A categorical approach for relativity theory

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Abstract

We give a categorical interpretation for a model we have previously developed, which combines aspects of the Galilei and the special relativity and that is based on a broad class of transformations - the Generalized Lorentz transformation. Particularly, we show how the standard Lorentz transformation of the special relativity satisfies a kind of universal property factoring out the Generalized Lorentz transformation through a certain redefinition of velocity. We also show how the Galilei transformation and the Generalized Lorentz transformation are related in terms of a multi-valued natural transformation that in some sense unify both relativities.

1 Introduction

In a previous work [1] we developed a model relating some kinematical aspects from the Galilei and the special relativity (SR), namely, given two inertial reference frames S and S' moving with relative velocity $\vec{v}$ we showed how the Galilean transformation of coordinate and velocity given by

$$
\begin{aligned}
&\vec{x}\,' = \vec{x} - \vec{v}\tau, \quad \tau' = \tau \\
&\vec{u}\,' = \vec{u} - \vec{v}
\end{aligned}
$$

with $\vec{u} := \frac{d\vec{x}}{d\tau}$, $\;\vec{u}\,' := \frac{d\vec{x}\,'}{d\tau}$ induce the corresponding coordinate and velocity transformations of the SR

$$
\begin{aligned}
&\vec{x}\,' = \vec{x} - (1-\gamma_{\widetilde{v}})\frac{\vec{x}\cdot\vec{\widetilde{v}}}{\widetilde{v}^2}\vec{\widetilde{v}} - \gamma_{\widetilde{v}} t\vec{\widetilde{v}}\,, \qquad t' = \gamma_{\widetilde{v}}\Big(t - \frac{\vec{x}\cdot\vec{\widetilde{v}}}{c^2}\Big) \\
&\vec{\widetilde{u}}\,' = \frac{\vec{\widetilde{u}} - \gamma_{\widetilde{v}}\vec{\widetilde{v}} - (1-\gamma_{\widetilde{v}})\frac{\vec{\widetilde{u}}\cdot\vec{\widetilde{v}}}{\widetilde{v}^2}\vec{\widetilde{v}}}{\gamma_{\widetilde{v}}\big(1 - \frac{\vec{\widetilde{u}}\cdot\vec{\widetilde{v}}}{c^2}\big)}
\end{aligned}
$$

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where $\vec{\tilde{u}} := \frac{d\vec{x}}{dt}$, $\vec{\tilde{u}}\,' := \frac{d\vec{x}\,'}{dt'}$ and $\gamma_{\tilde{v}} = 1/\sqrt{1 - \frac{\tilde{v}^2}{c^2}}$. We obtained this result in [1] by reevaluating the role played by the concept of absolute time of the Galilei relativity, which currently became superseded by the interpretation given by the SR. In fact, in SR there is no concept of absolute time, the latter being understood as any time variable transforming as $\tau = \tau'$. The closer we can get to such a transformation is when we consider the low speed limit $\tilde{v} \ll c$ of the relative motion between two inertial reference frames, a circunstance where both frames would register the same time for the ocurrence of an event, $t \simeq t'$, as we can see by neglecting terms of order $\geq \frac{\tilde{v}^2}{c^2}$ in the time transformation law, for example,

$$t' = \gamma_{\tilde{v}}\Big(t - \frac{\vec{x}\cdot\vec{\tilde{v}}}{c^2}\Big) = \Big(1 - \frac{1}{2}\frac{\tilde{v}^2}{c^2} + \dots\Big)(t - \frac{1}{c^2}\vec{x}\cdot\vec{\tilde{v}}) \simeq t\ .$$

We also obtain as limit cases for the other special relativity transformations the following form

$$\vec{x}\,' = \vec{x} - \vec{\tilde{v}}t, \quad \vec{\tilde{u}}' = \vec{\tilde{u}} - \vec{\tilde{v}}\ ,$$

which are similar to the Galilean laws since in this low velocity limit we also have $\vec{\tilde{u}} = \vec{u}$, $\vec{\tilde{v}} = \vec{v}$, as we have seen in [1].

There is, however, a possible way to give a concrete representation for the absolute time that goes beyond this low speed limit as envisaged by the SR. This is achieved by making some asssumptions borrowing elements from both relativities and that consists essentially on assuming there are two ways of registering the time, one based on the absolute time τ that obeys the laws of the Galilei relativity, and another based on the local time t that is suitable to the laws of the SR. As a consequence of these assumptions we end up with a class of transformations - the Generalized Lorentz transformation - that includes, as a particular case, the standard Lorentz transformation of the SR together with other transformations denoted by h that work as shifting the main elements of the Galilean relativity, for example, the Galilean coordinate system, the Galilean transformation, and so on to the corresponding elements of the Special relativity as shown in the diagram below (we review this construction in section 3)

$$\begin{tikzcd}
(\tau, \vec{x}) \arrow[r, "\text{Galilei}"] \arrow[d, "h"] & (\tau, \vec{x}\,') \arrow[d, "h'"] \\
(t, \vec{x}) \arrow[r, "\text{Lorentz}"] & (t', \vec{x}\,')\ .
\end{tikzcd} \tag{1}$$

It is the purpose of our work to investigate if the relations involving the kinematical aspects analyzed in our previous work [1] and represented schematically in diagram (1) reveal some sort of mathematical structure existing between the Galilei relativity and the

SR. In fact, all our effort here is to show that the vertical maps h, h' shown in diagram (1) define a *natural transformation* between some appropriate functors $\overline{G}, \overline{L} : \mathcal{S} \to \mathcal{C}$, where the categories $\mathcal{S}$ and $\mathcal{C}$ are introduced in order to characterize the distinct but complementary roles played in physics by the concepts of *inertial reference frame* and *coordinate system*. As we will see, these concepts become two stages on a process of modelling the physical world into a categorical way of thinking that elucidates, from a mathematical perspective, a connection between the formalisms of the Galilei relativity and the SR.

Our work is organized as follows. In section 2 we discuss the concepts of event, inertial reference frame, and coordinate system in a sense that allow us to categorize them, i.e. to define the categories of inertial reference frames and of coordinate systems. In section 3 we summarize the formalism we developed in [1], which shows a connection between the Galilei and the Generalized Lorentz transformation as indicated in diagram (1). In section 4 we introduce two particular ways to describe physical phenomena that constitutes the Galilean and the Lorentzian coordinate systems. In section 5 we develop a convenient categorical formulation for the elements introduced in sections 3 and 4 in order to reinterpret the model we developed in [1] and to clarify within a categorical perspective the unification of the Galilei relativity and the SR through a sort of multi-valued natural transformation uniting the Galilei and the Generalized Lorentz transformation.

In our work the term relativity reffers indistinctly to the Galilei relativity and to the special relativity. The constant c always refer to the speed of light in vacuum.

A complete and very readable reference for relativity is [3], [4]. A general reference for category is [5], [6].

2 Some basics concepts

Relativity is concerned with the description of physical occurrences in space and time and how different observers relate these descriptions among themselves. Then, we start with a brief exposition of some basic concepts such as event, inertial reference frame, and coordinate system.

An *event* is any occurrence of a physical phenomenon that may be described by the position *where* it takes place and the instant of time *when* it happens. In this definition it is implict that an event is any idealized physical phenomenon that occurs localized in space and time, i.e. it is any occurrence without extension and duration. In this sense a solid body is not an event since it has an extension, while a particle is not an event since it has a duration or, equivalently, it persists in time. However, single processes of creation and annihilation of a particle are both events.

