + Pow(2,-1)*(v1*v1)*ei1 + eo1 )
```

CSTA2 point embedding (§4.2):

```
def EV2(v):
    # Embed STA1 vector v as CSTA2 point.
    # STA1 vector v is converted to an STA2 vector v2.
    v2 = (v|e1)*e7 - ( (v|e2)*e8 + (v|e3)*e9 + (v|e4)*e10 )
    return ( v2 + Pow(2,-1)*(v2*v2)*ei2 + eo2 )
```

DCSTA point embedding (§5.2):

```
def EV(v):
    # Embed STA1 vector v as DCSTA point.
    return ( EV1(v)^EV2(v) )
```

6.1.6 Point projections

CSTA1 point projection (§4.2.6) to an STA1 vector:

```
def PV1(V1):
    # Project CSTA1 point to STA1 vector.
    # 1) Normalize point.
    # 2) Use multivector projection to project vector part.
    return Pow(scalar(-V1|ei1),-1)*(V1|I41)*I41.inv()
```

CSTA2 point projection (§4.2.6) to an STA1 vector:

```
def PV2(V2):
    # Project CSTA2 point to STA1 vector.
    # 1) Normalize point.
    # 2) Use multivector projection to project vector part.
    # 3) Convert into main STA1 space.
    v2 = Pow(scalar(-V2|ei2),-1)*(V2|I42)*I42.inv()
    return ( (v2|e7)*e1 + (-v2|e8)*e2 + (-v2|e9)*e3 + (-v2|e10)*e4 )
```

DCSTA point projection (§5.2.2) to an STA1 vector:

```
def PV(V):
    # Project DCSTA point V to an STA1 vector.
    # 1) Contract DCSTA point into CSTA1 point using ei2.
    # 2) Project CSTA1 point V1 to an STA1 vector.
    V1 = V|ei2
    return PV1(V1)
```

6.1.7 Symbolic vectors and points

Symbols for coordinates, parameters, and vectors:

```
w,x,y,z,c,t,g,b = symbols('w x y z c t g b')
pw,px,py,pz = symbols('p_w p_x p_y p_z')
rx,ry,rz = symbols('r_x r_y r_z')
nw,nx,ny,nz = symbols('n_w n_x n_y n_z')
vx,vy,vz = symbols('v_x v_y v_z')
v,v1,v2,V,V1,V2 = symbols('v v1 v2 V V1 V2')
```

The pw,px,py,pz are used as symbolic position coordinates for the center position of surface entities. The rx,ry,rz are used as symbolic radii parameters of implicit quadric surface functions. The nw,nx,ny,nz may be used as symbolic coordinates of a normalized unit vector **n**. The vx,vy,vz may be used to hold the velocity components of a velocity vector **v**. The symbol c is used as the symbolic speed of light, and symbol t is time.

Symbolic values, vectors, and points:

```
w = c*t
v = w*e1 + x*e2 + y*e3 + z*e4
v1 = v
v2 = w*e7 + x*e8 + y*e9 + z*e10
V1 = EV1(v)
V2 = EV2(v)
V = EV(v)
```

The embedding of the symbolic STA1 and STA2 vectors v1 and v2 are symbolic CSTA1 and CSTA2 points V1 and V2, respectively. The DCSTA embedding of a symbolic STA1 vector v is the symbolic DCSTA point V. In symbolic calculations, these symbolic point embeddings V1, V2, and V are useful to check results.

6.1.8 CSTA extraction elements

A CSTA point *value-extraction element* C_s (§4.3) extracts the value s from an embedded CSTA point $\boldsymbol{T}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ as

$$s \;=\; \boldsymbol{T}_{\mathcal{C}} \cdot C_s. \tag{6.1}$$

```
C11,C1t,C1w,C1x,C1y,C1z,C1t2 = symbols(
    'C1_t C1_1 C1_w C1_x C1_y C1_z C1_t2'
)
C21,C2t,C2w,C2x,C2y,C2z,C2t2 = symbols(
    'C2_t C2_1 C2_w C2_x C2_y C2_z C2_t2'
)
# CSTA1 (C1) point value-extraction elements
C11 = -ei1
C1w = e1
C1t = Pow(c,-1)*C1w
C1x = -e2
C1y = -e3
C1z = -e4
C1t2 = -2*eo1
# CSTA2 (C2) point value-extraction elements
C21 = -ei2
C2w = e7
C2t = Pow(c,-1)*C2w
C2x = -e8
C2y = -e9
C2z = -e10
C2t2 = -2*eo2
```

For example, in CSTA1, the element symbol `C1t2` extracts $\boldsymbol{t}^2_{\mathcal{M}}$ from an embedded point $\boldsymbol{T}_{\mathcal{C}}$.

6.1.9 CSTA differential elements

Using the commutator product ×, a CSTA differential element $D^{\mathcal{C}}_{\boldsymbol{n}}$ (§4.6.10) can take a partial derivative of an implicit surface function that is represented by a CSTA entity $\mathbf{E}$ as

$$\partial^{\mathcal{C}}_{\boldsymbol{n}}\mathbf{E} \;=\; \frac{\partial \mathbf{E}}{\partial \boldsymbol{n}} = D^{\mathcal{C}}_{\boldsymbol{n}} \times \mathbf{E}. \tag{6.2}$$

The simplest example is when $\mathbf{E}$ is any CSTA GIPNS 1-vector entity. The CSTA differential element $D^{\mathcal{C}}_{\boldsymbol{n}}$, for differentiating in the direction $\boldsymbol{n}$, can be a linear combination of the CSTA differential elements.

```
C1Dw,C1Dt,C1Dx,C1Dy,C1Dz,C2Dw,C2Dt,C2Dx,C2Dy,C2Dz = symbols(
    'C1Dw C1Dt C1Dx C1Dy C1Dz C2Dw C2Dt C2Dx C2Dy C2Dz'
)
# CSTA1 (C1) differential elements
C1Dw = C11*C1w.inv()
C1Dt = C11*C1t.inv()
C1Dx = C11*C1x.inv()
C1Dy = C11*C1y.inv()
C1Dz = C11*C1z.inv()
# CSTA2 (C2) differential elements
C2Dw = C21*C2w.inv()
C2Dt = C21*C2t.inv()
C2Dx = C21*C2x.inv()
C2Dy = C21*C2y.inv()
C2Dz = C21*C2z.inv()
```

6.1.10 DCSTA extraction elements

The DCSTA point value-extraction elements T_s (§5.2.3) are used to extract the value s from a DCSTA point $\boldsymbol{T}_{\mathcal{D}}$ as

$$s \;=\; \boldsymbol{T}_{\mathcal{D}} \cdot T_s. \tag{6.3}$$

The extraction elements are defined in code as:

```
(
    Tw,Tx,Ty,Tz,
    Tww,Txx,Tyy,Tzz,
    Txy,Tyz,Tzx,Twx,Twy,Twz,
    Twt2,Txt2,Tyt2,Tzt2,
    Tt,Ttt,Ttx,Tty,Ttz,Ttt2,
    T0,T1,Tt2,Tt4
) = symbols(
        ’Tw Tx Ty Tz ’
        ’Tww Txx Tyy Tzz ’
        ’Txy Tyz Tzx Twx Twy Twz ’
        ’Twt2 Txt2 Tyt2 Tzt2 ’
        ’Tt Ttt Ttx Tty Ttz Ttt2 ’
        ’T0 T1 Tt2 Tt4’
)
# Coordinates; linear extractions
Tw = Pow(2,-1)*((e1^ei2)+(ei1^e7))
Tt = Pow(c,-1)*Tw
Tx = -Pow(2,-1)*((e2^ei2)+(ei1^e8))
Ty = -Pow(2,-1)*((e3^ei2)+(ei1^e9))
Tz = -Pow(2,-1)*((e4^ei2)+(ei1^e10))
# Squares; quadratic extractons
Tww = e7^e1
Ttt = Pow(c,-2)*Tw
Txx = e8^e2
Tyy = e9^e3
Tzz = e10^e4
# Cross terms; quadratic extractions
Twx = Pow(2,-1)*((e1^e8)+(e2^e7))
Twy = Pow(2,-1)*((e1^e9)+(e3^e7))
Twz = Pow(2,-1)*((e1^e10)+(e4^e7))
Ttx = Pow(c,-1)*Twx
Tty = Pow(c,-1)*Twy
Ttz = Pow(c,-1)*Twz
Txy = Pow(2,-1)*((e8^e3)+(e9^e2))
Tyz = Pow(2,-1)*((e10^e3)+(e9^e4))
Tzx = Pow(2,-1)*((e10^e2)+(e8^e4))
# Coordinates * squared test vector; cubic extractions
Twt2 = (e1^eo2)+(eo1^e7)
Ttt2 = Pow(c,-1)*Twt2
Txt2 = (eo2^e2)+(e8^eo1)
Tyt2 = (eo2^e3)+(e9^eo1)
Tzt2 = (eo2^e4)+(e10^eo1)
# Unit scalar extraction
T1 = -ei
# Squared test vector; quadratic extraction
Tt2 = (eo2^ei1)+(ei2^eo1)
# Squared squared test vector; quartic extraction
Tt4 = -4*eo
```

6.1.11 DCSTA extraction pseudo-inverse elements

A DCSTA extraction pseudo-inverse element (§5.2.4) has the property

$$T_s \cdot T_s^+ \; = \; 1. \tag{6.4}$$

The extraction pseudo-inverse elements are defined in code as:

