Pi Formulas , Part 2

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abstract

In this note we give some formulas related to the constant Pi

π - FÓRMULAS

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Resumen. Se muestran algunas fórmulas que involucran la constante π.

1. INTRODUCCIÓN.

Se muestra una colección de fórmulas que involucran la constante $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

Algunas fórmulas clásicas son:

$$\left(\frac{\pi}{8}\right)^2 = \sum_{n=1}^{\infty}\left(\frac{n}{4n^2-1}\right)^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{15}\right)^2 + \left(\frac{3}{35}\right)^2 + \dots$$

$$\frac{\pi}{8} = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{15}\right)^2 + \left(\frac{3}{35}\right)^2 + \dots}$$

$$\left(\frac{8}{\pi}\right)^2 = 9 - \sum_{n=1}^{\infty}\left(\frac{n+1}{(2n+1)(2n+3)}\right)^2 \frac{1}{s_n s_{n+1}}$$

$$s_n = \sum_{k=1}^{n}\left(\frac{k}{4k^2-1}\right)^2$$

2. FÓRMULAS.

2.1. Para $0 < a < b < 1$, se tiene:

$$\frac{\pi^2(b-a)}{6} = \sum_{n=1}^{\infty}\frac{b^{n+1}-a^{n+1}+(1-a)^n-(1-b)^n}{n^2(n+1)} + $$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\left(b^{m+k+1}-a^{m+k+1}\right)}{m+k+1}$$

2.2. Para $0 < x < 1$, se tiene:

$$1-\frac{\pi^2}{6}(1-x)=\sum_{n=1}^{\infty}\frac{x^{n+1}-(1-x)^n}{n^2(n+1)}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^{m+k+1}}{m+k+1}$$

2.3. Para $0 < y < 1$, se tiene:

$$\frac{\pi^2}{12}\left(4-2y+y^2\right)+y-3=\sum_{n=1}^{\infty}\frac{y^{n+2}}{n^2(n+1)(n+2)}+\sum_{n=1}^{\infty}\frac{(1-y)^{n+1}}{n^2(n+1)^2}+$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{y^{m+k+2}}{(m+k+1)(m+k+2)}$$

2.4. Para $m=0,1,2,3,...$, se tiene:

$$\pi^{m+1}=\left(2\sqrt{3}\right)^{m+1}\sum_{n=0}^{\infty}\frac{a(m+1,n)}{3^n}$$

$$a(m+1,n)=(m+1)\sum_{k=0}^{n}\frac{(-1)^{n-k}a(m,k)}{2n+m+1}\quad,\quad a(1,n)=\frac{(-1)^n}{2n+1}$$

2.5. Para $0 < x < 1$, se tiene:

$$\frac{\pi^2}{16}\sqrt{2\left(1+x^2\right)}=\sqrt{2\left(1+x^2\right)}\left(\tan^{-1}(x)\right)^2+$$
$$+(1-x)\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n-m}(2m)!\,u^m v^{n-m}}{4^m(m!)^2(2m+1)(2n-2m+1)}$$

$$u=\frac{(1+x)^2}{2\left(1+x^2\right)}\quad,\quad v=\left(\frac{1-x}{1+x}\right)^2$$

2.6. Para $0 < y < \sqrt{2}-1$, se tiene:

$$\frac{\pi^2}{64}=\left(\tan^{-1}(y)\right)^2+$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n}{(2m+1)(2n-2m+1)}\left(\frac{\sqrt{2}-1+y}{1-\left(\sqrt{2}-1\right)y}\right)^{2m+1}\left(\frac{\sqrt{2}-1-y}{1+\left(\sqrt{2}-1\right)y}\right)^{2n-2m+1}$$

