Series on Knots and Everything - Vol. 22

ALEXEY STAKHOV

THE MATHEMATICS OF HARMONY

From Euclid to Contemporary Mathematics and Computer Science

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SCOTT OLSEN

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ALEXEY STAKHOV
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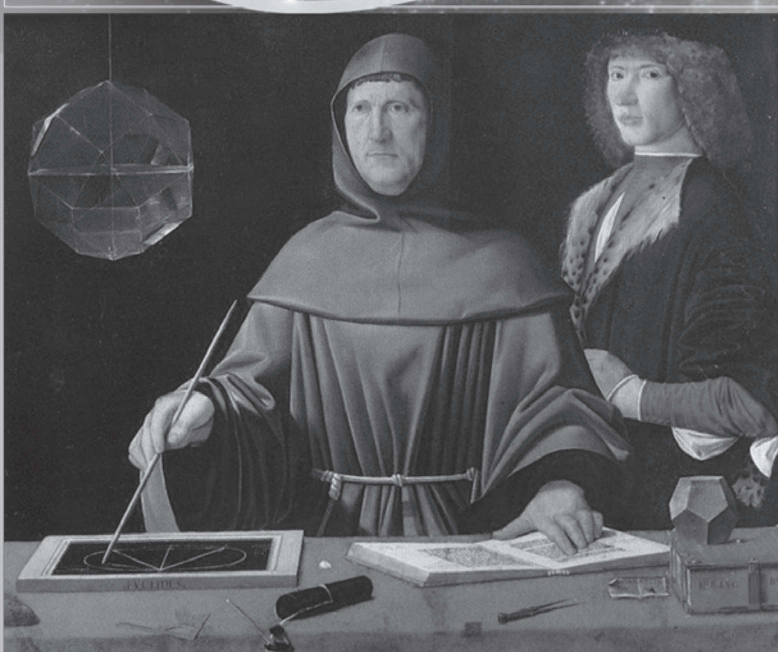

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With deep gratitude to my parents, my darling father, Peter Stakhov,
who was killed in 1941 during World War II (1941-1945),
and to my darling mother, Daria Stakh, who passed away in 2001,
and to my darling teacher Professor Alexander Volkov,
who passed away in 2007.

Preface

Academician Mitropolsky's commentary on the scientific research of Ukrainian scientist Professor Alexey Stakhov, Doctor of Engineering Sciences

I have followed the scientific career of Professor Stakhov for a long time—seemingly since the publication of his first book, *Introduction into Algorithmic Measurement Theory* (1977), which was presented by Professor Stakhov in 1979 at the scientific seminar of the Mathematics Institute of the Ukrainian Academy of Sciences. I became especially interested in Stakhov's scientific research after listening to his brilliant speech at a session of the Presidium of the Ukrainian Academy of Sciences in 1989. In his speech, Professor Stakhov reported on scientific and engineering developments in the field of "Fibonacci computers" that were conducted under his scientific supervision at Vinnitsa Technical University.

I am very familiar with Stakhov's scientific works as many of his papers were published in various Ukrainian academic journals at my recommendation. In April 1998, I invited Professor Stakhov to report on his scientific research at a meeting of the Ukrainian Mathematical Society. His lecture produced a positive reaction from the members of the society. At the request of Professor Stakhov, I wrote the introduction to his book, *Hyperbolic Fibonacci and Lucas Functions,* which was published in 2003 in small edition. In recent years, I have been actively corresponding with Professor Stakhov, and we have discussed many new scientific ideas. During these discussions I became very impressed with his qualifications and extensive knowledge in regard to his research in various areas of modern science. In particular, I am impressed by his knowledge in the field of mathematics history.

The main feature of Stakhov's scientific creativity consists of his unconventional outlook upon ancient mathematical problems. As an example, I shall begin with my review of his book *Introduction into Algorithmic Measurement Theory* (1977). This publication rewarded Professor Stakhov with recognition in the field of modern theoretical metrology. In this book, Professor Stakhov introduced a new mathematical direction in measurement theory—the Algorithmic Measurement Theory.

In 1993, I recommended a publication of an innovative paper, prepared by Professor Alexey Stakhov and Ivan Tkachenko, entitled "Fibonacci Hyperbolic Trigonometry," for publication in the journal *Reports of the Ukrainian Academy of Sciences.* The paper addressed a new theory of hyperbolic Fibonacci and Lucas functions. This paper demonstrated the uniqueness of Stakhov's scientific thinking. In fact, the classical hyperbolic functions were widely known and were used as a basis of non-Euclidean geometry developed by Nikolay Lobachevsky. It is quite peculiar that at the end of 20th century Ukrainian scientists Stakhov and Tkachenko discovered a new class of the hyperbolic functions based on the Golden Section, Fibonacci and Lucas numbers that has "strategic" importance for the development of modern mathematics and theoretical physics.

In 1999, I also recommended Stakhov's article "A Generalization of the Fibonacci Q-Matrix"—which was presented by the author in English—to be published in the journal *Reports of the Ukrainian Academy of Sciences* (1999, Vol. 9). In this article, Professor Stakhov generalized and developed a new theory of the Q-matrix which had been introduced by the American mathematician Verner Hoggatt—a founder of the Fibonacci-Association. Stakhov introduced a concept of the Q_p-matrices (p=0, 1, 2, 3...), which are a new class of square matrices (a number of such matrices is infinite). These matrices are based on so-called Fibonacci p-numbers, which had been discovered by Stakhov while investigating "diagonal sums" of the Pascal triangle. Stakhov discovered a number of quite unusual properties of the Q_p-matrices. In particular, he proved that the determinant of the Q_p-matrix or any power of that matrix is equal to +1 or -1. It is my firm belief that a theory of Q_p-matrices could be recognized as a new fundamental result in the classic matrix theory.

In 2004, *The Ukrainian Mathematical Journal* (Vol. 8), published Stakhov's article "The Generalized Golden Sections and a New Approach to Geometrical Definition of Number." In this article, Professor Stakhov obtained mathematical results in number theory. The following are worth mentioning:

1. A Generalization of the Golden Section Problem. The essence of this generalization is extremely simple. Let us set a non-negative integer (p=0, 1, 2, 3, ...) and divide a line segment AB at the point C in the following proportion:

$$\frac{CB}{AC}=\left(\frac{AB}{CB}\right)^p$$

We then get the following algebraic equation:

$$x^{p+1} = x^p + 1.$$

The positive roots of this algebraic equation were named the *Generalized Golden Proportions* or the *Golden p-proportions* t_p. Let's ponder upon this result. Within several millennia, since Pythagoras and Plato, mankind widely used the known classical Golden Proportion as some unique number. And at the end of the 20th century, the Ukrainian scientist Stakhov has generalized this result and proved the existence of the infinite number of the Golden Proportions; as all of them have the same right to express Harmony, as well as the classical Golden Proportion. Moreover, Stakhov proved that the golden p-proportions τ_p $(1 \le \tau_p \le 2)$ represented a new class of irrational numbers, which express some unknown mathematical properties of the Pascal triangle. Undoubtedly, such mathematical result has fundamental importance for the development of modern science and mathematics.

2. Codes of the Golden *p*-proportions. Using a concept of the golden p-proportion, Stakhov introduced a new definition of real number in the form:

$$A = \sum_i a_i \tau_p^i, \ (a_i \in \{0,1\})$$

He named this sum the "Code of the golden p-proportion." Stakhov proved that this concept, which is an expansion of the well-known Newton's definition of real number, could be used for the creation of a new theory for real numbers. Furthermore, he proved that this result could also be used for the creation of new computer arithmetic and new computers—*Fibonacci computers*. Stakhov not only introduced the idea of Fibonacci computers, but he also organized the engineering projects on the creation of such computer prototypes in the Vinnitsa Polytechnic Institute from 1977-1995. 65 foreign patents for inventions in the field of Fibonacci computers have been issued by the state patent offices of the United States, Japan, England, France, Germany, Canada, and other countries; these patents confirmed the significance of Ukrainian science and of Professor Stakhov's work in this important computer area.

In recent years, the area of Professor Stakhov's scientific interests has moved more and more towards the area of mathematics. For example, his lecture "The Golden Section and Modern Harmony Mathematics" delivered at the Seventh International Conference on Fibonacci Numbers and their Applications in Graz, Austria in 1996, and then repeated in 1998 at the Ukrainian Mathematical Society, established a new trend in Stakhov's scientific research. This lecture was impressive and it created wide discussion on Stakhov's new research.

Currently, Professor Stakhov is an actively working scientist who publishes his scientific papers in many internationally recognized journals. Most recently, he has published many fundamental papers in the international journals: *Computers & Mathematics with Applications*; *The Computer Journal*; *Chaos, Solitons & Fractals*; *Visual Mathematics*; and others. This fact demonstrates, undoubtedly, tremendous success not only for Professor Stakhov, but also for Ukrainian science.

Stakhov's articles are closing a cycle of his long-term research on the creation of a new direction in mathematics: **Mathematics of Harmony**. One may wonder what place in the general theory of mathematics this work may have. It seems to me that in the last few centuries as Nikolay Lobachevsky said, "Mathematicians have turned all their attention to the advanced parts of analytics, and have neglected the origins of Mathematics, and are not willing to dig the field that has already been harvested by them and left behind." As a result, this has created a gap between "Elementary Mathematics"—the basis of modern mathematical education—and "Advanced Mathematics." In my opinion, the Mathematics of Harmony developed by Professor Stakhov fills that gap. **Mathematics of Harmony** is a huge theoretical contribution to the development of "Elementary Mathematics," and as such should be considered of great importance for mathematical education.

It is imperative to mention that Professor Stakhov focuses his organizational work on stimulating research in the field of theory surrounding Fibonacci numbers and the Golden Section; he also assists in spreading knowledge among broad audiences inside the scientific community. In 2003, under Professor Stakhov's initiative and scientific supervision, the international conference on "Problems of Harmony, Symmetry, and the Golden Section in Nature, Science, and Art" was held. At this conference, Professor Stakhov was elected as President of the International Club of the Golden Section, confirming his official status as leader of a new scientific direction that is actively progressing the modern science.

Professor Stakhov proposed the discipline "Mathematics of Harmony and the Golden Section" for the mathematical faculties of pedagogical universities. In essence, this mathematical discipline can be considered the beginning of mathematical education reform—which is based on the principles of Harmony and the Golden Section. It should be noted that such discipline was delivered by Professor Stakhov during 2001-2002 for the students and faculty of physics and mathematics at Vinnitsa State Pedagogical University. I have no doubts about the usefulness of such discipline for future teachers in mathematics and physics. I believe that Professor Stakhov has the potential to write a textbook on this discipline for pedagogical universities, and also a textbook on *Mathematics of the Golden Section* for secondary schools.

It is clear to me that "Mathematics of Harmony," created by Professor Stakhov, has huge interdisciplinary importance as this mathematical discipline touches the bases of many sciences, including: mathematics, theoretical physics, and computer science. Stakhov suggested mathematical education reform based on the ideas of Harmony and the Golden Section. This reform opens the doors for the development of mathematical and general education curriculum. It would greatly contribute to the development of the new scientific outlook based on the principles of Harmony and the Golden Section.

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Honorable Professor: The Mathematics Institute of the National Academy of Sciences of Ukraine
Editor-in-Chief of the *Ukrainian Mathematical Journal*

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Introduction

Algebra and Geometry have one and the same fate. The very slow successes did follow after the fast ones at the beginning. They left science in a state very far from perfect. It happened, probably, because mathematicians paid the main attention to the higher parts of the Analysis. They neglected the beginnings and did not wish to develop those fields, which they finished once and left them from behind.

Nikolay Lobachevsky

Three "Key" Problems of Mathematics on the Stage of its Origin

1. The Main Stages of Mathematics Development

What is mathematics? What are its origin and history? What distinguishes mathematics from other sciences? What is the subject of mathematical research today? How does mathematics influence the development of other sciences? To answer these questions we refer to the book *Mathematics in its Historical Development* [1], written by the phenomenal Russian mathematician and academician, Andrew Kolmogorov. According to Kolmogorov's definition, mathematics is "a science about quantitative relations and spatial forms of real world."

Kolmogorov writes that "the clear understanding of mathematics, as a special science having its own subject and method, arose for the first time in Ancient Greece at 6-5 centuries BC after the accumulation of the big enough actual material."

Kolmogorov points out the following stages in mathematics development:

1. *Period of the "Mathematics origin,"* which preceded Greek mathematics.
2. *Period of the "Elementary Mathematics."* This period started during 6-5 centuries BC and ended in the 17th century. The volume of mathematical knowledge obtained up to the beginning of 17th century was, until now, the base of "elementary mathematics"—which is taught at the secondary and high school levels.
3. *The "Higher Mathematics" period,* started with the use of variables in Descartes' analytical geometry and the creation of differential and integral calculus.
4. *The "Modern Mathematics" period.* Lobachevsky's "imaginary geometry" is considered the beginning of this period. Lobachevsky's geometry was the beginning of the expansion of the circle of quantitative relations and spatial forms—which began to be investigated by mathematicians. The development of a similar kind of mathematical research gave mathematicians many new important features.

2. A "Count Problem"

Discussing the reasons of mathematical occurrence, Kolmogorov specifies two practical problems that stimulated the development of mathematics during its origin: *count* and *measurement*.

A *"count problem"* was the first ancient problem of mathematics. It is emphasized [1] that "on the earliest steps of culture development, the count of things led to the creation of the elementary concepts of natural number arithmetic. On the base of the developed system of oral notation, written notations arose, whereby different methods of the fulfillment of the four arithmetical operations for natural numbers were gradually developed."

The period that culminated in the origin of mathematics germinated the "key" mathematical discoveries. We are talking about the **positional principle of numbers representation**. It is emphasized in [2] that "the Babylonian sexagesimal numeral system, which arose approximately in 2000 BC, was the first numeral system based on the positional principle." This discovery underlies all early numeral systems created during the period of mathematics origin and the period of the elementary mathematics (including decimal and binary systems).

It is necessary to note that the positional principle of number representation and positional numeral systems (particularly the binary system), which were created in the period of mathematics origin, became one of the "key" ideas of modern computers. In this connection, it is also necessary to remember that multiplication and division algorithms, used in modern computers, were created by the ancient Egyptians (the method of doubling) [2].

However, the formation of the **natural number's concept** was the main result of arithmetic's development in the period of mathematics origin. Natural numbers are one of the major and fundamental mathematical concepts—without which the existence of mathematics is impossible. For studying the properties of natural numbers, the **number theory**—one of the fundamental mathematical theories—arose in this ancient period.

3. A "Measurement Problem"

Kolmogorov emphasizes in [1], that "the needs of measurement (of quantity of grain, length of road, etc.) had led to the occurrence of the names and designations of the elementary fractions and to the development of the methods of the fulfillment of arithmetic operations for fractions.... The measurement of areas and volumes, the needs of the building engineering, and a little bit later the needs of astronomy caused the development of geometry".

Historically, the first "theory of measurement" arose in ancient Egypt. It was the collection of rules, which the Egyptian land surveyors used. As the ancient Greeks testify, geometry—as a "science of Earth measurement"—had originated from these rules.

However, a discovery of the **"incommensurable line segments"** was the "key" discovery in this area. This discovery had been made in the 5th century BC in Pythagoras' scientific school at the investigation of the ratio of the diagonal to the side of a square. Pythagoreans proved that this ratio cannot be represented in the form of the ratio of two natural numbers. Such line segments were named incommensurable, and the numbers, which represented similar ratios, were named "irrationals." A discovery of the "incommensurable line segments" became a turning point in the development of mathematics. Owing to this discovery, the concept of **irrational numbers**, the second fundamental concept (after natural numbers) came into use in mathematics.

For overcoming the first crisis in the bases of mathematics, caused by the discovery of "incommensurable line segments," the Great mathematician Eudoxus had developed a **theory of magnitudes**, which was transformed later into **mathematical measurement theory** [3, 4], another fundamental theory of mathematics. This theory underlies all "continuous mathematics" including differential and integral calculus.

Influence of the "measurement problem" on the development of mathematics is so great that the famous Bulgarian mathematician L. Iliev had declared that "during the first epoch of mathematics development, from antiquity to the discovery of differential and integral calculus, mathematics, investigating first of all the measurement problems, had created Euclidean geometry and number theory" [5].

Thus, the two "key" problems of ancient mathematics, the *count problem* and the *measurement problem*, had led to the formation of the two fundamental concepts of mathematics: **natural numbers** and **irrational number**s—which, together with **number theory** and **measurement theory,** became the basis of "classical mathematics."

4. Mathematics. The Loss of Certainty

The book, *Mathematics: The Loss of Certainty* [6], written by American mathematician Morris Kline, had a huge influence upon the author and became a source of reflections about the nature and role of mathematics in modern science; it is a pleasure for the author to retell briefly the basic ideas of Morris Kline's book.

Since the origin of mathematics as an independent branch of knowledge (Greek mathematics), and during more than two millennia, mathematics was engaged in a search for truth and had achieved outstanding successes. It seemed that the vast amount of theorems about numbers and geometrical figures, which was proved in mathematics, is an inexhaustible source of absolute knowledge which never can change.

To obtain surprisingly powerful results, mathematicians had used a special *deductive method* which allowed them to get new mathematical results (theorems) from a small number of axiomatic principles, named by axioms. The nature of the deductive method guarantees a validity of the conclusion if the initial axioms are true. Euclid's *Elements* became the first great mathematical work in this area, which is a brilliant example of the effective application of the deductive method.

Euclidean geometry became the most esteemed part of mathematics—not only because the deductive construction of mathematical disciplines had begun with the Euclidean geometry—but, also its theorems completely corresponded to the results of physical research. It was considered a firm scientific axiom for many millennia. Euclidean geometry is the geometry of the physical world surrounding us. That is why the unusual geometries created at the beginning of the 19th century, named non-Euclidean geometries, became the first "blow" to the harmonious building of mathematical science. These unusual geometries had forced mathematicians to recognize that mathematical theories and theorems are not absolute truths in application to Nature. It was proved that new geometries are mathematically correct, that is, they could be geometrical models of the real world similar to Euclidean geometry, but then the following question arises: what geometry is a true model of the real world?

Finding the contradictions in Cantor's theory of infinite sets was another "blow" to mathematics. Comprehension of the "Tsarina of sciences" is not perfect regarding its structure; it lacks much, and it is subjected to monstrous contradictions, which can appear at any moment; it shocked mathematicians. The reaction of mathematicians to all of these events was ambiguous. Unfortunately, the majority of mathematicians had simply decided to ignore these contradictions. Instead, they fenced themselves off from the external world and concentrated their efforts on the problems arising within the modern field of mathematics, that is, mathematicians decided to break connections with natural sciences.

What was mathematics during several millennia? For previous generations, mathematics was first of all of the greatest creation of human intellect intended for nature's research. The natural sciences were the flesh and blood of mathematics and it fed mathematicians with their vivifying juices. Mathematicians willingly cooperated with physicists, astronomers, chemists, and engineers in searching for the solution to various scientific and technical problems. Moreover, many great mathematicians of the past were often outstanding physicists and astronomers. The mathematics was the "Tsarina" and simultaneously the "Servant" of natural sciences.

Morris Kline noticed that "pure" mathematics, which had completely disassociated from the inquiries of natural sciences, was never the center of attention and interest of the great mathematicians of the past. They considered "pure" mathematics as some kind of "entertainment," a rest from much more important and fascinating problems, which were put forward by natural sciences. In the 18th century, such abstract science like number theory had involved only a few mathematicians. For example, Euler, whose scientific interests had been connected with number theory, was the first to be a recognized specialist in mathematical physics. The Great mathematician Gauss did not consider number theory as the major branch of mathematics. Many of his colleagues suggested that he solves *The Great Fermat Theorem*. In one letter, Gauss noted that Fermat's hypothesis is an isolated mathematical problem, which is not connected with the most important mathematical problems, and consequently, it is not of particular interest.

Morris Kline specifies the various reasons that induced mathematicians to depart from studying the real world. Widening mathematical and natural-scientific research did not allow scientists to feel equally free in both mathematics and natural sciences. The problems, that stood before natural sciences - a solution to which the great mathematicians of the past participated actively -nowadays became more and more complex, and many mathematicians had decided to limit their activity to the problems of "pure" mathematics.