An *inertial reference frame* (or for brevity, a reference frame) is any material body very small in size and free of forces that can be used as a reference point relative to which we can determine the position of other material bodies. This reference frame relates to others in two ways: they are either at rest or in relative motion with constant velocity. For simplicity, reference frames that are at rest relative to each other will be considered as equivalent in the sense we may take any one of them as representing all. We denote reference frames by $S, S', S'', \ldots$.

A *coordinate system* (on a reference frame) consists on any system of rules and clocks attached to a reference frame and used for measuring lengths and intervals of time together with an analytical way that refers these measurements to a 4-upla of numbers (that may be achieved, for instance, by adopting rectangular or curvilinear coordinates), which provides a consistent way for registering events. Physically, the establishment of a coordinate system may be thought of as an *idealized* process where by taking a reference frame as a material body we can attach to it a system of three mutually perpendicular axis and by using the rules we form a grid in space, together with a system of clocks rigidly attached to each point of the grid that will register the instant of time of events ocurring at the position where the clock is placed.

On setting a coordinate system, depending on how we measure the time, we will fix our attention on two types of coordinates systems that we call the Galilean system and the Lorentzian system. We will analyze them in section 4.

In practical terms, it is the coordinate system that makes a reference frame effective for the task of recording events telling us how we measure the position and the time associated to an event.

3 The unified scheme for the Galilei and the Special relativity

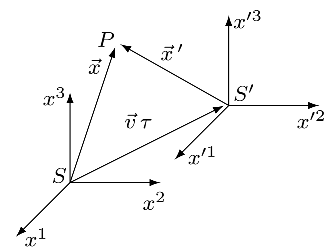
In a previous work [1], we developed a scheme unifying the Galilei and the special relativity It consisted essentially in describing time through two perspectives, one based on the Galilei relativity, where time has an *absolute* character, and another based on the special relativity, where time has a *local* character. The precise meaning of these terms will be given shortly after examining their transformation properties and how they are derived from assumptions taken from both relativities. Let us denote these two time variables by τ and t, which we call respectively the absolute and the local time.

3.1 The Axioms

We take as axioms the following assumptions:

I. Events are described relative to a reference frame S by specifying coordinates $\{\tau, t, \vec{x}\}$ with $\vec{x} = (x^1, x^2, x^3)$ standing for spatial coordinates. The multitude of all events form the *physical world*, which is represented as points of an abstract four-dimensional space labelled either as $x^\mu_G \equiv (x^0_G, x^i_G) := (\tau, \vec{x})$ or $x^\mu_L \equiv (x^0_L, x^i_L) := (t, \vec{x})$. In this sense, the use of coordinates x^μ_G or x^μ_L are just two forms of conceiving the physical world.

Relative to two frames S, S' moving with uniform velocity $\vec{v}$ as shown in the figure



an event P is described by coordinates sets $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$ where we assume additionaly

II. *The Galilei relativity law*

$$\vec{x}\,' = \vec{x} - \vec{v}\tau \ , \qquad \tau' = \tau \ .$$

III. *The invariance of the quadratic form* $Q(ct, x) := c^2t^2 - \vec{x}^2$ *of special relativity*

$$c^2t^2 - \vec{x}^2 = c^2t'^2 - \vec{x}\,'^{\,2} \ .$$

IV. *The relation between the local times* t *and* t' *is of the type*

$$t' = at + b\,\vec{v}\cdot\vec{x}$$

with a and b two arbitrary real parameters.

Remark: Given a reference frame we use the term physical world rather than *spacetime* to denote the abstract space formed by the ocurrences of all events as seen by an observer at rest in that frame. The distinction we do here is due to the fact we are not endowing the physical world with any a priori metric structure as it is assumed when one uses the term spacetime.

3.2 The coordinate transformations

Working with these axioms we have shown in [1] that we may fix $b = \frac{\sqrt{a^2-1}}{vc}$, and then obtain the transformation between $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$ as

$$(t, \vec{x}) \overset{L(a(v),\vec{v})}{\longrightarrow} (t', \vec{x}\,') : \begin{cases} \vec{x}\,' = \vec{x} - (1 - a(v))\frac{1}{v^2}\,\vec{x}\cdot\vec{v}\,\vec{v} - \sqrt{a(v)^2 - 1}\frac{1}{v}\,c\,t\,\vec{v} \\ t' = a(v)t - \sqrt{a(v)^2 - 1}\frac{1}{vc}\,\vec{x}\cdot\vec{v} \end{cases} \tag{2}$$

with

$$\tau = (1-a(v))\frac{\vec{x}\cdot\vec{v}}{v^2} + \sqrt{a(v)^2-1}\,\frac{c}{v}\,t = (1-a(v'))\frac{\vec{x}\,'\cdot\vec{v}\,'}{v'^2} + \sqrt{a(v')^2-1}\,\frac{c}{v'}\,t' = \tau' \tag{3}$$

and $\vec{v}\,' = -\vec{v}$, with $a(v)$ being an arbitrary real valued function whose only condition is that $|a(v)| > 1$. Note that $a(v') = a(v)$. We call transformation (2) the *Generalized Lorentz Transformation* (GLT). It satisfies the following universal property:

▶ *Given the GLT $L(a(v),\vec{v})$ depending on an arbitrary function $a(v)$ with $|a(v)| > 1$, it exists a Lorentz transformation $\mathcal{L}(\vec{\tilde{v}})$*

$$(t,\vec{x}) \xrightarrow{\mathcal{L}(\vec{\tilde{v}})} (t',\vec{x}\,') : \begin{cases} \vec{x}\,' = \vec{x} - (1-\gamma_{\tilde{v}})\frac{\vec{x}\cdot\vec{\tilde{v}}}{\tilde{v}^2}\vec{\tilde{v}} - \gamma_{\tilde{v}} t\vec{\tilde{v}} \\ t' = \gamma_{\tilde{v}}\big(t - \frac{\vec{x}\cdot\vec{\tilde{v}}}{c^2}\big) \end{cases} \tag{4}$$

with $\gamma_{\tilde{v}} = \frac{1}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}}$ and a unique transformation $\vec{v} \to \vec{\tilde{v}}$

$$\vec{\tilde{v}} = c\frac{\sqrt{a(v)^2-1}}{a(v)}\frac{\vec{v}}{v} \tag{5}$$

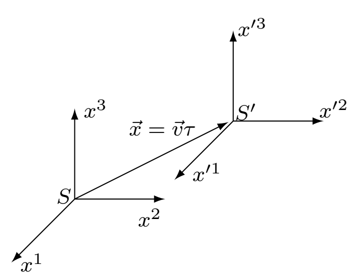
that renders the following diagram commutative

$$\begin{tikzcd} \mathbb{R}^4 \ni (t,\vec{x}) \arrow[r, "L(a(v),\vec{v})"] \arrow[dr, "\mathcal{L}(\vec{\tilde{v}})"'] & (t'(v),\vec{x}\,'(v)) \in \mathbb{R}^4 \arrow[d, "\vec{v}\to\vec{\tilde{v}}"] \\ & (t'(\tilde{v}),\vec{x}\,'(\tilde{v})) \in \mathbb{R}^4 \ \blacksquare \end{tikzcd} \tag{6}$$

Remarks:

1. We see from axiom II (and we also recover this again from equation (3)) that the absolute time τ associated to the occurrence of an event is the same for both observers, $\tau = \tau'$, and it is this feature that suggests us to call τ the absolute time.