```
(
    iTw,iTx,iTy,iTz,iTt,
    iTww,iTxx,iTyy,iTzz,iTtt,
    iTxy,iTyz,iTzx,iTwx,iTwy,iTwz,iTtx,iTty,iTtz,
    iT1,iTt2,iTt4 ) = symbols(
    ’i_Tww i_Txx i_Tyy i_Tzz i_Ttt ’
    ’i_Tw i_Tx i_Ty i_Tz iTt ’
    ’i_Txy i_Tyz i_Tzx i_Twx i_Twy i_Twz i_Ttx i_Tty i_Ttz ’
    ’i_T1 i_Tt2 i_Tt4’
)
iTw = Twt2
iTx = -Txt2
iTy = -Tyt2
iTz = -Tzt2
iTt = Pow(c,2)*Ttt2
iTww = -Tww
iTxx = -Txx
iTyy = -Tyy
iTzz = -Tzz
iTtt = -Pow(c,2)*Tww
iTxy = -2*Txy
iTyz = -2*Tyz
iTzx = -2*Tzx
iTwx = 2*Twx
iTwy = 2*Twy
iTwz = 2*Twz
iTtx = 2*Pow(c,2)*Ttx
iTty = 2*Pow(c,2)*Tty
iTtz = 2*Pow(c,2)*Ttz
iT1 = -Pow(4,-1)*Tt4
iTt2 = -Pow(2,-1)*Tt2
iTt4 = -Pow(4,-1)*T1
```

6.1.12 DCSTA differential elements

The DCSTA differential (§5.8.1) and pseudo-integral elements (§5.9) are:

```
(
    Dw,Dx,Dy,Dz,Dt,
    Iw,Ix,Iy,Iz,It
) = symbols(
    'D_w D_x D_y D_z D_t '
    'I_w I_x I_y I_z I_t'
)
# Differential elements
Dw = 2*Tw*Tww.inv()
Dx = 2*Tx*Txx.inv()
Dy = 2*Ty*Tyy.inv()
Dz = 2*Tz*Tzz.inv()
Dt = 2*Tt*Ttt.inv()
# Pseudo-integral elements
Iw = Pow(2,-1)*Tww*iTw
Ix = Pow(2,-1)*Txx*iTx
Iy = Pow(2,-1)*Tyy*iTy
Iz = Pow(2,-1)*Tzz*iTz
It = Pow(2,-1)*Ttt*iTt
```

In recent versions of the $\mathcal{GA}$lgebra module [2], the commutator product $A \times B$ is coded as `(A>>B)`, and the anti-commutator product $A \mathbin{\bar{\times}} B$ is coded as `(A<<B)`. The parentheses are required to ensure that the precedence rules for Python operators do not interfere. For example, the derivative of a DCSTA GIPNS 2-vector surface entity $\mathbf{\Omega}$ (§5.5) with respect to t is written $\partial_t \mathbf{\Omega} = \dot{\mathbf{\Omega}} = D_t \times \mathbf{\Omega}$, and if $\mathbf{\Omega}$ is assigned to variable `E`, then the derivative is coded as `(Dt>>E)` and evaluated symbolically as `(V|(Dt>>E))`. The operation `(A|B)` is the inner product.

6.1.13 DCSTA directional derivative operator

The DCSTA $\boldsymbol{n}$-directional derivative operator (§5.8.3) is defined in code as:

```
def Dn(w,x,y,z):
    n = sqrt(w**2 + x**2 + y**2 + z**2)
    return Pow(n,-1)*(w*Dw + x*Dx + y*Dy + z*Dz)
```

Only the direction of the spacetime vector

$$\boldsymbol{n} \;=\; w\mathbf{e}_1 + x\mathbf{e}_2 + y\mathbf{e}_3 + z\mathbf{e}_3 \tag{6.5}$$

is significant. The $\boldsymbol{n}$-directional derivative uses the norm-unit of $\boldsymbol{n}$, which is

$$\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} \;=\; \frac{\boldsymbol{n}}{\sqrt{n_w^2 + n_x^2 + n_y^2 + n_z^2}}. \tag{6.6}$$

The directional derivative of a DCSTA GIPNS 2-vector surface entity `E` is coded as `(Dn(w,` x,y,z)>>E).

6.1.14 DCSTA pseudo-integral operator

The DCSTA $\boldsymbol{n}$-directional pseudo-integral operator (§5.9) is defined in code as:

```
def In(w,x,y,z):
    n = sqrt(w**2 + x**2 + y**2 + z**2)
    return Pow(n,-1)*(w*Iw + x*Ix + y*Iy + z*Iz)
```

The directional pseudo-integral of a DCSTA GIPNS 2-vector surface entity E is coded as (In(w,x,y,z)>>E).

6.1.15 DCSTA GIPNS 2-vector surface entities

The following subsections define, in code, many of the same surface entities that are discussed in the paper on $\mathcal{G}_{8,2}$ DCGA [7]. The most general DCSTA GIPNS 2-vector surface entity $\mathbf{\Omega}$ (§5.5) is the linear combination of the value-extraction elements T_s (§6.1.10). The value-extraction elements can form a general DCSTA GIPNS 2-vector *quadric surface* entity $\mathbf{Q}$ that supports anisotropic dilations (§5.7.9). The value-extraction elements T_s (§5.2.3) can form particular cubic surfaces known as parabolic cyclides and particular quartic surfaces known as Dupin and Darboux cyclides that do not support anisotropic dilations. All of the DCSTA GIPNS 2-vector surfaces $\mathbf{\Omega}$ can be boosted (§5.7.3) into a velocity in spacetime, but only the quadric surface entities can correctly display length contraction or dilation effects.

6.1.16 DCSTA GIPNS 2-vector toroid

The DCSTA GIPNS 2-vector *toroid* is coded as:

```
def GIPNS_Toroid(R,r):
    # Torus centered at the origin circling the z-axis.
    # R is the major radius
    # r is the minor radius
    # R=0 degenerates into exactly -4*Sphere(0,r)
    # R=r=0 degenerates into exactly -4*eo
    # r=0 degenerates into non-standard circle radius R
    # Note, -Tt2 since signatures are negative
    return (
        Tt4 +
        -Tt2*2*(R**2 - r**2) +
        T1*(R**2 - r**2)**2 +
        (Txx + Tyy)*(-4)*R**2
    )
```

The Toroid is evaluated at $w=ct=0$ to obtain the same toroid as in $\mathcal{G}_{8,2}$ DCGA:

```
EV(c*0*e1+x*e2+y*e3+z*e4)|Toroid(R,r)
```

The cyclide surface that is formed from the toroid at other times $t \neq 0$ could be researched further. Most surfaces are evaluated at $t = 0$ to obtain a surface similar to those in DCGA.

6.1.17 DCSTA GIPNS 2-vector Dupin cyclide

The DCSTA GIPNS 2-vector *Dupin cyclide* is coded as:

```
def GIPNS_DupinCyclide(R,r1,r2):
    # DupinCyclide generalizes the torus.
    # Types of cyclide:
    #  Ring cyclide when (r1+r2)<2R
    #  Spindle cyclide when (r1+r2)>2R
    # Types of torus:
    #  Horn torus when (r1=r2)=R
    #  Ring torus when (r1=r2)<R
    #  Spindle torus when (r1=r2)>R
    #
    # R is major radius in the xy-plane.
    # r1 and r2 are minor radii.
    # r1 is the radius of sphere centered at x=+R.
    # r2 is the radius of sphere centered at x=-R.
    # When r1=r2, we get exactly a Toroid(R,r=r1=r2).
    # When r1+r2=2R, we get the union of two spheres
    # that touch in a tangent point, exactly.
    #
    # Note: -Tt2 since signatures are negative.
    a = R
    u = (r1+r2)*Pow(2,-1)
    c = (r1-r2)*Pow(2,-1)
    b = sqrt(a**2-c**2)
    return (
        Tt4 +
        -2*(b**2-u**2)*Tt2 +
        (b**2-u**2)**2*T1 +
        -4*(a**2*Txx - 2*a*c*u*Tx + c**2*u**2*T1) +
        -4*b**2*Tyy
    )
```

The `DupinCyclide` is evaluated at $t = 0$ to obtain the same Dupin cyclide as in $\mathcal{G}_{8,2}$ DCGA:

```
EV(c*0*e1+x*e2+y*e3+z*e4)|DupinCyclide(R,r1,r2)
```

6.1.18 DCSTA GIPNS 2-vector horned Dupin cyclide

The DCSTA GIPNS 2-vector *horned Dupin cyclide* is coded as:

```
def GIPNS_HornedDupinCyclide(R,r1,r2):
    # Compared to DupinCyclide, just exchange values of
    # u and c to get horned Dupin cyclide.
    # For r1=r2: symmetrical, with horn points on y-axis.
    # For (r1+r2)<2R: horned ring cyclide.
    # For (r1+r2)>2R: horned spindle cyclide.
    # For (r1+r2)=2R: union of two spheres exactly.
    a = R
    u = (r1+r2)*Pow(2,-1)
    c = (r1-r2)*Pow(2,-1)
    b = sqrt(a**2-u**2)
    return (
        Tt4 +
        -2*(b**2-c**2)*Tt2 +
        (b**2-c**2)**2*T1 +
        -4*(a**2*Txx - 2*a*c*u*Tx + c**2*u**2*T1) +
        -4*b**2*Tyy
    )
```

6.1.19 DCSTA GIPNS 2-vector ellipsoid

The DCSTA GIPNS 2-vector *ellipsoid* is coded as:

```
def GIPNS_Ellipsoid(px,py,pz,rx,ry,rz):
    # Axis-aligned ellipsoid.
    return (
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -Tx*2*px*Pow(rx**2,-1) +
        -Ty*2*py*Pow(ry**2,-1) +
        -Tz*2*pz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

An ellipsoid, or any other *quadric surface* entity (§5.5), that has an initial position (p_x, p_y, p_z) can be boosted using a translated-boost (§5.7.4). After the boost operation(s) on a quadric surface entity, the boosted entity can be evaluated at any time t, where the entity has a moving position and displays a length contraction effect.

6.1.20 DCSTA GIPNS 2-vector elliptic cylinder, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned elliptic cylinder* is coded as:

```
def GIPNS_ECylinderX(px,py,pz,rx,ry,rz):
    # x-axis aligned elliptic cylinder.
    return (
        T1*(py**2*Pow(ry**2,-1)+pz**2*Pow(rz**2,-1)-1) +
        Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

6.1.21 DCSTA GIPNS 2-vector elliptic cylinder, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned elliptic cylinder* is coded as:

```
def GIPNS_ECylinderY(px,py,pz,rx,ry,rz):
    # y-axis aligned elliptic cylinder.
    return (
        T1*(px**2*Pow(rx**2,-1)+pz**2*Pow(rz**2,-1)-1) +
        Txx*Pow(rx**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

6.1.22 DCSTA GIPNS 2-vector elliptic cylinder, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned elliptic cylinder* is coded as:

```
def GIPNS_ECylinderZ(px,py,pz,rx,ry,rz):
    # z-axis aligned elliptic cylinder.
    return (
        T1*(px**2*Pow(rx**2,-1)+py**2*Pow(ry**2,-1)-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1)
    )
```

6.1.23 DCSTA GIPNS 2-vector elliptic cone, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned elliptic cone* is coded as:

```
def GIPNS_ConeX(px,py,pz,rx,ry,rz):
    # x-axis aligned elliptic cone.
    return (
        -T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        -Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

6.1.24 DCSTA GIPNS 2-vector elliptic cone, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned elliptic cone* is coded as:

```
def GIPNS_ConeY(px,py,pz,rx,ry,rz):
    # y-axis aligned elliptic cone.
    return (
        T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        2*py*Ty*Pow(ry**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

6.1.25 DCSTA GIPNS 2-vector elliptic cone, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned elliptic cone* is coded as:

```
def GIPNS_ConeZ(px,py,pz,rx,ry,rz):
    # z-axis aligned elliptic cone.
    return (
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        2*pz*Tz*Pow(rz**2,-1)
    )
```

6.1.26 DCSTA GIPNS 2-vector elliptic paraboloid, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned elliptic paraboloid* is coded as:

```
def GIPNS_ParaboloidX(px,py,pz,rx,ry,rz):
    # x-axis aligned elliptic paraboloid.
    return (
        -2*pz*Tz*Pow(rz**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -Tx*Pow(rx,-1) +
        Tzz*Pow(rz**2,-1) +
        Tyy*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*px*Pow(rx,-1)
    )
```

6.1.27 DCSTA GIPNS 2-vector elliptic paraboloid, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned elliptic paraboloid* is coded as:

```
def GIPNS_ParaboloidY(px,py,pz,rx,ry,rz):
    # y-axis aligned elliptic paraboloid.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1) +
        -Ty*Pow(ry,-1) +
        Txx*Pow(rx**2,-1) +
        Tzz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        T1*py*Pow(ry,-1)
    )
```

6.1.28 DCSTA GIPNS 2-vector elliptic paraboloid, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned elliptic paraboloid* is coded as:

```
def GIPNS_ParaboloidZ(px,py,pz,rx,ry,rz):
    # z-axis aligned elliptic paraboloid.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -Tz*Pow(rz,-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

6.1.29 DCSTA GIPNS 2-vector hyperbolic paraboloid

The DCSTA GIPNS 2-vector *z-axis aligned hyperbolic paraboloid* is coded as:

```
def GIPNS_HParaboloidZ(px,py,pz,rx,ry,rz):
    # z-axis aligned hyperbolic paraboloid.
    # A saddle-like shape
    #  that "saddles" x-axis
    #  and "straddles" y-axis
    #  with "up" direction as z-axis.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        2*py*Ty*Pow(ry**2,-1) +
        -Tz*Pow(rz,-1) +
        Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

6.1.30 DCSTA GIPNS 2-vector hyperboloid of one sheet

The DCSTA GIPNS 2-vector *hyperboloid of one sheet* is coded as:

```
def GIPNS_Hyperboloid1(px,py,pz,rx,ry,rz):
    # z-axis aligned hyperboloid of one sheet.
    # An hourglass-like shape that
    #  is elliptic in the xy-plane
    #  and hyperbolic in xz and yz planes.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        2*pz*Tz*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

6.1.31 DCSTA GIPNS 2-vector hyperboloid of two sheets

The DCSTA GIPNS 2-vector *hyperboloid of two sheets* is coded as:

```
def GIPNS_Hyperboloid2(px,py,pz,rx,ry,rz):
    # z-axis aligned hyperboloid of two sheets.
    # A shape like two dishes that
    #  are elliptic in the xy-plane
    #  and hyperbolic in xz and yz planes.
    return (
        Tx*2*px*Pow(rx**2,-1) +
        Ty*2*py*Pow(ry**2,-1) +
        -Tz*2*pz*Pow(rz**2,-1) +
        -Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

6.1.32 DCSTA GIPNS 2-vector parabolic cylinder, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned parabolic cylinder* is coded as:

```
def GIPNS_PCylinderX(px,py,pz,rx,ry,rz):
    # Cylinder along x-axis with
    # constant parabola cross-section in yz-plane.
    return (
        -2*py*Ty*Pow(ry**2,-1) +
        -Tz*Pow(rz,-1) +
        Tyy*Pow(ry**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

6.1.33 DCSTA GIPNS 2-vector parabolic cylinder, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned parabolic cylinder* is coded as:

```
def GIPNS_PCylinderY(px,py,pz,rx,ry,rz):
    # Cylinder along y-axis with
    # constant parabola cross-section in xz-plane.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -Tz*Pow(rz,-1) +
        Txx*Pow(rx**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