Abstraction, generalizations, specialization, and *axiomatization* are the basic directions of activity chosen by "pure" mathematicians. This activity led to the situation where, nowadays, mathematics and natural sciences go different ways. New mathematical concepts are developing without any attempt to find their applications. Moreover, mathematicians and representatives of natural sciences do not understand each other today—owing to the excessive specialization in fields and often mathematicians do not understand each other.

What can resolve this situation? Morris Kline emphasizes that researchers should return to nature and natural sciences, which were the original objectives of mathematics. Ultimately, common sense should win. The mathematical world should search for a distinction not between "pure" and applied mathematics, but between the mathematics; whereby, its purpose is to find a solution to reasonable problems. Mathematicians should not indulge someone's personal tastes and whims as our quests in mathematics is purposeful and never-ending because mathematics is rich in content that is empty, alive, and bloodless.

5. A "Harmony Problem"

As is known, returning to the past is a fruitful source of cognition to the present. The return to the sources of mathematics, to its history, is one of the important directions to overcome the crisis of contemporary mathematics. In returning to ancient science, particularly Greek science, we should pay attention to an important scientific problem, which was the focus of ancient science starting with Pythagoras and Plato.

We are talking about the "harmony problem." What is the harmony? Well-known Russian philosopher Shestakov in his remarkable book *Harmony as an Aesthetic Category* [7] emphasizes that "in the history of aesthetic doctrines, the diversified types of understanding of harmony were put forward. The concept of "harmony" is multiform and used extremely widely. It meant the natural organization of nature and space, a beauty of the human physical and moral world, principles of art works' design or the law of aesthetic perception." Among the various types of harmony (mathematic, aesthetic, artistic), which arose during the development of science and aesthetics, we will first be interested in *mathematical harmony*. In this sense, harmony is understood as equality or proportionality of the parts between themselves and the parts with the whole. In the Great Soviet Encyclopedia, we can find the following harmony definition, which expresses the mathematical understanding of the harmony: "The harmony of an object is a proportionality of the parts and the whole, a merge of the various components of the object to create a uniform organic whole. In harmony, the internal order and the measure of the object had obtained external revealing."

In the present book we concentrate our attention on *mathematical harmony*. It is clear that the mathematical understanding of harmony accepts, as a rule, the mathematical kind, and it is expressed in the form of certain numerical proportions. Shestakov emphasizes [7] that mathematical harmony "fixes attention on its quantitative side and is indifferent to qualitative originality of the parts forming conformity... The mathematical understanding of the harmony fixes, first of all, quantitative definiteness of the harmony, but it does not express aesthetic quality of the harmony, its expressiveness, connection with a beauty."

6. The Numerical Harmony of the Pythagoreans

Pythagoreans, for the first time, put forth the idea of harmonious organization of the universe. According to Pythagoreans, "harmony is an internal connection of the things, without which the Cosmos could not exist." At last, according to Pythagoras, harmony has numerical representation, namely that harmony is connected with the concept of number. The Pythagoreans had created the doctrine about the creative essence of number and their number theory had a qualitative character. Aristotle, in his "Metaphysics", emphasizes this feature of the Pythagorean doctrine:

> "The so-called Pythagoreans, studying mathematical sciences, for the first time have moved them forward and, basing on them, began to consider mathematics as the beginnings of all things... Because all things became like to numbers, and numbers occupied first place in all nature, they assumed that the elements of numbers are the beginning of all things and that all universe is harmony and number."

Pythagoreans recognized that a form of the universe should be harmonious, and all elements of the universe are connected with harmonious figures. Pythagoras taught that the Cube originates the Earth, the Tetrahedron the Fire, the Octahedron the Air, the Icosahedron the Water, the Dodecahedron the sphere of the universe, that is, the Ether.

The Pythagorean doctrine about the numerical harmony of the universe had influenced the development of all subsequent doctrines about the nature and essence of harmony. It was reflected upon and developed in the works of great thinkers. In particular, the Pythagorean doctrine underlies *Plato's cosmology*. Plato developed the Pythagorean doctrine; specifically emphasizing the cosmic importance of harmony. He remained firmly convinced that world harmony can be expressed in numerical proportions. The influence of Pythagoreans is especially traced in Plato's "Timaeus"; whereby, Plato developed the doctrine about proportions and analyzed the role of Regular Polyhedrons (Platonic Solids), from which—in his opinion—God had created the world.

The main conclusion, which follows from the Pythagorean doctrine, consists of the following. Numerical or mathematical harmony is objective property of the universe, it exists irrespective of our consciousness and is expressed in the harmonious organization of all in the real world starting from cosmos and finishing by microcosm.

7. A "Harmony Problem" in Euclid's *Elements*

We ask how Pythagoras and Plato's harmonious ideas were reflected in antique mathematics. To answer this question we analyze the greatest mathematical work of Greek mathematics: the *Elements* of Euclid. As is known, the *Elements* of Euclid is not an original work. A significant part of *Elements* was written by Pythagorean mathematicians. Their contribution to the theory of proportions—in which all ancient science and culture is based—is especially great. As the further progression of science had shown, the Pythagoreans, using numerical representations, did not leave the real world, but rather came nearer to it.

The 13th, and final, Book of Euclid's *Elements* is devoted to the theory of the *regular polyhedrons,* which is expressed in ancient science as *universe harmony*. The regular polyhedrons were used by Plato in his Cosmology and therefore they were named *Platonic Solids*. This fact originated the widespread hypothesis formulated by Proclus—one of the most known commentators of Euclid's *Elements*. According to Proclus' opinion, Euclid created the *Elements* not with the purpose to present geometry as axiomatic mathematical science, but with the purpose to give the full systematized theory of Platonic Solids, in passing having covered some advanced achievements of the ancient mathematics. Thus, the main goal of the *Elements* was a description of the theory of Platonic Solids described in the final book of *Elements*. It would not be out of place to remember that seemingly, the most important material of a scientific book is placed into the final Chapter of the book. Consequently, the placement of the Platonic Solids theory in the final book of the *Elements* is indirect proof surrounding the validity of Proclus' hypothesis; meaning that Pythagoras' Doctrine about the numerical harmony of the universe got its brightest embodiment in the greatest mathematical work of the ancient science: Euclid's *Elements*.

In order to develop a complete theory of the Platonic Solids, in particular the *Dodecahedron*, Euclid formulated in Book II the famous Theorem II,11 about the *division in the extreme and mean ratio* (DEMR), which is known in modern science under the name of the *golden section*. DEMR penetrated all Books of Euclid's *Elements,* and it had been used by Euclid for the geometric construction of the following "harmonic" geometric figures: equilateral triangle with the angles 72°, 72° and 36° (the "golden" equilateral triangle), regular pentagon and then the Dodecahedron based on the golden section. Taking into consideration Proclus' hypothesis, and a role of the DEMR in Euclid's *Elements,* we can put forward the following unusual hypothesis: **Euclid's *Elements* was the first attempt to create the "Mathematical Theory of Harmony" which was the main idea of Greek science.**

It is clear that the formulation of the division in the extreme and mean ratio (the golden section) can be considered as the "key" mathematical discovery in the field of the "harmony problem." The Great Russian philosopher Alexey Losev wrote in one of his articles that: "From Plato's point of view, and generally from the point of view of all antique cosmology, the universe is a certain proportional whole that is subordinated to the law of harmonious division, the Golden Section."

Thus, we have to add the "harmony problem" to the list of the "key" problems of mathematics regarding the stage of its origin. Such approach leads us to the original view on the history of mathematics. This idea underlies the present book.

During its historical development, the “classical mathematics” had lost Pythagoras’ and Plato’s “harmonious idea” embodied by Euclid in his *Elements*. As the outcome, mathematics had been divided into a number of mathematical theories (geometry, number theory, algebra, differential and integral calculus, etc.), which sometimes have very weak correlations. Unfortunately, a significance of the “golden mean” had been belittled in modern mathematics and theoretical physics. For many modern mathematicians, the “golden section” reminds us of a “beautiful fairy tale,” which has no relation to serious mathematics.

8. Fibonacci Numbers

Nevertheless, despite the negative relation of “materialistic” mathematics to the “golden mean,” its theory continued to develop. The famous Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ..., had been introduced into mathematics during the 13th century by the famous Italian mathematician Leonardo from Pisa (Fibonacci) at the solution of the *rabbits reproduction problem*. It is necessary to note that the method of recursive relations—one of the most powerful methods of combinatorial analysis—follows directly from Fibonacci’s discovery. Later, the Fibonacci numbers had been found in many natural objects and phenomena, in particular, the botanical phenomenon of phyllotaxis.

9. The First Book on the Golden Mean in the History of Science

During the Italian Renaissance, interest in the “golden mean” arose with new force. Of course, the universal genius of the Italian Renaissance Leonardo da Vinci could not pass the division of the extreme and mean ratio (the golden section). There is an opinion that Leonardo had introduced into the Renaissance culture by the name of the “golden section.” Leonardo da Vinci had influenced the book *Divina Proportione* [8], which was published by Italian mathematician Luca Paccioli in 1509. This unique book was the first mathematical book on the “golden mean” in history. The book was illustrated with 60 brilliant geometric figures drawn by Leonardo da Vinci; additionally, the book had a great influence on Renaissance culture.

10. Johannes Kepler and the Golden Section

In the 17th century, astronomer and mathematician Johannes Kepler created the original geometrical model of Solar system based on Platonic Solids. Kepler had expressed his admiration of the golden section with the following words: "Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first, we may compare to a measure of gold; the second we may name a precious stone."

11. Fibonacci Numbers and the Golden Section in 19th Century Science

After Kepler's death, interest in the golden section, considered one of the two "treasures of geometry," decreased; whereby, such strange oblivion continued for two centuries. Active interest in the golden section revived in mathematics in the 19th century. During this period, many mathematical works were devoted to Fibonacci numbers and the golden mean, and according to the witty saying of one mathematician: they "started to reproduce as Fibonacci's rabbits." French mathematicians Lucas and Binet became the leaders of this type of research in 19th century. Lucas had introduced into mathematics the name "Fibonacci Numbers," and also the famous Lucas numbers (1, 3, 4, 7, 11, 18, ...). Binet had deduced the famous Binet formulas, which connect the Fibonacci and Lucas numbers with the golden mean.

During this time, the German mathematician Felix Klein tried to unite together all branches of mathematics on the base of the Regular Icosahedron, the Platonic Solid—dual to the Dodecahedron. Klein treats the Regular Icosahedron based on the golden section as the main geometric object, from which the branches of the five mathematical theories follow, namely, geometry, Galois' theory, group theory, invariant theory, and differential equations. Klein's main idea is extremely simple: "Each unique geometrical object is somehow or another connected to the properties of the Regular Icosahedron."

12. The Golden Section and Fibonacci Numbers in Science of the 20th and 21st Centuries

In the second half of the 20th century the interest in Fibonacci numbers and the golden mean in mathematics had revived with new force, and the revival expanded into the 21st century with many original books [9-57] being published that were devoted to the golden mean, Fibonacci numbers, and other related topics, which is evidence of the increasing interest in the golden mean and Fibonacci numbers in modern science. Prominent mathematicians Gardner [12], Vorobyov [13], Coxeter [14], and Hoggatt [16] were the first researchers who felt new tendencies growing in mathematics. In 1963, the group of American mathematicians had organized the Fibonacci Association and they started publishing the mathematical journal *The Fibonacci Quarterly*. Owing to the activity of the Fibonacci Association and the publications of the special books by Vorobyov [13], Hoggatt [16], Vaida [28], Dunlap [38], and other mathematicians, a new mathematical theory—the **"Fibonacci numbers theory"**—appeared in contemporary mathematics. This theory has its own interesting mathematical history, which is presented in the book *A Mathematical History of the Golden Number,* written by the prominent Canadian mathematician Roger Herz-Fishler [40].

In 1992 a group of the Slavic scientists from Russia, Ukraine, Belarus, and Poland had organized the so-called **Slavic "Golden" Group**. Resulting from the initiative of this group, the International symposiums of **"The Golden Section and Problems of System Harmony"** had been held in Kiev, Ukraine in 1992 and 1993, and then again in Stavropol, Russia from 1994-1996.

The golden mean, pentagram, and Platonic Solids were widely used by astrology and other esoteric sciences, and this became one of the reasons for the negative reaction of "materialistic" science towards the golden mean and Platonic Solids. However, all attempts of "materialistic" science and mathematics to forget and completely disregard the "golden mean" and Platonic Solids and to throw them out along with astrology and esoteric sciences on the "dump of the doubtful scientific concepts," had failed. Mathematical models based on the golden mean, Fibonacci numbers, and Platonic Solids had proved to be "enduring," and they began to appear unexpectedly in different areas of nature. Already, Johannes Kepler had found Fibonacci's spirals on the surface of the phyllotaxis objects. The research of the phyllotaxis objects growth made by the Ukrainian architect Oleg Bodnar [37, 52] demonstrated that the geometry of phyllotaxis objects is based on a special class of hyperbolic functions—the "golden" hyperbolic functions. In 1984, the Byelorussian, philosopher Eduardo Soroko, had formulated the "Law of structural harmony of systems" [25]. This law confirmed a general character of self-organized processes in the system of any nature; it demonstrated that all self-organized systems are based on the generalized golden p-proportions. Shechtman's quasi-crystals, based on the Platonic icosahedron, and fullerenes (Nobel Prize of 1996), were based on the Archimedean truncated icosahedron, had confirmed Felix

Klein's great prediction about the fundamental role of the icosahedron in science and mathematics [58]. Ultimately, Petoukhov's "golden" genomatrices [59] did completed the list of modern outstanding discoveries based on the golden mean, Fibonacci numbers, and the regular polyhedra.

It is possible to assume that the increasing interest in the golden mean and Fibonacci numbers in modern theoretical physics and computer science is one of the main features of 21st century science. Prominent theoretical physicist and engineering scientist Mohammed S. El Nashie is a world leader in this field [60-72]. El Nashie's discovery of the golden mean in the famous physical two-slit experiment—which underlies quantum physics—became a source of many important discoveries in this area, in particular, the *E*-infinity theory. In this respect, we mention the works of El Nashie's numerous followers working in theoretical physics [73-83]. It is also necessary to note the contribution of Slavic researchers to this important area. The book [53] written by the Byelorussian physicist Vasyl Pertrunenko is devoted to the applications of the golden mean in quantum physics and astronomy. In 2006, the book *Metaphysics of the 21st century* [57], edited by the famous Russian physicist and theorist Y.S. Vladimirov was published. The book [57] consists of three chapters and the last chapter was devoted to the golden mean applications in modern science. This chapter begins with two important articles [59, 84]. Stakhov's article [84] is devoted to the substantiation of "Harmony Mathematics" as a new interdisciplinary direction of modern science. Petoukhov's article [59] is devoted to the description of the important scientific discovery: the "golden" genomatrices; which reaffirms the deep mathematical connection between the golden mean and genetic code. The famous Russian physicist Professor Vladimirov (Moscow University) finishes his book Metaphysics [85] with the following words: "It is possible to assert that in the theory of electroweak interactions there are relations that coincide with the 'Golden Section' that play an important role in the various areas of science and art."

In the second half of the 20th century multiple interesting mathematical discoveries in the area of golden mean applications in computer science and mathematics had been made [86-119]. In 1956, the young American mathematician George Bergman made an important mathematical discovery in the field of number systems [86]. We are talking about the number system with irrational base (the golden mean) described in [86]. Modern mathematicians had been so anxious of overcoming the crisis in the basis of mathematics that they simply had not noticed Bergman's discovery, which is, without doubt, one of the greatest mathematical discoveries in the field of number systems after the discovery by Babylonians of the positional principle of number representation. Bergman's number system was generalized by Alexey Stakhov who developed in the book [24] more general class of the number systems with irrational radices named "Codes of the golden proportion." Alexey Stakhov, in the article [105], developed a new approach to geometric definition of real numbers that is of great importance for number theory. In his article [87], and then in the book [20], Stakhov developed the so-called Fibonacci codes. The codes of the golden proportion and Fibonacci codes became a source of the Fibonacci computer project [30] developed in the Soviet Union. This computer project was an original project, which was defended by 65 patents issued by the State Patenting Departments of the United States, Japan, England, Germany, France, Canada, and other countries [120-131]. Parallel with Soviet computer science, work continued on Fibonacci computers in the United States [132-135]. In the works [44, 103, 113, 114], a new class of square matrices, the generalized Fibonacci matrices and the so-called "golden" matrices, was developed. This led to a new kind of theory of coding and cryptography [44, 113, 114].

A new class of hyperbolic functions, the hyperbolic Fibonacci and Lucas functions, introduced by Alexey Stakhov, Ivan Tkachenko, and Boris Rozin [51, 98, 106, 116, 119], was another important modern mathematical discovery.

The beginning of the 21st century is characterized by a number of the interesting events; all of which have a direct relation to Fibonacci numbers and the golden mean. First of all, it is necessary to note that three International Conferences on Fibonacci Numbers and their Applications were held in the 21st century (Arizona, USA, 2002; Braunschweig, Germany, 2004; California, USA, 2006). In 2003, the international conference **Problems of Harmony, Symmetry, and the Golden Section in Nature, Science and Art** was held in Vinnitsa, Ukraine following the initiative of the Slavic "Golden" Group, which had transformed into the **International Club of the Golden Section.** In 2005, the Academy of Trinitarizm (Russia) and the International Club of the Golden Section, had organized the **Institute of the Golden Section**.

Intersecting the 20th and 21st centuries, Western and Slavic scientists had published a number of scientific books in the field of the golden mean and its applications. The most interesting of them are the following:

Dunlap R.A. *The Golden Ratio and Fibonacci Numbers* (1997) [38].

Herz-Fishler Roger. *A Mathematical History of the Golden Number* (1998) [40].

Vera W. de Spinadel. *From the Golden Mean to Chaos* (1998) [42].

Gazale Midhat J. *Gnomon. From Pharaohs to Fractals* (1999) [45].

Kappraff Jay. *Connections. The Geometric Bridge between Art and Science* (2001) [47].

Kappraff Jay. *Beyond Measure. A Guided Tour Through Nature, Myth, and Number* (2002) [50].

Shevelev J.S. *Meta-language of the Living Nature* (2000) (Russian)[46].

Petrunenko V.V. *The Golden Section in Quantum States and its Astronomical and Physical Manifestations* (2005) (Russian) [53].

Bodnar O.J. *The Golden Section and Non-Euclidean Geometry in Science and Art* (2005) (Ukrainian) [52].

Soroko E. M. *The Golden Section, Processes of Self-organization and Evolution of System. Introduction into General Theory of System Harmony* (2006) (Russian) [56].

Stakhov A.P., Sluchenkova A.A.. Scherbakov I.G. *The da Vinci Code and Fibonacci Series* (2006) (Russian) [55].

Olsen Scott. *The Golden Section: Nature's Greatest Secret* (2006) [54].

This list confirms a great interest in the golden mean in 21st century science.

13. The Lecture: "The Golden Section and Modern Harmony Mathematics"

By the end of the 20th century, the development of the "Fibonacci numbers theory" was widening intensively. Many new generalizations of Fibonacci numbers and the golden section had been developed [20]. Different unexpected applications of Fibonacci numbers and the golden section particularly in theoretical physics (the hyperbolic Fibonacci and Lucas functions [51, 98, 106]), computer science (Fibonacci codes and the codes of the golden proportion [20, 24, 87, 89, 102]), botany (the law of the spiral biosymmetries transformation [37]), and even philosophy (the law of structural harmony of systems [25]) were obtained. It became clear that the new results in this area were far beyond the traditional "Fibonacci numbers theory" [13, 16, 28]. Moreover, it became evident that the name "Fibonacci numbers theory" considerably narrows the subject of this scientific direction—which studies mathematical models of system harmony. Therefore, the idea to unite the new results in the theory of the golden mean and Fibonacci numbers and their applications under the flag of the new interdisciplinary direction of the modern science, named "Harmony Mathematics," appeared. Such idea had been presented by Alexey Stakhov in the lecture **"The Golden Section and Modern Harmony Mathematics"** at the Seventh International Conference on Fibonacci Numbers and their Applications in Graz, Austria in July 1996. The lecture was later published in the book *Applications of Fibonacci Numbers* [100].