2. There is an important distinction to be made between $\vec{v}$ and $\vec{\tilde{v}}$. Consider in axiom II a sucession of events represented by the movement of the origin of frame S' as seen by frame S as shown in the figure below



We have $\vec{x}\,' = 0$ and then $\vec{x} = \vec{v}\tau$ which gives $\vec{v} := \frac{d\vec{x}}{d\tau}$. Then, identifying $\vec{\tilde{v}} := \frac{d\vec{x}}{dt}$ and using (3) we obtain the relation between $\vec{v}$ and $\vec{\tilde{v}}$ as given in (5). Now, since the parameter a in the GLT is an arbitrary function of v the explicit form between $\vec{\tilde{v}}$ and $\vec{v}$ will be determined only when we fix a particular form for $a(v)$. However, no matter what the form of $a(v)$ is, when it is expressed in terms of $\vec{\tilde{v}}$, we obtain that $a(\tilde{v}) = 1/\sqrt{1-\frac{\tilde{v}^2}{c^2}}$ and this is what gives to the ordinary Lorentz transformation $\mathcal{L}(\vec{\tilde{v}})$ a universal character as manifested in the diagram (6).

3. The GLT incorporates aspects of both relativities. In fact, with respect to the SR we notice the GLT includes the ordinary Lorentz transformation as a particular case, for if we impose that $\vec{\tilde{v}} = \vec{v}$ then equation (5) fixes $a(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and with this choice the GLT $L(a(v), \vec{v})$ becomes $\mathcal{L}(\vec{v})$

$$\mathcal{L}(\vec{v}) : \begin{cases} \vec{x}\,' = \vec{x} - (1-\gamma_v)\frac{1}{v^2}\,\vec{x}\cdot\vec{v}\,\vec{v} - \gamma_v\, t\,\vec{v} \\ t' = \gamma_v(t - \frac{\vec{x}\cdot\vec{v}}{c^2})\ . \end{cases} \tag{7}$$

As for the Galilei relativity we observe that axioms II, III, IV essentially turn the Galilei relativity law $\vec{x}\,' = \vec{x} - \vec{v}\tau$ into the form given by equation (2) upon identifying τ as given in (3).

3.3 The velocity transformation

There is a similar universal property for the velocity transformation. Let us consider $\vec{u} = \frac{d\vec{x}}{d\tau}$ and $\vec{\tilde{u}} = \frac{d\vec{x}}{dt}$. Then,

$$\vec{\tilde{u}} = \frac{d\vec{x}}{dt} = \frac{d\vec{x}}{d\tau}\frac{d\tau}{dt} \Rightarrow \vec{u} = \left[(1-a)\frac{\vec{\tilde{u}}\cdot\vec{v}}{v^2} + \sqrt{a^2-1}\frac{c}{v}\right]\vec{u}$$

or equivalently

$$\vec{\tilde{u}} = \frac{\sqrt{a^2-1}}{\left[1-(1-a)\frac{\vec{u}\cdot\vec{v}}{v^2}\right]}\frac{c}{v}\vec{u}\ . \tag{8}$$

Considering in a similar way $\vec{u}\,' = \frac{d\vec{x}\,'}{d\tau}$ and $\vec{\tilde{u}}\,' = \frac{d\vec{x}\,'}{dt}$ we obtain

$$\vec{\tilde{u}}\,' = \frac{\sqrt{a^2-1}}{\left[1+(1-a)\frac{\vec{u}\,'\cdot\vec{v}}{v^2}\right]}\frac{c}{v}\vec{u}\,'\ . \tag{9}$$

Now, taking the derivative relative to the absolute time in axiom II we obtain the velocity law of the Galilei relativity, $\vec{u}\,' = \vec{u} - \vec{v}$, and using (8, 9) in this expression we obtain the *Generalized Lorentz Transformation for Velocity* (GLTV), which we denote by $L^*(a(v), \vec{v})$

$$\vec{\tilde{u}} \stackrel{L^*(a(v),\vec{v})}{\longrightarrow} \vec{\tilde{u}}\,' = \frac{\vec{\tilde{u}} - \sqrt{a^2-1}\frac{c}{v}\vec{v} - (1-a)\frac{\vec{\tilde{u}}\cdot\vec{v}}{v^2}\vec{v}}{a - \sqrt{a^2-1}\frac{1}{cv}\vec{\tilde{u}}\cdot\vec{v}}\ . \tag{10}$$

We also have a universal property for the velocity that has the form

▶ *Given the GLTV $L^*(a(v), \vec{v})$ depending on an arbitrary function $a(v)$ with $|a(v)| > 1$, it exists a Lorentz transformation for velocity $\mathcal{L}^*(\vec{\vec{v}})$*

$$\vec{\vec{u}} \xrightarrow{\mathcal{L}^*(\vec{\vec{v}})} \vec{\vec{u}}\,' = \frac{\vec{\vec{u}} - \gamma_{\vec{\vec{v}}}\vec{\vec{v}} - (1-\gamma_{\vec{\vec{v}}})\frac{\vec{\vec{u}}\cdot\vec{\vec{v}}}{\vec{\vec{v}}^2}\vec{\vec{v}}}{\gamma_{\vec{\vec{v}}}\left(1 - \frac{\vec{\vec{u}}\cdot\vec{\vec{v}}}{c^2}\right)} \tag{11}$$

and a unique transformation $\vec{v} \to \vec{\vec{v}}$ (5) that renders the following diagram commutative

$$\begin{tikzcd}
\mathbb{R}^3 \ni \vec{\vec{u}} \arrow[r, "L^*(a(v),\vec{v})"] \arrow[dr, "\mathcal{L}^*(\vec{\vec{v}})"'] & \vec{\vec{u}}\,'(\vec{v}) \in \mathbb{R}^3 \arrow[d, "\vec{v}\to\vec{\vec{v}}"] \\
& \vec{\vec{u}}\,'(\vec{\vec{v}}) \in \mathbb{R}^3 \ \blacksquare
\end{tikzcd} \tag{12}$$

Here, considering the possible values for the velocities $\vec{u}$ and $\vec{\vec{u}}$ as vectors in $\mathbb{R}^3$ we also have a commutative diagram of the type

$$\begin{tikzcd}
\mathbb{R}^3 \ni \vec{u} \arrow[r, "\text{Galilei}"] \arrow[d, "h^*"] & \vec{u}\,' \in \mathbb{R}^3 \arrow[d, "h'^*"] \\
\mathbb{R}^3 \ni \vec{\vec{u}} \arrow[r, "\text{Lorentz}"] & \vec{\vec{u}}\,' \in \mathbb{R}^3
\end{tikzcd} \tag{13}$$

with h^* and h'^* given by (8, 9). Contrarily to the case of the coordinate transformation we will not pursue in our work the task of giving a categorical interpretation for the diagram (13).

4 The Galilean and Lorentzian systems

In section 2 we defined a coordinate system on a reference frame S in terms of a set of rules and clocks allowing us to register the ocurrence of events. Since we assumed two forms for registering the time we are expected to have two distinguished coordinate systems for the same reference frame, each one providing a convenient description for the physical world.