2.7. Para $0<x<\frac{\sqrt{2}-1}{\sqrt{2}+1}$, se tiene:

$$\frac{\pi^2}{64}=\left(\tan^{-1}\left(\left(\sqrt{2}+1\right)x\right)\right)^2+$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n}{(2m+1)(2n-2m+1)}\left(\sqrt{2}\,\frac{1+x}{1-x}-1\right)^{2m+1}\left(\sqrt{2}\,\frac{1-x}{1+x}-1\right)^{2n-2m+1}$$

2.8.

$$\pi=8\sqrt{2-\sqrt{2+\sqrt{2}}}\,\exp\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n2^{8n}}\right)$$

$$\pi=\frac{8}{7}\sqrt{2+\sqrt{2+\sqrt{2}}}\,\exp\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)7^{2n}}{n2^{8n}}\right)$$

2.9. Para $m=1,2,3,...$, se tiene:

$$\frac{\pi^{2m}}{a_m+b_m\sqrt{2}+c_m\sqrt{2+\sqrt{2}}+d_m\sqrt{2}\sqrt{2+\sqrt{2}}}=2^{6m}\exp\left(2m\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n2^{8n}}\right)$$

$$\begin{pmatrix}a_{m+1}\\b_{m+1}\\c_{m+1}\\d_{m+1}\end{pmatrix}=\begin{pmatrix}2&0&-2&-2\\0&2&-1&-2\\-1&0&2&0\\0&-1&0&2\end{pmatrix}\begin{pmatrix}a_m\\b_m\\c_m\\d_m\end{pmatrix}\quad,\quad\begin{pmatrix}a_0\\b_0\\c_0\\d_0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

2.10. Para $m = 1,2,3,...$, se tiene:

$$\frac{\pi^{2m}}{a_m + b_m\sqrt{2} + c_m\sqrt{2+\sqrt{2}} + d_m\sqrt{2}\sqrt{2+\sqrt{2}}} = \frac{2^{6m}}{7^{2m}} exp\left(2m \sum_{n=1}^{\infty} \frac{\zeta(2n) 7^{2n}}{n 2^{8n}} \right)$$

$$\begin{pmatrix} a_{m+1} \\ b_{m+1} \\ c_{m+1} \\ d_{m+1} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 2 \\ 0 & 2 & 1 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_m \\ b_m \\ c_m \\ d_m \end{pmatrix} \quad , \quad \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.11. Para $|x| < \frac{1}{4}, |y| < \frac{1}{4}$, se tiene:

$$\frac{1}{4} ln\left(1 + (tan(x\pi))^2\right) ln\left(1 + (tan(y\pi))^2\right) - xy\pi^2 =$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2n-1} \frac{(-1)^n (tan(x\pi))^m (tan(y\pi))^{2n-m}}{m(2n-m)}$$

$$\frac{y\pi}{2} ln\left(1 + (tan(x\pi))^2\right) + \frac{x\pi}{2} ln\left(1 + (tan(y\pi))^2\right) =$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2n} \frac{(-1)^{n-1} (tan(x\pi))^m (tan(y\pi))^{2n-m+1}}{m(2n-m+1)}$$

2.12.

$$\frac{1}{4} ln\left(\frac{4}{3}\right) ln\left(8 - 4\sqrt{3}\right) - \frac{\pi^2}{72} = \sum_{n=1}^{\infty} \sum_{m=1}^{2n-1} \frac{(-1)^n \left(2 - \sqrt{3}\right)^{2n-m}}{\left(\sqrt{3}\right)^m m(2n-m)}$$

$$\frac{\pi}{24}\left(6\,ln(2)+2\,ln\left(2-\sqrt{3}\right)-ln(3)\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{2n}\frac{(-1)^{n-1}\left(2-\sqrt{3}\right)^{2n-m+1}}{\left(\sqrt{3}\right)^{m}m(2n-m+1)}$$

2.13.