After 1996, the author continued to develop and deepen this idea [101-119]. However, the creation of "Harmony Mathematics" was a result of collective creativity; the works of other prominent researchers in the field of the golden section and Fibonacci numbers Martin Gardner [12], Nikolay Vorobyov [13], H. S. M. Coxeter [14], Verner Hoggat [16], George Polya [17], Alfred Renyi [23], Stephen Vaida [28], Eduardo Soroko [25, 56], Jan Grzedzielski [26], Oleg Bodnar [37, 52], Nikolay Vasutinsky [31], Victor Korobko [43], Josef Shevelev [46], Sergey Petoukhov [59], Roger Herz-Fishler [40], Jay Kappraff [47, 50], Midhat Gazale [45], Vera W. de Spinadel [42], R.A. Dunlap [38], Scott Olsen [54], Mohammed S. El Nashie [60-72], and other scientists had influenced the author's research in this field.

"Harmony Mathematics," in its origin, goes back to the Euclidean problem of "division in the extreme and mean ratio" (the golden section) [40]. Harmony Mathematics is a continuation of the traditional "Fibonacci numbers theory" [13, 16, 28]. What are the purposes of this new mathematical theory? Similar to "classical mathematics," which is defined sometimes as the "science about models" [5], we can consider Harmony Mathematics as the **"science about the models of harmonic processes"** in the world surrounding us.

14. Two Historical Ways of Mathematics Development

In research, returning to the origin of mathematics, we can point out the two ways of mathematics development, which arose in the ancient mathematics. The first way was based on the "**count problem"** and the "**measurement problem"** [1]. In the period of mathematics origin two fundamental discoveries were made. **The positional principle of number representation** [2] was used in all known numeral systems, including the Babylonian sexagecimal, decimal, and binary systems. Ultimately, the development of this direction culminated in the formation of the concept of **natural numbers**; it also led to the creation of **number theory**—the first fundamental theory of mathematics. **Incommensurable line segments** discovered by Pythagoreans led to the discovery of **irrational numbers** and the creation of **measurement theory** [3, 4]—which was the second fundamental theory of mathematics. Ultimately, natural and irrational numbers became those basic mathematical concepts that underlie all mathematical theories of "classical mathematics," including number theory, algebra, geometry, and differential and integral calculus. Theoretical physics and computer science are the most important applications of "classical mathematics" (see Fig. I.1).

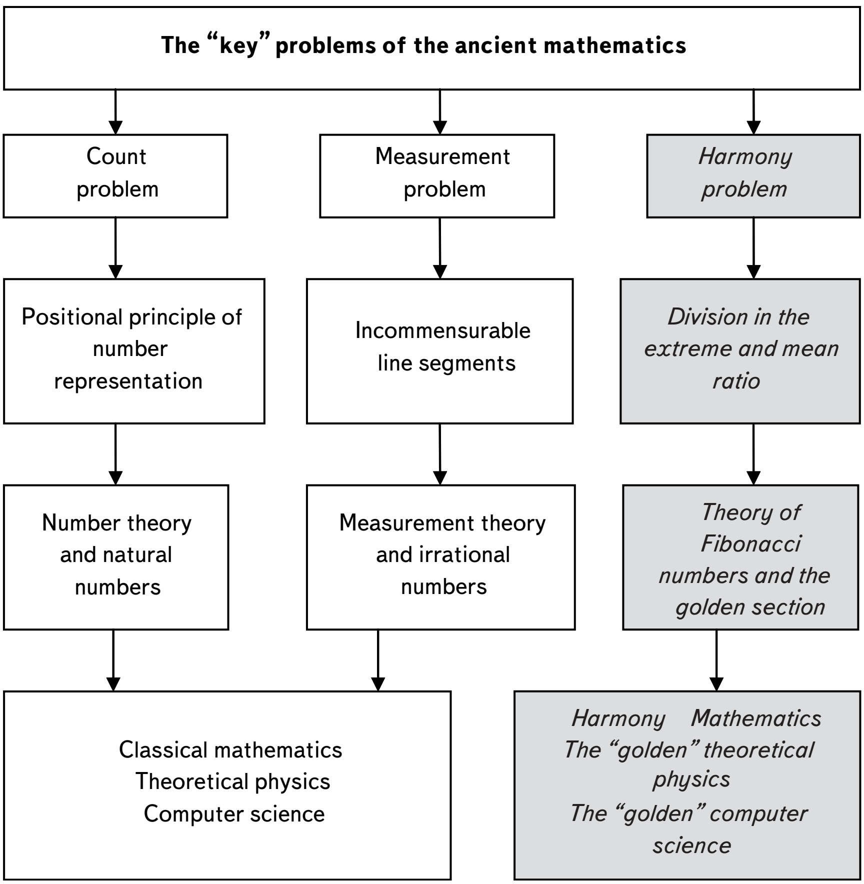

Figure I.1. Three "key" problems of the ancient mathematics and new directions in mathematics, theoretical physics and computer science

However, parallel with the "classical mathematics" in the ancient science another mathematical theory—Harmony Mathematics—had started to develop. Harmony Mathematics originated from another "key" idea of antique science—the "Harmony problem." The Harmony Problem triggered the "Doctrine about Numerical Harmony of the Universe" that was developed by Pythagoras.

A division in the extreme and mean ratio (the golden section) was the "key" mathematical discovery in this area [40]. The development of this idea resulted in the **Fibonacci numbers theory** [13, 16, 28] in modern mathematics. However, the extension of the Fibonacci numbers theory and its applications coupled with the generalization of Fibonacci numbers and the golden section produced the concept of **"Harmony Mathematics"** [100] as a new interdisciplinary direction of modern science and mathematics. This can result in the creation of the **"golden" theoretical physics**, based on the "golden" hyperbolic models of nature [51, 98, 106, 116, 118], and the **"golden" computer science**, based on new computer arithmetic [20, 24, 30, 87, 89, 94, 104] and a new theory of coding and cryptography [55, 113, 114].

15. The Main Goal of the Present Book

It seems that the dramatic history of the DEMR (the golden section) - that continued over several millennia - has ended as a great triumph for the golden section in the beginning of the 21st century. Many outstanding scientific discoveries that are based on the golden section (quasi-crystals, fullerenes , "golden" genomatrices and so on) gave reason to conclude that **the golden section may be considered as some kind of "metaphysical knowledge," "pre-number," or "universal code of Nature," which could become the basis for the future development of science; particularly, theoretical physics, genetics, and computer science.** This idea is the main concept of the book [57] and the articles [59, 84]. These scientific facts demand reappraisal of the role of the golden section in contemporary mathematics.

The main purpose of the present book is to revive the interest in the golden section and Pythagoras, Plato, and Euclid's "harmonic idea" in modern mathematics, theoretical physics, and computer science. It also strives to demonstrate that the Euclidean problem of the "division in extreme and mean ratio" (the golden section) is a powerful and fruitful source of many fundamental ideas and concepts of contemporary mathematics, theoretical physics, and computer science. We consider different generalizations of the golden mean, in particular, the *generalized golden p-proportions* (p=0, 1, 2, 3, ...) and the *generalized golden means of the order m* (*m* is a positive real number) as *fundamental mathematical constants* similar to the numbers π and *e*. We show that this approach resulted in: a new class of elementary functions—the *hyperbolic Fibonacci and Lucas functions;* a new class of the recursive numerical sequences—the *generalized Fibonacci and Lucas p-numbers* (p=0, 1, 2, 3, ...) and the *generalized Fibonacci and Lucas numbers of the order m* (*m* is a positive real number); and it also led to a new class of square matrices—the *Fibonacci and "golden" matrices.* Also, this approach resulted in a new measurement theory, *algorithmic measurement theory,* in a new class of number systems with irrational radices that are *codes of the golden p-proportions.* Additionally, a new kind of computer arithmetic, the *Fibonacci and "golden" arithmetic* and the *ternary mirror-symmetric arithmetic,* was developed, as well as a *new coding theory based on the Fibonacci matrices* and a new kind of cryptography—*the "golden" cryptography.*

The book consists of three parts. Part I "Classical Golden Mean, Fibonacci numbers, and Platonic Solids" consists of three chapters, Chapter 1 "The Golden Section", Chapter 2 "Fibonacci and Lucas Numbers", and Chapter 3 "Regular Polyhedrons". Part I is popular introduction into the Fibonacci numbers theory and its applications. Part I is intended for a wide audience including mathematics teachers of secondary schools, students of colleges and universities. Also, Part I can attract attention to the representatives of various branches of modern science and art that are interested in both creative and practical applications of the golden mean, Fibonacci numbers, and Platonic Solids.

Part II "Mathematics of Harmony" consists of three chapters, Chapter 4 "Generalizations of Fibonacci Numbers and the Golden Mean," Chapter 5 "Hyperbolic Fibonacci and Lucas Functions," and Chapter 6 "Fibonacci and Golden Matrices". Part II calls for special knowledge in mathematics and is intended, first of all, for mathematicians and scientists in theoretical physics.

Part III "Applications in Computer Science" consists of five chapters, Chapter 7 "Algorithmic Measurement Theory", Chapter 8 "Fibonacci Computers", Chapter 9 "Codes of the Golden Proportion" , Chapter 10 "Ternary Mirror-Symmetrical Arithmetic," and Chapter 11 "A New Coding Theory Based on Matrix Approach." Part III is intended for mathematicians and specialists in computer science.

Note that Parts II and III are, in the main, results of original researches obtained by the author in about 40 years of scientific work.

Acknowledgements

This book is a result of my life-time research in the field of the Golden Section, Fibonacci numbers, "Mathematics of Harmony," and their applications in modern science. I have met along the way a lot of extraordinary people who helped me to broaden and deepen my scientific knowledge and professional skills. More than 40 years ago I read the remarkable brochure *Fibonacci Numbers* written by the Russian mathematician **Nikolay Vorobyov**. This brochure was the first mathematical work on Fibonacci numbers published in the second half of the 20th century. It determined my scientific interests in Fibonacci numbers. In 1974, I met with Professor Vorobyov in St. Petersburg (formerly Leningrad) and narrated him about my scientific achievements in this area. He gave me as a keepsake his brochure *Fibonacci Numbers* with the following inscription: *"To highly respected Alexey Stakhov with Fibonacci's greetings."*

I have great gratitude to my teacher, the outstanding Ukrainian scientist, Professor **Alexander Volkov** (1924-2007), under whose leadership I defended my PhD dissertation in 1966 and DrSci dissertation in 1972. These dissertations were the first step in my research, which led me to the new scientific direction - Mathematics of Harmony.

I appreciated a true value of mathematics as a "method of thinking" after an influential collaboration with **Igor Vitenko**, a graduate of the mathematical faculty of the Lvov University, who, without any doubts, would have been known as an outstanding mathematician and glory of Ukrainian science if he hadn't passed away so early in life. His death in October 1974—which happened as a result of suicide—was a big loss. Also, I consider the outstanding Russian philosopher **George Chefranov** as my teacher of philosophy. Our evening walks, along Chekhov's street of Taganrog city, were accompanied by hot discussions of philosophical problems of science, which became a major philosophical university for me.

In my stormy scientific life I met many fine people, who could understand me, my enthusiasm, and appreciate my scientific direction. With deep gratitude, I recall a meeting in the Austrian city of Graz in 1976 with the exceptional Austrian mathematician **Alexander Aigner**. His review of my lecture at Graz University was the beginning of international recognition of my scientific direction. The German scientist and academician **Dr. Volker Kempe** who now lives in Austria, also impacted on my career. Thanks to Dr. Kempe, I was honored to be invited to Dresden Technical University as a visiting

professor in 1988. Another remarkable person who had a great influence on my work was the Ukrainian mathematician and academician **Yury Mitropolsky** (1917-2008). His influence on my research pertinent to the history of mathematics and other topics, such as the application of "Harmony Mathematics" to the contemporary mathematics and to the mathematical education, is inestimable contribution. Thanks to the support of Yury Mitropolski, I was able to publish many scientific articles in various Ukrainian academic journals.

My arrival in Canada in 2004 set the arena for the next stages of development in my scientific researches. Thanks to support from the famous physicist Professor **Mohammed El Nashie**, the editor-in-chief of the international magazine *Chaos, Solitons and Fractals*, I was able to publish many articles in this remarkable interdisciplinary journal. Within two years, I published 14 fundamental articles in *Chaos, Solitons and Fractals*, which closed my cycle of research in the field of "Mathematics of Harmony." These articles attracted attention of the international scientific community to my latest scientific results; also, these articles paved the way for making new professional contacts with world-renowned Western scientists. Scientific discussion with outstanding scientists in the field of the Golden Section, such as the Argentinean mathematician **Vera W. de Spinadel**—author of the book *From the Golden Mean to Chaos* (1998), the American mathematician **Louis Kauffman**—editor of the *Knots and Everything Series* from World Scientific, the American researcher **Jay Kappraff**—author of two remarkable books *Connections: The geometric bridge between Art and Science* (2001) and *Beyond Measure: A Guided Tour Through Nature, Myth, and Number* (2002), have considerably influenced on my research in the field of the Golden Section and Mathematics of Harmony. Thanks to the support from Professors **Louis Kauffman** and **Jay Kappraff**, this book was accepted by World Scientific for publication. I would like to express my gratitude to the initiative of Professor **Vera W. de Spinadel**, who helped with acceptance of my lecture, the "Three 'Key' Problems of Mathematics on the Stage of its Origin and the "Harmony Mathematics" as Alternative Way of Mathematics Development," as the plenary lecture at the Fifth Mathematics & Design International Conference (July 2007, Brazil).

Furthermore, I would like to express special thanks to **Dr. Scott Olsen**, the American philosopher and prominent researcher in the golden mean and author of the book *The Golden Section: Nature's Greatest Secret* (2006), for his invaluable assistance in the scientific and English editing of my book. Also, I would like to express my greatest gratitude to the Russian businessman and scientist **Dr. Ivan Raylyan**, a great enthusiast of the Golden Section, for his

financial assistance, which provided me with all of the best technology, informational tools, and pecuniary means to produce a camera-ready manuscript.

During my Canadian period of my scientific life I published a number of important articles in the field of the Harmony Mathematics together with **Boris Rozin.** I would like to express to him great gratitude for scientific collaboration.

I would like to express my special thanks to **Alexander Soypher** for his invaluable assistance in overall design of the book and illustrations, cover design and color appendix “Museum of Harmony and Golden Section.“

Lastly, this book would never have been written without self-denying support of my wife **Antonina,** who always creates perfect conditions for my scientific work in any countries I have been; and who has been sailing with me for more than 47 years on my “Golden” journey. In addition, I would like to express my special appreciation to my daughter **Anna Sluchenkova** for her critical remarks, and her invaluable help in the English translation and editing of the book, and, especially, for her work in preparing illustrations, and coordination and final preparation of camera-ready manuscript. Without her support this book would never been published.

Alexey Stakhov

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Appendix Museum of Harmony and Golden Section

Symmetry in Nature

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Pentagonal symmetry in nature

Sea Urchin (on the background) underwater photo credit Cornelia Kraus

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Comb Sand Star credit Peter Bryant, source: nathistoc.bio.uci.edu/Echinos/Astropecten.htm

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Golden spirals in nature

Barred Spiral Galaxy NGC 1300 credit NASA, ESA, and The Hubble Heritage Team (STScI/AURA), source: hubblesite.org

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Nautilus shell see credit for Figure 2.6-a

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Fibonacci numbers and phyllotaxis

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Golden Section in architecture

United Nations Building source: www.inetours.com/New_York/Pages/photos/UN-Headquarters-building.html

The Pentagon, US Department of Defense building credit US Department of Defense, source: www.defenselink.mil/photos/newsphoto.aspx?newsphotoid=1309

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Golden Section in Contemporary Abstract and Applied Art

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Chapter 1

The Golden Section

1.1. Geometric Definition of the Golden Section

1.1.1. *A Problem of the Division in the Extreme and Mean Ratio (DEMR)*

The Elements of Euclid is one of the best known mathematical works of ancient science. Written by Euclid in the 3rd century B.C., it contains the main theories of ancient mathematics: elementary geometry, number theory, algebra, the theory of proportions and ratios, methods of calculations of areas and volumes, etc. Euclid, in this work, systematized a 300-year period of development of Greek mathematics, and this work created a strong base for the further development of mathematics. The information about Euclid himself is extremely scanty. Except for several jokes, we only know that he taught at the mathematical school in Alexandria. *The Elements* of Euclid surpassed all works of his predecessors in the field of geometry, and during more than two millennia, *The Elements* remained the basic work for the teaching of "Elementary Mathematics." The 13 books of *The Elements* are dedicated to the knowledge of geometry and arithmetic of the Euclidean epoch.

From *The Elements* of Euclid, the following geometrical problem, which was named the problem of "Division in Extreme and Mean Ratio" (DEMR), came to us [40]. This problem was formulated in Book II of *The Elements* as follows:

Theorem II.11 (the area formulation of DEMR)**.** To divide a line AB into two segments, a larger one AC and a smaller one CB so that:

$$S(AC)=R(AB,CB). \qquad (1.1)$$

Note that $S(AC)$ is the area of a square with a side AC and $R(AB,CB)$ is the area of a rectangle with sides AB and CB.

We can rewrite the expression (1.1) in the following form:

$$(AC)^2 = AB \times CB. \tag{1.2}$$

Now, divide both parts of the expression (1.2) by AC and then by CB. The expression (1.2) then takes the form of the following proportion:

$$\frac{AB}{AC} = \frac{AC}{CB}. \tag{1.3}$$

This form is well-known to mathematicians as the "golden section."

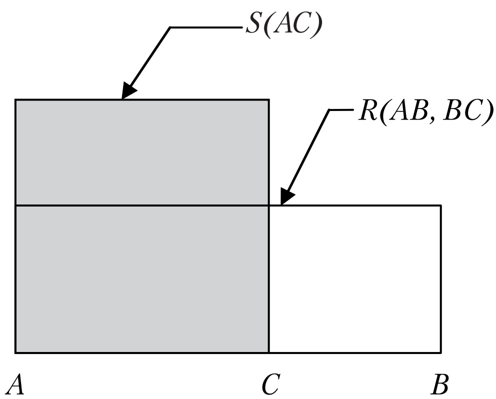


Figure 1.1. A geometrical interpretation of Theorem II.11 (*The Elements* of Euclid)

We can interpret a proportion (1.3) geometrically (Fig. 1.2): divide a line AB at the point C into two segments, a larger one AC and a smaller one CB, so that the ratio of the larger segment AC to the smaller segment CB is equal to the ratio of the line AB to the larger segment AC.

Figure 1.2. The division of a line in extreme and mean ratio (the golden section)

Denote a proportion (1.3) by x. Then, taking into consideration that $AB = AC + CB$, the proportion (1.3) can be written in the following form:

$$x = \frac{AB}{AC} = \frac{AC+CB}{AC} = 1 + \frac{CB}{AC} = 1 + \frac{1}{\frac{AC}{CB}} = 1 + \frac{1}{x},$$

whence we obtain the following algebraic equation:

$$x^2 = x + 1. \tag{1.4}$$

It follows from the "geometrical meaning" of the proportion (1.3) that the required solution of Eq. (1.4) has to be a positive number; it also follows that a positive root of Eq. (1.4) is a solution of the problem. If we denote this root by τ, then we obtain:

$$\tau = \frac{1+\sqrt{5}}{2}. \tag{1.5}$$

This number is called the *Golden Proportion, Golden Mean, Golden Number,* or *Golden Ratio*.

We can write the following identity, which connects the powers of the golden ratio:

$$\tau^n = \tau^{n-1} + \tau^{n-2} = \tau \times \tau^{n-1}. \quad (1.6)$$

The approximate value of the golden proportion is:
$\tau \approx 1.6180339887498948482045868343656381177203091 80....$

Do not be astonished by this number! Do not forget that this number is an irrational one! In this book, we will use the following approximate value: $\tau \approx 1.618$ or even $\tau \approx 1.62$.

This surprising number, which possesses unique algebraic and geometrical properties, became an aesthetic canon of ancient Greek art and Renaissance art.

Why did Euclid formulate Theorem II.11? As is shown in [40], by using this theorem, Euclid introduced the geometric construction of the golden triangle, pentagram, and dodecahedron (these geometric figures will be discussed below).

