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Random dynamics of dikes: On the possibility of 0.5 m random changes in water level related to 0.1 kPa monotone barometric pressure increase.

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Abstract. In this paper, random dynamic systems theory is applied to time series ($\Delta t = 5$ minutes) of measurement of water level, W, temperature, T, and barometric pressure, P, in sea dikes. The time series were obtained from DDSC and are part of DMC systems dike maintenance program of the Ommelanderzeedijk in northern Netherlands. The result of numerical analysis of dike (W, T, P) time series is that after the onset of a more or less monotone increase in barometric pressure, an unexpected relatively sharp increase or decrease in water level can occur. The direction of change is related to random factors shortly before the onset of the increase. From numerical study of the time series, we found that $\Delta W_{max} \approx \pm 0.5$ mNAP[1]. The randomness in the direction of change is most likely explained by the random outcome of two competitive processes shortly before the onset of a continuous barometric pressure increase. The two processes are pore pressure compaction and expulsion of water by air molecules. An important cause of growing barometric pressure increase can be found in pressure subsidence following a decrease in atmospheric temperature. In addition, there is a diurnal atmospheric tide caused by UV radiation fluctuations. This can give an additional $\Delta P_{tide} \approx \pm 0.1$ kPa barometric fluctuation[2] in the mid latitudes $(30°N - 60°N)$.

1. Introduction

Random dynamic theory is the stochastic variant of the study of deterministic dynamic systems. The theory of random dynamic systems (RDS) extends and unites probability theory and dynamical system theory *Arnold* [1998]. In an RDS model the ergodic theorem is involved. Its foundation is in stochastic differential equations (SDE) and/or random differential equations (RDE). On page 76-77 of *Chueshov* [2001] the relation between stochastic and random differential equations in the Wong-Zakai theorem is presented.

The RDS is in fact the solution (a solution) of the SDE or RDE and shows interesting features such as absorbing sets and/or attractor sets in the solution space of the differential equation. There is a vast literature on attractor sets. We mention e.g. *Cao* [2010], *Crauel* [2001], *Gess* [2013], *Crauel* [1997], *Crauel* [1994], *Chueshov* [2001] and *Arnold* [1998]. In the present analysis we will not go into the possibility of attractor sets in the RDS model of water content data. However, the behavior of the data might possibly be understood in terms of attractor sets and much of our theoretical concepts can be found in the cited literature on attractors.

The theoretical underpinning of the present application of RDS to time series in dikes can be found in the description of stochastic porous media and its nonlinear diffusion processes *Hilfer* [2000], *Gess* [2011]. Our data from the DMC system of the Ommelanderzeedijk (sea dike) lacks sufficient spatial differentiation in order to check the theoretical assumption thoroughly. Nevertheless, given the nature of the time series, stochastic porous media seem to make sense.

Before we enter the details of our model it is noted here that there is equivalence between models employing Markov chains and RDS modeling of time series in dikes. This is demonstrated in theorem 2.1.4, page 53 *Arnold* [1998]. Hence, models of dike behavior that may appear very different from our present stochastics are still closely associated to it. For a discussion of the associated topic of measure attractors and Markov attractors see *Crauel* [2008].

The essence of Random Dynamics is a functional operator $\varphi(t, \omega)$, parametrized with time $t \in \mathbb{T}$ and elementary random events $\omega \in \Omega$ on some initial value x of states-pace resulting in the (time series) solution $x_t(\omega)$ over a measurable dynamic system, DS, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_\tau)_{\tau \in \mathbb{T}})$. We have the sample space denoted by Ω and $\omega \in \Omega$. $\theta_t \omega = \omega_t$ is the probability process at time $t \in \mathbb{T}$, with $\mathbb{T} \subset \mathbb{R}$ a time interval. In the paper we follow the general custom to have a time dependence indicated by an index, see e.g. *Arnold* [1998]. The θ_t is measurable in the probabilistic sense and $\theta_0 = id$ the identity operation. The semi flow property $\theta_{s+t} = \theta_s \circ \theta_t$, with $\circ$ the (topological) composition of θ operators, stands at the basis of the RDS. See *Arnold* [1998] page 536. A semi flow ensures a consistent probability process with a propagation in time but independent of time. Perhaps that a many sided dice with temperature and pressure on each side of the dice is a good approximative concept to understand a (discrete) measurable dynamic system of $\omega \sim (T, P)$ time series. Note for completeness that the $\sim$ indicates a relation, it is not a proportionality. If ω stands for the outcome of a throw of a dice at $t = 0$, then $\theta_t \omega$ stands for the outcome of a throw t seconds later. Furthermore, $\mathcal{F}$ is the associated σ-algebra and $\mathbb{P}$ is the θ independent probability measure over $\mathcal{F}$. The $\theta_t \omega$ represents the $\theta_t \omega \sim (T_t, P_t)$ time series.