$$\frac{\pi}{4}-\frac{ln(2)}{2}-\sum_{n=2}^{\infty}(-1)^{n}\int_{0}^{1}tan^{-1}\left(x^{n}\right)dx=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k+1}}{(2n+1)((2n+1)k+1)}$$

$$\frac{\pi}{6\sqrt{3}}-\frac{1}{2}ln\left(\frac{4}{3}\right)+\sum_{n=2}^{\infty}\int_{0}^{1/\sqrt{3}}tan^{-1}\left(x^{n}\right)dx=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n}\left(1/\sqrt{3}\right)^{(2n+1)k+1}}{(2n+1)((2n+1)k+1)}$$

2.14. Para $0<x<1$, se tiene:

$$\frac{\pi}{8}ln\left(1+x^{2}\right)+\frac{tan^{-1}(x)}{2}ln\left(\frac{2}{(1+x)^{2}}\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{2n}\frac{(-1)^{n-1}x^{m}\left(\frac{1-x}{1+x}\right)^{2n-m+1}}{m(2n-m+1)}$$

2.15. Para $n\in\mathbb{N}-\{1\}=\{2,3,4,...\}$, se tiene:

$$\frac{\pi}{4}=\int_{1}^{\sqrt[n]{1+\sqrt[n]{1+...}}}\frac{nx^{n-1}-1}{1+x^{2}-2x^{n+1}+x^{2n}}dx$$

$$\frac{\pi}{6}=\int_{1}^{\sqrt[n]{\sqrt{\frac{1}{3}}+\sqrt[n]{\sqrt{\frac{1}{3}}+...}}}\frac{nx^{n-1}-1}{1+x^{2}-2x^{n+1}+x^{2n}}dx$$

$$\frac{\pi}{3}=\int_{1}^{\sqrt[n]{\sqrt{3}+\sqrt[n]{\sqrt{3}+\ldots}}}\frac{nx^{n-1}-1}{1+x^{2}-2x^{n+1}+x^{2n}}dx$$

$$\frac{\pi}{2}=\int_{1}^{\infty}\frac{nx^{n-1}-1}{1+x^{2}-2x^{n+1}+x^{2n}}dx$$

$$\frac{\pi}{4}=\int_{0}^{\frac{1}{2}+\left(\frac{1}{2}+\left(\frac{1}{2}+\ldots\right)^{n}\right)^{n}}\frac{1-nx^{n-1}}{1+x^{2}-2x^{n+1}+x^{2n}}dx+\int_{0}^{\frac{1}{3}+\left(\frac{1}{3}+\left(\frac{1}{3}+\ldots\right)^{n}\right)^{n}}\frac{1-nx^{n-1}}{1+x^{2}-2x^{n+1}+x^{2n}}dx$$

2.16. Para $m=1,2,3,\ldots$, se tiene:

$$\frac{2^{2m+3}}{\pi^{2}}\frac{\left(2^{m}+1\right)^{2}}{\left(2^{m}-1\right)^{4}}\prod_{n=1}^{\infty}\frac{\left(2^{m+2}n\right)^{4}\left(\left(2^{m+2}n\right)^{2}-\left(2^{m}+1\right)^{2}\right)^{2}}{\left(\left(2^{m+2}n\right)^{2}-\left(2^{m}-1\right)^{2}\right)^{4}}=$$

$$=\frac{2\left(2+\overbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}^{m-radicales}\right)}{\left(2-\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}_{m-radicales}\right)^{2}}=\sum_{n=0}^{\infty}\left(2n+1\right)\left(\frac{1}{2}\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}_{m-radicales}\right)^{n}$$