1.1.2. *The Origin of the Concept and Title of the Golden Section*

Authorities vary over who introduced both the concept and terminology for the golden section. According to [40], the concept of the DEMR (or the golden section) was introduced in *The Elements* (see Theorem II.11). However, *The Elements* is not a completely original work. There is an opinion that the majority of theorems presented in *The Elements* are scientific results obtained by the Pythagoreans. Roger Herz-Fischler wrote [40]: "Many authors point to references according to which the pentagram was the symbol of the Pythagoreans, and from this, they deduce that the Pythagoreans were probably acquainted with DEMR." Herz-Fischler notes that the famous historians of mathematics, Heath and Van der Waerden, supported this point of view. However, Euclid did not use the term "golden section" in his works; instead, he used the term DEMR. In [40], we can trace the history of terminology for the DEMR throughout the ages. As follows from [40], the terminology of DEMR was used by Euclid, Fibonacci (1220), Zumberti (1516), Gryaneus (1533), Candalla (1566), Billingsley (1570), Commandino (1572), Clavius (1579), and Barrow (1722). However, many mathematicians used the term "Middle and Two Ends," in particular: Abu Kamil (850-930), al-Biruni (973-1050), Gerhard of Cremona (12th century), Adelard (12th century), Campanus of Novara (13th century), and Billingsley (1570). In addition to the definitions of the "division in the extreme and mean ratio" and "proportion having a middle and two ends," other definitions, namely: "divina proportione," "proportionally divided," "continuous proportion," "medial section," "the golden number," and "the golden section" were used. The term "divina proportione" was the title of Pacioli's book published in 1509. This term was used by Kepler in his 1608 letter. We can find the term "proportionally divided" in Clavius' 1574 edition of *The Elements*. Kepler, in his *Mysterium Cosmographycum* used the term "proportional division" for the DEMR. The term "continuous proportion" was used by Euclid and we can also find it used in Kepler's 1597 letter. The term "medial section" was used by Leslie (1820) in connection with the DEMR.

The "golden number," the "golden section," or the "golden mean" are the most popular names for the DEMR. As is emphasized in [40], Tannery (1882) used the term "Section d'Or" and Cantor (1894) used a title of the "golden shnitt." Also noted in [40], for the first time, the term "golden schnitt" (the "golden section") was used by Ohm in the second edition of his book *Pure Elementary Mathematics* (1835).

However, other points of view exist about the origin of the term "the golden section" — mainly in Russian literature. Edward Soroko, who is one of the most authoritative Slavic researchers of the "golden section," wrote in his book [25]:

"The title of the "Golden Section" ("Sectio Aurea") takes its origin from Claudia Ptolemey who gave this title for the number 0.618, after he had been convinced that the growth of a person with perfect constitution is divided naturally in this ratio. The given title was fixed and then became popular due to Leonardo da Vinci who often used this title."

Unfortunately, the claim about the involvement of Leonardo da Vinci to the introduction of the title of the golden section has been questioned because there are no direct references to this title in his works. However, we should not forget that the great Italian mathematician Luca Pacioli was first to introduce the title "divine proportion" for the DEMR. Leonardo da Vinci actively participated in Pacioli's book *De Divina Proportione* as he illustrated this unique mathematical work — which was the first book on the golden mean in world history. This means that, without any doubt, Leonardo da Vinci knew the concept of the "divine proportion."

Very often, the golden mean is denoted by the Greek letter *F* (the number PHI). This letter is the first one in the name of the well-known Greek sculptor Phidias, who widely used the golden section in his sculptural works. Remember that Phidias (5th century B.C.), together with Polyclitus, were considered to be two of the most famous and authoritative masters of ancient Greek sculpture of the Classical epoch. He became famous thanks to his supervision over designing the Acropolis. He created the enormous bronze statue of Athena ("Winner in Battle") in commemoration of the victory over the Persians. Also, he created two grandiose statues from gold and ivory: Athena Parthenos ("Maidens") for the Parthenon and Zeus for the Olympian temple of Zeus (about 430 B.C.), which is considered one of the "Seven Miracles of the World." Despite the monumental character of his sculptures — which were unprecedented by their sizes among all Greek sculptures of that time (for example, the 9-meter sculpture of Athena Parthenos or the 13-meter sculpture of the Olympian Zeus) — they were constructed with strict steadiness and harmony based on the golden mean.

1.1.3. *A Way of Geometrical Construction of the Golden Section*

The golden section arises often in geometry. From the Euclidean *Elements,* we know the following form of geometric construction of the golden section by using only a pair of compasses and a ruler (Fig. 1.3).

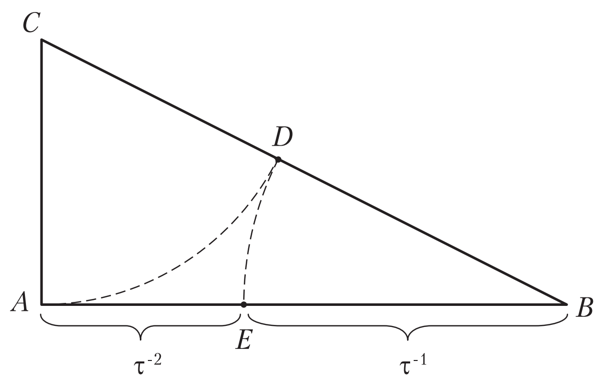


Figure 1.3. The geometric construction of the golden section

Construct a right triangle ABC with the sides AB=1 and AC=1/2. Then, according to the Pythagorean Theorem, we have: $CB=\sqrt{1+(1/2)^2}=\sqrt{5}/2$. By drawing the arc AD with the center at the point C before its intersection with the segment CB at the point D, we obtain the segment

$$BD=CB-CD=\left(\sqrt{5}-1\right)/2=\tau^{-1}. \qquad (1.7)$$

By drawing the arc DE with the center at the point B before its intersection with the segment AB at the point E, we obtain a division of the segment AB at the point E by the golden section because

$$\frac{AB}{EB}=\frac{EB}{AE}=\tau \quad \text{or} \quad AB=1=EB+AE=\tau^{-1}+\tau^{-2}. \qquad (1.8)$$

Thus, the simple right triangle with leg ratio of 1:2, well known to ancient science, could be the basis for the discoveries of *the Pythagorean Theorem, the Golden Section* and *Incommensurable Line Segments* — the three great mathematical discoveries attributed to Pythagoras.

1.1.4. *A "Double" Square*

Many mathematical regularities, we can say, did "lie on the surface," and they needed only to be seen by a person with analytical, logical thinking that was inherent in antique philosophers and mathematicians. It is possible that ancient mathematicians could find the golden proportion by investigating the so-called elementary rectangle with the side ratio of 1:2. This is named a "double" or "two-adjacent" square because it consists of two squares (Fig. 1.4).

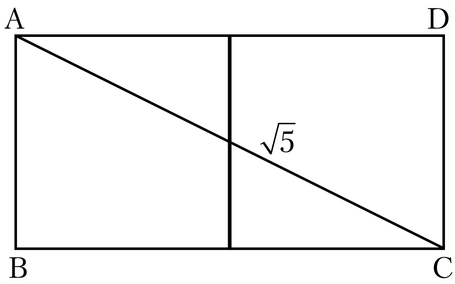


Figure 1.4. A rectangle with a side ratio of 2:1 (a "double" square)

Let $AB = 1$ and $AD = 2$. Then, if we calculate the diagonal *AC* of the "double" square, then according to the Pythagorean Theorem we get: $AC = \sqrt{5}$.

If we take the ratio of the sum $AB + AC$ relative to the larger side *AD* of the "double" square, we come to the golden mean, because:

$$(AB+AC)/AD = \left(1+\sqrt{5}\right)/2. \qquad (1.9)$$

It is paradoxical that the Pythagorean Theorem is very well-known to each schoolboy while the golden mean is familiar to very few. The main purpose of this book is to tell about this wonderful discovery of antique science to everyone who preserved the feeling to be surprised and admired, and we will show to modern scientists the far, not trivial, applications of the golden mean in many fields of modern science. We will tell our readers about this unique mathematical discovery, which during the past millenia has attracted the attention of outstanding scientists, mathematicians and philosophers of the past, like: Pythagoras, Plato, Euclid, Leonardo da Vinci, Luca Pacioli, Johannes Kepler, Zeising, Florensky, Ghyka, Corbusier, Eisenstein, American mathematician Verner Hogatt — founder of the Fibonacci Association, and the Great scientist Alan Turing — a founder of modern computer science.

1.2. Algebraic Properties of the Golden Mean

1.2.1. *Remarkable Identities of the Golden Mean*

What a "miracle" of nature and mathematics is the golden mean? Why does our interest in it not wither, but, on the contrary, increases with each century? To ponder this issue, we suggest that our readers focus all of their mathematical knowledge and plunge into this fascinating aspect of the world of mathematics. Only then there is the possibility of enjoying and understanding the wonderful mathematical properties, beauty and harmony of this unique phenomenon — the golden mean.

Let us start from the algebraic properties of the golden mean. It follows from (1.4) — which is very simple and nevertheless a rather surprising property of the golden mean. If we substitute the root τ (the golden mean) for x in Eq. (1.4), then we will get the following remarkable identity for the golden mean:

$$\tau^2=\tau+1. \tag{1.10}$$

Let us be convinced that the identity (1.10) is valid. For this purpose, it is necessary to carry out elementary mathematical transformations over the left-hand and right-hand parts of the identity (1.10) and to prove that they coincide.

In fact, we have for the right-hand part:

$$\tau+1=\frac{1+\sqrt{5}}{2}+1=\frac{3+\sqrt{5}}{2}.$$

On the other hand,

$$\tau^2=\left(\frac{1+\sqrt{5}}{2}\right)^2=\frac{1+2\sqrt{5}+5}{4}=\frac{3+\sqrt{5}}{2},$$

from whence the validity of the identity (1.10) follows.

If we divide all terms of the identity (1.10) by τ we come to the following expression for τ:

$$\tau=1+\frac{1}{\tau}. \tag{1.11}$$

This can be represented in the following form:

$$\tau-1=\frac{1}{\tau}. \tag{1.12}$$

Now, analyze the identity (1.12). It is well known that any number a has its own *inverse number* $1/a$. For example, the fraction 0.1 is an inverse number to 10. A traditional algorithm to get the inverse number $1/a$ from the initial number a consists of the division of the number 1 by the number a. In general case, this is a very complex procedure. Try, for example, to get the inverse number of $\boldsymbol{a}$ = **357821093572**. This can be fulfilled only by the use of modern computer.

Consider the golden mean $\tau=\left(1+\sqrt{5}\right)/2$, which is an irrational number. How can we get the inverse number $1/\tau$? The formula (1.12) gives a very simple answer to this question. To solve this problem, it is enough to subtract 1 from the golden mean τ. In fact, on the one hand,

$$\frac{1}{\tau}=\frac{2}{1+\sqrt{5}}=\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})}=\frac{2(1-\sqrt{5})}{(1)^2-(\sqrt{5})^2}=\frac{\sqrt{5}-1}{2}.$$

While on the other hand, as follows from (1.12), the inverse number $1/\tau$ can be found in the following manner:

$$\frac{1}{\tau}=\tau-1=\frac{1+\sqrt{5}}{2}-1=\frac{\sqrt{5}-1}{2}.$$

However, we will get greater "aesthetic pleasure" if we carry out the following transformations of the identity (1.10). Multiply both parts of the identity (1.10) by τ, and then divide them by τ^2. The outcome we get is two new identities:

$$\tau^3=\tau^2+\tau \tag{1.13}$$

and

$$\tau=1+\tau^{-1}. \tag{1.14}$$

If we continue to multiply both parts of the identity (1.13) by τ, then divide both parts of the identity (1.14) by τ^2, and continue this procedure ad infinitum, we will come to the following graceful identity that connects the adjacent degrees of the golden mean:

$$\tau^n=\tau^{n-1}+\tau^{n-2}, \tag{1.15}$$

where n is an integer taking its values from the set $\{0,\pm1,\pm2,\pm3,...\}$.

The identity (1.15) can be expressed verbally as follows: "Any member of the golden series (golden powers) is the sum of the previous two golden powers."

This property of the golden mean is truly unique! It is very difficult to believe that the following identity is "absolutely true":

$$\tau^{100}=\left(\frac{1+\sqrt{5}}{2}\right)^{100}=\left(\frac{1+\sqrt{5}}{2}\right)^{99}+\left(\frac{1+\sqrt{5}}{2}\right)^{98}, \tag{1.16}$$

however, its validity follows from the validity of the general identity (1.15).

Moreover, the following identity is also true:

$$\tau^{100}=\left(\frac{1+\sqrt{5}}{2}\right)^{100}=\left(\frac{1+\sqrt{5}}{2}\right)^{99}+\left(\frac{1+\sqrt{5}}{2}\right)^{97}+\left(\frac{1+\sqrt{5}}{2}\right)^{96}. \tag{1.17}$$

There are an infinite number of identities similar to (1.17), and all of them follow from the general identity (1.15).

1.2.2. *The "Golden" Geometric Progression*

Consider the following sequence of golden mean degrees:

$$\left\{...,\tau^{-n},\tau^{-(n-1)},...,\tau^{-2},\tau^{-1},\tau^{0}=1,\tau^{1},\tau^{2},...,\tau^{n-1},\tau^{n},...\right\}. \tag{1.18}$$

The sequence (1.18) has very interesting properties. On the one hand, the sequence (1.18) is a geometric progression because each of its element is equal to the preceding one multiplied by the number τ, a *Denominator* or *Constant Multiplier* in the geometric progression (1.18), that is,

$$\tau^n = \tau \times \tau^{n-1}. \tag{1.19}$$

On the other hand, according to (1.15), the sequence (1.18) has the property of "additivity" because each element is equal to the sum of the two preceding elements. Note, that the property (1.15) is characteristic only for the geometrical progression with the denominator τ, and such geometrical progression is named the *Golden Progression*, *Golden Series* or *Golden Powers*.

In this connection, any *Logarithmic Spiral* corresponds to a certain geometrical progression of the type (1.18), the opinion of many researchers is that the property (1.15) distinguishes the golden progression (1.18) among other geometrical progressions, and this fact is a reason for the wide-spread prevalence of the golden logarithmic spiral in forms and structures of nature.

1.2.3. *A Representation of the Golden Mean in the Form of Continued Fraction*

Let us now prove another astonishing property of the golden mean based on the identity (1.11). If we substitute the expression $1+1/\tau$ for t in the right-hand part of (1.11), then we get the representation t in the form of the following continued fraction

$$\tau = 1 + \cfrac{1}{1 + \cfrac{1}{\tau}}.$$

If we continue such substitution ad infinitum, we get the continued fraction of the following form:

$$\tau = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}. \tag{1.20}$$

A representation of (1.20) in mathematics is called a *Continued* or *Chain* fraction. Note that the theory of continued fractions is one of the most important topics of modern mathematics.

1.2.4. *A Representation of the Golden Mean in "Radicals"*

Consider the identity (1.10). If we extract a square root from the right-hand and left-hand parts of the identity (1.10), then we will get the following representation for τ:

$$\tau = \sqrt{1+\tau}. \tag{1.21}$$

If we substitute the expression $\sqrt{1+\tau}$ for τ in the right-hand part of (1.21), then we get the following representation for τ:

$$\tau=\sqrt{1+\sqrt{1+\tau}}. \tag{1.22}$$

If we continue such substitution ad infinitum, we will get another remarkable representation of the golden mean τ in "radicals":

$$\tau=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}. \tag{1.23}$$

Every mathematician intuitively aspires to express mathematical results in the simplest and most compact form, and if he discovers such an expression, it gives him "aesthetic pleasure." In this respect, mathematical creative work (aspiration for "aesthetic" expression of mathematical results), is similar to the creative activity of a composer or poet, because their main aim is to find perfect musical or poetic forms that give rise to aesthetic pleasure. Note that the formulas (1.20) and (1.23) give us an aesthetic pleasure, arousing a feeling of rhythm and harmony when we begin to think of the infinite repeatability of the same simple mathematical elements in formulas for τ.

1.3. The Algebraic Equation of the Golden Mean

1.3.1. *Polynomials and Equations*

Since ancient times, mathematicians paid special attention to the study of polynomials and the solutions of algebraic equations, and this important mathematical problem promoted the development of algebra. As is well known, a *Polynomial* of *n-th degree* is represented by the following expression:

$$a_n x^n+\ldots+a_2 x^2+a_1 x+a_0,\ \text{where} \tag{1.24}$$

In other words, a polynomial is the sum of the integer-valued degrees taken with certain coefficients. For example, a decimal notation of a number is, in essence, some representation of this number in the form of a polynomial of the number 10, for instance, $365=3\times\left(10^2\right)+6\times(10)+5$. If x is a variable, then a polynomial gives any *Polynomial Function*, the range of which coincides with the range of the variable x. The polynomials of the first, second, third, and fourth degrees are called *Linear, Quadratic, Cubic,* and *Biquadrate* polynomials, respectively. The *Algebraic Equation* (in the standard form) is a statement written with algebraic designations that some polynomial function is equal to zero for some values of a variable x. For example,

$x^2-5x+6=0$ is an algebraic equation. The values of the variable, for which the polynomial becomes equal to 0, are named *roots* of this polynomial. For example, the polynomial x^2-5x+6 has two roots, 2 and 3, because $2^2-5\times 2+6=0$ and $3^2-5\times 3+6=0$. Note that in the polynomial x^2-5x+6 the variable x represents an arbitrary number from the range of the given polynomial function, but the algebraic equation $x^2-5x+6=0$, in contrast, means that the variable x can only be the number 2 or 3, the roots of this algebraic equation.

Algebraic equations have always served as a powerful means to solutions of practical problems. The exact language of mathematics has allowed mathematicians to simply express the facts and relations, which, when presented in ordinary language, can seem confusing and complicated. The unknown values, denoted by some algebraic symbols, for example x, can be found if we formulate the problem in mathematical language through the form of equations. Methods of an equation's solution are the basis for the *Equation Theory* in mathematics.

Now, recall the basic data of linear and square algebraic equations that are well-known to us from secondary school. The linear equation, in general form, can be written as $ax+b=0$, where a and b are some numbers and $a\neq 0$. This equation has one solution: $x=-b/a$; that is, the linear equation has only one root. The quadratic equation has the following form:

$$ax^2+bx+c=0, \text{ where } a\neq 0. \tag{1.25}$$

The rules for the solution of algebraic equations of the first and second degree were well-known in antiquity. For example, as we recall from secondary school mathematics, we know the following formula for the roots of the quadratic equation (1.25):

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{1.26}$$

Remember that the values of the roots depend on the *Determinant D* of the quadratic equation (1.25) — which is defined by the following formula:

$$D=b^2-4ac. \tag{1.27}$$

If the determinant D is positive, the formula (1.26) gives exactly two real roots. If $D=0$, then $x=-b/2a$, and we say that Eq. (1.25) has two equal roots. If the determinant D is negative, we have to introduce an imaginary unit i, which is defined by $i=\sqrt{-1}$, and for this case, both of these roots are complex.

1.3.2. *Quadratic Irrationals*

The discovery of irrational numbers is the greatest mathematical discovery of Greek mathematics. This discovery was made by Pythagoreans in the 5th century

B.C. during the investigation of the ratio of two geometric segments: a diagonal and a side of a square. By using *Reductio ad Absurdum,* that is, the *Indirect Method of Contradiction*, the Pythagoreans proved that the ratio of the diagonal to the side of the square, which is equal to $\sqrt{2}$, cannot be represented in the form of the ratio of two natural numbers. Such geometric segments were named *Incommensurable*, and the numbers, which express similar ratios, were named *Irrational Numbers*.

The discovery of *Incommensurable Segments* resulted in the first crisis of the foundations of mathematics, and, ultimately, became a turning point in the development of mathematics. One legend says that in honor of this discovery, Pythagoras had carried out a "hecatomb," namely, he had sacrificed 100 bulls to the gods. In the beginning, the Pythagoreans attempted to keep the new discovery a secret. According to legend, Hippias, one of the Pythagoreans, broke the oath and divulged the secret of this discovery. Later, he perished during a ship-wreck that the Pythagoreans considered to be a punishment from the gods for the disclosure the discovery's secret. The influence of this discovery on the development of science can be compared to the discovery of Lobachevsky's geometry in the first half of the 19th century, and with the discovery of Einstein's Relativity Theory at the beginning of the 20th century. Thanks to this discovery, mathematics received an entirely new mathematical concept — the concept of *Irrational Numbers*.

The title "rational" is from the Latin word "ratio," which is a translation of the Greek word "logos." The numbers, which can be represented as the ratios of two integers, were called "rational." In contrast to rational numbers, those numbers, which express the ratios of the incommensurable line segments, had been named irrational (from the Greek word "alogos").