Let S be a inertial reference frame. We then define:

⌟ A *Galilean coordinate system* in S is a coordinate system in which the points of the physical world are described as $(x_G^0, x_G^i) = (\tau, \vec{x})$.

⌟ A *Lorentzian coordinate system* in S is a coordinate system in which the points of the physical world are described as $(x_L^0, x_L^i) = (t, \vec{x})$.

From (3) we have

$$t = \frac{v}{c\sqrt{a(v)^2 - 1}}\left\{\tau - (1 - a(v))\frac{\vec{x}\cdot\vec{v}}{v^2}\right\}, \tag{14}$$

and since $\vec{x}_G = \vec{x}_L = \vec{x}$, the relation between these two coordinates systems defined for the same frame S is determined essentially from (14). Then, we define the transformation

$$(x_G^0, x_G^i) \xrightarrow{h_{\vec{v}}} (x_L^0, x_L^i) : \begin{cases} x_L^0 = \frac{v}{c\sqrt{a(v)^2-1}}\left\{x_G^0 - (1-a(v))\frac{1}{v^2}\vec{x}_G \cdot \vec{v}\right\} \\ \vec{x}_L = \vec{x}_G \ . \end{cases} \tag{15}$$

Given another frame S' endowed with Galilean and Lorentzian coordinates denoted by $(x_G'^0, x_G'^i) = (\tau', \vec{x}\,')$, $(x_L'^0, x_L'^i) = (t', \vec{x}\,')$ with S' moving with velocity $\vec{v}$ relative to S we also have defined the transformation $h_{\vec{v}\,'}$: $(x_G'^0, x_G'^i) \xrightarrow{h_{\vec{v}\,'}} (x_L'^0, x_L'^i)$ (with $\vec{v}\,' = -\vec{v}$). The maps $h_{\vec{v}}$ and $h_{\vec{v}\,'}$ are such that the following diagram is commutative

$$\begin{tikzcd} (x_G^0, x_G^i) \arrow[r, "G(\vec{v})"] \arrow[d, "h_{\vec{v}}"] & (x_G'^0, x_G'^i) \arrow[d, "h_{\vec{v}\,'}"] \\ (x_L^0, x_L^i) \arrow[r, "L(a(v),\vec{v})"] & (x_L'^0, x_L'^i) \end{tikzcd} \tag{16}$$

i.e. they satisfy

$$h_{\vec{v}\,'} \circ G(\vec{v}) = L(a(v), \vec{v}) \circ h_{\vec{v}} \tag{17}$$

where $G(\vec{v})$ is the Galilei transformation as given in axiom II

$$(x_G^0, x_G^i) \xrightarrow{G(\vec{v})} (x_G'^0, x_G'^i) : \begin{cases} x_G'^0 = x_G^0 \\ \vec{x}\,'_G = \vec{x}_G - \vec{v} x_G^0 \end{cases}$$

and $L(a(v), \vec{v})$ is the Generalized Lorentz transformation (2)

$$(x_L^0, x_L^i) \xrightarrow{L(a(v),\vec{v})} (x_L'^0, x_L'^i) : \begin{cases} \vec{x}\,'_L = \vec{x}_L - (1-a(v))\frac{1}{v^2}\,\vec{x}_L \cdot \vec{v}\,\vec{v} - \sqrt{a(v)^2-1}\frac{1}{v}\,c\,x_L^0\,\vec{v} \\ x_L'^0 = a(v)x_L^0 - \sqrt{a(v)^2-1}\frac{1}{vc}\,\vec{x}_L \cdot \vec{v}\ . \end{cases} \tag{18}$$

Remarks:

1. Equation (14) is written in terms of the velocity $\vec{v}$ between the frames S and S', therefore it suggests the local time t depends on the state of motion of the frames. A similar interpretation for the local time was invstigated by Horwitz, Arshansky, and Elitzur in their work [2] (pg. 1163), which we refer the reader for further details. Using a different argument than ours and limiting themselves to the framework of SR alone they also infer that *"in relativity then, the time at which an event occurs depends on the state of motion of the frame (and the clocks attached to it)"*. Here, from the form of the expression we obtained in (14) it would be more precise to say that the time at which an event occurs depends on the state of motion of the frame relative to another frame, which is explicitly represented by the relative velocity $\vec{v}$ between the frames as shown in (14).

2. We notice that on setting these two coordinate systems for the physical world, the coordinate x^0 has a dimension of time, while the space coordinate $\vec{x}$ has the dimension of length. In many treatments it is desireable to have all coordinates with the same dimension, which is usually done identifying x^0 as $c\tau$ or ct with c being the speed of light. However, since this convention of having space and time with the same dimension has no effect in our framework we will treat them as being dimensionless.

3. From (14) we assume the local times x_L^0 and $x_L'^0$ appearing in the form of the GLT (18) depends respectively on $\vec{v}$ and $\vec{v}\,'$ with $\vec{v}\,' = -\vec{v}$, which suggest us to use the following notation for the GLT $(x_L^0, x_L^i, \vec{v}) \overset{L(a(v),\vec{v})}{\longrightarrow} (x_L'^0, x_L'^i, \vec{v}\,')$. The inverse transformation would then be written in the form

$$(x_L'^0, x'^i_L, \vec{v}\,') \overset{L(a(v'),\vec{v}\,')}{\longrightarrow} (x_L^0, x_L^i, \vec{v}) : \begin{cases} \vec{x}_L = \vec{x}\,'_L - (1 - a(v'))\frac{\vec{x}\,'_L \cdot \vec{v}\,'}{v'^2}\,\vec{v}\,' - \sqrt{a(v')^2 - 1}\frac{1}{v'}\,c\,x_L'^0\,\vec{v}\,' \\ x_L^0 = a(v')x_L'^0 - \sqrt{a(v')^2 - 1}\frac{1}{v'c}\,\vec{x}\,'_L \cdot \vec{v}\,' \,. \end{cases}$$

We will use this notation for the GLT in the next sections.

5 Relativity under a categorical perspective

Now, we will interpret the physical elements we have previously introduced using the language of categories. We do this in four stages that we now describe.

5.1 The category of inertial reference frames

In this first stage we focus on the notion of inertial reference frame.

We define the category $\mathcal{S}$ of inertial reference frames as follows.

⌟ $\mathrm{Obj}_{\mathcal{S}}$ is the class of all reference frames $S, S', S'' \ldots$

⌟ $\mathrm{Morf}_{\mathcal{S}}(S, S')$ is a set with only one element that we denote by $S \xrightarrow{\vec{v}_{SS'}} S'$ and we understand as the relative velocity of frame S' as seen by frame S. For ease of notation we also write the morphism $S \xrightarrow{\vec{v}_{SS'}} S'$ as $\vec{v}_{SS'}$.

⌟ Composition $\circ$ is defined by

$$\left(S' \xrightarrow{\vec{v}_{S'S''}} S''\right) \circ \left(S \xrightarrow{\vec{v}_{SS'}} S'\right) = \left(S \xrightarrow{\vec{v}_{SS''}} S''\right) . \tag{19}$$

Here the velocity $\vec{v}_{SS''}$ is the velocity of the frame S'' as seen by the frame S and we identify $\vec{v}_{SS''}$ in terms of the Galilei relativity law: $\vec{v}_{SS''} = \vec{v}_{SS'} + \vec{v}_{S'S''}$.

⌟ To the object S we have the identity $S \xrightarrow{\vec{v}_{SS}} S$ where $\vec{v}_{SS} = 0$.