An RDS functional transformation operator of the initial x has to follow certain rules. A random dynamical system with time domain $\mathbb{T}$ and statespace X, refers to a pair (θ, φ) consisting of a measurable dynamical system $\mathrm{DS}(\Omega, \mathcal{F}, \mathbb{P}, (\theta_\tau)_{\tau\in\mathbb{T}})$ and a cocycle $\varphi : \mathbb{T} \times \Omega \times X \to X$. The reader is referred to *Arnold* [1998] and *Chueshov* [2001] for more details. If one holds the RDS $\varphi(t, \omega)$ then one holds a solution to the SDE or RDE. We quote here the important cocycle property of the RDS φ

$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \tag{1}$$

In the cocycle for φ the "echo" of the semiflow in the $\mathrm{DS}(\Omega, \mathcal{F}, \mathbb{P}, (\theta_\tau)_{\tau\in\mathbb{T}})$ shows. We may translate it such that the statespace control variables are consistently influenced in their temporal changes by the random fluctuations in the DS.

In the present case we are interested in the effect of random fluctuations in causal variables, here, temperature and barometric pressure time series on the control variables, here the time series of water content in a dike. In our study we employ, initially, temporal random differential equations in a continuous time domain. The $\omega \sim (T, P)$ dice is therefore only a conceptual approximation containing the measured time series. The continuous development can be approximated with linear interpolation. The DMC systems configuration is such that there are two water level measurement series $W_t(\omega) = (W_{1,t}(\omega), W_{2,t}(\omega))$ and two temperature measurements in the embankment of the dike $T = (T_1, T_2)$. There is only one barometric pressure time series P_t.

$$\frac{d}{dt} W_t(\omega) = f(t, \omega, W_t(\omega)) \tag{2}$$

and $\omega \sim (T_1, T_2, P)$. The RDS can be written as, $\varphi(t, T_1, T_2, P)(W_0)$. In our discussion we often will employ the equivalent $\varphi(t, \omega)(x)$ or even the $\varphi(t, \omega, x)$ notation when it suits the discussion. The difficulty of (2) lies in the fact that the progresion of ω has important influence on the φ but the time change of ω is not a simple function of time but instead is driven by a measurable dynamic system operator θ_t "in time".

2. A cocycle from a (W_1, W_2, T_1, T_2, P) RDE "Ansatz"

2.1. $\omega \in \Omega$ dependence

Starting from the general form in (2) it is a long way to reach a cocycle φ such as in (1) given the data in time series (W_1, W_2, T_1, T_2, P). The proces to reach a cocycle φ is to simply start with a linear approximation to the possible SDE(s) or RDE(s) behind the (W_1, W_2, T_1, T_2, P) time series. If we admit in the Ansatz linearity then looking at (2) the f on the right hand side of (2) is a 1×2 linear vector function. In the Asatz we may take e.g.

$$f(t, \omega, x_t(\omega)) = M(\omega)x_t(\omega) + b(\omega) \tag{3}$$

Here, $M(\omega)$ is a 2×2 matrix depending on ω and the time series $x_t(\omega)$ is a 1×2 vector $\sim (W_{1,t}, W_{2,t})$. The t dependence of f resides in this Ansatz form in the x_t. The $b(\omega)$ is a constant-of-time 1×2 vector.

Let us, for the sake of the example in the computation, first look at the estimation of the differential equation for $W_{1,t}$. Then, with the discrete time series in the function f a a_1 multiplication for $x_{1,t} \sim W_{1,t}$ and a_2 for $x_{2,t} \sim W_{2,t}$, together with b_1 are estimated with the use of the "glm" statistical estimation function from R. Because we may note that a_1 and a_2 are estimated coefficients and *not* functions we may use the $a_j, (j = 1, 2)$, in a further attempt to create an Ansatz for matrix M and vector b. In fact, in this particular Ansatz, we concentrated on $M = M(\omega)$ and note that $\omega \sim (T_1, T_2, P)$ without further "periodicity" data such as sea tide or rainfall or.. . Although we had no further periodicity data it is wise to employ "time" as an indicator for periodicity. All the more because the φ was erected for $N = 100$ sample points of the time series prior to a selected sample point (corresponding to a certain date-time label that can be selected as one pleases in the numerical experiments). For completeness, $N = 100$ consecutive data points were used showing time-date labels less than the sampled starting point are employed.