Unlike a linear equation, a quadratic equation of the kind (1.25) with rational coefficients can have irrational roots called *quadratic irrationals*. Book 10 of *The Elements* is devoted to the study of the quadratic irrationals.

Scientists of India, the Middle East, and many European mathematicians of the Middle Ages studied quadratic irrationals. It was proved that any number of the kind $\sqrt{N}$ for any integer N, which is not a full square, is an irrational number as well as a number $\sqrt[3]{N}$, where N is not a cube, etc. As they say, similar irrationals can be represented in *Radicals*. In 16th century, Italian mathematicians Tartalja, Cardano, and Ferrari found a general formula for the roots of the cubic and biquadrate algebraic equations that represented their irrational roots in radicals. Despite the fact that attempts to find general formulas for the roots of the algebraic equations of the fifth and more degrees appeared unsuccessful, this research direction resulted in new mathematical discoveries, which were connected with the mathematical works of Niels Abel and Evariste Galois.

1.3.3. *Poor "Studious" Niels Abel*

Niels Henrik Abel
(1802-1829)

The history of science and mathematics contains many tragic pages. One of them is the life and scientific work of the Norwegian mathematician Niels Abel, whose mathematical research involves algebra and solutions to algebraic equations.

For a long time, mathematicians assumed that an irrational root of any algebraic equation with rational factors can be expressed in radicals. However, in the 19th century, Abel proved that the irrational roots of the general equations of the 5th and more degrees cannot be expressed in radicals.

On the big square in the city of Oslo, the capital of Norway, a majestic monument rises up above the cityscape. On the granite stone of the monument, a young man is carved with a spiritualized face struggling against two disgusting monsters. The monument is of the well-known Norwegian mathematician Niels Henrik Abel. What do these monsters symbolize? Some mathematicians joke that these monsters represent the equations of the fifth degree and the elliptic functions conquered by Abel. Others consider that it is allegory: the sculptor wished to embody the social injustice with which Abel struggled all his life.

The sad life of this Norwegian mathematical genius is typical for scientific geniuses-not just from his country or of his time. Unfortunately, the destiny of many mathematical geniuses unfolds tragically. It is connected with the fact that many of great mathematicians are often not understood, and their stunning mathematical discoveries are not recognized by their contemporaries, unfortunately, they are only recognized some time after their death.

Abel was born in 1802 in Northwest Norway in the small fishing town of Finnej, where there were neither mathematicians, nor mathematical text-books available to him. Very little is known about the first years of his childhood. He was enrolled in school in Oslo at the age of thirteen. Abel studied lightly and obtained good marks excelling in mathematics. He also liked to play chess and visit the theatre. However, within three years of school, a sudden change opened up his world. Instead of a cruel mathematics teacher who beat pupils, the new mathematics teacher, Holmboe, arrived at the school. He knew his subject very well, and he spiked the interest of his pupils. Holmboe gave each pupil the opportunity to act independently, and encouraged them to take the first steps paid special attention towards a more fulfilling mathematics education. Soon, Abel not only took a great interest in mathematics, but also found that he was capable of solving mathematical problems of which other people were incapable.

Abel's family lived in humiliating poverty so he was attending a school free of charge. In 1820, Abel's father died, and the family remained without any financial means. Their situation was desperate. Niels thought about returning to his native city and searching for a job, but some of his professors paid attention to the young man's talent and helped him to enroll in the local university. They managed to obtain grants for Niels Abel to preserve his unique mathematical talent in science, and for traveling abroad. Abel's visits to Berlin, Paris, and to other large mathematical centers of that time contributed greatly to his future mathematical works. Unfortunately, the young mathematician's discoveries - which surpassed much the science of the time - resulted in a lack of understanding and underestimation by his contemporaries. Abroad, as well as in his native land, Abel experienced great financial difficulties and a constant feeling of intolerable loneliness. His attempts to get scientific recognition were unsuccessful. He sent his works to the Parisian Academy where they were forwarded to the French mathematician, Cauchy. Unfortunately, these works were lost. Also his letter to the German mathematician Gauss was left unanswered.

A monument of Niels Abel in Oslo

The young mathematician, who made a revolution in mathematics, returned home to remain the same poor and unknown "studious" Abel. He could not find a job. Being ill by tuberculosis, and "poor as a church mouse," by his own words, Abel, in a condition of the cheerless melancholy, died on April 6, 1829, at 26.

The proof of insolvability in radicals of the algebraic equations of the fifth and greater degrees was his most important mathematical discovery, made at just 22. In the opinion of the mathematician Ermit, "Abel left such a rich heritage for mathematicians that they will develop it during the next 500 years."

There is a custom, to which new results and discoveries are named after the scientists who made them. Now, anyone holding a book on higher mathematics can see that Abel's name is immortalized in various areas of mathematics. There are several theorems and mathematical results bearing Abel's name:

Abelian integrals, Abelian equations, Abelian groups, Abelian formulas, and Abelian transformations. Abel would be surprised to learn that his work has had a profound influence upon the development of mathematics.

1.3.4. *Mathematician and Revolutionist Evariste Galois*

The destiny of one more genius, the French mathematician Evariste Galois, is not less tragic. He was born on October 26, 1811, in the small town of Bourg-la-Reine near Paris, France.

Evariste Galois
(1811-1832)

For the first twelve years of his life his mother, who was a fluent reader of Latin and classical literature, was responsible for his education. In October 1823, he entered the Lycee Louis-le-Grand. At Louis-le-Grand, Galois enrolled in the mathematics class of Louis Richard. He worked more and more on his own researches and less and less on his schoolwork. He studied Legendre's *Geometrie* and the treatises of Lagrange. His teacher Richard reported, "This student works only in the highest realms of mathematics." In April 1829 Galois published his first mathematics paper on continued fractions in the *Annales de Mathematiques*. Later he submitted articles on the algebraic solution of equations to the Academie des Sciences. Cauchy was appointed as referee of Galois' paper. On December 29, 1829 Galois has received his Baccalaureate degree. In the development of Abel's work that he learned from *Bulletin de Ferussac*, and by the Cauchy's advice, he submitted a new article *On the Condition that an Equation be Soluble by Radicals* in February 1830. The paper was sent to Fourier, the secretary of the Paris Academy, to be considered for the Grand Prize in mathematics. Since Fourier died in April 1830, Galois' paper was never found and considered for the prize. Further Galois' works on the theory of elliptic functions and Abelian integrals were initiated by Abel and Jacobi's works. With support from Jacques Sturm, he published three papers in *Bulletin de Ferussac* in April 1830.

Galois enthusiastically participated in revolutionary activity, and eventually found himself in a prison for several times. In March 1832, being in a prison, he felt ill with cholera and was sent to the pension Sieur Faultrier where he fell in love with Stephanie-Felice du Motel, the daughter of the resident physician. In May 1832 his stormy life finished: he was killed in a duel. The reason for the duel was not clear but certainly linked with Stephanie. Before the duel, he wrote a resume of his mathematical discoveries and transferred his notes to one of his friends, requesting him to pass it on to one of the leading mathematicians. The note ended with the following words: "You will publicly ask Jacobi or Gauss to give a conclusion not about validity, but about the value of these theorems. Then, I hope, people, who can decipher all this mess, will be found." From what is known, Galois' letter reached neither Jacobi nor Gauss. The mathematical community only learned of Galois' works in 1846, when the French mathematician Liouville published most of Galois' works in his mathematical journal. All of Galois' works were presented in a small format of 60 pages. These works contained a presentation of group theory — now considered to be the "key" theory of modern algebra and geometry, the first classification of irrationals defined by algebraic equations. This doctrine is now referred to as *Galois Theory*.

1.3.5. *The "Golden" Algebraic Equations of n-th Degree*

And now, after such fundamental mathematical training, we again turn to the equation of the golden proportion (1.4). Clearly, it is the quadratic algebraic equation of the type (1.25) with the factors:

$$a = 1; b = -1; c = -1. \tag{1.28}$$

By using (1.27) and (1.28), we can calculate the determinant of Eq. (1.4):

$$D = 5.$$

It follows from here that Eq. (1.4) has two real roots:

$$x_1 = \tau = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = -\frac{1}{\tau} = \frac{1-\sqrt{5}}{2}. \tag{1.29}$$

The root x_1 coincides with the golden mean τ.

Typically, the main problem of the theory of algebraic equations is to find their roots. We know the elementary algebraic equation (1.4) with the root equal to the golden mean. With this equation, the following question arises: are there algebraic equations of the highest degrees with the root equal to the golden mean? And if so, what do they look like? To answer this question we will use the following reasoning concerning the initial equation of the golden mean (1.4).

Multiply both parts of Eq. (1.4) by x; as a result we get the following equality:

$$x^3 = x^2 + x. \tag{1.30}$$

Equation (1.4) can be written in the following form:

$$x = x^2 - 1.$$

If we substitute x^2 - 1 for x in the expression (1.30), we get the following algebraic equation of the third degree:

$$x^3 = 2x^2 - 1. \tag{1.31}$$

On the other hand, if we substitute $x+1$ for x^2 in (1.30), then we get one more equation of the third degree:

$$x^3 = 2x + 1. \tag{1.32}$$

Thus, we have obtained two new algebraic equations of the third degree. Now, prove that the golden mean is a root of Eq. (1.31). With this end, we substitute the golden mean $\tau = \left(1+\sqrt{5}\right)/2$ for x to the left-hand and right-hand parts of Eq. (1.31) and show that both parts of this equation coincide. In fact, by using the identity (1.15), we can obtain for the left-hand part:

$$\tau^3 = \tau^2 + \tau = \frac{3+\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} = \frac{4+2\sqrt{5}}{2} = 2+\sqrt{5}.$$

We can also obtain for the right-hand part of this equation:

$$2\tau^2 - 1 = 2\times\frac{3+\sqrt{5}}{2} - 1 = 2+\sqrt{5}.$$

Hence, the golden mean τ is, in fact, a root of Eq. (1.31), that is, Eq. (1.31) is a golden one. By analogy, we can prove that Eq. (1.32) is a golden one as well.

Let us derive the golden algebraic equation of the fourth degree, for this purpose we multiply both parts of the equality (1.30) by x; as a result we will get the following equality:

$$x^4 = x^3 + x^2. \tag{1.33}$$

We then use the expression (1.4) for x^2 and the expressions (1.31) or (1.32) for x^3. If we substitute them into the expression (1.33), we obtain two new golden algebraic equations of the fourth degree:

$$x^4 = 3x^2 - 1 \tag{1.34}$$

$$x^4 = 3x + 2. \tag{1.35}$$

The analysis of Eq. (1.35) resulted in unexpected result. It is found, that this equation describes the energetic state of the butadiene molecule, a valuable chemical substance used in the production of rubber. American scientist Richard Feynman, Nobel Prize Laureate, expressed his enthusiasm concerning Eq. (1.35) in the following words: "What miracles exist in mathematics! According to my theory, the golden proportion of the ancient Greeks gives the minimal energy condition of the butadiene molecule."

This fact, at once, raises our interest in the golden equations of the higher degrees, which, probably, describe energy conditions of the molecules of other chemical substances. These equations can be obtained, if we will consistently consider the equalities of the kind $x^n = x^{n-1} + x^{n-2}$. As an example, we can derive the following golden equations of the higher degrees:

$x^5 = 5x^2 - 2 = 5x + 3;$

$x^6 = 8x^2 - 3 = 8x + 5;$

$x^6 = 13x^2 - 5 = 13x + 8.$

The analysis of these equations demonstrates that Fibonacci numbers (1, 1, 3, 5, 8, 13, 21, 34, 55) — which are addressed in Chapter 2 — are the numerical factors in the right-hand part of these equations. It is easy to show, that in the general case, the golden algebraic equation of the n-th degree can be expressed in the following form:

$$x^n = F_n x^2 - F_{n-2} = F_n x + F_{n-1}, \tag{1.36}$$

where F_n, F_{n-1}, F_{n-2} are Fibonacci numbers.

Note, once again, that the main mathematical property of all equations of the kind (1.36) is the fact that all of them have a common root — the golden mean.

Thus, our simple reasoning resulted in a small mathematical discovery: we have found an infinite number of new golden algebraic equations given by (1.36) — which have the golden mean as a common root.

If we substitute in Eq. (1.36) its root $\tau = \left(1+\sqrt{5}\right)/2$ for x, then we obtain the following remarkable identities that connect the golden mean with Fibonacci numbers:

$$\tau^n = F_n \tau^2 - F_{n-2} = F_n \tau + F_{n-1}. \tag{1.37}$$

Below we show that, for instance, the18th, 19th, and 20th Fibonacci numbers are equal to the following:

$$F_{18} = 2584, F_{19} = 4181, F_{20} = 6765. \tag{1.38}$$

Then, taking into consideration (1.38) and using (1.36), we can write the following algebraic equations of the 20th degree:

$$x^{20} = 6765x^2 - 2584 \tag{1.39}$$

$$x^{20} = 6765x + 4181. \tag{1.40}$$

It is difficult to imagine that the golden mean $\tau = \left(1+\sqrt{5}\right)/2$ is a root of the equations (1.39) and (1.40). However, this fact follows from the theory of golden algebraic equations given by (1.36), and by looking at the equations (1.36), (1.39) and (1.40). Once again, we are convinced of the greatness of mathematics — which allows us to express the complex scientific information in such compact form.

1.4. The Golden Rectangles and the Golden Brick

1.4.1. *A Golden Rectangle with a Side Ratio of τ*

Many different definitions of the golden rectangles exist. First, we will study the golden rectangle with a side ratio τ (Fig. 1.5). A rectangle in Fig. 1.5 is a golden one because the ratio of its larger side to the smaller side is equal to the golden mean, that is:

$$AB:BC=\tau=\frac{1+\sqrt{5}}{2}.$$

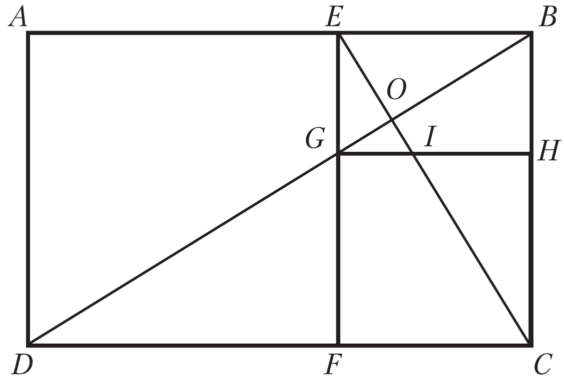


Figure 1.5. Golden rectangle with a side ratio of τ

Consider the golden rectangle with the sides $AB=\tau$ and $BC=1$. We can find points *E* and *F* on the line segments *AB* and *DC*, which divide the segments *AB* and *DC* in the golden section.

It is clear that if $AE=DF=1$, then $EB=AB-AE=\tau-1=1/\tau$.

Now, we draw the line segment *EF*, which is called a *Golden Line*. It is clear that the golden line *EF* divides the golden rectangle *ABCD* into two new rectangles *AEFD* and *EBCF*. It follows from the above geometric considerations that the rectangle *AEFD* is a square.

Now, let us consider the rectangle *EBCF*. As its big side *BC* = 1 and its small side $EB=1/\tau$, hence their ratio $BC:EB=\tau$. It means that the rectangle *EBCF* is a golden one also. Thus, the golden line *EF* divides the initial golden rectangle *ABCD* into a square *AEFD* and a new golden rectangle *EBCF*.

Now, draw the diagonals *DB* and *EC* of the golden rectangles *ABCD* and *EBCF*. It follows from the similarity of the triangles *ABD, FEC, BCE* that the point *G* divides the diagonal *DB* in the golden section. Then, we draw a new golden line *GH* in the golden rectangle *EBCF*. It is clear that the golden line *GH* divides the golden rectangle *EBCF* into a square *GHCF* and a new golden rectangle *EBHG*. By repeating this procedure infinitely, we will get an infinite sequence of squares and golden rectangles converging in the limit to the point *O*.

Note that such endless repetition of the same geometric figures, that is, squares and golden rectangles, gives rise to aesthetic feelings of harmony and beauty. It is considered, that this fact is a reason why many rectangular forms (match boxes, lighters, books, suitcases) frequently are golden rectangles. Below we will discuss the applications of the golden rectangle to architecture and painting.

For example, we widely use credit cards in our daily life, however we do not pay attention that in many cases our credit cards have a form that is, or approximates to the golden rectangle (Fig. 1.6).

Figure 1.6. A credit card has a form of the golden rectangle

1.4.2. *A Golden Rectangle with a Side Ratio of* τ^2

We look back to Theorem II.11 of *The Elements* of Euclid. Consider, once again, a division in the extreme and mean ratio represented in Fig. 1.3, and according to Theorem II.11, we should construct the line segments *AB, AC,* and *CB,* which form the golden mean (Fig. 1.2), a rectangle for the condition (1.2). We name the rectangle a *Euclidean rectangle*. If we designate the lengths of the line segments *AB, AC* and *CB* as $AB = a, AC = b$ and $CB = c,$ then, we can rewrite the expression (1.2) as follows:

$$a \times c = b^2. \tag{1.41}$$

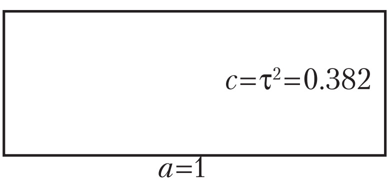


Figure 1.7. The Euclidean rectangle

Taking into account (1.41), we can represent the Euclidean rectangle as is shown in Fig. 1.7.

It follows from Fig. 1.7 that a ratio of the larger side to the smaller one in the Euclidean rectangle is equal to the ratio of the initial segment to the smaller segment in Theorem II.11 of *The Elements* here, according to (1.41), its area is equal to a square of the larger segment.

If we take the segment of the length 1 as the initial segment $(a = 1)$, then the lengths of the larger segment (b) and smaller one (c) in the DEMR will always be a proper fraction, and the expression (1.41) can be written in the following form:

$$c = b^2. \tag{1.42}$$

The following formulation of Theorem II.11 for the case of the line segment of the length 1 comes directly from (1.42).

Theorem II.11 for the segment of the length 1. We divide the line segment of the length 1 into two unequal parts in golden ratio; thus, the length of the smaller line segment will be equal to the square of the larger line segment.

If the initial segment AB in Fig. 1.1 will be the line segment of length 1, that is, $AB=1$, then the larger segment $AC=\tau^{-1}$, and the smaller segment $CB=\tau^{-2}$. Taking into account this fact, we can say that the Euclidean rectangle in Fig. 1.7 is a new golden rectangle with a side ratio $AB:CB=\tau^2$. It follows from this consideration that Euclid, in his Theorem II.11, not only formulated the DEMR (the golden section), but he also discovered a new golden rectangle with a side ratio τ^2.

Note that Theorem II.11, for the segment with length 1, presents the following well-known property of the golden mean:

$$1 = \tau^{-1} + \tau^{-2} = 0.618 + 0.382. \tag{1.43}$$

The identity (1.43) is a partial case of the identity (1.6), and it expresses the famous **"Principle of the Golden Proportion"** that has been widely used in human culture since antiquity.

1.4.3. *The "Golden" Brick of Gothic Architecture*

Again, we will use the Euclidean rectangle in Fig. 1.8, where the larger side is $AB=\tau$ and the smaller side is $AD=\tau^{-1}$. Draw the Euclidean rectangle in Fig. 1.7 with a diagonal DB.

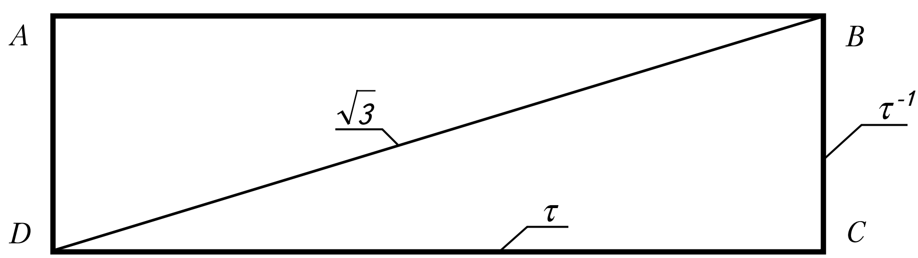


Figure 1.8. A calculation of the diagonal in the Euclidean rectangle

By using the Pythagorean Theorem, we can write:

$$DB^2 = BC^2 + DC^2 = \tau^2 + \tau^{-2}. \tag{1.44}$$

Calculate numerical values for τ^2 and τ^{-2}. In fact, we have:

$$\tau^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2}. \tag{1.45}$$

On the other hand, we have:

$$\tau^{-2} = \left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{5-2\sqrt{5}+1}{4} = \frac{3-\sqrt{5}}{2}. \tag{1.46}$$

Taking into account (1.45) and (1.46), we can rewrite the expression (1.44) as follows:

$$DB^2 = 3. \tag{1.47}$$

It comes from (1.47) that

$$DB = \sqrt{3}. \tag{1.48}$$

As is shown in [26], the Euclidean rectangle in Fig. 1.8, together with the classical golden rectangle in Fig. 1.5, can be used for the construction of a special kind of rectangular parallelepiped — also called a *Golden Brick* (Fig. 1.9). Consider this parallelepiped for the case $AB = DC = 1$.