From a physical perspective if $\vec{v}_{SS'} \neq 0$ a morphism $S \xrightarrow{\vec{v}_{SS'}} S'$ relates different inertial reference frames S, S' that are considered equally well-suited to describe the physical world.

5.2 The category of coordinate systems

In this second stage we endow the inertial reference frames previously introduced with a *quantitative* way to describe the physical world allowing us to register events representing the occurence of physical phenomena.

We define the category $\mathcal{C}$ of coordinate systems as follows.

$\lrcorner$ Obj$_{\mathcal{C}}$ is the class of coordinate systems of the type $(x^0_G, \vec{x}_G)$, $(x'^0_G, \vec{x}\,'_G)\ldots$, $(x^0_L, \vec{x}_L, \vec{v})$, $(x'^0_L, \vec{x}\,'_L, \vec{v}\,'), \ldots$

We use the notation $(x^0_L, \vec{x}_L, \vec{v})$, instead of $(x^0_L, \vec{x}_L)$, in order to recall that the local time x^0_L is registered by a clock that is rigidly attached to a frame and its value depends on the state of motion of the frame as shown in (14).

$\lrcorner$ The morphisms of $\mathcal{C}$ are formed from the maps below, whose form we have defined in section 4:

$(x^0_G, \vec{x}_G) \xrightarrow{G(\vec{v})} (x'^0_G, \vec{x}\,'_G)$

$(x^0_G, \vec{x}_G) \xrightarrow{h_{\vec{v}}} (x^0_L, \vec{x}_L, \vec{v})$

$(x^0_L, \vec{x}_L, \vec{v}) \xrightarrow{h^{-1}_{\vec{v}}} (x^0_G, \vec{x}_G)$

$(x^0_L, \vec{x}_L, \vec{v}) \xrightarrow{L(a(v), \vec{v} = -\vec{v}\,')} (x'^0_L, \vec{x}\,'_L, \vec{v}\,')$

and their compositions whenever they are defined, for instance,

$$\left[(x'^0_G, \vec{x}\,'_G) \xrightarrow{G(\vec{v}\,')} (x''^0_G, \vec{x}\,''_G)\right] \circ \left[(x^0_G, \vec{x}_G) \xrightarrow{G(\vec{v})} (x'^0_G, \vec{x}\,'_G)\right] = (x^0_G, \vec{x}_G) \xrightarrow{G(\vec{v}'')} (x''^0_G, \vec{x}\,''_G)$$

with $\vec{v}\,'' = \vec{v} + \vec{v}\,'$, and

$$\left[(x'^0_L, \vec{x}\,'_L, \vec{v}\,') \xrightarrow{L(a(v'), \vec{v}\,' = -\vec{v})} (x^0_L, \vec{x}_L, \vec{v})\right] \circ \left[(x^0_L, \vec{x}_L, \vec{v}) \xrightarrow{L(a(v), \vec{v} = -\vec{v}\,')} (x'^0_L, \vec{x}\,'_L, \vec{v}\,')\right] = Id_{(x'^0_L, \vec{x}\,'_L, \vec{v}\,')}\ ,$$

and so on. We will call *coordinate transformations* the morphisms of $\mathcal{C}$ between coordinate systems of the same type that are represented either by the Galilei or by the Generalized Lorentz transformations.

5.3 Introducing the Galilei and the Lorentz functors $\overline{G}, \overline{L} : \mathcal{S} \to \mathcal{C}$

The category $\mathcal{C}$ is too large for our purposes, therefore in this third stage we define two functors in order to distinguish within the category $\mathcal{C}$ two classes of coordinate systems and coordinate transformations that form the core of relativity: the Galilei transformation and the Lorentz transformation.

5.3.1 The Galilei functor

The *Galilei functor* $\overline{G} : \mathcal{S} \to \mathcal{C}$ is defined as

$\lrcorner$ Obj$_{\mathcal{S}} \xrightarrow{\overline{G}}$ Obj$_{\mathcal{C}}$

$S \xrightarrow{\overline{G}} \overline{G}(S) := (x^0_G, \vec{x}_G)$

$\lrcorner$ $\mathrm{Morf}_{\mathcal{S}}(S, S') \xrightarrow{\overline{G}} \mathrm{Morf}_{\mathcal{C}}(\overline{G}(S), \overline{G}(S'))$

$S \xrightarrow{\vec{v}_{SS'}} S' \xrightarrow{\overline{G}} \overline{G}(S \xrightarrow{\vec{v}_{SS'}} S') := (x^0_G, \vec{x}_G) \xrightarrow{G(\vec{v}_{SS'})} (x'^0_G, \vec{x}\,'_G)$

$\lrcorner$ $\overline{G}$ is a covariant functor as it is immediately seen from (19)

$$\overline{G}\Big(\big(S' \xrightarrow{\vec{v}_{S'S''}} S''\big) \circ \big(S \xrightarrow{\vec{v}_{SS'}} S'\big)\Big) = \overline{G}(S \xrightarrow{\vec{v}_{SS''}} S'') = (x^0_G, \vec{x}_G) \xrightarrow{G(\vec{v}_{SS''})} (x''^0_G, \vec{x}\,''_G)$$

and

$$\begin{aligned}(x^0_G, \vec{x}_G) \xrightarrow{G(\vec{v}_{SS''})} (x''^0_G, \vec{x}\,''_G) &= \big[(x'^0_G, \vec{x}\,'_G) \xrightarrow{G(\vec{v}_{S'S''})} (x''^0_G, \vec{x}\,''_G)\big] \circ \big[(x^0_G, \vec{x}_G) \xrightarrow{G(\vec{v}_{SS'})} (x'^0_G, \vec{x}\,'_G)\big] \\ &= \overline{G}(S' \xrightarrow{\vec{v}_{S'S''}} S'') \circ \overline{G}(S \xrightarrow{\vec{v}_{SS'}} S')\end{aligned}$$

i.e. $\overline{G}$ satisfies

$$\overline{G}\Big(\big(S' \xrightarrow{\vec{v}_{S'S''}} S''\big) \circ \big(S \xrightarrow{\vec{v}_{SS'}} S'\big)\Big) = \overline{G}(S' \xrightarrow{\vec{v}_{S'S''}} S'') \circ \overline{G}(S \xrightarrow{\vec{v}_{SS'}} S') \, .$$

Physically, the role of the Galilei functor $\overline{G}$ is to discriminate in $\mathcal{C}$ a class of coordinate systems where the events are registered as points $(\tau, \vec{x}) \in \mathrm{Obj}_{\mathcal{C}}$, with τ being the absolute time, together with a class of morphisms that are the Galilei transformations $(\tau, \vec{x}) \xrightarrow{\overline{G}(\vec{v}_{SS'})} (\tau', \vec{x}\,')$.

5.3.2 The Lorentz functor

The *Lorentz functor* $\overline{L} : \mathcal{S} \to \mathcal{C}$ is defined as follows.

$\lrcorner$ $\mathrm{Obj}_{\mathcal{S}} \xrightarrow{\overline{L}} \mathrm{Obj}_{\mathcal{C}}$

$S \xrightarrow{\overline{L}} \overline{L}(S) := (x^0_L, \vec{x}_L, \{\vec{v}\})$

where

$$(x^0_L, \vec{x}_L, \{\vec{v}\}) := \{(x^0_L, \vec{x}_L, \vec{v}) : \vec{v} \in \mathrm{Obj}_{\mathcal{S}}\} \tag{20}$$

constitute a class of objects in $\mathcal{C}$ with x^0_L defined as in (14).