Hence, if the a_j are used as guiding lines for the function $M(\omega)$ then in a linear estimation the size relevant coefficients for $\omega \sim (T_1, T_2, P)$ are influenced by t. There is of course no random variable "time" so in random studies time resumes its basic role as an integration variable in the algorithm and is neglected as random factor. We then aim to minimalize with least squares the functional relation for (c_1, c_2, c_3, d) coefficients in

$$S(c_1, c_2, c_3, d) = \sum_{n=1}^{N} \{\hat{a}_1 - u_1(P) + (c_1 u_2(T_1) + c_2 u_3(T_2) + c_3 u_4(t) + d)\}^2 \tag{4}$$

The functions u_k, with $k = 1, 2, 3, 4$ are transformations of (P, T_1, T_2, t), introduced for computational purposes. The $\hat{a}_1$ is the size $N = 100$ array containing the glm estimated a_1 and we assume that the (c_1, c_2, c_3, d) numerically are taken in relation to a unit weighted barometric pressure in the linear Ansatz function. Of course, some criticism may be raised against this method of estimation but we note that we do not pretend to hold the final truth about the random differential equation generating the $N = 100$ time series with the Ansatz. Moreover, we note that for each next $N = 100$ an in principle, different RDE or SDE is allowed to generate the time series (W_1, W_2, T_1, T_2, P).

2.2. Ansatz φ

The next step is to formally solve the differential equation that was more or less patched together from the (W_1, W_2, T_1, T_2, P) time series data. If we look at $x_{1,t}$ we then may notice

$$\frac{d}{dt}x_t = M_{1,1}(\omega)x_{1,t} + M_{1,2}(\omega)x_{2,t} + b_1(\omega) \tag{5}$$

The $b_1(\omega)$ in (5) stays a glm estimate, hence a number not a "function", in this Ansatz. The M coefficients can be estimated such as in (4). The equation in (5) can be formally solved with basic means. This makes our employed algorithm at that point fairly easy. It is noted that ω may progress in time but is not a function of time. Suppose $R_t = M_{1,2}x_{2,t}+b_1$ where $x_{2,t}$ can be obtained from $W_{2,t}$. Then the solution for some arbitrary $N = 100$ consecutive sample points equals

$$x_{1,t}(\omega) = e^{M_{1,1}(\omega)t}\left(x_{1,t_0} + \int_{t_0}^{t} R_\tau e^{-M_{1,1}(\omega)\tau}d\tau\right) = \tilde{\varphi}(t-t_0,\omega)(x_{1,t_0}) \tag{6}$$

Here, $\tilde{\varphi}(t-t_0,\omega)(x_{1,t_0})$ is the initial estimate from the Ansatz of the RDS. We note that $\tilde{\varphi}(0,\omega) = 1$ for arbitrary $\omega \in \Omega$. In our computations we have $\tilde{\varphi}$ as a multiplication of the initial x_{1,t_0}. So, $\tilde{\varphi}(t-t_0,\omega,x_{1,t_0}) = \tilde{\varphi}(t-t_0,\omega)(x_{1,t_0}) = \tilde{\varphi}(t-t_0,\omega)\cdot x_{1,t_0}$. The separator dot between $\tilde{\varphi}(t-t_0,\omega)$ and x_{1,t_0} that indicates the multiplication will be suppressed in the following. It is noted that higher Taylor terms can provide a more complete generator $\tilde{\varphi}$ function.

3. Cocycle φ

3.1. Preliminaries

The next obvious question to be raised is of course whether or not we have a cocycle φ. Given the patchwork employed in the Ansatz we cannot expect that our multiplication function from (6) has the cocycle property formulated in (1). Furthermore we note that in the algorithm that we employed the ω was, similar to time, an index to a matrix Φ. The multi indexed matrix Φ was supposed to have $N \times N \times N$ entries where $\Phi : \mathbb{T}_{disc} \times \Omega_{disc} \times \Omega_{disc} \to \mathbb{R}$. Note, $\mathbb{T}_{disc}$ is the discrete series of times (steady growth with 5 minute interval) and Ω_{disc} based on the measured triples (T_1, T_2, P). The concept of two ω indices is a compromise between computational capacity and the wish to have "other than measured" combinations between barometric pressure P on the one hand and in-dike temperature pairs (T_1, T_2) on the other. So in the computations we have the water level estimated generator $\tilde{\varphi}(t - t_0, \omega, \omega')$ expressed in the discrete representation of the matrix $\Phi(t_n, o_1, o_2)$ and $t_n = 1, 2..N$, together with, o_1 and o_2 in $1, 2, ...N$. If the need is there to obtain a function from Φ then cubic spline interpolation functions from R in addition to R's numerical integration routines are employed.