6.1.34 DCSTA GIPNS 2-vector parabolic cylinder, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned parabolic cylinder* is coded as:

```
def GIPNS_PCylinderZ(px,py,pz,rx,ry,rz):
    # Cylinder along z-axis with
    # constant parabola cross-section in xy-plane.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -Ty*Pow(ry,-1) +
        Txx*Pow(rx**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py*Pow(ry,-1)
    )
```

6.1.35 DCSTA GIPNS 2-vector hyperbolic cylinder, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned hyperbolic cylinder* is coded as:

```
def GIPNS_HCylinderX(px,py,pz,rx,ry,rz):
    # Cylinder along x-axis with
    # constant hyperbola cross-section in yz-plane
    # opening up and down the y-axis.
    return (
        -Ty*2*py*Pow(ry**2,-1) +
         Tz*2*pz*Pow(rz**2,-1) +
         Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
         T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

6.1.36 DCSTA GIPNS 2-vector hyperbolic cylinder, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned hyperbolic cylinder* is coded as:

```
def GIPNS_HCylinderY(px,py,pz,rx,ry,rz):
    # Cylinder along y-axis with
    # constant hyperbola cross-section in xz-plane
    # opening up and down the z-axis.
    return (
        -Tz*2*pz*Pow(rz**2,-1) +
         Tx*2*px*Pow(rx**2,-1) +
         Tzz*Pow(rz**2,-1) +
        -Txx*Pow(rx**2,-1) +
         T1*pz**2*Pow(rz**2,-1) +
        -T1*px**2*Pow(rx**2,-1) +
        -T1
    )
```

6.1.37 DCSTA GIPNS 2-vector hyperbolic cylinder, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned hyperbolic cylinder* is coded as:

```
def GIPNS_HCylinderZ(px,py,pz,rx,ry,rz):
    # Cylinder along z-axis with
    # constant hyperbola cross-section in xy-plane
    # opening up and down the x-axis.
    return (
        -Tx*2*px*Pow(rx**2,-1) +
         Ty*2*py*Pow(ry**2,-1) +
         Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
         T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        -T1
    )
```

6.1.38 DCSTA GIPNS 2-vector parallel planes pair, perpendicular to x-axis

The DCSTA GIPNS 2-vector *parallel planes pair* $\perp x$-axis is coded as:

```
def GIPNS_PPlanesPairX(px1,px2):
    # Parallel planes pair, x=px1 and x=px2.
    return ( Txx - (px1+px2)*Tx + px1*px2*T1 )
```

6.1.39 DCSTA GIPNS 2-vector parallel planes pair, perpendicular to y-axis

The DCSTA GIPNS 2-vector *parallel planes pair* $\perp y$-axis is coded as:

```
def GIPNS_PPlanesPairY(py1,py2):
    # Parallel planes pair, y=py1 and y=py2.
    return ( Tyy - (py1+py2)*Ty + py1*py2*T1 )
```

6.1.40 DCSTA GIPNS 2-vector parallel planes pair, perpendicular to z-axis

The DCSTA GIPNS 2-vector *parallel planes pair* $\perp z$-axis is coded as:

```
def GIPNS_PPlanesPairZ(pz1,pz2):
    # Parallel planes pair, z=pz1 and z=pz2.
    return ( Tzz - (pz1+pz2)*Tz + pz1*pz2*T1 )
```

6.1.41 DCSTA GIPNS 2-vector non-parallel planes pair, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned non-parallel planes pair* is coded as:

```
def GIPNS_XPlanesPairX(py,pz,ry,rz):
    # The non-parallel planes pair aligned with x-axis
    # is a type of cylinder with constant cross section
    # that is a pair of lines in the yz-plane. The lines
    # intersect at (py,pz), and the slopes of the two
    # lines are +rz/ry and -rz/ry.
    return (
        -2*py*Ty*Pow(ry**2,-1) +
        2*pz*Tz*Pow(rz**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1)
    )
```

6.1.42 DCSTA GIPNS 2-vector non-parallel planes pair, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned non-parallel planes pair* is coded as:

```
def GIPNS_XPlanesPairY(px,pz,rx,rz):
    # The non-parallel planes pair aligned with y-axis
    # is a type of cylinder with constant cross section
    # that is a pair of lines in the xz-plane. The lines
    # intersect at (px,pz), and the slopes of the two
    # lines are +rz/rx and -rz/rx.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        2*pz*Tz*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        -T1*pz**2*Pow(rz**2,-1)
    )
```

6.1.43 DCSTA GIPNS 2-vector non-parallel planes pair, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned non-parallel planes pair* is coded as:

```
def GIPNS_XPlanesPairZ(px,py,rx,ry):
    # The non-parallel planes pair aligned with z-axis
    # is a type of cylinder with constant cross section
    # that is a pair of lines in the xy-plane. The lines
    # intersect at (px,py), and the slopes of the two
    # lines are +ry/rx and -ry/rx.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        2*py*Ty*Pow(ry**2,-1) +
        Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1)
    )
```

6.1.44 CSTA1 GIPNS 1-vector hyperplane

The CSTA1 GIPNS 1-vector *hyperplane* (§4.4.3) is coded as:

```
def GIPNS_HPlane1(p,n):
    # CSTA1 1-vector hyperplane
    # p is any point on the hyperplane
    # n is the normal vector
    # the magnitude of n is not significant
    return ( n + (p|n)*ei1 )
```

6.1.45 CSTA2 GIPNS 1-vector hyperplane

The CSTA2 GIPNS 1-vector *hyperplane* (§4.4.3) is coded as:

```
def GIPNS_HPlane2(p,n):
    # CSTA2 1-vector hyperplane
    # p is any point on the hyperplane
    # n is the normal vector
    # the magnitude of n is not significant
    p2 = (p1|e1)*e7 + (-p1|e2)*e8 + (-p1|e3)*e9 + (-p1|e4)*e10
    n2 = (n1|e1)*e7 + (-n1|e2)*e8 + (-n1|e3)*e9 + (-n1|e4)*e10
    return ( n2 + (p2|n2)*ei2 )
```

6.1.46 DCSTA GIPNS 2-vector hyperplane

The DCSTA GIPNS 2-vector *hyperplane* (§5.3.2) is coded as:

```
def GIPNS_HPlane(p,n):
    # DCSTA 2-vector hyperplane
    # p is any point on the hyperplane
    # n is the normal vector
    # the magnitude of n is not significant
    return ( GIPNS_HPlane1(p,n)^GIPNS_HPlane2(p,n) )
```

6.1.47 CSTA1 GIPNS 1-vector hyperhyperboloid of one sheet

The CSTA1 GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* (§4.4.5) is coded as:

```
def GIPNS_HPSphere1(p,r):
    # CSTA1 1-vector hyperpseudosphere (hyperhyperboloid)
    # p is the STA center point
    # r is the radius
    # imaginary radius r=I*|r| makes imaginary hyperpseudosphere
    return ( EV1(p) + Pow(2,-1)*r**2*ei1 )
```

6.1.48 CSTA2 GIPNS 1-vector hyperhyperboloid of one sheet

The CSTA2 GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* (§4.4.5) is coded as:

```
def GIPNS_HPSphere2(p,r):
    # CSTA1 1-vector hyperpseudosphere (hyperhyperboloid)
    # p is the STA center point
    # r is the radius
    # imaginary radius r=I*|r| makes imaginary hyperpseudosphere
    return ( EV2(p) + Pow(2,-1)*r**2*ei2 )
```

6.1.49 DCSTA GIPNS 2-vector hyperhyperboloid of one sheet

The DCSTA GIPNS 2-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* (§5.3.2) is coded as:

```
def GIPNS_HPSphere(p,r):
    # DCSTA 2-vector hyperpseudosphere (hyperhyperboloid)
    # p is the STA center point
    # r is the radius
    # imaginary radius r=I*|r| makes imaginary hyperpseudosphere
    return ( GIPNS_HPSphere1(p,r)^GIPNS_HPSphere2(p,r) )
```

6.1.50 CSTA1 GIPNS 2-vector plane

The CSTA1 GIPNS 2-vector *plane* (§4.4.10) is coded as:

```
def GIPNS_Plane1(p,da,db):
    # p is any STA1 point on the plane
    # da is STA1 direction one of plane
    # db is STA1 direction two of plane
    # The STA1 plane bivector B is da^db
    p1 = p
    B1 = da^db
    N1 = Pow(sqrt(scalar(B1|(e1*B1.rev()*e1))),-1)*B1
    D1 = ((1*N1*1)|I41.inv())
    return ( D1 - ((p1|D1)^ei1) )
```

6.1.51 CSTA2 GIPNS 2-vector plane

The CSTA2 GIPNS 2-vector *plane* (§4.4.10) is coded as:

```
def GIPNS_Plane2(p,da,db):
    # p is any STA1 point on the plane
    # da is STA1 direction one of plane
    # db is STA1 direction two of plane
    p2 = (p|e1)*e7+(-p|e2)*e8+(-p|e3)*e9+(-p|e4)*e10
    da2 = (da|e1)*e7+(-da|e2)*e8+(-da|e3)*e9+(-da|e4)*e10
    db2 = (db|e1)*e7+(-db|e2)*e8+(-db|e3)*e9+(-db|e4)*e10
    B2 = da2^db2
    N2 = Pow(sqrt(scalar(B2|(e7*B2.rev()*e7))),-1)*B2
    D2 = (N2|I42.inv())
    return ( D2 - ((p2|D2)^ei2) )
```

6.1.52 DCSTA GIPNS 4-vector standard plane

The DCSTA GIPNS 4-vector *standard plane* (§5.3.6) is coded as:

```
def GIPNS_Plane(p,da,db):
    # p is any STA1 point on the plane
    # da is STA1 direction one of plane
    # db is STA1 direction two of plane
    return ( GIPNS_Plane1(p,da,db)^GIPNS_Plane2(p,da,db) )
```

The standard plane can be intersected with all other DCSTA GIPNS surface entities.

6.1.53 CSTA1 GIPNS 3-vector line

The CSTA1 GIPNS 3-vector *line* (§4.4.11) is coded as:

```
def GIPNS_Line1(p,d):
    # p is any STA1 point on the line
    # d is the STA1 direction of line
    dc = e1*d*e1
    d1 = Pow(sqrt(scalar(d|dc)),-1)*d
    D1 = d1|(-I41)
    return ( D1 + ((p|D1)^ei1) )
```

6.1.54 CSTA2 GIPNS 3-vector line

The CSTA2 GIPNS 3-vector *line* (§4.4.11) is coded as:

```
def GIPNS_Line2(p,d):
    # p is any STA1 point on the line
    # d is the STA1 direction of line
    p2 = (p|e1)*e7+(-p|e2)*e8+(-p|e3)*e9+(-p|e4)*e10
    dc = e1*d*e1
    d1 = Pow(sqrt(scalar(d|dc)),-1)*d
    d2 = (d1|e1)*e7+(-d1|e2)*e8+(-d1|e3)*e9+(-d1|e4)*e10
    D2 = d2|(-I42)
    return ( D2 + ((p2|D2)^ei2) )
```

6.1.55 DCSTA GIPNS 6-vector standard line

The DCSTA GIPNS 6-vector *standard line* (§5.3.7) is coded as:

```
def GIPNS_Line(p,d):
    # p is any STA1 point on the line
    # d is the STA1 direction of line
    return ( GIPNS_Line1(p,d)^GIPNS_Line2(p,d) )
```

The standard line can be intersected with all other DCSTA GIPNS surface entities.

6.1.56 CSTA1 plane-line intersection

The CSTA1 plane-line intersection (§4.5.5) is coded as:

```
def GIPNS_PlaneLineIntersection1(p,l):
    # Intersect GIPNS_Plane1 p and GIPNS_Line1 l
    plwedge = (p^l)
    if plwedge != 0: return ei1
    plmeet = (((p|I41.inv())^(l|I41.inv()))|I41)
    if plmeet == 0: return l
    return ((e1*plmeet*e1)|p)^l
```

6.1.57 CSTA2 plane-line intersection

The CSTA2 plane-line intersection (§4.5.5) is coded as:

```
def GIPNS_PlaneLineIntersection2(p,l):
    # Intersect GIPNS_Plane2 p and GIPNS_Line2 l
    plwedge = (p^l)
    if plwedge != 0: return ei2
    plmeet = (((p|I42.inv())^(l|I42.inv()))|I42)
    if plmeet == 0: return l
    return ((e7*plmeet*e7)|p)^l
```

6.1.58 CSTA1 GOPNS 2-vector point pair decomposition

The decomposition of a CSTA1 GOPNS 2-vector *point pair* (§4.5.3) is coded as:

```
def GOPNS_PointPairDecomp1(pp,pm):
    # pp is a CSTA1 GOPNS 2-vector point pair
    # pm is -1 or 1 to select a point of the pair
    # returns a CSTA1 null 1-vector point entity
    return ( (pp + pm*sqrt(scalar(pp|pp)))*(-ei1|pp).inv() )
```

6.1.59 CSTA2 GOPNS 2-vector point pair decomposition

The decomposition of a CSTA2 GOPNS 2-vector *point pair* (§4.5.3) is coded as:

```
def GOPNS_PointPairDecomp2(pp,pm):
    # pp is a CSTA2 GOPNS 2-vector point pair
    # pm is -1 or 1 to select a point of the pair
    # returns a CSTA2 null 1-vector point entity
    return ( (pp + pm*sqrt(scalar(pp|pp)))*(-ei2|pp).inv() )
```

6.1.60 CSTA1 GOPNS 2-vector flat point projection

The projection of the point of a CSTA1 GOPNS 2-vector *flat point* (§4.5.5) is coded as:

```
def GOPNS_FlatPointProj1(fp):
    # fp is a CSTA1 GOPNS 2-vector flat point
    # returns the STA1 vector projection of the point
    E = eo1^ei1
    return ( -(fp|eo1)*Pow(scalar(E|fp),-1) - eo1 )
```

6.1.61 CSTA2 GOPNS 2-vector flat point projection

The projection of the point of a CSTA2 GOPNS 2-vector *flat point* (§4.5.5) is coded as:

```
def GOPNS_FlatPointProj2(fp):
    # fp is a CSTA2 GOPNS 2-vector flat point
    # returns the STA2 vector projection of the point
    E = eo2^ei2
    return ( -(fp|eo2)*Pow(scalar(E|fp),-1) - eo2 )
```

6.1.62 SA1, STA1, and CSTA1 2-versor rotor

The CSTA1 2-versor spatial *rotor* (§2.6) is coded as:

```
def Rotor1(axis,angle):
    # Spatial rotor in SA1, STA1, and CSTA1, where
    #  axis is SA1 vector axis of rotation and
    #  angle is scalar angle of rotation in degrees.
    ax1 = Pow(norm(axis),-1)*axis
    ang = pi*Pow(180,-1)*angle
    return (
        cos(ang*Pow(2,-1)) +
        sin(ang*Pow(2,-1))*(ax1|(-I31))
    )
```

6.1.63 SA2, STA2, and CSTA2 2-versor rotor

The CSTA2 2-versor spatial *rotor* (§2.6) is coded as:

```
def Rotor2(axis,angle):
    # Spatial rotor in SA2, STA2, and CSTA2, where
    #  axis is SA1 vector axis of rotation and
    #  angle is scalar angle of rotation in degrees.
    ax1 = Pow(norm(axis),-1)*axis
    ax2 = (-ax1|e2)*e8 + (-ax1|e3)*e9 + (-ax1|e4)*e10
    ang = pi*Pow(180,-1)*angle
    return (
        cos(ang*Pow(2,-1)) +
        sin(ang*Pow(2,-1))*(ax2|(-I32))
    )
```

6.1.64 DCSTA 4-versor rotor

The DCSTA 4-versor spatial *rotor* (§5.7.6) is coded as:

```
def Rotor(axis,angle):
    # Spatial rotor in DCSTA, where
    #  axis is SA1 vector axis of rotation and
    #  angle is scalar angle of rotation in degrees
    return ( Rotor1(axis,angle)^Rotor2(axis,angle) )
```

6.1.65 CSTA1 2-versor line rotor

The CSTA1 2-versor *line rotor* (§4.6.5) for the rotation around a line is coded as:

```
def LRotor1(p,d,a):
    # Rotor around a line l by angle a in degrees
    # line l is given by STA1 point p and direction d
    l = GIPNS_Line1(p,d)
    t = Rational(1,2)*pi*Pow(180,-1)*a
    return ( cos(t) + sin(t)*(-e1|l) )
```

6.1.66 CSTA2 2-versor line rotor

The CSTA2 2-versor *line rotor* (§4.6.5) for the rotation around a line is coded as:

```
def LRotor2(p,d,a):
    l = GIPNS_Line2(p,d)
    t = Rational(1,2)*pi*Pow(180,-1)*a
    return ( cos(t) + sin(t)*(-e7|l) )
```

6.1.67 DCSTA 4-versor line rotor

The DCSTA 4-versor *line rotor* (§5.7.6) for the rotation around a line is coded as:

```
def LRotor(p,d,a):
    return LRotor1(p,d,a)^LRotor2(p,d,a)
```

6.1.68 STA1 and CSTA1 2-versor hyperbolic rotor (boost operator)

The CSTA1 2-versor spacetime *hyperbolic rotor (boost operator)* (§3.2.3) is coded as:

```
def HRotor1(b,d):
    # STA1 and CSTA1 boost operator, where
    #  0<=b<=1 is scalar natural speed of boost and
    #  d is SA1 direction vector of boost velocity
    v1 = Pow(sqrt(scalar(-d|d)),-1)*d
    r = atanh(b)
    return ( cosh(Pow(2,-1)*r) + sinh(Pow(2,-1)*r)*(v1^e1) )
```

The `b` is the natural speed $\beta_{\mathbf{v}}$. The `d` is the SA1 spatial velocity direction $\hat{\mathbf{v}}$ that is normalized as `v1`. The spatial velocity of the boost is $\mathbf{v} = \beta_{\mathbf{v}} c\hat{\mathbf{v}} = \|\mathbf{v}\|\hat{\mathbf{v}}$ relative to an observer $\boldsymbol{o} = ct\mathbf{e}_1$. The `r` is the rapidity $\varphi_{\mathbf{v}} = \operatorname{atanh}(\beta_{\mathbf{v}})$.

6.1.69 STA2 and CSTA2 2-versor hyperbolic rotor (boost operator)

The CSTA2 2-versor spacetime *hyperbolic rotor (boost operator)* (§4.6.8) is coded as:

```
def HRotor2(b,d):
    # STA2 and CSTA2 boost operator, where
    #  0<=b<=1 is scalar natural speed of boost and
    #  d is SA1 direction vector of boost velocity
    v1 = Pow(sqrt(scalar(-d|d)),-1)*d
    v2 = (-v1|e2)*e8 + (-v1|e3)*e9 + (-v1|e4)*e10
    r = atanh(b)
    return ( cosh(Pow(2,-1)*r) + sinh(Pow(2,-1)*r)*(v2^e7) )
```

6.1.70 DCSTA 4-versor hyperbolic rotor (boost operator)

The DCSTA 4-versor spacetime *hyperbolic rotor (boost operator)* (§5.7.3) is coded as:

```
def HRotor(b,d):
    # DCSTA boost operator, where
    #  0<=b<=1 is scalar natural speed of boost and
    #  d is SA1 direction vector of boost velocity
    return ( HRotor1(b,d)^HRotor2(b,d) )
```

For an *anisotropic dilation* (§5.7.9) of a *quadric surface* `Q` by factor d in direction `d`, then speed `b` should be set to $\beta_{\mathbf{v}} = \sqrt{1-d^2}$, which may be an imaginary number.

The anisotropic dilation of Q by a dilation factor d in an SA1 direction v=vx*e2+vy*e3+vz*e4 is coded as:

```
((
    HRotor(sqrt(1-d**2),v)*
    Q*
    HRotor(sqrt(1-d**2),v).rev()
)|IDS)*IDS.inv()
```

The projection using IDS is the space projection (§5.7.2) into the $\mathcal{G}_{2,8}$ anti-DCGA, which discards imaginary components that are by-products of the directed scaling operation. A good example to try can use Q=Ellipsoid(px,py,pz,rx,ry,rz).

6.1.71 CSTA1 2-versor translator

The CSTA1 2-versor spacetime *translator* (§4.6.4) is coded as:

```
def Translator1(d):
    # CSTA1 spacetime translator, where
    #  d is an STA1 spacetime displacement vector.
    d1 = d
    return ( 1 - Pow(2,-1)*(d1^ei1) )
```

6.1.72 CSTA2 2-versor translator

The CSTA2 2-versor spacetime *translator* (§4.6.4) is coded as:

```
def Translator2(d):
    # CSTA2 spacetime translator, where
    #  d is an STA1 spacetime displacement vector.
    d2 = (d|e1)*e7 + (-d|e2)*e8 + (-d|e3)*e9 + (-d|e4)*e10
    return ( 1 - Pow(2,-1)*(d2^ei2) )
```

6.1.73 DCSTA 4-versor translator

The DCSTA 4-versor spacetime *translator* (§5.7.7) is coded as:

```
def Translator(d):
    # DCSTA spacetime translator, where
    #  d is an STA1 spacetime displacement vector.
    return ( Translator1(d)^Translator2(d) )
```

6.1.74 CSTA1 2-versor isotropic dilator

The CSTA1 2-versor spacetime *isotropic dilator* (§4.6.6) is coded as:

```
def Dilator1(d):
    # CSTA1 isotropic dilator, where
    #  d is the scalar dilation factor.
    # Note: dilation factor d=0 is not generally valid.
    return ( Pow(2,-1)*(1+d) + Pow(2,-1)*(1-d)*(ei1^eo1) )
```

6.1.75 CSTA2 2-versor isotropic dilator

The CSTA2 2-versor spacetime *isotropic dilator* (§4.6.6) is coded as:

```
def Dilator2(d):
    # CSTA2 isotropic dilator, where
    #  d is the scalar dilation factor.
    # Note: dilation factor d=0 is not generally valid.
    return ( Pow(2,-1)*(1+d) + Pow(2,-1)*(1-d)*(ei2^eo2) )
```

6.1.76 DCSTA 4-versor isotropic dilator

The DCSTA 4-versor spacetime *isotropic dilator* (§5.7.8) is coded as:

```
def Dilator(d):
    # DCSTA isotropic dilator, where
    #  d is the scalar dilation factor.
    # Note: dilation factor d=0 is not generally valid.
    return ( Dilator1(d)^Dilator2(d) )
```

The *anisotropic dilator* (§5.7.9) on quadric surface entities is implemented using the *hyperbolic rotor* (§6.1.70).

6.2 Example computations

6.2.1 Reframe to new observer in STA

The observer position is $\boldsymbol{o}t = ct\mathbf{e}_1$, and an observable $\boldsymbol{v}t = (\boldsymbol{o} + \mathbf{v})t$ moves relative to $\boldsymbol{o}$, where

$$\mathbf{v} \;=\; \beta_{\mathbf{v}} c\mathbf{e}_2 = \frac{1}{2}c\mathbf{e}_2. \tag{6.7}$$

We want to passively transform observable $\boldsymbol{o}$, which is the conventional coordinate time t observer, relative to the rest frame of the observable $\boldsymbol{v}$ with proper time τ. However, we prefer not to passively transform t into τ, and prefer to get a velocity subtraction in the frame of $\boldsymbol{o}$. Solution: use a passive boost operation (§3.2.3) on $\boldsymbol{o}$, followed by a spacetime contraction.

```
o_rel_v = (
    HRotor1( Rational(1,2), e2 ).rev()*
    ( c*t*e1 )*
    HRotor1( Rational(1,2), e2 )
)
normalized = c*t*Pow( scalar(o_rel_v|e1), -1 )*o_rel_v
normalized
```

$$\text{normalized} \;=\; ct\mathbf{e}_1 - \frac{ct}{2}\mathbf{e}_2. \tag{6.8}$$

Observable $\boldsymbol{o}$ is seen to be moving with velocity $\mathbf{v}=-\frac{1}{2}c\mathbf{e}_2$ relative to observable $\boldsymbol{v}$. The spacetime contraction is a normalization of the conventional observer component $\boldsymbol{o}$ into the normalized spacetime velocity value $c\boldsymbol{\gamma}_0$ or spacetime position value $ct\boldsymbol{\gamma}_0$.

6.2.2 Collinear velocity addition in STA

A particle moves with velocity $\mathbf{u}=\frac{3}{4}c\mathbf{e}_2$ relative to another particle moving with velocity $\mathbf{v}=\frac{1}{2}c\mathbf{e}_2$ relative to an observer $\boldsymbol{o}=c\mathbf{e}_1$. The two velocities $\mathbf{u}$ and $\mathbf{v}$ are collinear, and if we simply add the velocities, we may conclude that $\mathbf{u}$ relative to $\boldsymbol{o}$ is a velocity $\mathbf{v}+\mathbf{u}=\frac{5}{4}c\mathbf{e}_2$. However, this speed is greater than light speed c, which according to the physical theory of special relativity is an impossible speed. Velocities cannot be simply added, and we must use a reframe operation to reframe $\mathbf{u}$ relative to $\boldsymbol{o}$. Relative to $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$, the particle moving with velocity $\mathbf{u}$ is written $\boldsymbol{u}t=\boldsymbol{o}t+\mathbf{u}t=ct\mathbf{e}_1+\mathbf{u}t$, where *this* $\boldsymbol{o}$ *is* $\boldsymbol{v}$ as the observer and *this time* t is its proper time. We want this $\boldsymbol{u}$ reframed relative to observer $\boldsymbol{o}$ of $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$. The solution is to apply to $\boldsymbol{u}$ the operation for the reverse of the reframe relative to $\boldsymbol{v}$ that goes back to relative to its $\boldsymbol{o}$, and this reframe is also seen as *boosting* the particle $\boldsymbol{u}=\boldsymbol{o}+\mathbf{u}$ by the velocity $\mathbf{v}$ relative to $\boldsymbol{o}$.