$\lrcorner$ $\mathrm{Morf}_{\mathcal{S}}(S, S') \xrightarrow{\overline{L}} \mathrm{Morf}_{\mathcal{C}}(\overline{L}(S), \overline{L}(S'))$

$S \xrightarrow{\vec{v}_{SS'}} S' \xrightarrow{\overline{L}} \overline{L}(S \xrightarrow{\vec{v}_{SS'}} S') := (x^0_L, \vec{x}_L, \{\vec{v}\}) \xrightarrow{\overline{L}(\vec{v}_{SS'})} (x'^0_L, \vec{x}\,'_L, \{\vec{v}\,'\})$

where

$$(x^0_L, \vec{x}_L, \{\vec{v}\}) \xrightarrow{\overline{L}(\vec{v}_{SS'})} (x'^0_L, \vec{x}\,'_L, \{\vec{v}\,'\}) := (x^0_L, \vec{x}_L, \vec{v}_{SS'}) \xrightarrow{L(a(v_{SS'}), \vec{v}_{SS'})} (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S}) \, . \tag{21}$$

Remarks:

1. We notice from the action of $\overline{L}$ in the objects of $\mathcal{S}$ that we are adopting a sligthly modified concept of a functor with $\overline{L}$ becoming a *multi-valued* map. Here, the velocities

$\{\vec{v}\}$ in $\overline{L}(S) = (x^0_L, \vec{x}_L, \{\vec{v}\})$ correspond to all morphisms $\{S \xrightarrow{\vec{v}} S'\}$ in which S is an initial object in $\mathcal{S}$.

2. As defined above, the Lorentz functor $\overline{L}$ acts on the objects of $\mathcal{S}$ that are the domain or codomain of a morphism $S \overset{\vec{v}_{SS'}}{\to} S'$ by choosing one specific object among $\overline{L}(S) = (x^0_L, \vec{x}_L, \{\vec{v}\})$ and $\overline{L}(S') = (x'^0_L, \vec{x}\,'_L, \{\vec{v}\,'\})$, in this case, $(x^0_L, \vec{x}_L, \vec{v}_{SS'})$ and $(x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S})$. In this sense, when the objects of $\mathcal{S}$ are considered as the domain or codomain of a morphism the functor $\overline{L}$ acts on them bringing with the respective object a "knowledge" of the morphism to which the considered object is a domain or codomain, thus selecting a single representative of the class (20).

In order to define the composition of morphisms and to show the functoriality of $\overline{L}$ we need a prescription that is based on the following equivalence between objects and morphisms of $\mathcal{C}$.

$\lrcorner$ Two objects $(x^0_L, \vec{x}_L, \vec{v})$, $(x^0_L, \vec{x}_L, \vec{v}\,') \in \mathrm{Obj}_{\mathcal{C}}$ are *equivalent* if there is a common frame S and two other frames S', S'' such that we can identify $\vec{v} = \vec{v}_{SS'}$, $\vec{v}\,' = \vec{v}_{SS''}$ and we have

$$(x^0_L, \vec{x}_L, \vec{v}\,' = \vec{v}_{SS''}) = h_{\vec{v}_{SS''}} h^{-1}_{\vec{v}_{SS'}} (x^0_L, \vec{x}_L, \vec{v} = \vec{v}_{SS'}) \tag{22}$$

as it is indicated by the diagram below

$$\begin{tikzcd}[column sep=small]
 & \overline{G}(S) = (x^0_G, \vec{x}_G) \arrow[dl, "h_{\vec{v}_{SS'}}"'] \arrow[dr, "h_{\vec{v}_{SS''}}"] & \\
(x^0_L, \vec{x}_L, \vec{v} = \vec{v}_{SS'}) \arrow[rr] & & (x^0_L, \vec{x}_L, \vec{v}\,' = \vec{v}_{SS''}) \,.
\end{tikzcd} \tag{23}$$

$\lrcorner$ Also, two morphisms $(x^0_L, \vec{x}_L, \vec{v}) \xrightarrow{f} (x'^0_L, \vec{x}\,'_L, \vec{v}\,') \in \mathrm{Morf}_{\mathcal{C}}\big((x^0_L, \vec{x}_L, \vec{v}), (x'^0_L, \vec{x}\,'_L, \vec{v}\,')\big)$ and $(x^0_L, \vec{x}_L, \vec{v}\,'') \xrightarrow{g} (x'^0_L, \vec{x}\,'_L, \vec{v}\,''') \in \mathrm{Morf}_{\mathcal{C}}\big((x^0_L, \vec{x}_L, \vec{v}\,''), (x'^0_L, \vec{x}\,'_L, \vec{v}\,''')\big)$ are *equivalent* if there are frames S, S' such that the diagram below is commutative

$$\begin{tikzcd}
 & (x^0_L, \vec{x}_L, \vec{v}) \arrow[r, "f"] & (x'^0_L, \vec{x}\,'_L, \vec{v}\,') \arrow[dr, "h^{-1}_{\vec{v}\,'}"] & \\
G(S) = (x^0_G, \vec{x}_G) \arrow[ur, "h_{\vec{v}}"] \arrow[dr, "h_{\vec{v}\,''}"'] & & & G(S') = (x'^0_G, \vec{x}\,'_G) \\
 & (x^0_L, \vec{x}_L, \vec{v}\,'') \arrow[r, "g"] & (x'^0_L, \vec{x}\,'_L, \vec{v}\,''') \arrow[ur, "h^{-1}_{\vec{v}\,'''}"'] &
\end{tikzcd} \tag{24}$$

or, equivalently, if there exists maps $h_{\vec{v}}, h_{\vec{v}\,'}, h_{\vec{v}\,''}, h_{\vec{v}\,'''}$ satisfying

$$f = h_{\vec{v}\,'} h^{-1}_{\vec{v}\,'''}\, g\, h_{\vec{v}\,''} h^{-1}_{\vec{v}} \,.$$

Now, we analyze how to compose morphisms under $\overline{L}$. Let us consider the maps

$$(x^0_L, \vec{x}_L, \{\vec{v}\}) \overset{\overline{L}(\vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \{\vec{v}\,'\}) = (x^0_L, \vec{x}_L, \vec{v}_{SS'}) \overset{L(a(v_{SS'}), \vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S})$$
$$(x'^0_L, \vec{x}\,'_L, \{\vec{v}\,'\}) \overset{\overline{L}(\vec{v}_{S'S''})}{\longrightarrow} (x''^0_L, \vec{x}\,''_L, \{\vec{v}\,''\}) = (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S''}) \overset{L(a(v_{S'S''}), \vec{v}_{S'S''})}{\longrightarrow} (x''^0_L, \vec{x}\,''_L, \vec{v}_{S''S'}) \,.$$