3.2. Transformation to a.s. cocycle

Obviously the discrete Φ and its continuous (interpolated) equivalent $\tilde{\varphi}$ from the Ansatz are not cocyclic. Let us present the continuous form testing parameter and describe the functional transformations leading to an "almost sure" cocycle. Suppose we define

$$U_s^k(t, \underline{\omega}) = \frac{\varphi_\lambda^k(t + s, \underline{\omega})}{\varphi_\lambda^k(t, \theta_s \underline{\omega}) \varphi_\lambda^k(s, \underline{\omega})} \tag{7}$$

The $\underline{\omega}$ represents the two probability processes, the index λ on the phi shows linear transformation while the superscript $k = 1, 2...$ indicates the number of transformations taken. We fix an arbitrary initial time in the $N = 100$ array to be s. If we test it in the discrete space (i.e. based on Φ) we check whether

$$\frac{1}{N^3} \sum_{n=1}^{N} \sum_{o_1=1}^{N} \sum_{o_2=1}^{N} U_s^k(t_n, o_1, o_2) \approx 1 \tag{8}$$

together with the check if the standard deviation over t_n, o_1 and o_2, i.e. a N^3 size array, σ_U, is "small enough". In the practice of our computations thus far (8) was always very closely met and we also found that *always* $\sigma_U < 1$, ranging from $\sigma_U \approx 0.6$ to $\sigma_U \approx 1 \times 10^{-5}$. On average we see $\sigma_U \approx 0.001$ to 0.003. Randomly selecting t and s indices in the Φ_λ^k showed that the cocycle with the stopping criterion (8) and $\sigma_U < 1$, is "almost surely" perfect.

We note that there is the danger of a meaningless perfect cocycle where $U = 1/(1 \cdot 1)$. It is noted that therefore the variance of the U array, defined in (7), must be unequal to zero. Up until now we did not find any cocycle U such that *all* Φ entries are almost equal to unity. However, there are in the final Φ matrix not too large sections where 1 is closely approximated from below and from above. Considering the approximative perfection measure we may conclude that in most of the important computations, there is no transformation to the trivial cocycle. This numeric fact adds to the claim of ± 0.5 mNAP variations at some sample point instances of the time series

3.3. The reason for the cocycle

Apart from the existence of the attractor set which needs a cocycle projection between causal variables T and P and control variables W, there is another reason for insisting on a cocycle structure. The almost sure perfect non-trivial cocycle is the hallmark for the existence of an SDE or RDE. In *Arnold* [1998] in Theorem 2.2.13 on page 66 and also in Theorem 2.3.30 on page 87, it is demonstrated that a perfect cocycle is 1-1 related to either a random differential equation or a stochastic differential equation.

Hence we conclude that starting from an imperfect linear Ansatz and demonstrating numerically that an a.s. perfect cocycle can be obtained which is not by necessity trivial, we have an equivalent description of the dike time series as if we would have done an a.s correct guess at the initial SDE or RDE that generates the time series (W_1, W_2, T_1, T_2, P) for the $N = 100$ data points.

In other words, we have a valid model in Φ^k_λ or its continuous equivalent $\varphi(t, \underline{\omega})$. Hence, we are allowed to meaningfully "play" with the causal variables to see what will happen in the development of the water level time series.

4. Barometric pressure exercises improving the functional form of the cocycle

The variable selected for further numeric experimentation is barometric pressure. Numerical experiments indicated an interesting phenomenon associated to the possibility to maintain a cocycle Φ, suppressing the λ and k indices, after manipulation with the pressure time series. In the first place the temporal index of $\Phi(t_n, o_1, o_2)$ is "mixed" with the time index of the barometric pressure, $P[n] = P(t_n) = P_{t_n}$. When nothing is changed then after cocycle transformation we see the relation in figure - 1. [Insert figure 1 about here.] So, using the array $P[\,] = AtPressure[\,]$ to match the time index of Φ the following transformation numerical exercise with the data can be performed.

$$\begin{array}{l}
for(o_1\, in\, seq(1, N))\{\\
\quad for(o_2\, in\, seq(1, N))\{\\
\quad\quad as.function(fPHI < -splinefun(P[\,], \Phi[, o_1, o_2],\\
\quad\quad\quad\quad method = c("fmm")\,, ties = mean))\\
\quad\quad for(n\, in\, seq(1, N))\{\\
\quad\quad\quad \Phi'[n, o_1, o_2] < -fPHI(P'[n])\\
\quad\quad \}\\
\quad \}\\
\}
\end{array} \tag{9}$$

Starting from the "raw" model and using this transformation, it is now possible to perform numerical experiments with uniformly varying the barometric pressure. Referring to (9), $P'[n] = P[n] + \delta P$, for $n = 1, 2...N$ together with the cubic spline interpolation, fmm method, a new matrix $\Phi'(t_n, o_1, o_2)$ is created. Initially δP added uniformly 0.5 kPa to the P_{t_n} time series, $n = 1, 2, ...N$.