```
u_rel_o = (
    HRotor1( Rational(1,2), e2 )*
    ( c*t*e1 + Rational(3,4)*c*t*e2 )*
    HRotor1( Rational(1,2), e2 ).rev()
)
normalized = c*t*Pow( scalar(u_rel_o|e1), -1 )*u_rel_o
normalized
```

$$\text{normalized} \;=\; ct\mathbf{e}_1+\frac{10}{11}ct\mathbf{e}_2. \tag{6.9}$$

The result is relativistic velocity addition, where the boost of velocities does not exceed the speed of light c.

6.2.3 Velocity addition in STA

The velocities $\mathbf{u}$ and $\mathbf{v}$ need not be collinear, and the same operation of the previous section (§6.2.2) for collinear velocities can be applied to reframe any velocity $\mathbf{u}$ relative to $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ into $\mathbf{u}$ relative to $\boldsymbol{o}$. The result is the so-called *velocity-addition formula*, which could also be called the *velocity boost formula*,

$$\begin{aligned}
\boldsymbol{u}_{\mathrm{rel}\,\boldsymbol{v}} \rightarrow \boldsymbol{u}_{\mathrm{rel}\,\boldsymbol{o}} \;&=\; \boldsymbol{o}+\mathbf{u}_{\mathrm{rel}\,\boldsymbol{o}} && (6.10)\\
&=\; \boldsymbol{o}+\mathbf{u}\oplus\mathbf{v} && (6.11)\\
& c\boldsymbol{\gamma}_0+\frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\,\mathbf{u}^{\perp\hat{\mathbf{v}}}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} && (6.12)
\end{aligned}$$

where $\boldsymbol{o}=c\boldsymbol{\gamma}_0$, and the $\mathcal{G}_{0,3}$ SA metric gives

$$\begin{aligned}
\hat{\mathbf{v}} \;&=\; \frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{\mathbf{v}}{\sqrt{-\mathbf{v}^2}} && (6.13)\\
\hat{\mathbf{v}}^2 \;&=\; -1 && (6.14)\\
\mathbf{u}\cdot\mathbf{v} \;&=\; -\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta_{\mathbf{uv}}). && (6.15)
\end{aligned}$$

The notation $\mathbf{u}\oplus\mathbf{v}$ can be read "$\mathbf{u}$ boosted by $\mathbf{v}$" since this is the actual operation, but this may be backwards compared to some other literature. In general, $\mathbf{u}\oplus\mathbf{v}\neq\mathbf{v}\oplus\mathbf{u}$. Some other identities are

$$\begin{aligned}\mathbf{u}=\mathbf{u}^{\|\hat{\mathbf{v}}}+\mathbf{u}^{\perp\hat{\mathbf{v}}} &= (\mathbf{u}\cdot\hat{\mathbf{v}}+\mathbf{u}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}}^{-1}=(-\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}+(\hat{\mathbf{v}}\wedge\mathbf{u})\cdot\hat{\mathbf{v}} && (6.16)\\ \beta_{\mathbf{v}} &= \frac{\|\mathbf{v}\|}{c} && (6.17)\\ \gamma_{\mathbf{v}} &= \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}. && (6.18)\end{aligned}$$

When the boost velocity approaches light speed $\|\mathbf{v}\|\to c$, we get

$$\mathbf{u}\oplus\mathbf{v} = \frac{\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})\hat{\mathbf{v}}+c\hat{\mathbf{v}}}{1+\frac{\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})}{c}}=\frac{c(\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})+c)\hat{\mathbf{v}}}{c+\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})}=c\hat{\mathbf{v}}=\mathbf{v}. \qquad (6.19)$$

For collinear $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u}\oplus\mathbf{v}=\alpha\mathbf{v}\oplus\mathbf{v}=\mathbf{v}\oplus\mathbf{u} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{\|\mathbf{u}\|\|\mathbf{v}\|}{c^2}} \qquad (6.20)$$

where as the boost velocity approaches light speed $\|\mathbf{v}\|\to c$,

$$\mathbf{u}\oplus\mathbf{v} \to \frac{c(\|\mathbf{u}\|+c)\hat{\mathbf{u}}}{c+\|\mathbf{u}\|}=c\hat{\mathbf{u}}=\mathbf{v}. \qquad (6.21)$$

For perpendicular $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u}\oplus\mathbf{v} = \sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}+\mathbf{v}=\frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}+\mathbf{v} \qquad (6.22)$$

where as the boost velocity approaches light speed $\|\mathbf{v}\|\to c$, $\mathbf{u}/\gamma_{\mathbf{v}}\to 0$ and $\mathbf{u}\oplus\mathbf{v}\to\mathbf{v}$. The velocity-addition formula is derived and discussed more in [9] and (§3.2.3).

6.2.4 Boost of an ellipsoid entity for contraction effect

Any DCSTA GIPNS 2-vector quadric surface entity can be boosted into a velocity in spacetime. Boosting sets the quadric surface into motion at constant velocity and gives the surface a length contraction effect. As an example of the contraction effect, we can boost an ellipsoid to a natural speed $\beta_{\mathbf{v}}=\sqrt{1-d^2}$ for the dilation factor d. A good example is to choose $d=1/2$ to squeeze the ellipsoid into one-half its length in the direction of the velocity.

```
moving_ellipsoid = (
    HRotor( sqrt(1-Rational(1,2)**2), e2 )*
    GIPNS_Ellipsoid(0,0,0,10,10,10)*
    HRotor( sqrt(1-Rational(1,2)**2), e2 ).rev()
)
print( N(V|moving_ellipsoid) )
```

The `moving_ellipsoid` is evaluated at a symbolic point `V`. The full symbolic output can be long, therefore numeric output has been generated using `N()`. The result is printed in plain text using `print()`. Output of this form can be graphed using *Mayavi*. For graphing, it works well to use natural units, where $c=1$, so that the graph can be near the origin. Mayavi seems to work best if graphing can be limited to a small cube around the origin that is about ±20 units on each axis. If Mayavi is installed and working, a small `mayavi.py` python file can be created to graph this output (copied into `surface`) as:

```
from __future__ import division
from numpy import *
from mayavi import mlab

mlab.figure(bgcolor=(1,1,1))
x, y, z = mgrid[-20:20:100j, -20:20:100j, -20:20:100j]

# axes
cylx = y**2 + z**2 - 1/10
cyly = x**2 + z**2 - 1/10
cylz = y**2 + x**2 - 1/10
mlab.contour3d(x,y,z,cylx,contours=[0],opacity=0.25,color=(1,0,0))
mlab.contour3d(x,y,z,cyly,contours=[0],opacity=0.25,color=(0,1,0))
mlab.contour3d(x,y,z,cylz,contours=[0],opacity=0.25,color=(0,0,1))

# function for rendering a dot somewhere
def dotat(px,py,pz):
    blackdot = (x-px)**2 + (y-py)**2 + (z-pz)**2 - 1/sqrt(5)
    mlab.contour3d(
        x, y, z, blackdot, contours=[0],
        opacity=0.5, color=(0,0,0)
    )
    return