The faces of the golden brick in Fig. 1.9 are the golden rectangles with geometric ratios based on the golden mean. The face *ABCD* is a classical golden rectangle with the side ratio of τ. This means that the edge is $AD = \tau^{-1}$. The face *ABGF* is also a classical golden rectangle with the side ratio of τ. This means that the edge $AF = \tau$. Finally, the rectangle *BCHG* is the Euclidean rectangle with the side ratio of τ^2 (Fig. 1.7). This means that the edges are $BG = \tau$ and $BC = \tau^{-1}$. Note that the diagonal $CG = \sqrt{3}$.

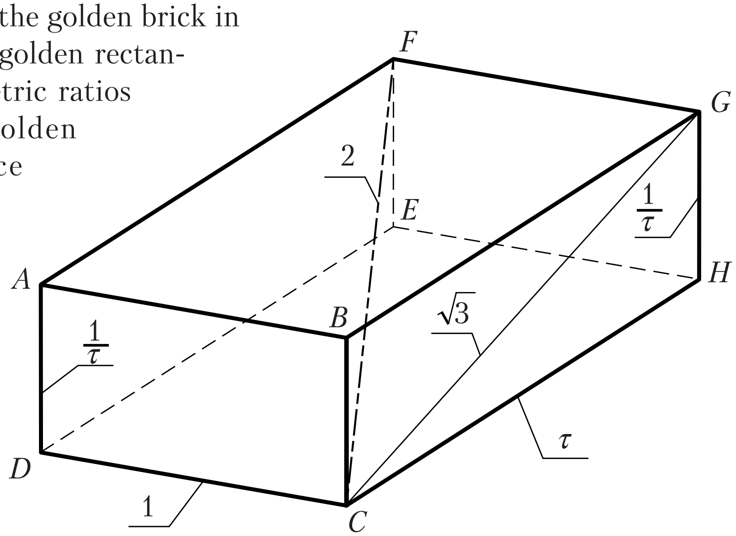


Figure 1.9. The golden brick

By using the Pythagorean Theorem, we can calculate the diagonal *CF* of the golden brick:

$$CF = \sqrt{FG^2 + CG^2} = \sqrt{1 + (\sqrt{3})^2} = 2.$$

In the book [26], evidence is suggested that the golden bricks were widely used in the Gothic castles as a form of basic building blocks. In [26], the hypothesis is drawn that the surprising strength of the Gothic castles is bound up with the use of the golden bricks during the construction of architectural monuments of gothic style.

1.5. Decagon: Connection of the Golden Mean to the Number π

The quantity of irrational (incommensurable) numbers is infinite. However, some of these numbers take a special place in the history of mathematics — specifically, in the history of material and spiritual culture. Their significance involves the fact that they express proportions and relations — which have universal character — and often appear in the most unexpected places. The irrational number $\sqrt{2}$ — which is equal to the ratio of the diagonal to the side of a square — is the first of them. Recall that the discovery of the so-called "incommensurable line segments" is connected directly with this number. This discovery caused the most dramatic period in ancient mathematics. It resulted in the development of the theory of irrational numbers and ultimately to the creation of modern "continuous" mathematics.

Two mathematical constants — the number π, which expresses the ratio of a circle length to its diameter, and "Euler's Number" e, the base of natural logarithms - are the next important irrational numbers. Their significance consists in the fact that they originate the main classes of the "elementary functions": trigonometric functions (the number π) and also exponential function e^x, logarithmic function $\log_e x$, and lastly hyperbolic functions (the number e). The numbers π and e, that is, two of the most important constants of mathematical analysis, are connected by the following astonishing formula:

$$1+e^{ix}=1, \tag{1.49}$$

where $i=\sqrt{-1}$ is an imaginary unit.

This identity is not recognized by everyone at first sight. However, in considering this identity, which connects among themselves the fundamental mathematical constants, the numbers π and e, it is difficult to refrain from pondering the mystical character of this mathematical formula (1.49).

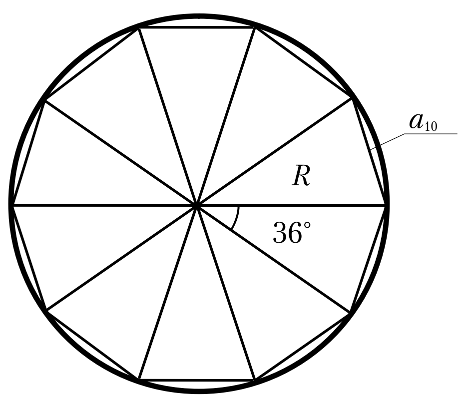


Figure 1.10. Decagon

The golden mean τ refers also to a category of the fundamental mathematical constants, but this poses a question. Is there some connection between these mathematical constants, for example, between the numbers π and τ? The analysis of a regular *Decagon* (Fig. 1.10) gives the intriguing answer to this question.

Consider a circle with a radius R and a decagon inscribed inside the circle (Fig. 1.9). It is known from geometry that the side a_{10} of the decagon is connected to the radius R by the following formula:

$$a_{10} = 2R\sin 18°. \tag{1.50}$$

If we carry out some trigonometric transformations by using the formulas, well-known from secondary school trigonometry, we obtain the following outcomes:

(1) The side of the decagon inscribed inside the circle with radius R is equal to a large part of the radius R divided in the golden section, that is: $a_{10} = R/\tau$.

(2) The golden mean τ is connected with the number π by the following correlation:

$$\tau = 2\cos 36° = 2\cos(\pi/5). \tag{1.51}$$

This formula, which is obtained as an outcome of mathematical analysis of the decagon, is one more confirmation of the fact that the golden mean together with the number π, shares the status of being amongst main mathematical constants of nature.

Consider the following from the area of nuclear physics. By studying the laws of the nuclear kernel, Byelorussian physicist Vasily Petrunenko, in the book *The Golden Section of Quantum States and its Astronomical and Physical Manifestations* (2005) [53], came to the following conclusion: a huge stability of the nuclear kernel is due to the wave's multiplicities of the golden mean underlying their organization. Thus, he proved that the regular decagon underlies the structure of the nuclear kernel!

1.6. The Golden Right Triangle and the Golden Ellipse

1.6.1. *The Golden Right Triangle*

The right triangle based upon the golden section is used widely in architecture. Consider the right triangle ABC with the side ratio $AC : CB = \sqrt{\tau}$ (Fig. 1.11).

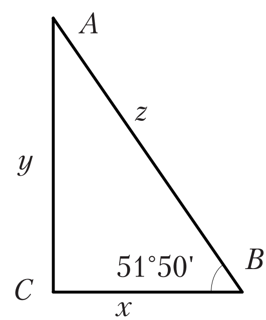


Figure 1.11. The golden right triangle

If we designate x, y, and z as the side lengths of the right triangle ABC and also take into account that the ratio $y : x = \sqrt{\tau}$, then according to the Pythagorean Theorem, the length z of the side AB can be calculated by the formula:

$$z = \sqrt{x^2 + y^2}. \qquad (1.52)$$

If we take $x = 1, y = \sqrt{\tau}$, then we have:

$$z = \sqrt{1+\tau} = \sqrt{\tau^2} = \tau.$$

The right triangle with the side ratios $\tau : \sqrt{\tau} : 1$ is called a *Golden Right Triangle*.

Below, we will show that the golden right triangle is the main geometric idea of the Great Pyramid in Giza.

1.6.2. *The "Golden" Ellipse*

The golden ellipse [26] is formed with the help of the two golden rhombi (*ACBD* and *ICJD*) inscribed into the ellipse (Fig. 1.12). The golden rhombi *ACBD* and *ICJD* consist of four right triangles of the kind *OCB* or *OCJ*, which are the golden right triangles (Fig. 1.11).

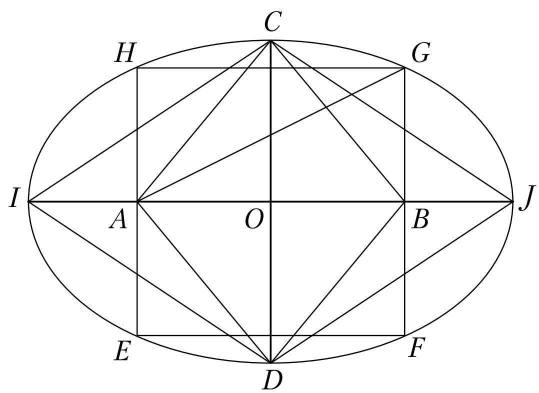


Figure 1.12. The golden ellipse

Now, consider the basic geometric correlations of the golden ellipse in Fig. 1.12. Let a focal distance *AB* of the ellipse in Fig. 1.12 be equal to $AB = 2$. The following correlation follows from the ellipse definition:

$$AC + CB = AG + GB. \qquad (1.53)$$

The following correlations, which connect the sides of the golden right triangles *OCB* and *OCJ*, follow from the definition of the golden right triangle. As the smaller leg *OB* of the golden right triangle is equal to 1, that is, $OB = 1$, it follows from the definition of the golden right triangle that the values for the hypotenuse and the larger leg of the triangle *OCB* are:

$$CB = \tau \quad \text{and} \quad OC = \sqrt{\tau}. \qquad (1.54)$$

Let us consider the golden right triangle *OCJ*. Because its smaller leg $OC = \sqrt{\tau}$, then, according to the definition of the golden right triangle, we have:

$$OJ = OC \times \sqrt{\tau} = \sqrt{\tau} \times \sqrt{\tau} = \tau \quad \text{and} \quad CJ = \tau \times OC = \tau \times \sqrt{\tau}. \qquad (1.55)$$

There is also the well-known correlation that connects the half-axes *OI* and *OC* with the line segment *OB* of the ellipse in Fig. 1.12:

$$OC^2 = OI^2 - OB^2. \tag{1.56}$$

We know that the correlation (1.56) is true for the figure in Fig. 1.11. In fact, if we substitute in (1.56) the numerical values of the line segments $OI = \tau, OC = \sqrt{\tau}$ and $OB = 1$, we can obtain:

$$\left(\sqrt{\tau}\right)^2 = \tau^2 - 1 \quad \text{or} \quad \tau = \tau^2 - 1$$

that is equivalent to the identity (1.6). This means that the correlation (1.56) is valid and the figure in Fig. 1.11 is an ellipse.

In the opinion of the Polish scientist Jan Grzedzielski, author of the book *Energeticzno-Geometryczny Kod Phzyrody* [26], the golden ellipse can be used as a geometric model for the spreading of the light in optic crystals, that is, the geometrical correlations of the golden ellipse give optimal conditions for the attainability by photons of the focuses with minimal energetic losses.

1.7. The "Golden" Isosceles Triangles and Pentagon

1.7.1. *Construction of the "Golden" Isosceles Triangle and Regular Pentagon in The Elements*

Theorem II.11 played an important role in *The Elements* by Euclid. Using this theorem, Euclid constructs the isosceles triangle and regular pentagon, which are then used for the construction of the dodecahedron (see Chapter 3). First, we construct an isosceles triangle whose angles at its base are double relative to the vertex angle (Fig. 1.13). We can do this by drawing the line *AB* and then divide it in the golden section at the point *C*. Then, we draw a circle with the cenetr *A* and the radius *AB*. We mark on the circle the point *D* so that *AC*=*CD*=*BD*. The triangle *ABD* has the property that its angles *B* and *D,* at the base *BD,* are double relative to its vertex angle *A*. Note that the vertex angle *A*=36° and the angles *B*=*D*=72°. As *C* is the golden section point, *AC*=*CD*=*BD*, and *AD*=*AD,* this means that the ratio of the hips *AB* and *AD* to the base *BD* of the isosceles triangle *ABD* is equal to the golden mean, that is:

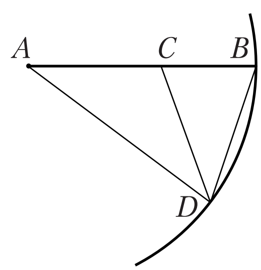


Figure. 1.13. A geometric construction of the golden isosceles triangle

$$\frac{AB}{BD}=\frac{AD}{BD}=\tau \tag{1.57}$$

The isosceles triangle with a golden ratio (1.57) is aptly called the *Golden Isosceles Triangle* (Fig. 1.13).

By using the triangle *ABD,* we draw a circle through *A*, *B,* and *D* (Fig. 1.14). Then we bisect the angle *ADB* by the line *DE*, which meets the circle at the point *E*. Note that the line *DE* passes through the point *C*, which divides a line *AB* in the golden section. Similarly we can find a point *F* and then we can draw the pentagon *AEBDF*.

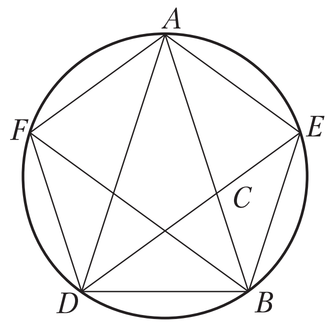


Figure 1.14. A geometric construction of a regular pentagon

1.7.2. *A Regular Pentagon*

Let us consider a regular pentagon presented in Fig. 1.15.

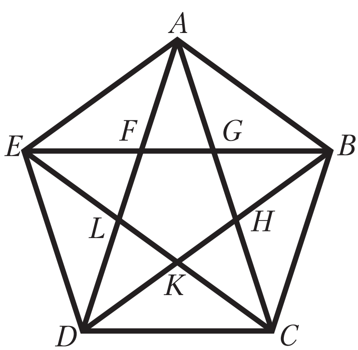


Figure 1.15. Regular pentagon and pentagram

If we draw in the regular pentagon all diagonals, we obtain a pentagonal star called a *Pentagram* or *Pentacle*. Note that the word "pentagon" originates from the Greek word *pentagonon* and the word "pentagram" originates from the Greek word "pentagrammon" ("pente" is five and "grammon" is a line). A "pentagram" is a regular pentagon with golden isosceles triangles constructed on each side. It is proved that the crossing points of diagonals in the pentagon are always the points of the golden section. Thus, they form a new regular pentagon: *FGHKL*. In this new pentagon, we can draw diagonals at their crossing points to form another pentagon. This process can be continued infinitely. Thus, the pentagon *ABCDE* consists of an infinite number of the pentagons formed by the crossing points of diagonals. This infinite repetition of the same geometrical figures creates a feeling of rhythm and harmony, which is perceived by our intellect.

The regular pentagon in Fig. 1.15 is a rich source of golden sections. If we assume that a side of the inner pentagon *FGHKL* is equal to 1, that is, $FG = GH = HK = KL = 1,$ then we get the following properties of the regular pentagon and pentagram *ABCDE* based on the golden section:

1. All five triangles, *AGF, BHG, CKH, DLK,* and *EFL,* are golden isosceles triangles; here the lengths of the ten segments
$AG = AF = BH = BG = CK = CH = DL = DK = EF = EL = \tau$.
2. All five sides of the regular pentagon $ABCDE$ are equal to τ^2, that is, $AB = BC = CD = DE = EA = \tau^2$.
3. All line segments $AH = GC = AL = DF = EK = LC = \tau^2$.
4. All five diagonals of the regular pentagon $ABCDE$ are equal to τ^3, that is, $AC = AD = BE = BD = EC = \tau^3$.
5. The lengths of the segments *AC, AH, HC,* and *GH* are in geometric progression because
$AC = \tau^3, AH = \tau^2, HC = \tau, GH = \tau^0 = 1$.

Some interesting information about the pentagram (pentacle) is presented in Dan Brown's book *The Da Vinci Code*. The five-pointed star is clearly a pre-Christian symbol connected with the worship and dedication of nature. The ancient people divided the world into two halves, male and female. They had gods and goddesses maintaining the balance of forces. When a male's and female's beginnings are balanced, then harmony is reigning in the world. When the balance is broken, chaos appears. The pentacle symbolizes the female half of the Universe. Historians, who study religions, name this symbol "the sacred female beginning," or "the Sacred Goddess." The five-pointed star symbolizes Venus, the Goddess of love and beauty. The Goddess Venus and the planet Venus are the same. The Goddess takes her place in the night sky and is known under many names: Venus, East Star, Ishtar, and Astarte, all of them symbolizing the powerful female beginnings connected with nature and mother earth.

Over an eight year period, the planet Venus describes a pentacle on the big circle of the heavenly sphere. Ancient people noticed this unique phenomenon and they were so impressed, that Venus and its pentacle became symbols for perfection and beauty. Today, only a few people know that the Olympic Games follow the half cycle of Venus; fewer people know that the five-pointed star missed out on becoming a symbol for the Olympic Games, because of a last moment modification: the five acute ends of the star were replaced with the five rings. Ironically, the organizers apparently believed that their chosen symbol better reflected the spirit and harmony of the Olympic Games.

1.7.3. *The Pentagon*

The Pentagon (Fig. 1.16) is the headquarters of the United States Department of Defense and is located in Arlington, Virginia. It is now considered to be the symbol of the U.S. military. The Pentagon is often used met-

Figure 1.16. The Pentagon

onymically to refer to the Department of Defense — rather than the building itself. The Pentagon is the largest-capacity office building in the world, and it is one of the world's largest buildings in terms of floor area. It houses approximately 23,000 military and civilian employees plus approximately an additional 3,000 non-defense support personnel. It has five sides, five floors above ground (plus two basement levels), and five ring corridors per floor with a total of 17.5 miles (28 km) of corridors.

1.7.4. *The "Golden Cup" and the Golden Isosceles Triangle*

The regular pentagon in Fig. 1.15 includes a number of remarkable figures used widely in works of art. In ancient art, the law of the golden cup (Fig. 1.17) was widely known. It was used by the antiquity's sculptors and masters of golden things. The shaded part of the pentagram in Fig. 1.17 gives a schematic representation of the golden cup.

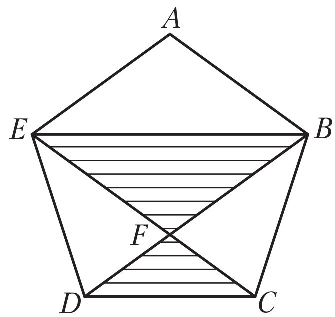


Figure 1.17. The golden cup

The regular pentagon in Fig. 1.15 consists of five golden isosceles triangles. Each golden triangle has an acute angle $A = 36°$ at the top and two acute angles $B = D = 72°$ at the base of the triangle (Fig. 17). The main feature of the golden triangle in Fig. 1.18 consists of the fact that the ratio of the hip $AC=AD$ to the base DC is equal to the golden mean.

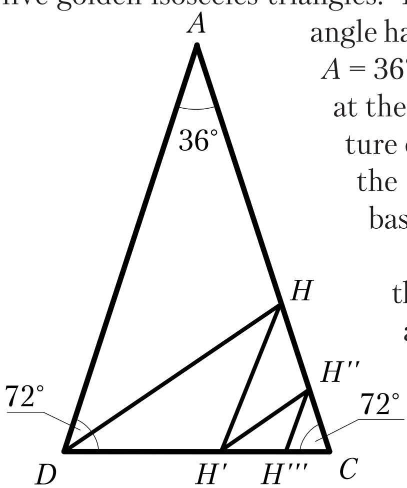


Figure 1.18. The golden isosceles triangle

In studying the pentagram in Fig. 1.15 and the golden triangle in Fig. 1.18, the Pythagoreans admired the fact that the bisector DH coincides with the diagonal DB of the pentagon (Fig. 1.15), and it divides the side AC in the golden section at the point H. Moreover, the new golden triangle DHC arises. If we draw the bisector of the angle H to the point H', and continue this process ad infinitum, we will get an infinite sequence of the golden triangles. Furthermore, the case of the golden rectangle in Fig. 1.5 and the pentagon in Fig. 1.15, the infinite repetition of the same geometric figure (the golden triangle) after the drawing of the next bisector causes an aesthetic feeling of rhythm and harmony.