Here, the codomain of $L(a(v_{SS'}),\vec{v}_{SS'})$ is $(x'^0_L,\vec{x}\,'_L,\vec{v}_{S'S})$ and the domain of $L(a(v_{S'S''}),\vec{v}_{S'S''})$ is $(x'^0_L,\vec{x}\,'_L,\vec{v}_{S'S''})$. As we see from (14), if the velocities are different we cannot compose the maps $L(a(v_{SS'}),\vec{v}_{SS'})$, $L(a(v_{S'S''}),\vec{v}_{S'S''})$ directly, since $x'^0_L(\vec{v}_{SS'})$ and $x'^0_L(\vec{v}_{S'S''})$ are not the same. Then, we prescribe that the composition $\overline{L}(\vec{v}_{S'S''})\circ\overline{L}(\vec{v}_{SS'})$ is defined by

$$\overline{L}(\vec{v}_{S'S''})\circ\overline{L}(\vec{v}_{SS'}) := L(a(v_{S'S''}),\vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}),\vec{v}_{SS'}) \tag{25}$$

which in arrow form corresponds to

$$\begin{aligned}
&(x^0_L,\vec{x}_L,\{\vec{v}\}) \overset{\overline{L}(\vec{v}_{S'S''})\overline{L}(\vec{v}_{SS'})}{\longrightarrow} (x''^0_L,\vec{x}\,''_L,\{\vec{v}\,''\}) = \\
&\quad = (x^0_L,\vec{x}_L,\vec{v}_{SS'}) \xrightarrow{L(a(v_{S'S''}),\vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}),\vec{v}_{SS'})} (x''^0_L,\vec{x}\,''_L,\vec{v}_{S''S'})\ .
\end{aligned} \tag{26}$$

Explicitly we have

$$\begin{aligned}
x''^0_L &= \frac{\sqrt{a^2(v_{SS'})-1}}{\sqrt{a^2(v_{S'S''})-1}}\left\{a(v_{S'S''})\frac{v_{S'S''}}{v_{SS'}}-\big(1-a(v_{S'S''})\big)\frac{\vec{v}_{SS'}\cdot\vec{v}_{S'S''}}{v_{SS'}v_{S'S''}}\right\}x^0_L + \\
&\quad + \frac{\big(1-a(v_{SS'})\big)}{\sqrt{a^2(v_{S'S''})-1}}\left\{a(v_{S'S''})\frac{v_{S'S''}}{cv^2_{SS'}}-\big(1-a(v_{S'S''})\big)\frac{\vec{v}_{SS'}\cdot\vec{v}_{S'S''}}{cv_{S'S''}v^2_{SS'}}\right\}\vec{x}_L\cdot\vec{v}_{SS'} + \\
&\quad + \frac{\big(1-a(v_{S'S''})\big)}{\sqrt{a^2(v_{S'S''})-1}}\,\frac{1}{cv_{S'S''}}\,\vec{x}_L\cdot\vec{v}_{S'S''}
\end{aligned} \tag{27}$$

$$\begin{aligned}
\vec{x}\,''_L &= \vec{x}_L-\sqrt{a^2(v_{SS'})-1}\,\frac{c}{v_{SS'}}\,x^0_L(\vec{v}_{SS'}+\vec{v}_{S'S''}) + \\
&\quad -\big(1-a(v_{SS'})\big)\frac{1}{v^2_{SS'}}\vec{x}_L\cdot\vec{v}_{SS'}(\vec{v}_{SS'}+\vec{v}_{S'S''})\ .
\end{aligned} \tag{28}$$

Having defined the above prescription on how to compose maps, we need a second prescription in order to be able to characterize $\overline{L}$ as a functor as we now discuss.
From (19) we have that

$$\begin{aligned}
\overline{L}\Big(\big(S' \overset{\vec{v}_{S'S''}}{\longrightarrow} S''\big)\circ\big(S \overset{\vec{v}_{SS'}}{\longrightarrow} S'\big)\Big) &= \overline{L}(S \overset{\vec{v}_{SS''}}{\longrightarrow} S'') \\
&= (x^0_L,\vec{x}_L,\{\vec{v}\}) \overset{\overline{L}(\vec{v}_{SS''})}{\longrightarrow} (x''^0_L,\vec{x}\,''_L,\{\vec{v}\,''\}) \\
&= (x^0_L,\vec{x}_L,\vec{v}_{SS''}) \overset{L(a(v_{SS''}),\vec{v}_{SS''})}{\longrightarrow} (x''^0_L,\vec{x}\,''_L,\vec{v}_{S''S})
\end{aligned} \tag{29}$$

We have also seen that

$$\begin{aligned}
&\overline{L}\big(S' \overset{\vec{v}_{S'S''}}{\longrightarrow} S''\big)\circ\overline{L}\big(S \overset{\vec{v}_{SS'}}{\longrightarrow} S'\big) = \\
&= \big((x'^0_L,\vec{x}\,'_L,\{\vec{v}\,'\}) \overset{\overline{L}(\vec{v}_{S'S''})}{\longrightarrow} (x''^0_L,\vec{x}\,''_L,\{\vec{v}\,'\})\big)\circ\big((x^0_L,\vec{x}_L,\{\vec{v}\}) \overset{\overline{L}(\vec{v}_{SS'})}{\longrightarrow} (x'^0_L,\vec{x}\,'_L,\{\vec{v}\,'\})\big) \\
&= (x^0_L,\vec{x}_L,\vec{v}_{SS'}) \xrightarrow{L(a(v_{S'S''}),\vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}),\vec{v}_{SS'})} (x''^0_L,\vec{x}\,''_L,\vec{v}_{S''S'})\ .
\end{aligned} \tag{30}$$

The morphisms given in (29), (30) cannot be directly compared since the domain/codomain of one differs from the domain/codomain of the other. However the maps $L(a(v_{SS''}), \vec{v}_{SS''})$ and $L(a(v_{S'S''}), \vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}), \vec{v}_{SS'})$ are equivalent because we have a similar diagram as the one shown in (24) that has the form

$$\begin{tikzcd}
(x^0_L, \vec{x}_L, \vec{v}_{SS''}) \arrow[rr, "L(a(v_{SS''}),\vec{v}_{SS''})"] & & (x''^0_L, \vec{x}\,''_L, \vec{v}_{S''S}) \arrow[d, "h^{-1}_{\vec{v}_{S''S}}"] \\
G(S) = (x^0_G, \vec{x}_G) \arrow[u, "h_{\vec{v}_{SS''}}"] \arrow[d, "h_{\vec{v}_{SS'}}"] & & G(S'') = (x''^0_G, \vec{x}\,''_G) \\
(x^0_L, \vec{x}_L, \vec{v}_{SS'}) \arrow[rr, "L(a(v_{S'S''}),\vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}),\vec{v}_{SS'})"] & & (x''^0_L, \vec{x}\,''_L, \vec{v}_{S''S'}) \arrow[u, "h^{-1}_{\vec{v}_{S''S'}}"]
\end{tikzcd} \tag{31}$$

or,

$$L(a(v_{SS''}), \vec{v}_{SS''}) = h_{\vec{v}_{S''S}}h^{-1}_{\vec{v}_{S''S'}}L(a(v_{S'S''}), \vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}), \vec{v}_{SS'})h_{\vec{v}_{SS'}}h^{-1}_{\vec{v}_{SS''}} \tag{32}$$

Then, we prescribe that the functorial property of $\overline{L}$

$$\overline{L}\Big(\big(S' \overset{\vec{v}_{S'S''}}{\longrightarrow} S''\big) \circ \big(S \overset{\vec{v}_{SS'}}{\longrightarrow} S'\big)\Big) = \overline{L}\big(S' \overset{\vec{v}_{S'S''}}{\longrightarrow} S''\big) \circ \overline{L}\big(S \overset{\vec{v}_{SS'}}{\longrightarrow} S'\big) \tag{33}$$

is, in fact, a statement of equivalent maps in $\mathcal{C}$ as prescribed in diagram (24).