Subsequently it was checked whether this new Φ' obtained from the uniform transformation $P' = P + 0.5$ could, again, be transformed to an a.s. cocycle. This numerical hypothesis was correct. For Φ' an equivalent of (8) for *each* initial sample point, using $N = 100$ data points consecutively before the sample point, turned out to be valid. Here the monotinicty in sample point $N_{smp} = 3130$ is presented in figure-2 [Insert figure-2 about here.]. Moreover, for the associated standard deviation for *each* sampled starting point in time, again based on $N = 100$ consecutive data points before sample point, we found $\sigma_{U'} < 1$. The transformation of the Φ - barometric pressure relation was however remarkably changed into a more or less monotone decreasing or increasing relation in pressure and $\varphi(\cdot, \underline{\omega})$. The next step was to subtract again $\delta P = 0.5$ kPa and to see if we then returned to the original Φ - barometric pressure relation after cocycle transformation.

The outcome of this numerical experiment was that the subtraction of $\delta P = 0.5$ kPa after the addition of $\delta P = 0.5$ kPa in *all* cases provided an a.s. perfect non-trivial cocycle. In addition, the form of the Φ - barometric pressure relation again in all cases became increasingly more monotone, or what is the same, a function of $C^\infty(\mathbb{R})$. It also showed either an initial increase or decrease in the initial part of Φ versus barometric pressure P function. The result for the next transformation step can be found in figure -3 [Insert figure-3 about here.]. This monotonicity plus initial behavior remains more or less undisturbed by the "factor" $\underline{\omega}$, described by o_1 and o_2 indices in the matrix. Although it has to be noted that every now and then, for small jumps along the y-axis representing the

φ, the initial increase turned into initial decrease for the initial section of the $\Phi(\cdot, o_1, o_2)$ versus barometric pressure $P(\cdot)$ relation when the (o_1, o_2) pair changed.

Because we have an a.s. perfect non-trivial cocycle in the numerics it can be claimed that the monotone relation between Φ, hence φ, and barometric pressure also describes a solution of the hypothetical SDE/RDE behind the time series (W_1, W_2, T_1, T_2, P). This claim, again, is based on *Arnold* [1998], Theorem 2.2.13 on page 66 and Theorem 2.3.30 on page 87.

5. Random behavior after monotone increase in barometric pressure

In this section we study the importance of random factors in the prediction of increase or decrease of water level in monotone pressure increase.

5.1. Mathematical preliminaries

Given the numerical proof of an a.s. description of a SDE responsible for the time series when we have an a.s. perfect non-trivial cocycle φ, we are allowed to formally write a Stratonovich SDE associated to $(W_1, W_2, T_1, T_2, P)_t$, $t \in \mathbb{T}$. Hence, in x notation

$$dx_t = F(x_t, \circ dt) \tag{10}$$

Here it is important to note that the right hand side of (10), i.e. the $F(x_t, t, \underline{\omega})$ is a semi martingale helix, see *Arnold* [1998] page 78 definition 2.3.15 and page 74 definition 2.3.8 for the forward version. For our purpose we may state that a semi martingale helix contains a deterministic part (bounded variation) and a probabilistic part (martingale). A solution of the Stratonovich SDE in (10) also is a semi martingale as can be seen from the forward Stratonovich integral (*Arnold* [1998] page 81) from the SDE. We have formally for initial x

$$G_s(x,t,\underline{\omega}) = \int_s^t F_s(\varphi_{s,u}(\underline{\omega})x, \circ d^+u) \tag{11}$$

In this equation the compound notation $\varphi_{s,u}(\underline{\omega}) = \varphi(u,\underline{\omega}) \circ \varphi(s,\underline{\omega})^{-1}$ and $F_s(\cdot,t,\underline{\omega}) = F(\cdot,t,\underline{\omega}) - F(\cdot,s,\underline{\omega})$ is used. Because, φ is the RDS generator function related to the solution of the SDE, we may write for initial x

$$G_s(x,t,\underline{\omega}) = \varphi(t,\underline{\omega}) \circ \varphi(s,\underline{\omega})^{-1}x - x \tag{12}$$

which also reads, $G_s(x,t,\underline{\omega}) = \varphi_{s,t}(\underline{\omega})x - x$. Interestingly, *Arnold* [1998] on page 88 also formulates the "inverse" integration, with, again, x initial value of the x time series,

$$F_s(x,t,\underline{\omega}) = \int_s^t G_s(\varphi_{s,u}(\underline{\omega})^{-1}x, \circ d^+u) \tag{13}$$

If we then in a meaningful sense want to use the right hand of (12) into the Stratonovich integral of (13) then let us make use of a notational device $1\left((\varphi_{s,u}(\underline{\omega})y - y), \circ d^+u\right)$ referring to the right hand side of $G_s(y, \circ d^+u)$ fully expressed in (12). So, e.g. $\int_s^t 1(f(u,\underline{\omega}), \circ d^+u)$ is the Stratonovich integral of a semi martingale helix $F(f(u,\underline{\omega}),u,\underline{\omega}) = f(u,\underline{\omega})$ integrated for $s \leq u \leq t$. We note that because Stratonovich makes use of triangular area, when a nonstochastic function is integrated, a Stratonovich integral closely approximates a Rieman outcome.