2.17. Para $n=1,2,3,\ldots$, se tiene:

$$I\left(n\right)=\int_{0}^{1}\left(1-x\right)^{n-1}\tan^{-1}\left(x\right)dx=\left(n-1\right)!\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}\left(2m\right)!}{\left(2m+n+1\right)!}$$

$$I(1)=\frac{\pi}{4}-ln(2)$$

$$I(2)=\frac{1}{2}-\frac{1}{2}ln(2)$$

$$I(3)=\frac{5}{6}-\frac{\pi}{6}-\frac{1}{3}ln(2)$$

$$I(4)=\frac{5}{6}-\frac{\pi}{4}$$

$$I(5)=\frac{23}{60}-\frac{\pi}{5}+\frac{2}{5}ln(2)$$

$$I(6)=-\frac{79}{180}+\frac{2}{3}ln(2)$$

$$I(7)=\frac{2}{7}\pi-\frac{134}{105}+\frac{4}{7}ln(2)$$

2.18. Para $a=e^{-e^{-e^{-\cdots}}}=0.56714329...$, se tiene:

$$\frac{\pi}{4}=\int_0^a \frac{(1+x)e^x}{1+x^2e^{2x}}dx=\int_a^1 \frac{1-ln(x)}{x^2+(ln(x))^2}dx$$

2.19. Para $m=1,2,3,...$, se tiene:

$$\pi q_m\sqrt{3}+3p_m\left(2\,ln(2)-ln(3)\right)=2^{2m+3}\sum_{n=1}^{\infty}\left(H_m-H_{2n+m}\right)\binom{2n+m}{2n}\left(-\frac{1}{3}\right)^n$$

$$9q_m\left(2\ln(2)-\ln(3)\right)-p_m\pi\sqrt{3}=$$
$$=2^{2m+3}\sum_{n=0}^{\infty}\left(H_m-H_{2n+m+1}\right)\binom{2n+m+1}{2n+1}\left(-\frac{1}{3}\right)^n$$

$$\begin{array}{ll} p_{m+2}=6p_{m+1}-12p_m & ,\ p_1=2\ ,\ p_2=0 \\ q_{m+2}=6q_{m+1}-12q_m & ,\ q_1=2\ ,\ q_2=8 \end{array}$$

donde $H_n=\sum_{k=1}^{n}\frac{1}{k}$

2.20. Para $p\in\mathbb{N}$, se tiene:

$$\sum_{n=1}^{\infty}sen\left(n^{-2p}\right)=\frac{2^{2p-1}B_p\pi^{2p}}{(2p)!}+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{(-1)^n}{(2n+1)!\,(k-n+1)^{(4n+2)p}}$$

donde B_p son los números de Bernoulli.

$$\sum_{n=1}^{\infty}sen\left(n^{-2}\right)=\frac{\pi^2}{6}+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{(-1)^n}{(2n+1)!\,(k-n+1)^{(4n+2)}}$$

2.21.

$$1+\sum_{n=2}^{\infty}(-1)^{n-1}Li_n\left(\frac{1}{2n-1}\right)=\frac{\pi}{4}+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{(-1)^n}{(k-n+2)^{n+1}(2n+1)^{k-n+2}}$$

donde $Li_n(x)$ es la función Polylogaritmo.

2.22. Para $m\in\mathbb{N}-\{1\}$, se tiene:

$$\frac{\pi}{4}+\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{(-1)^n}{(k-n+2)^m(2n+1)^{k-n+2}}=\sum_{n=0}^{\infty}(-1)^n Li_m\left(\frac{1}{2n+1}\right)$$

donde $Li_m(x)$ es la función Polylogaritmo.

2.23. Para $m \in \mathbb{N} - \{1\}$, se tiene:

$$\frac{\pi}{4} + \sum_{n=2}^{\infty} tan^{-1}\left(\frac{1}{n^m}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} Li_{(2k+1)m}(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \zeta\left((2k+1)m\right)$$

donde $Li_s(x)$ es la función Polylogaritmo y $\zeta(x)$ es la función zeta de Riemann.

2.24. Para $m \in \mathbb{N}$, se tiene:

$$\sum_{n=2}^{\infty} (-1)^n tan^{-1}\left(\frac{1}{n^m}\right) = \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} Li_{(2k+1)m}(-1) =$$
$$= \frac{\pi}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \zeta_a\left((2k+1)m\right)$$

donde $Li_s(x)$ es la función Polylogaritmo y $\zeta_a(x)$ es la función zeta alternada.