Physically, the role of the Lorentz functor $\overline{L}$ is to discriminate in $\mathcal{C}$ another class of coordinate systems where the events are registered as $(t, \vec{x})$, with t being the local time of the SR, together with a class of morphisms that are the Generalized Lorentz transformations $(t, \vec{x}) \overset{L(a(v_{SS'}),\vec{v}_{SS'})}{\longrightarrow} (t', \vec{x}\,')$.

Remark: The difficulty we found in defining the composition $\overline{L}(\vec{v}_{S'S''}) \circ \overline{L}(\vec{v}_{SS'})$ in (25) is a direct consequence of relation (14) which tells us that $(x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S})$ and $(x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S''})$ are not the same. Here, if we restrict ourselves to the SR and consider no dependence of x'^0_L with the state of motion of the frame, we could have defined $\overline{L}(\vec{v}_{S'S''}) \circ \overline{L}(\vec{v}_{SS'})$ with no recourse to the maps appearing in (22). This is the case for the ordinary Lorentz transformation of SR $\mathcal{L}(\vec{v})$ given by (7), where there is no restriction on composing two Lorentz transformations of the type $(x^0_L, \vec{x}_L) \overset{\mathcal{L}(\vec{v})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L) \overset{\mathcal{L}(\vec{v}\,')}{\longrightarrow} (x''^0_L, \vec{x}\,''_L)$.

However, as it is well-known, the composition of two ordinary Lorentz transformations does not produce a Lorentz transformation, therefore, if we had taken $\overline{L}(S \overset{\vec{v}_{SS'}}{\longrightarrow} S') = (x^0_L, \vec{x}_L) \overset{\mathcal{L}(\vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,')$, $\overline{L}$ would not verify the functorial property
$\overline{L}\Big(\big(S \overset{\vec{v}_{SS'}}{\longrightarrow} S'\big) \circ \big(S' \overset{\vec{v}_{S'S''}}{\longrightarrow} S''\big)\Big) = \overline{L}\big(S \overset{\vec{v}_{SS'}}{\longrightarrow} S'\big) \circ \overline{L}\big(S' \overset{\vec{v}_{S'S''}}{\longrightarrow} S''\big)$.

5.4 Unifying the Galilei and the Lorentz transformation through a natural transformation

In this fourth stage we give a precise meaning to the maps $h_{\vec{v}}$, seeing them as related to a natural transformation between the functors $\overline{G}, \overline{L} : \mathcal{S} \to \mathcal{C}$.

Given the functors $\overline{G}, \overline{L} : \mathcal{S} \to \mathcal{C}$ we define a multi-valued map

$$\begin{aligned} h : \mathrm{Obj}_{\mathcal{S}} &\to \mathrm{Morf}_{\mathcal{C}} \\ S &\to h_S \subset \mathrm{Morf}_{\mathcal{C}}(\overline{G}(S), \overline{L}(S)) \\ & \quad\; h_S := \{(x_G^0, \vec{x}_G) \xrightarrow{h_{\vec{v}}} (x_L^0, \vec{x}_L) : \vec{v} \in \mathrm{Morf}_{\mathcal{S}}\} \end{aligned} \tag{34}$$

and such that for any $S \xrightarrow{\vec{v}} S' \in \mathrm{Morf}_{\mathcal{S}}(S, S')$ there are $h_{\vec{v}} \in h_S$, $h_{\vec{v}\,'} \in h_{S'}$ $(\vec{v}\,' = -\vec{v})$ as given in (15) that makes the diagram (16) commutative.

Then, we identify h as a kind of multi-valued natural transformation between the functors $\overline{G}, \overline{L}$. Here, we also have a similar behaviour as the one shown by the Lorentz functor $\overline{L}$, since the action of h on an object that is the domain or codomain of a morphism $S \xrightarrow{\vec{v}} S'$ associates particular elements in h_S and $h_{S'}$, for example, $(x_G^0, \vec{x}_G) \xrightarrow{h_{\vec{v}}} (x_L^0, \vec{x}_L, \vec{v})$ and $(x_G'^0, \vec{x}\,'_G) \xrightarrow{h_{\vec{v}\,'}} (x_L'^0, \vec{x}\,'_L, \vec{v}\,')$.

It follows from the commutativity of the diagram (16), together with the verification that the maps $h_{\vec{v}}, h_{\vec{v}\,'}$ are invertible, that for any $\vec{v}$, $\vec{v}\,' = -\vec{v}$ we have

$$\mathrm{Lorentz} = h_{\vec{v}\,'} \circ \mathrm{Galilei} \circ h_{\vec{v}}^{-1} \tag{35}$$

which completes the unification of both the Galilei relativity and the SR, since one transformation originates the other and vice-versa.

6 Conclusion

In this work we showed how the Galilei and the Lorentz transformations are conected by a sort of natural transformation h that is related to the existence of the absolute time τ as we saw in (15). This relation carries a significant physical meaning that allow us to extend our conception of time by considering the absolute time τ and the local time t as genuine time variables, rather than assuming τ as existing only as a limit case of the local time in the low velocity limit as it is assumed by the special relativity. The fact that the $h_{\vec{v}}$ relates two coordinate systems for the *same* frame S was the key element for us to interpret h as a natural transformation of the type $h : \mathrm{Obj}_{\mathcal{S}} \to \mathrm{Morf}_{\mathcal{C}}$, which was the leading principle guiding us on how to build the categorical framework we developed. However, some adjustments were necessary to be made since the definitions of $h : \mathrm{Obj}_{\mathcal{S}} \to \mathrm{Morf}_{\mathcal{C}}$ and $\overline{L} : \mathcal{S} \to \mathcal{C}$ are sligthly different from the usual definitions of natural transformation and functor because h and $\overline{L}$ are multivalued maps associating to a frame $S \in \mathrm{Obj}_{\mathcal{S}}$ respectively a family of maps h_S (34), and objects $\overline{L}(S)$ (20). However, when the object is a domain/codomain of a morphism, $S \xrightarrow{\vec{v}_{SS'}} S'$, they select one representative of h_S and $\overline{L}(S)$. Further modifications could be necessary to be made if we intend to give a categorical interpretation for the diagram involving the transformation of velocities in both relativities (13).

Now, this categorical framework unifying the Galilei relativity and the special relativity offers a new perspective for investigation. For example, the transition from the special relativity to the general relativity is performed as a shift from the infinitesimal line element $ds^2 \equiv \eta_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - d\vec{x}^2$ to the more general one $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Then, introducing this line element as a new data in our physcial world it suggest us to search what new structure could be endowed to the Galilei relativity so that we would have a diagram like

$$
\begin{tikzcd}
\text{Galilei Relativity} \arrow[r, "?"] \arrow[d] & \text{“General Galilei Relativity”} \arrow[d, "?"] \\
\text{Special Relativity}, ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \arrow[r] & \text{General Relativity}, ds^2 = g_{\mu\nu}dx^\mu dx^\nu
\end{tikzcd}
$$

commutative. Here, our main idea is to think on what could be a “General Galilean relativity” model, built on the basis of making the unknow arrows in the above diagram satisfy the same construction as seen in the diagram (1).

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