Returning to (11) we see that using (12)

$$\varphi_{s,t}(\underline{\omega})x - x = \int_s^t F_s(\varphi_{s,u}(\underline{\omega})x, \circ d^+u) \tag{14}$$

whereas $F_s(\varphi_{s,u}(\underline{\omega})x, \circ d^+u)$ in (14) reads according to (13)

$$F_s(\varphi_{s,u}(\underline{\omega})x, u, \underline{\omega}) = \int_s^t G_s(\varphi_{s,v}(\underline{\omega})^{-1} \circ \varphi_{s,u}(\underline{\omega})x, \circ d^+v) \tag{15}$$

If we take $z = \varphi_{s,u}(\underline{\omega})x$, then

$$F_s(z, u, \underline{\omega}) = \int_s^t G_s(\varphi_{s,v}(\underline{\omega})^{-1}z, \circ d^+v) \tag{16}$$

or in the $1(\cdot, ..)$ notation

$$F_s(z, u, \underline{\omega}) = \int_s^u 1\left((\varphi_{sv}(\underline{\omega}) \circ \varphi_{s,v}(\underline{\omega})^{-1}z - \varphi_{s,v}(\underline{\omega})^{-1}z), \circ d^+v\right) \tag{17}$$

Because $\varphi_{sv}(\underline{\omega}) \circ \varphi_{s,v}(\underline{\omega})^{-1}z - \varphi_{s,v}(\underline{\omega})^{-1}z = z - \varphi_{s,v}(\underline{\omega})^{-1}z$, which in turn also may read as $\varphi_{s,v}(\underline{\omega})^{-1} \circ (\varphi_{sv}(\underline{\omega})z - z)$, the following double Stratonovich arises from (14) with $z = \varphi_{s,u}(\underline{\omega})x$

$$\varphi_{s,t}(\underline{\omega})x - x = \int_s^t \left(\int_s^u 1\left((\varphi_{su}(\underline{\omega})x - \varphi_{vs}(\underline{\omega}) \circ \varphi_{su}(\underline{\omega})x), \circ d^+v\right), \circ d^+u \right) \tag{18}$$

Where, $\varphi_{sv}(\underline{\omega})^{-1} = \varphi_{vs}(\underline{\omega})$ and the u integral is Stratonovich over the v integral which, in turn, refers to a semi martingale. Hence, taking $\underline{\omega}$ dependence implicitly and employ the Stratonovich integral over he components

$$\varphi_{s,t}x - x = \int_s^t 1\left((u-s)\varphi_{su}x, \circ d^+u\right) - \int_s^t \left(\int_s^u 1\left(\varphi_{vs} \circ \varphi_{su}x, \circ d^+v\right), \circ d^+u \right) \tag{19}$$

Component wise splitting of the expression in (18) is allowed looking at the definition of forward Stratonovich in *Arnold* [1998] page 81. Suppose we take, $t = s + \tau$ then $\varphi_{st}(\underline{\omega}) = \varphi(\tau + s, \underline{\omega}) \circ \varphi(s, \underline{\omega})^{-1}$. Then, $\varphi(\tau + s, \underline{\omega}) = \varphi(\tau, \theta_s\underline{\omega}) \circ \varphi(s, \underline{\omega})$. This gives, using $\theta_s\underline{\omega} = \underline{\omega}_s$,

$$\varphi(\tau, \underline{\omega}_s)x - x = \int_0^\tau 1(u\varphi(u, \underline{\omega}_s)x, \circ d^+u) - \int_0^\tau \left(\int_0^u 1\left(\varphi_{v+s,s} \circ \varphi(u, \underline{\omega}_s)x, \circ d^+v\right), \circ d^+u \right) \tag{20}$$

With, $\varphi_{v+s,s}(\underline{\omega}) = \varphi(s, \underline{\omega}) \circ \varphi(v + s, \underline{\omega})^{-1}$ and from the cocycle for $\varphi(v + s, \underline{\omega})$, we see that via, $\varphi(v + s, \underline{\omega})^{-1} = \varphi(s, \underline{\omega})^{-1} \circ \varphi(v, \theta_s\underline{\omega})^{-1}$ it follows that $\varphi_{v+s,s}(\underline{\omega}) = \varphi(v, \theta_s\underline{\omega})^{-1}$. Because $\varphi(0, \cdot) = 1$, it follows using the cocyle that $\varphi(-v, \theta_v\underline{\omega}_s) \circ \varphi(v, \underline{\omega}_s)$ provides the left inverse of $\varphi(v, \underline{\omega}_s)$. It can be demonstrated that $\varphi(-v, \theta_v \underline{\omega}_s)$ is the right inverse when we note that $\varphi(v, \theta_{-v}\underline{\omega}_{s+v}) \circ \varphi(-v, \underline{\omega}_{s+v}) = 1$ and $\theta_{-v}\underline{\omega}_{s+v} = \underline{\omega}_s$. So we need a two-sided time development, see *Arnold* [1998].