2.25. Para $m \in \mathbb{N}$, se tiene:

$$\frac{\pi}{6} + \sum_{n=2}^{\infty} tan^{-1}\left(\frac{\left(1/\sqrt{3}\right)^n}{n^m}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} Li_{(2k+1)m}\left(\frac{1}{3^k\sqrt{3}}\right)$$

donde $Li_s(x)$ es la función Polylogaritmo.

2.26. Para $m \in \mathbb{N}$, se tiene:

$$\frac{\pi}{4} + \sum_{n=2}^{\infty}\left(tan^{-1}\left(\frac{1}{n^m 2^n}\right) + tan^{-1}\left(\frac{1}{n^m 3^n}\right)\right) =$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(Li_{(2k+1)m}\left(\left(\frac{1}{2}\right)^{2k+1}\right) + Li_{(2k+1)m}\left(\left(\frac{1}{3}\right)^{2k+1}\right)\right)$$

donde $Li_s(x)$ es la función Polylogaritmo.

2.27.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(e^{1/(2n-1)}-1\right)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^{m-1}}{(n-m+2)!(2m-1)^{n-m+2}}$$

2.28.

$$1+\frac{1}{2}\sum_{n=1}^{\infty}ln\left(\frac{2n+1+(-1)^n}{2n+1-(-1)^n}\right)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^m}{(2n-2m+3)(2m+1)^{2n-2m+3}}$$

2.29.

$$\frac{\pi}{4}=\frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\left(-\frac{1}{\sqrt{2}}\right)^k\frac{k!}{\left(\frac{k+1}{2}\right)_{k+1}}-\frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{\sqrt{2}}\right)^k\frac{k!}{\left(\frac{k+2}{2}\right)_{k+1}}$$

$$\frac{\pi}{4}=\sqrt{2}F\left(\frac{1}{2},1,1;\frac{5}{6},\frac{7}{6};\frac{2}{27}\right)-\frac{1}{4}F\left(1,1,\frac{3}{2};\frac{4}{3},\frac{5}{3};\frac{2}{27}\right)$$
$$-\frac{1}{2}F\left(\frac{1}{2},1,1;\frac{2}{3},\frac{4}{3};\frac{2}{27}\right)+\frac{\sqrt{2}}{15}F\left(1,1,\frac{3}{2};\frac{7}{6},\frac{11}{6};\frac{2}{27}\right)$$

2.30.

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{2^{2n}B_n}{(2n)!}\sum_{k=0}^{2n-1}\frac{\binom{2n-1}{k}}{2k+1}+\int_0^1\frac{1}{tan\left(1+x^2\right)}dx$$

$$\frac{\pi}{4}=-\sum_{n=1}^{\infty}\frac{2\left(2^{2n-1}-1\right)B_n}{(2n)!}\sum_{k=0}^{2n-1}\frac{\binom{2n-1}{k}}{2k+1}+\int_0^1\frac{1}{sen\left(1+x^2\right)}dx$$

$$\frac{\pi}{4}=-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}B_n}{(2n)!}\sum_{k=0}^{2n-1}\frac{\binom{2n-1}{k}}{2k+1}+\int_0^1\frac{1}{th\left(1+x^2\right)}dx$$

$$\frac{\pi}{4}=-\sum_{n=1}^{\infty}\frac{(-1)^{n}\,2\left(2^{2n-1}-1\right)B_n}{(2n)!}\sum_{k=0}^{2n-1}\frac{\binom{2n-1}{k}}{2k+1}+\int_0^1\frac{1}{sh\left(1+x^2\right)}dx$$

B_n son los números de Bernoulli

2.31.