1.7.5. *Pentagonal Symmetry in Nature*

In nature, the forms based on pentagonal symmetry (starfishes, sea hedgehogs, flowers) are widespread. The flowers of a water lily, a dog rose, a hawthorn, a carnation, a pear, a cherry, an apple-tree, a wild strawberry, and many other plants, consist of five-petals. Below, in Fig. 1.19, we can see examples of nature's structures based on the pentagonal symmetry.

The presence of five fingers on a human's hand and, the five bone rudiments on the paws of many animals are an additional confirmation of the widespread occurrence of pentagonal forms in the morphology of flora and the biological world.

Figure 1.19. Pentagonal symmetry in nature

1.8. The Golden Section and the Mysteries of Egyptian Culture

1.8.1. *Phenomenon of Ancient Egypt*

In the early 20th century, in Saqqara (Egypt), archaeologists opened a crypt in which the Egyptian architect Hesire was buried. The wood panels covered by a magnificent thread were extracted from the crypt together with different material objects. In total, there were the 11 panels in the crypt; among them, only 5 of the panels were preserved; the remaining panels were completely destroyed by moisture inside the crypt. On the salvaged panels the architect Hesire was depicted with various objects and figures that had symbolic significance at the time (Fig. 1.20).

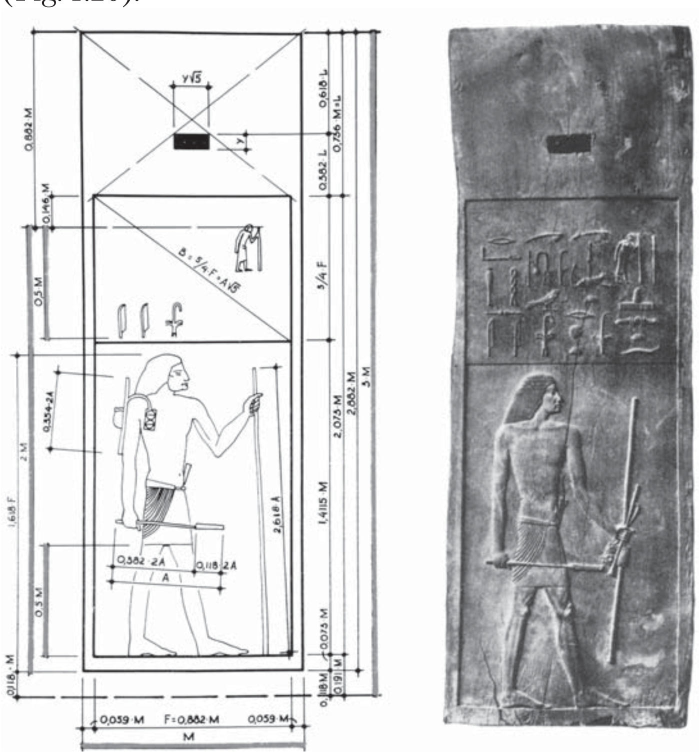


Figure 1.20. Hesire panels

For a long time, the purpose of the Hesire panels remained vague. At first, the Egyptologists considered these panels to be merely false doors. However, in the mid-20th century, the situation surrounding the panels began to clarify. In the early sixties of the 20th century the Russian architect Shevelev observed that the sticks the architect holds in his hands on one of the panels relate between themselves as $1:\sqrt{5}$, that is, as the ratio of the small side of the 1:2 rectangle ("double square") to its diagonal. Basing on this observation, another Russian architect Shmelev made a very precise geometrical analysis of the Hesire panels. This analysis led him to sensational discovery [36].

Having explored the Hesire panels, Igor Shmelev came to the conclusion: "Now, after the comprehensive and justified analysis, by using the proportion method, we have good ground to assert that the Hesire panels are the harmony rules encoded in geometric language... So, in our hands we have the concrete evidence, which shows us by "plain text" the highest standard of creative thought of the Ancient Egyptian intellectuals. The artist who created the panels with amazing accuracy, exquisite refinement and masterly ingenuity demonstrated the rule of the Golden Section in its broadest range of variations. It gave birth to the "GOLDEN SYMPHONY" presented by the ensemble of highly artistic works, which testifies not only the ingenious talents of their creator, but also convincingly proves that the author was aware of the secret of harmony. This genius was the "Golden Affair Craftsman" by the name Hesire."

But who was Hesire? Ancient texts inform us that Hesire was "a Chief of Destius and a Chief of Boot, a Chief of Doctors, a Scribe of Pharaoh, a Priest of Gor, a Main Architect of Pharaoh, a Supreme Chief of South Tens, and a Carver."

Analyzing the regalia of Hesire, Shmelev paid particular attention to the fact that Hesire was the Priest of Gor. Gor was considered by the Ancient Egyptians as the God of Harmony, and therefore, to be the Priest of Gor he had to execute the functions of the keeper of Harmony.

As it follows from his name, Hesire was elevated to the rank of God of Ra (the God of the Sun). Shmelev assumes that Hesire was elevated to this rank rewarded for his "development of aesthetic ... principles in the canon system reflecting the harmonic fundamentals of the Universe ... The orientation to the harmonic principle discovered by the Ancient Egyptian civilization was the way to the unprecedented flowering of culture; this flowering falls into the period of Zoser, pharaoh when the system of written signs was completely implemented. Therefore it is possible to assume that Zoser's pyramid became the first experimental pyramid, which was followed by construction of the unified complex of the Great pyramids in Giza according to the program designed under Hesire's supervision" [36].

Thus, Shmelev states: "It is only necessary to recognize that the Ancient Egypt civilization is the super-civilization explored by us extremely superficially and it demands a qualitatively new approach to the development of its richest heritage... The outcomes of researches of Hesire's panels demonstrate that the sources of modern science and culture are in boundless layers of history feeding creativity of the craftsmen of our days by great ideas, which for long time inspired the aspirations of the outstanding representatives of the mankind. And our purpose is to keep a unity of a connecting thread" [36].

1.8.2. *The Mysteries of the Egyptian Pyramids*

This infinite never-ending uniform sea of sand, with infrequent dried bushes, and the barely visible tracks from a camel covered by sand, is a wasteland under an incandescent sun. Everything around seems dull and covered by fine sand.

Suddenly, as if beset by a mirage, to our amazement we see the pyramids (Fig. 1.21) whose fancy stone figures are directed toward the Sun. They astonish our imagination by their vast size and perfect geometric forms.

A Pharaoh's authority in Ancient Egypt was tremendous; divine honors were given to him, and a Pharaoh was called the "Great God." The God-Pharaoh was a progenitor of the country and a judge of people's fate. The cult of the dead pharaoh had great significance in the Egyptian religion. The monumental pyramids (in addition to their more profound philosophical and mathematical significance) were constructed for the preservation of a pharaoh's body, spirit, and for extolling his authority. Constructed by human hands, they deserve to be included amongst the Seven Wonders of the World.

The pyramids had multiple functions. They served not only as vaults of a Pharaoh's mortal remains, but they also were the attributes of his majesty, power, riches of country, the monuments of culture, the country's history, and items of information about the pharaoh, his people, and life.

It is clear that the pyramids held deep "scientific knowledge" embodied in their forms, sizes, and orientation of terrain. Each part of a pyramid and each element of its form were selected carefully to demonstrate the high level of knowledge of the pyramid creators. They were constructed for millennia, "for all time," and for this reason, the Arabian proverb says: "All in the World are afraid of a Time. However, a Time is afraid of the Pyramids."

Fig. 1.21. *Cheops'* Pyramid in Giza

Among the gigantic Egyptian pyramids, the Great Pyramid of the Pharaoh Cheops (Khufu) is of special interest (Fig. 1.21).

Before the beginning of the analysis of the form and dimensions of Cheops' Pyramid it is necessary to introduce the

Egyptian measurement system. The ancient Egyptians used three units of measurement: the "elbow" (466 mm) which equaled 7 "palms" (66.5 mm), which, in turn, was equal to 4 "fingers" (16.6 mm). Let us analyze the dimensions of Cheops' Pyramid (Fig. 1.22) employing considerations given in the remarkable book of Ukrainian scientist Nikolay Vasutinsky [31].

The majority of researchers believe that the length of the side of a pyramid's base is, for example, GF is equal to $L = 233.16$ m. This value corresponds almost precisely to 500 "elbows." The full accordance to the 500 "elbows" will be, if we take the length of the "elbow" equal to 0.4663 m. The altitude of the Pyramid (H) was estimated by researchers variously from 146.6 m up to 148.2 m and depending on the adopted pyramid's altitude, all ratios of its geometric construction will change considerably. What is a cause for the differences in the estimation of a pyramid's altitude? Strictly speaking, Cheops' pyramid is truncated. Today, its apex is approximately 10×10, but 1 century ago, it was 6×6. Apparently, the top of the pyramid was dismantled.

In estimating a pyramid's height, it is necessary to consider physical factors, such as "shrinkage" of its construction. Under the enormous pressure (reaching 500 tons on 1 m^2 of the ground) exerted by its weight, the height of most pyramids would have decreased in comparison to its initial designed value. What was the initial height of the Pyramid? This height can be reconstructed if we can discover the main or key "geometrical idea" of the Pyramid.

In 1837, English Colonel Howard Vyse measured the inclination angle of the Pyramid faces: it appeared t*o be* $\alpha = 51°51'$. A majority of researchers recognize this value today. The indicated value of the inclination angle corresponds to the tangent equal to 1.27306. This value corresponds to the ratio of the pyramid's altitude AC, which is half of its base CB (Fig. 2.22), that is, $AC / CB = H / (L/2) = 2H / L$.

Researchers were in a big surprise! If we consider the square root of the golden mean, that is, $\sqrt{\tau}$, we obtain the following outcome $\sqrt{\tau} = 1.272$. Comparing this value with the value of $tg\alpha = 1.27306$, we see that these values are very close. If we accept the angle $\alpha = 51°50'$, that is, we decrease it by one arc minute, the value of $tg\alpha$ will become

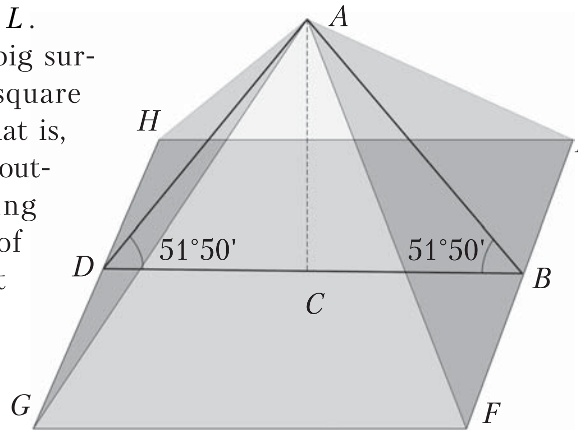


Figure 1.22. A geometric model of *Cheops'* pyramid

equal to 1.272, that is, it will be exactly equal to the value of $\sqrt{\tau}$. It is necessary to note that in 1840, Colonel Vyse repeated his measurements and corrected the value of the angle to $\alpha = 51°50'$. These measurements led researchers to the following hypothesis: the ratio $AC/CB = 1.272$ underlies the triangle ACB of Cheops' Pyramid! This ratio is characteristic for the above golden right triangle (Fig. 1.11). Then, if we accept the hypothesis that the golden right triangle is the main geometrical idea of Cheops' Pyramid, it is then possible to calculate the initial height of Cheops' Pyramid. It is equal to $H = (L/2) \times \sqrt{\tau} = 148.28$ m.

Now, we deduce other relationships for Cheops' Pyramid following from this golden hypothesis. In particular, let us find the ratio of the external area of the Pyramid to its base. For this purpose, we accept the length of the leg CB for the unit, that is, $CB = 1$, however, the length of the side of the Pyramid base is $GF = 2$, and the area of the base $EFGH$ will be equal to $S_{EFGH} = 4$. Now, let us calculate the area of the lateral side of Cheops Pyramid. As the altitude AB of the triangle AEF is equal to τ, then the area of each lateral side will be equal to $S_D = \tau$, and the common area of all four lateral sides of the pyramid will be equal to 4τ, while the ratio of the external area of the pyramid to its base will be equal to the golden mean. This is the main geometrical secret of Cheops Pyramid!

Analyses of other pyramids confirm that the Egyptians always intended to embody their most significant mathematical knowledge in the pyramids they built. In this respect, the Pyramid of Chephren (Khafre) is also rather interesting. The measurements of Chephren's Pyramid show that the inclination angle of the lateral sides is equal to 53°12', which corresponds to leg's ratio of the right triangle: 3:4. This leg's ratio corresponds to the well-known right triangle with the side ratio: 3:4:5, this triangle is called the "perfect," "sacred," or "Egyptian" triangle. According to testimony from historians, the "Egyptian" triangle had a magical sense. Plutarch wrote that the Egyptians compared the universe to the "sacred" triangle; they symbolically ascribed the vertical leg to the husband, the base to the wife, and the hypotenuse to the child born from them.

According to the Pythagorean Theorem for the triangle 3:4:5 we have: $3^2+4^2 = 5^2$. Possibly, this famous theorem was perpetuated in Chephren's pyramid based on the 3:4:5 triangle. It is difficult to find a more appropriate example to demonstrate The Pythagorean Theorem — which evidently was well-known to the Egyptians long before its re-discovery by Pythagoras.

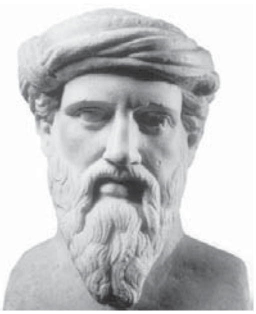
Thus, the ingenious architects of the Egyptian Pyramids sought to astonish future generations and fulfilled this goal by selecting the golden right triangle as the "main geometrical idea" of Cheops' Pyramid, and the "sacred" or "Egyptian" right triangle as the "main geometrical idea" of Chephren's Pyramid.

1.9. The Golden Section in Greek Culture

1.9.1. *Pythagoras*

Pythagoras is one of the most celebrated people in the history of science. His name is known to each person who has studied geometry and been introduced to the Pythagorean Theorem. The famous philosopher, scientist, religious and ethical reformer, influential politician, "demigod" — in the eyes of his followers, Phythagoras personifies ancient wisdom. Coins with his image, which were made in 430-420 B.C., testify to the exclusive popularity of Pythagoras. In the 5th century B.C., this was unprecedented! Pythagoras was the first, among the Greek philosophers, to have a special book dedicated to him.

His scientific school is internationally known. He had organized it in Croton, a Greek colony in northern Italy. The "Pythagorean School," or "Pythagorean Union," was simultaneously a philosophical school, a political party, and a religious brotherhood. Entrance to the "Pythagorean Union" was very demanding. Each, who entered the "Pythagorean Union," would be refused personal property for the benefit of the Union; they undertook to not spill blood, not eat meat, and protect the secret of their doctrine. The members of the Union were prohibited to train others for money.



Pythagoras
(Born c.569 B.C., died c.475 B.C.)

The Pythagorean doctrine focused upon harmony, geometry, number theory, astronomy, and other topics, but the Pythagoreans, most of all, appreciated the results that were obtained in the theory of harmony because it confirmed their idea that "numbers determine everything." Some ancient scientists assume that the concept of the golden section was borrowed by Pythagoras from the Babylonians.

Many great mathematical discoveries were attributed to Pythagoras — some perhaps undeservedly. For example, the famous geometric "theorem of squares" (Pythagorean Theorem), which sets a ratio between the sides of a right triangle, was known to the Egyptians, the Babylonians, and the Chinese long before Pythagoras.

However, discovery of "incommensurable line segments" is considered to be the main mathematical discovery of the Pythagoreans. In investigating the ratio of a diagonal to the side of a given square, the Pythagoreans proved that its ratio cannot be expressed as the ratio of two natural numbers, that is, this ratio is “irrational.” This discovery caused the first crisis in the history of mathematics, as the Pythagorean doctrine about integer-valued basis of all existing things could no longer be accepted as true. Therefore, the Pythagoreans attempted to keep the discovery a secret; they created a legend about the death of one of the Pythagoreans — who divulged the secret of this discovery.

Also attributed to Pythagoras were a number of the other geometrical discoveries, namely, the theorem about the sum of the interior angles of a triangle, and the problem about segmentation of the plane into regular polygons (triangles, squares and hexagons). There is a view that Pythagoras discovered some or all of the primary regular “spatial” figures, that is, the five regular polyhedrons.

Why was Pythagoras so popular already during his life? The answer to this question follows from interesting facts from his biography. According to legend, Pythagoras went to Egypt and lived there for 22 years to learn from the wisdom of the priest-scientists. After studying all Egyptian sciences, including mathematics, he moved to Babylon, where he lived for 12 years and was introduced to the scientific knowledge of Babylonian priests. Legends also attribute Pythagoras with a visit to India. It is possible because Ionia and India-at that time-had business relations. He returned home about 530 B.C. Pythagoras attempted to organize his philosophical school. However, due to undetermined reasons, he abandoned Samos and settled in Croton-a Greek colony in North Italy. Here, Pythagoras organized his school.

Thus, the outstanding role of Pythagoras in the development of Greek science consists in fulfillment of the historical mission of knowledge transmission from the Egyptian and Babylonian priests to the culture of ancient Greece. Thanks to Pythagoras, who was, without any doubt, one of the leading thinkers of his time, the Greek science could obtain new knowledge in the fields of philosophy, mathematics, and natural sciences. This new scientific knowledge contributed to the rapid development and augmentation of Greek science in ancient Greek culture.

In developing the idea about the role of Pythagoras in the historical development of the Greek science, Igor Shmelev, in the brochure *Phenomenon of Ancient Egypt* [36], wrote:

“His world renowned name the Croton teacher obtained after the rite of “consecration.” This name is compounded of two halves and it means “The Prophet of Harmony” because the “Pythians” in Ancient Greece were pagan priests who predicted a future and Gor in Ancient Egypt personified harmony. So in the last years of their civilization, the Egyptian priests transmitted their secret knowledge to the representatives of the new civilization, symbolically cementing in one person the Union of man's and woman's origins, the bastion of Harmony."

It is known that the *Pentagram* has always attracted special attention from the Pythagoreans; they considered it the symbol of health. There is a legend surrounding the pentagram: when a guest in a foreign land, one of the Pythagoreans lay on his deathbed, unable to pay the man helping him. He asked the man to draw the pentagram on his dwelling, hoping that this symbol would be seen by one of the Pythagoreans. Some years later, one of the Pythagoreans saw this symbol and the kind host received a rich compensation.

Already, the fact that the Pythagoreans chose the "pentagram," brimming with golden sections, as the main symbol of their Fraternity, is another confirmation that the Pythagoreans knew and esteemed the golden section. The Russian researcher Alexander Voloshinov wrote [137]:

"The Pythagoreans gave special attention to the pentagram, the five-pointed star that is formed by diagonals of the regular pentagon. In the pentagram, the Pythagoreans found all proportions well-known in antiquity: arithmetic, geometric, harmonic, and also the well-known golden proportion, or the golden ratio. A perfection of mathematical forms of the pentagram finds a reflection in the perfection of its form itself. The pentagram is proportional and, hence, beautiful. Probably owing to the perfect form and the wealth of mathematical forms, the pentagram was chosen by the Pythagoreans as their secret symbol and a symbol of health. Thanks to the Pythagoreans, the five-pointed star is today a symbol for many states and is on the flags of many countries of the world."

1.9.2. *The Idea of Harmony in Greek Culture*

The idea of harmony based on the golden mean became one of the most fruitful ideas in Greek art. Nature, taken in a broad sense, includes the human creative patterns of art, where the same laws of rhythm and harmony dominate. Aristotle states:

"Nature aims for the contrasts and from them, instead of similar things, Nature forms a consonance .... This combines a male with a female and thus the first public connection is formed through the connection of contrasts, instead of similarities. Also in art, apparently, by imitating Nature the artist acts in the same way. Namely the painter makes the pictures conform to the originals of Nature by mixing white, black, yellow and red paints. Music creates the unique harmony by mixing different voices, high and low, long and short, in congregational singing. Grammar creates its entire art from the mixture of vowels and consonants."