5.2. Barometric pressure increase

From equation (20) it follows that a "naive" $\frac{\partial}{\partial \tau}$, denoted by dot, on the left and right hand and a subsequent $0 < \tau \to 0$ limit process provides , noting any $\underline{\omega}$ can be $\underline{\omega}_s$,

$$\lim_{0<\tau\to 0} \dot{x}_\tau = -\lim_{0<\tau\to 0} I_0^\tau(\varphi x) \tag{21}$$

Here,

$$I_0^\tau(\varphi) = \int_0^\tau 1\left(\varphi(-v, \theta_v \underline{\omega}_s) \circ \varphi(\tau, \underline{\omega}_s)x, \circ d^+ v\right) \tag{22}$$

and a "negative time" progression occurs in the integral. When a multiplicative topology, such as here is used the limit $0 < \tau \to 0$ unifies $\varphi(\tau, \underline{\omega}) \to 1$. Hence, the logarithmic derivative in $\tau = 0$ stands on the left hand side of (22) while the right hand side also can be written as

$$I_{0^-}^0(\varphi) = \lim_{0<\tau\to 0} \int_{-\tau}^0 1\left(\varphi(v, \theta_{-v}\underline{\omega}_s), \circ d^+ v\right) \tag{23}$$

From our computations it follows that the φ-barometric pressure curves can be approximated with a quadratic polynomial for a certain interval $0 < t < t_{max}$. Some coefficients (a_0, a_1, a_2) estimates from the data are presented in Tables 1, 2 and 3 below. If $\frac{a_1}{a_0} > 0$ then the curve is ascending in $0 < \tau < \tau_{max}$ and as a consequence the $I_{0^-}^0(\varphi) < 0$ nonzero. If $\frac{a_1}{a_0} < 0$ then the curve is descending in $0 < \tau < \tau_{max}$ and as a consequence the $I_{0^-}^0(\varphi) > 0$ but not vanishing to zero on a small time scale.

5.3. Numerical results

Hence, we see that the initial behavior of the Φ barometric pressure curve is, the other way around, determined by the outcome of the Stratonovich integral on an infinitesimal small time interval prior to the onset of the monotone approximately linear increase $P_\tau \propto P_0\tau$ with time τ. Defining a multiplicative factor in $\Delta W = x_0(1 - \frac{\varphi(\tau_{0.1kPa})}{\varphi(0)})$, with $\tau_{0.1kPa}$ the time it takes for an increase of 0.1 kPa on the horizontal barometric pressure axis. From computational experiments we found that the maximal *decrease* ΔW caused by $I^0_{0^-}(\varphi) > 0$ is in the order of 0.5 mNAP (at $N_{smp} = 2365$) and a maximal *increase* at $I^0_{0^-}(\varphi) < 0$ of ΔW in the order of 0.1 mNAP (at $N_{smp} = 2356$). If we go 5 minutes before and look at $N_{smp} = 2355$, then $I^0_{0^-}(\varphi) < 0$ causes $\Delta W \approx 0.009$. If we look at $N_{smp} = 2357$, i.e. 5 minutes later than $N_{smp} = 2356$, we still have $I^0_{0^-}(\varphi) < 0$ but the computations show a difference $\Delta W \approx 0.001$. If the numerical experiments correspond to real behavior of the dike then we could say that the jump of 0.1 mNAP is surrounded by lesser (or negligible) upward movements in water level because we think, in the physics of the soil, the $I^0_{0^-}(\varphi) < 0$ remains.

The question is what is causing a nonzero $I^0_{0^-}(\varphi)$. Obviously in a vanishingly small interval, the deterministic contribution to the $\neq 0$ of $I^0_{0^-}(\varphi)$ disappears. So the conclusion is that the $I^0_{0^-}(\varphi) \neq 0$ is related to the random factors that can still vary erratically in a small temporal interval.

Of course the vanishingly small time interval is only a mathematical ideal. In practice it means that stochastic influences are still active on time scales where deterministic influences are negligible. In other words the martingale seems to dominate the behavior of the water content in the dike shortly before a monotone barometric pressure increase sets in.

6. Conclusion

The main conclusion from the computational and mathematical stochastic analysis of dike time series is that monotone almost sure perfect not trivial cocycles can be found in the time series of water content related to fluctuations in temperature and barometric pressure. Temperature is measured inside the dike whereas barometric pressure is measured outside the dike.

If there occurs a more or less monotone increase in barometric pressure and there is a 0.1 kPa increase we uncovered the possibility of a maximum lowering of the order of 0.5 mNAP while maximum increase of 0.1 mNAP apparently also may occur. The raising or lowering response is uncontrollable and stochastic. We can put forth the hypothesis that the outcome of the competition of expulsion by air molecules versus compaction by decreasing pore size, co-occur during an increase in barometric pressure. The outcome of this competition is determined by "a coin flip".