$$\int_0^{\infty}\left(e^{\left(1+x^2\right)^{-1}}-1\right)dx=\pi\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{2n+1}(n+1)!}$$

$$\int_0^{\infty}ln\left(\frac{2+x^2}{1+x^2}\right)dx=\frac{\pi}{1+\sqrt{2}}$$

$$\int_0^{\infty}ln\left(1+\frac{2}{x^2}\right)dx=\pi\sqrt{2}$$

$$\int_0^{\infty}sen\left(\frac{1}{1+x^2}\right)dx=\pi\sum_{n=0}^{\infty}\frac{(-1)^n\binom{4n}{2n}}{2^{4n+1}(2n+1)!}$$

$$\int_0^{\infty}\left(1-cos\left(\frac{1}{1+x^2}\right)\right)dx=\pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\binom{4n-2}{2n-1}}{2^{4n-1}(2n)!}$$

$$\int_0^{\infty} sh\left(\frac{1}{1+x^2}\right)dx = \pi\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}}{2^{4n+1}(2n+1)!}$$

$$\int_0^{\infty}\left(ch\left(\frac{1}{1+x^2}\right)-1\right)dx = \pi\sum_{n=1}^{\infty}\frac{\binom{4n-2}{2n-1}}{2^{4n-1}(2n)!}$$

2.32.

$$\frac{1}{\pi}=\frac{1}{3}-8\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1}$$

$$\frac{1}{\pi}=\frac{1}{3}-4\left(e^{2\pi}-1\right)\sum_{n=1}^{\infty}\frac{n(n+1)e^{2\pi n}}{\left(e^{2\pi n}-1\right)\left(e^{2\pi(n+1)}-1\right)}$$

$$\frac{1}{\pi}=\frac{1}{3}-2sh(\pi)\sum_{n=1}^{\infty}\frac{n(n+1)}{sh(n\pi)sh((n+1)\pi)}$$

$$\frac{1}{\pi}=\frac{1}{3}-4sh(\pi)\sum_{n=1}^{\infty}\frac{n(n+1)}{ch((2n+1)\pi)-ch(\pi)}$$

$$\frac{1}{\pi}=\frac{1}{3}-4\sum_{n=1}^{\infty}\frac{ne^{-\pi n}}{sh(n\pi)}$$

$$\frac{1}{\pi}=\frac{1}{3}-4\sum_{n=1}^{\infty}\left(\frac{1}{sh(n\pi)}-\frac{1}{sh((n+1)\pi)}\right)\sum_{k=1}^{n}ke^{-k\pi}$$

$$\frac{1}{\pi}=\frac{1}{3}-4\left(1-e^{-\pi}\right)\sum_{n=1}^{\infty}e^{-n\pi}\sum_{k=1}^{n}\frac{k}{sh(k\pi)}$$

$$\frac{1}{\pi}=\frac{1}{3}-8\left(e^{\pi}-1\right)\sum_{n=1}^{\infty}\frac{e^{n\pi}}{\left(e^{n\pi}+1\right)\left(e^{(n+1)\pi}+1\right)}\sum_{k=1}^{n}\frac{k}{e^{k\pi}-1}$$

$$\frac{1}{\pi}=\frac{1}{3}-8\left(e^{\pi}-1\right)\sum_{n=1}^{\infty}\frac{e^{n\pi}}{\left(e^{n\pi}-1\right)\left(e^{(n+1)\pi}-1\right)}\sum_{k=1}^{n}\frac{k}{e^{k\pi}+1}$$

2.33.

$$\pi=\left(\frac{4}{3}\right)^{2}\prod_{n=1}^{\infty}\left(\left(\frac{(n+2)^{2}}{(n+1)(n+3)}\right)^{2}\prod_{k=0}^{n-1}p_n\left(\frac{n-k}{n-k+1}\right)^{\frac{1}{(k+1)(k+2)}}\right)$$

$$p_n=\left(\frac{(k+2)\left(2(k+1)(n-k)+k+2\right)}{(k+1)\left(2(k+2)(n-k)+k+3\right)}\right)^{2}$$