# plot some dots
dotat(5,0,0)
dotat(0,10,0)
dotat(0,0,10)

# Set the light speed (units per second)
#  Use a small unit or else boosted moving surfaces move
#  out of graphing range after only a few time units.
c = 1
# Set the time.
#  Boosted surfaces move natural-speed units per time unit.
#  At t=20, a surface at speed c=1 moves out of graphing range.
t = 0
# The numerical printed output, copied into here:
surface = (
    0.03*c**2*t**2 - 0.0692820323027551*c*t*x +
    0.04*x**2 + 0.01*y**2 + 0.01*z**2 - 0.999999999999999
)
# Mayavi rendering function
mlab.contour3d(
    x, y, z, surface, contours=[0], opacity=0.5,
    color=(0.0, 1.0, 1.0)
)
mlab.show()
```

The `mayavi.py` file is saved and then run from a command line as:

```
ipython mayavi.py
```

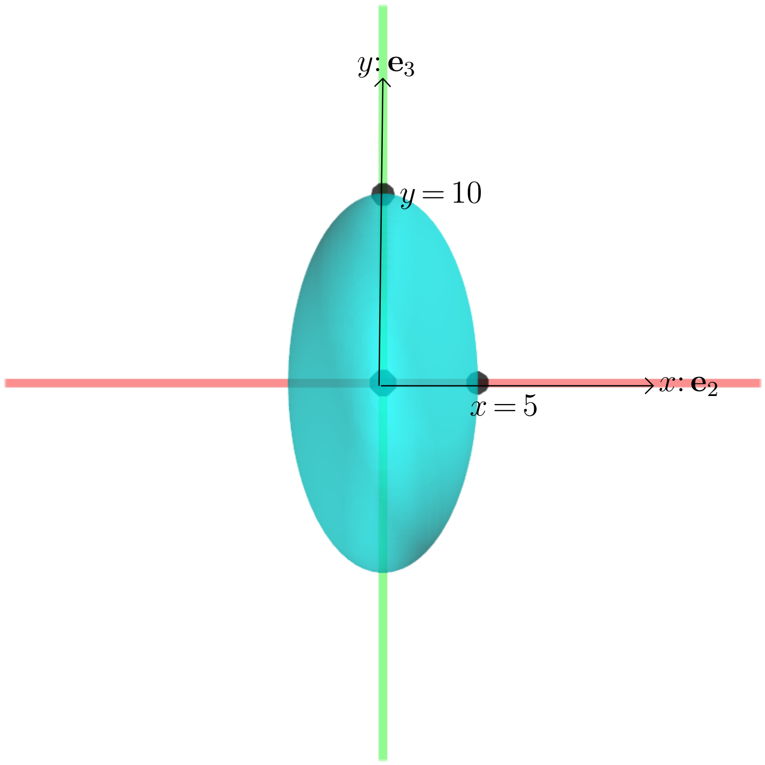

Figure 6.1. Ellipsoid (sphere $r=10$) boosted to $\beta_{\mathbf{v}}=\sqrt{1-\left(\frac{1}{2}\right)^2}$ in x-direction

Figure 6.1 shows a boosted ellipsoid at time $t=0$. The ellipsoid was initially at the origin and spherical with radius $r=r_x=r_y=r_z=10$. The spherical ellipsoid was boosted into a natural speed $\beta_{\mathbf{v}}=\sqrt{1-\left(\frac{1}{2}\right)^2}$ for a dilation factor $d=\frac{1}{2}$ in the x-direction $\hat{\mathbf{v}}=\boldsymbol{\gamma}_1=\mathbf{e}_2$. The boosted sphere is squeezed by the boost into an ellipsoid that is length-contracted to half-size in the x-direction with $r_x=5$, while the y and z directions hold their sizes with $r_y=r_z=10$. As the time t is increased, the boosted sphere moves toward the right along the x-axis. For natural units $c=1$, the boosted sphere moves $\beta_{\mathbf{v}}=\sqrt{3}/2\approx 0.866$ units per time unit.

6.2.5 Boost of an ellipsoid entity for dilation

As an example of dilating a quadric surface by dilation factor $d=2$ using the boost operation, we can take a spherical ellipsoid with radius $r=5$ centered at $5(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$ and boost it into an imaginary natural speed $\beta_{\mathbf{v}}=\sqrt{1-d^2}$ in the unit direction $\frac{1}{\sqrt{3}}(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$. Following the boost for dilation, the result is projected onto the spatial subalgebra using the $\mathcal{G}_{2,8}$ anti-DCGA pseudoscalar $\mathbf{I}_{\mathcal{DS}}$. The projection discards imaginary time components and resets the entity at time $t=0$. The dilation is coded as:

```
dilated_ellipsoid = ((
    HRotor( sqrt(1-2**2), e2+e3+e4 )*
    GIPNS_Ellipsoid(5,5,5,5,5,5)*
    HRotor( sqrt(1-2**2), e2+e3+e4 ).rev()
)|IDS)*IDS.inv()
print( N(V|dilated_ellipsoid) )
```

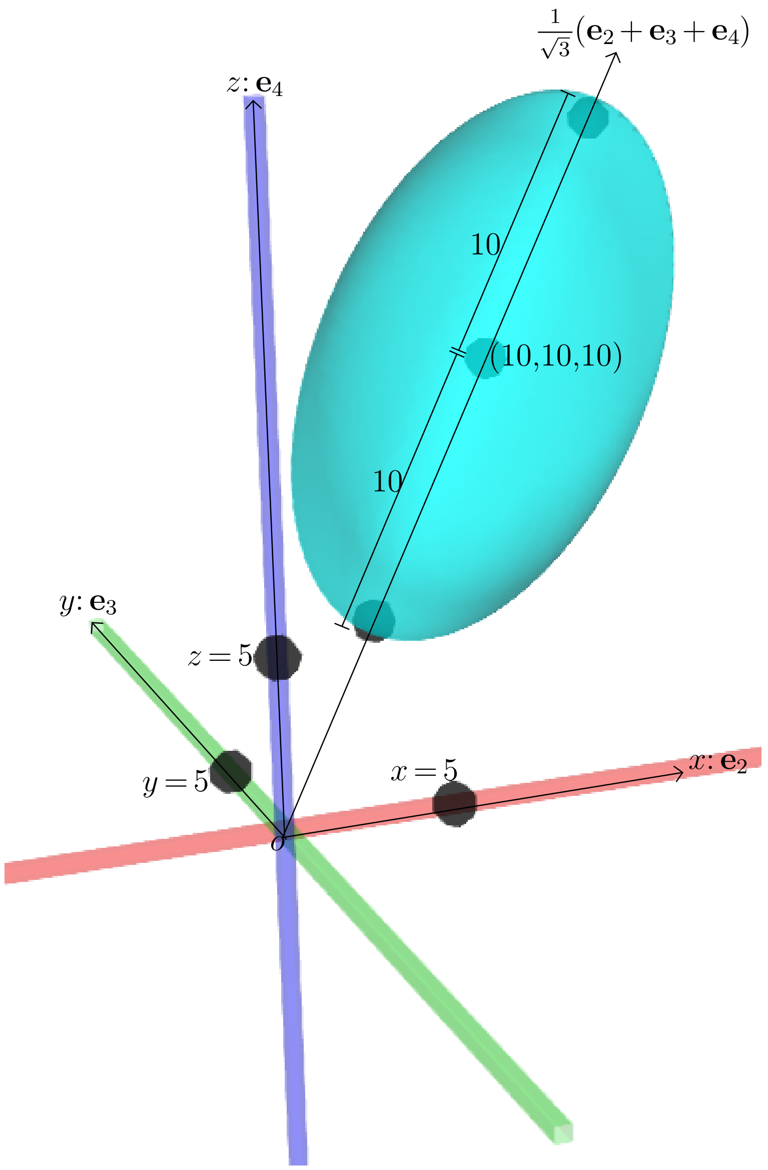

Figure 6.2. Ellipsoid (sphere $r=5$) dilated by factor $d=2$ in direction $\frac{1}{\sqrt{3}}(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$

Figure 6.2 shows the graph of the dilated ellipsoid. In the unit direction $\frac{1}{\sqrt{3}}(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$ of dilation by factor $d=2$, the spherical ellipsoid is dilated from a diameter of 10 into a diameter of 20. The spherical diameter remains 10 orthogonal to the direction of dilation. The center point $(5,5,5)$ of the original spherical ellipsoid is dilated into $(10,10,10)$ as the new center point of the dilated ellipsoid.

7 Conclusion

The $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) has been presented in this paper as a straightforward extension of the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) [7][5][6][8]. DCSTA is a large, complicated algebra and this paper may contain some mistakes and has probably overlooked some things that should have been discussed. Nevertheless, this author feels that this paper substantially conveys the basic ideas and concepts of DCSTA. Certainly, much further research can be done into DCSTA and its applications.

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