Let us furthermore note that monotone increase in barometric pressure measurements is most likely caused by susbsidence in atmospheric pressure when the atmospheric temperature decreases. In addition, there exists the possibility of a small diurnal change of atmospheric pressure from the atmospheric tide caused by UV heating and cooling effects related to the rotation of the earth towards or away from the face of the sun *LeBlanc* [2011]. Of course the "delta" of atmospheric tide can be small if change is uniformly distributed over 24h. On the other hand we found effects in the range of the atmospheric tide of 0.1 kPa. It depends on how the pressure changes develop in the atmospheric tide.

In a more jump like development caused by clouds or other unexpected conditions the 0.1 kPa can set in more quickly. It is nototed that the atmospheric pressure subsidence caused by a cooling down of the atmosphere tends to coincide with a more clear sky and is most likely the important factor for level variation caused by barometric pressure increase.

On the practical side, for maintenance of the water content in a dike and in combination with other conditions like high tide or meteorological water, a sub-critical level for adding water to or subtracting water from the dike incorporating the effect of pressure increase would contribute to its safety. The pressure effect is not the most critical but, combined with other factors, criticality can be reached earlier than expected because of the pressure effect. Moreover, it is possible that submarine groundwater discharges, which are also effective in a delta environment *Taniguchi et al* [2008], can with a nonzero probability be furthered by the influence of the found barometric pressure increase on water height in a dike.

Notes

1. NAP indicates New Amsterdam water level which is a zero determining water level well known in the Netherlands.

2. 1Pa=1Pascal=1$Nm^{-2} \approx 10 kgs^{-2}m^{-1}$.

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Figure 1. Maximum pressure - generator φ relation in the raw model for sample point $N = 3130$.

Table 1. coefficients for $0 < \tau < \tau_{max}$ with sample point $N_{smp} = 2350$

Parameters	Value	Estim stdev	t-value	$Pr(\|t\| >)$	signif
a_0	1.008	5.831×10^{-6}	172802.3	$< 2 \times 10^{-6}$	***
a_1	-1.313×10^{-2}	2.694×10^{-5}	-487.4	$< 2 \times 10^{-6}$	***
a_2	6.02×10^{-3}	2.608×10^{-5}	231.0	$< 2 \times 10^{-6}$	***

[a] Quadratic polynomial $\varphi(\tau) = a_0 + a_1\tau + a_2\tau^2$ for $0 < \tau < \tau_{max}$ estimated with nls from R.

[b] Initially the $\Phi[\cdot, o_1, o_2] \sim P[\cdot]$ curve has a downward slope

Figure 2. Maximim pressure - generator φ relation in the model for sample point $N = 3130$ after uniform adding $\delta P = 0.5$kPa and cocycle transformation.

Table 2. coefficients for $0 < \tau < \tau_{max}$ with sample point $N_{smp} = 3130$

Parameters	Value	Estim stdev	t-value	$Pr(\|t\| >)$	signif
a_0	0.9651742	0.0004	2607.93	$< 2 \times 10^{-6}$	***
a_1	0.1156603	0.00171	67.65	$< 2 \times 10^{-6}$	***
a_2	-0.0798858	0.0017	-48.27	$< 2 \times 10^{-6}$	***

[a] Quadratic polynomial $\varphi(\tau) = a_0 + a_1\tau + a_2\tau^2$ for $0 < \tau < \tau_{max}$ estimated with nls from R.

[b] Initially the $\Phi[\cdot, o_1, o_2] \sim P[\cdot]$ curve has an upward slope

Figure 3. Maximum pressure - generator φ relation in the model for sample point $N = 3130$ after uniform subtracting $\delta P = 0.5$kPa from the previous, thereby returning to the original pressure distribution in time, and subsequent cocycle transformation. Maximum-y is 1.00097. Minimum-y is 0.96114.

Table 3. coefficients for $0 < \tau < \tau_{max}$ with sample point $N_{smp} = 3145$

Parameters	Value	Estim stdev	t-value	$Pr(\|t\| >)$	signif
a_0	1.000	1.600×10^{-10}	6250395144	$< 2 \times 10^{-6}$	***
a_1	-1.538×10^{-3}	7.393×10^{-10}	-2080852	$< 2 \times 10^{-6}$	***
a_2	9.157×10^{-4}	7.157×10^{-10}	1279499	$< 2 \times 10^{-6}$	***

[a] Quadratic polynomial $\varphi(\tau) = a_0 + a_1\tau + a_2\tau^2$ for $0 < \tau < \tau_{max}$ estimated with nls from R.

[b] Initially the $\Phi[\cdot, o_1, o_2] \sim P[\cdot]$ curve has a downward slope