2.34.

$$\frac{\pi\left(3\sqrt{3}ch(\pi)+sh(\pi)\right)}{4\left(ch(\pi)\right)^{2}-1}=\frac{9}{10}+180\sum_{n=1}^{\infty}\frac{(-1)^{n}}{\left(9n^{2}+8\right)^{2}+36}$$

$$\frac{\pi\left(3sh(\pi)-\sqrt{3}ch(\pi)\right)}{4\left(ch(\pi)\right)^{2}-1}=\frac{6}{5}+30\sum_{n=1}^{\infty}\frac{(-1)^{n}\left(9n^{2}+8\right)}{\left(9n^{2}+8\right)^{2}+36}$$

2.35. Para $k,m\in\mathbb{N}$, se tiene:

$$\frac{(2k+1)_{2m}B_k}{\pi^{2m}2^{2m}B_{k+m}}=1+\sum_{n=1}^{\infty}\left(\frac{H_{n+1,2k}}{H_{n+1,2k+2m}}-\frac{H_{n,2k}}{H_{n,2k+2m}}\right)$$

B_k son los números de Bernoulli

$$H_{k,r}=\sum_{n=1}^{k}\frac{1}{n^r}$$

2.36. Para $0<x<1$, se tiene:

$$\frac{\pi}{4}\tan^{-1}\left(\frac{x^2+2x-1}{1+2x-x^2}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(H_{2n}-\frac{H_n}{2}\right)\left(x^{2n}-\left(\frac{1-x}{1+x}\right)^{2n}\right)$$

$$H_n=\sum_{k=1}^{n}\frac{1}{k}$$

2.37. Para $m,p\in\mathbb{N}$,se tiene:

$$\pi^{2p}\frac{2^{2p-1}B_p}{(2p)!}=\sum_{n=1}^{\infty}m\tan^{-1}\left(\frac{1}{mn^{2p}}\right)+\int_0^1\sum_{n=1}^{\infty}\frac{x^2}{n^{2p}\left(x^2+m^2n^{4p}\right)}dx$$

$$\int_0^1\sum_{n=1}^{\infty}\frac{x^2}{n^{2p}\left(x^2+m^2n^{4p}\right)}dx=\sum_{n=1}^{\infty}\int_0^{n^{-2p}}\frac{x^2}{x^2+m^2}dx$$

B_p , son los números de Bernoulli

2.38. Para $0<y<1,p\in\mathbb{N}$, se tiene:

$$\pi^{2p}\frac{2^{2p-1}B_p}{(2p)!}=(1-y)\ln\prod_{n=1}^{\infty}\left(1+n^{-2p}\right)+\sum_{n=1}^{\infty}\int_0^{n^{-2p}}\frac{x+y}{x+1}dx$$

$$\sum_{n=1}^{\infty}\int_0^{n^{-2p}}\frac{x+y}{x+1}dx=\int_0^1\sum_{n=1}^{\infty}\frac{x+yn^{2p}}{n^{2p}\left(x+n^{2p}\right)}dx$$

B_p , son los números de Bernoulli

2.39. Para $p \in \mathbb{N}$,se tiene:

$$\pi^{2p}\frac{2^{2p-1}B_p}{(2p)!} = ln\prod_{n=1}^{\infty}\left(1+n^{-2p}\right)+2\sum_{n=1}^{\infty}\int_0^{n^{-p}}\frac{x^3}{x^2+1}dx$$

$$\sum_{n=1}^{\infty}\int_0^{n^{-p}}\frac{x^3}{x^2+1}dx = \int_0^1\sum_{n=1}^{\infty}\frac{x^3}{n^{2p}\left(x^2+n^{2p}\right)}dx$$

B_p , son los números de Bernoulli

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