To take the subject and eliminate all that is superfluous is the plan of the ancient Greek artist. This is the main idea of Greek art, where the golden section lay at the center of the canon of aesthetics. The theory of proportion is the foundation of Greek art, and, the problem of proportionality could not have escaped the thought of Pythagoras. Among the Greek philosophers, Pythagoras was the first who attempted mathematically to understand the essence of musical or harmonic ratios or intervals. Pythagoras knew that the intervals of the octave can be expressed by natural numbers, which fit the corresponding oscillations of the string, and these numerical relations were put forth by Pythagoras as the basis of his theory of musical harmony. Pythagoras was credited with possessing knowledge of arithmetic, geometric, harmonic proportions, and the law of the golden section. Pythagoras paid special attention to the golden section by choosing the pentagram as the distinctive symbol of the "Pythagorean Union." By developing further the Pythagorean doctrine about harmony, Plato analyzed the five regular polyhedrons (the so-called Platonic solids) and emphasized their ideal beauty.

As to the main requirements of beauty, Aristotle proposed order, proportionality, and size limitation. The order arises when certain relations and proportions hold between the whole and its parts. In music, Aristotle recognized the octave as the most beautiful consonance; he considered that the ratio of oscillations between the basic tone and its octave is expressed by the first natural numbers: 1:2. He also maintained that in poetry, the rhythmic relations of a verse are based on small numerical relations, and by this, a beautiful impression is expressed. Except for the simplicity based upon the commensurability of separate parts and the whole, Aristotle, as well as his mentor Plato, recognized that the highest beauty of perfect figures and proportions was based upon the golden section.

The ancient Harmony theory, based on the principle of the division in extreme and mean ratio (the golden section), became the "launching pad," upon which, subsequently, a harmony concept was developed in the science and art of European culture.

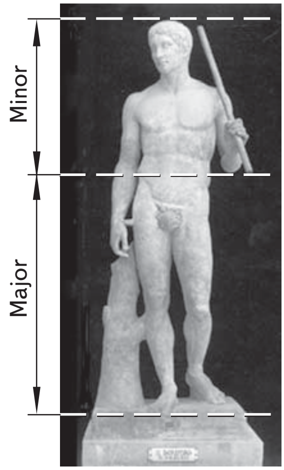


Figure 1.23. *Doryphorus* by Polyclitus

1.9.3. *The Golden Section in Greek Sculpture*

For a long time, the creations by the great Greek sculptors Phidias, Polyclitus, Myron, and Praxiteles were considered to be the standard of beauty and harmonious construction based on the human body. The statue of *Doryphorus,* created by Polyclitus in the 5th century B.C., serves as one of the greatest achievements of classical Greek Art (Fig. 1.23). This statue is considered to be the best example for analysis of the proportions of the ideal human body established by ancient Greek sculptors. It is especially important because the name "Canon" was ascribed to this famous sculpture. The harmonic analysis of the *Doryphorus* as given in the book *Proportionality in Architecture* (1933) [10], written by the Russian architect G.D. Grimm, indicates the following connections of the famous statue to the golden mean :

1. The first golden cut division of the *Doryphorus* figure with its overall height $M^0=1$, relating segments M^1 and M^2, is at the navel.
2. The second division of the lower part of the torso, relating M^2 and M^3, passes through the line of his knee.
3. The third division, relating M^3 and M^4, passes through the line of his neck.

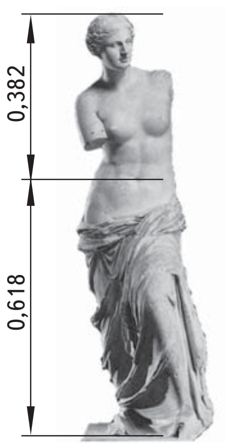


Figure 1.24. *Venus de Milo*

The *Venus de Milo* (Fig. 1.24), a statue of the goddess *Aphrodite*, is one of the best-known monuments of Greek sculptural art. This statue was created by Agesandr in the 2nd century B.C. The Goddess *Aphrodite* is represented half-naked, so that her clothing — which wrap up her legs and the bottom of her torso, as if it were a pedestal for the open hands, which are showing in movement (Fig. 1.24). During the Hellenic epoch, Aphrodite was one of the most favorite goddesses. From the Island Melos, Aphrodite is strict and restrained. It is thought that she stood on the high pedestal and looked at over the spectators. The *Venus de Milo* is a pearl of the Louvre; it is a standard of female beauty in ancient Greece.

1.10. The Golden Section in Renaissance Art

1.10.1. *The Idea of the "Divine Harmony" in the Renaissance Epoch*

The idea of Harmony belonged to such ancient conceptual ideas, which attributed to the interest of the church. According to Christian doctrine, the universe was created by God and submitted to its will unconditionally. The Christian God, at the creation of the universe, was guided by mathematical principles. In Renaissance culture, Catholic doctrine included the mathematical plan to which God created the universe.

Figure 1.25. *Holy Family* by Michelangelo

In the opinion of the American historian of mathematics, Morris Klein, the close merge of the religious doctrine about God as the creator of the universe, and the antique idea of numerical harmony of the universe, became one of the major staples of the Renaissance culture [6]. Mainly, the objective of Renaissance science is presented in the following text from the well-known astronomer Jogannes Kepler:

"The overall objective of all researches of the external world should be to discover the rational order and harmony, which was embodied by the God in the Universe and then was presented for us by the God on mathematics language."

The art of the Renaissance period (especially paintings) is substantially connected to topics from the Bible. The picture *Holy Family* by Michelangelo is a bright example of such topical art. This picture is a fairly recognized masterpiece of West-European art. The picture is often named "Tondo Doni" because, firstly, the picture belonged to the Doni family in Florence, and secondly, it had a round form (in English "tondo"). After the harmonic analysis of this picture, researchers found that a compositional construction of the picture is based on the pentacle (Fig. 1.25).

Figure 1.26. *Crucifixion* by Rafael Santi

The picture *Crucifixion,* by Rafael Santi (1483-1520), is another example of a picture based on the topics of the Bible. The harmonic analysis of this picture (Fig. 1.26) showed that the compositional plan of the picture is based on the golden isosceles triangle (Fig. 1.18).

1.10.2. *"Vitruvian Man" by Leonardo da Vinci*

During the age of the Renaissance, attempts to create an ideal model of a harmoniously developed human body were continuing. It is known, that a sweep of human hands approximately is equal to a human growth. It means that the human figure can be inserted into a square and a circle. The ideal human figures, created by Leonardo da Vinci and Durer, are widely known. For a long time, the opinion existed that the "pentagonal" or "five-fold" symmetry, which is characteristic for flora and animals, is shown in the structure of the human body. The human body can be considered as a sample of the pentagram, where the human head, two hands, and two legs are, as if, beams of the pentagonal star. Such model found a reflection in the constructions of Leonardo da Vinci and Durer, in particular, in the figure of the well-known *Vitruvian Man* by Leonardo da Vinci (Fig. 1.27).

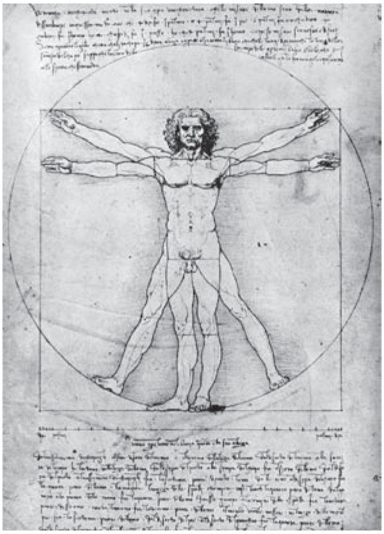


Figure 1.27. *Vitruvian Man* by Leonardo da Vinci

1.10.3. *"Mona Lisa" by Leonardo da Vinci*

The Renaissance, in the cultural history of the Western and Central European countries, is a transitive epoch, whereby it progressed from the Medieval culture to the culture of a new era. A humanistic world outlook and a return toward the antique cultural heritage, as though, "Renaissance" of the ancient culture, are the most typical features of this epoch. The Renaissance is characterized by large scientific shifts in the field of natural sciences. The close connection to art is a specific feature of science of this epoch, and this feature was sometimes expressed by one creative person. Leonardo da Vinci, the outstanding artist, scientist, and engineer of the Renaissance, was a brilliant example of a many-sided person. By his nature, Leonardo da Vinci had enviable health, was good-looking man, tall, and blue-eyed. Leonardo was born on the 15th of April under the Star of Mars, and possibly, therefore, he possessed a huge force and man's valour. He sang marvelously by composing melodies and verses for his listeners. He played many different musical instruments; moreover, he created new musical instruments.

Leonardo da Vinci

But art and science were the main spheres of Leonardo's creative work — where Leonardo showed his genius talent. For Leonardo's art works, his contemporaries and descendants gave such definitions, as "genius," "divine," "great," but the same words can be used to describe Leonardo's scientific discoveries. He invented a military tank, a helicopter, an underwater ship, a parachute, an automatic weapon, a diving helmet, an elevator, and so on. He solved the most complicated problems of acoustics, botany, and cosmology; he invented an hour pendulum one century earlier than Galileo, and he developed a mechanics theory — among other things. In his relationship with Universe Harmony, Leonardo expressed the following words: "All earth, mountains, woods, and seas form a whole, in which each thing feeds the other one, all things are cooperated and interconnected, supported, but at the same time destroyed and updated."

Seemingly, each visitor of the Louvre in Paris tries to find the famous "Mona Lisa" ("Jokonda") by Leonardo da Vinci. The Great Artist drew this portrait tensely and long. He made many sketches; he paid attention to the pose of Mona Lisa, to the turn of her head, and to the position of her hands. Italian artist Giorgio Vazari (1511-1574) tells in his "Biographies," that during the painting of Mona Lisa, Leonardo invited singers, musicians, and clowns to his studio to support the cheerful mood of the young woman and to have an opportunity to watch a changeable expression of her face. And only after four years of intense work he, at last, could present to cultural community his world-famous "Mona Lisa" ("Jokonda").

It is considered, that a secret of Jokonda's charm is in the variability of her smile. There is an opinion that the woman represented in the picture had lost a child, several months ago. The dark color of her clothes speaks about this. She sits in a quiet pose combining her hands in her lap, but her face is full of imperceptible movement: her lips tremble in light smile, and her smiling eyes attract spectators attentively and derisively. In those days, the lips, slightly opened in corners of the mouth, were considered as an attribute of elegance. Jokonda's hardly appreciable smile, gentle and mysterious, lights up the picture. Mysteriousness of the image is strengthened by the background, a mountainous silvery-blue landscape.

What is the reason for Jokonda's charm? The search for the answer to this question continues. By creating the masterpiece, Leonardo used a secret known for many portraitists: the vertical axis of the picture passes through the center of the left eye what should cause the feeling of excitation of the spectator, that is, in the picture the artist used a "Principle of Symmetry," But, possibly, the reason is in another. The picture of the genius artist attracted attention of the researchers. After precise analysis, they found two golden triangles - one with its base stand-

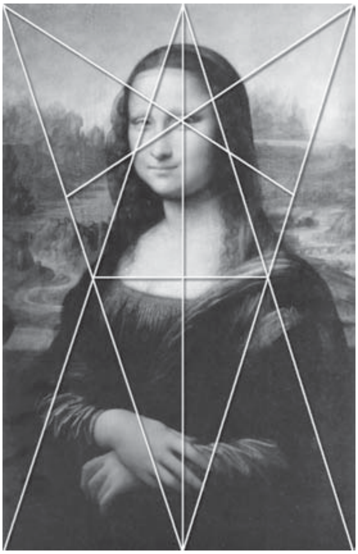


Figure 1.28. Harmonic Analysis of *Mona Lisa ("Jokonda")* by Leonardo da Vinci

ing on the bottom of the picture, another with its base touching the top of the picture and the top touching the bottom of the picture, and their common height crosses the center of Jokonda's left eye. (Fig. 1.28). Further harmonious analysis of the picture showed, that the center of Jokonda's left eye is on the crossing of the two bisectors of the upper golden triangle, which, on the one hand, bisect the angles at the base of the upper golden triangle, and, on the other hand it divides the sides of this golden triangle in the golden section. Thus, Leonardo used in the picture not only the "Principle of Symmetry," but also the "Golden Section Principle."

What is considered the apex of Leonardo's creativity, this picture is thought as the crystallization of his genius, innermost thoughts, and inspiration. Very little is known about Mona Lisa, except for several insignificant facts, therefore, it is difficult to answer very important questions often asked and discussed: whether she was simply a beautiful model for Leonardo or she was his muse and even his love. There are some facts, which confirm the last assumption, and this, probably, explains a special magic of the picture. But it is clear that Mona Lisa was that woman who inspired Leonardo to create this unique masterpiece inspiring thousands of people over many centuries.

A huge number of legends are connected with this well-known picture. We will begin from the history of its creation. Giorgio Vazari, in his "Biographies," wrote the following: "Leonardo undertook to execute for Franchesco Del Jokondo the portrait of his wife Mona Lisa." Some researches assume that Vasari, apparently, was mistaken. The newest investigations showed that in the picture another woman, not Mona Lisa, the wife of the Florentine nobleman Del Jokondo, was presented. Many researchers speculate why Del Jokondo refused the portrait of his wife, but in reality the portrait became the property of the artist, and this fact is an additional argument favoring the fact that Leonardo represented another woman and not Mona Lisa. However, there is another legend why Mona Lisa's portrait was found at Leonardo — simply Leonardo had drawn two copies of this famous picture.

Probably, the creation of this picture is connected with some secret in Leonardo's life. Leonardo's riddle begins with his birth. As is known, Leonardo was an illegally born son of a woman, about which almost nothing is known. It is known only that her name was Katerina and that she was the owner of a tavern. Not much more information is known about Leonardo's father. Mr. Pero, Leonardo's father, was 25 years old when Leonardo was born. He was the notary, and he possessed impressing male merits: he had lived till the age of 77s, had four wives (three of them he buried), fathered 12 children and the last child was born when he was 75 years old.

In the Renaissance, a relationship with illegitimate children was tolerated. Leonardo, at once, was recognized by the family of his father. However, in the house of his father he was taken away after awhile. Soon after his birth, Leonardo was sent together with Katerina to the village Anhiano located near to the city of Vinci. He remained there about four years. During these years, Mr. Pero married his first wife — a 16-year-old girl — who had a higher social position than Leonardo's mother.

The young wife had turned out fruitless. Probably, for this reason, Leonardo at the age of almost 5 was taken in his father's city house, where at once he was cared by his numerous relatives: grandfather, grandmother, father, uncle, and foster mother.

During his life, Leonardo always remembered his native mother Katerine who surprisingly resembled Mona Lisa, the wife of the Florentine merchant Jokondo, and, probably, this fact became the main reason of Leonardo's desire to create Mona Lisa's portrait. He embodied into this picture everything that was cheerful, light, and clear to him. He embodied all of a son's love to his mother Katerine in this well-known picture, which defined the development of painting for many centuries forward, and we should thank God that Leonardo da Vinci met Mona Lisa during the final stage of his creative way.

1.11. *De Divina Proportione* by Luca Pacioli

1.11.1. *Luca Pacioli*

The spiritual heritage of Greece, Rome, and Byzantium were marvelously combined in the new Renaissance lead by its Titans of gigantic intellect and artistic talent. "Titan" is the most appropriate term for the likes of Leonardo da Vinci, Michelangelo, Nicholas Copernicus, Albrecht Durer, Raphael, Bramante and many others. Luca Pacioli, the Italian mathematician of the

Luca Pacioli
(1445 - 1514)

Renaissance, collaborated with Leonardo da Vinci, was the seminal contributor to the field now known as accounting, takes a place of pride amongst the Titans. He was also called Luca di Borgo after his birthplace, Borgo Santo Sepolcro, Tuscany, where he was born in 1445.

Luca Pacioli is rightfully called "The Father of Accounting" and "The Unsung Hero of the Renaissance." In fact, the Fransican Friar Luca Pacioli was one of the most remarkable people of his epoch, but unfortunately was one of the least well-known. This is surprising because his work as a master of math is brilliant and revolutionary and continues to affect us.

Although we know little of Pacioli's early life, there is evidence that he began to study in the art studio of the artist Piero della Francesca. Amongst his contemporaries, Pierro della Francesca was known as a mathematician and geometer, as well as, an artist. Today he is chiefly appreciated for his art. Through Piero, Pacioli gained access to the Count of Urbino and the library of Federico where he gained access to thousands of books. This allowed Luca to expand and deepen his knowledge of mathematics. Piero also would later introduce Pacioli to his new mentor, Leon Battista Alberti, who became the second great man in Luca Pacioli's life. The following words of Alberti fell deeply into Luca's consciousness: "We may conclude Beauty to be such a Consent and Agreement of the Parts with the Whole in which it is found, as to Number, Finishing and Collocation, as Congruity, that is to say, the principal Law of Nature requires. This is what Architecture chiefly aims at, and by this she obtains her Beauty, Dignity and Value."

Alberti introduced Luca to Pope Paul II. Pope Paul encouraged Luca to dedicate his life to God and become a monk. When Alberti died in 1472, Luca took the vows of a Franciscan Minor.

In 1475 Pacioli began to work at the University of Perugia, receiving a professorial chair in 1477. He remained at the University for six years while becoming the first lecturer holding a chair in math in this University. While there Pacioli wrote a mathematical manuscript dedicated to the "Youth of Perugia." After he left Perugia, he took up more traveling and wandered throughout Italy, but in 1486 was called back to the University of Perugia by the Franciscans. It was at this time that he started calling himself "Magister," or "Master" — the equivalent of what we today refer to as a full professorship.

Luca wrote his famous work *Summa de Arithmetica, Geometria, Proportioni et Proportionalita,* when he was 49 years of age. Luca did this out of the belief that mathematics was being poorly taught at this time.

One section of the book that was entitled *Particularis de Computis et Scripturis* was dedicated to accounting. This book was referred to by some as "a catalyst that launched the past into the future." Luca was the first person to explain the double-entry system, or the Venetian method. This was absolutely revolutionary and far ahead of its time, sealing for him the title "The Father of Accounting." The *Summa* was recognized for many accomplishments and it became the most widely read mathematical work in all of Italy.

After Leonardo Da Vinci read Pacioli's *Summa,* he arranged for Luca to come to the Court of Duke Lodovico Maria Sforzo to tutor him in mathematical perspective and proportion. Luca joined Leonardo at the Sforza Court in 1496, whence began a seven-year relationship that produced two enduring masterpieces. Under the direct influence of Leonardo da Vinci, Luca Pacioli began to write the book *De Divina Proportione*, published in 1509. The book had a noticeable influence on his contemporaries and was one of the first fine examples of the Italian art of book-printing. The historical significance of this book is that it was the first mathematical book solely dedicated to the golden mean. Da Vinci used his artistic skills to illustrate Luca's *De Divina Proportione,* the second important Pacioli manuscript. At the same time, Luca taught Leonardo perspective and proportionality. This knowledge would remain with him forever, and help him to create one of his greatest masterpieces, "*The Last Supper.*" Painted on the back wall of the dining hall at the Dominican convent of Santa Maria delle Grazie in Italy, it instantly became one of the most famous works of the Italian Renaissance (if not of all time).

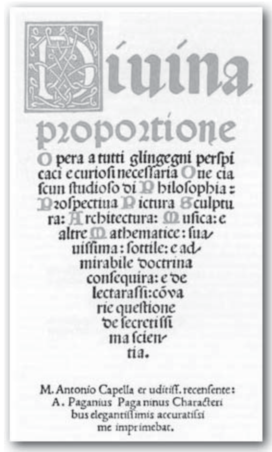

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Figure 1.29.
Luca Pacioli's book
De Divina Proportione

De Divina Proportione consists of three parts: the first part sets out the properties of the golden ratio, the second part is dedicated to the five regular polyhedrons, and the third part describes the applications of the Golden Section in architecture. Using Plato's *Republic*, *Timeaus*, and *Laws* Pacioli sequentially infers the twelve different properties of the Golden Section. Pacioli characterizes them using such epithets as "exceptional," "remarkable," "almost supernatural," etc. Considering his view (and that of the ancients) that the Golden Section is the universal ratio that expresses a perfection of Beauty in Nature and Art, he names it the *Divine Proportion* and recognizes it as a "thinking tool", the "aesthetic canon", and the "Fundamental Principle of the Cosmos".

This book is one of the first mathematical works that attempted to provide a solid scientific basis fot the Christian doctrine about God, the creator of the Cosmos. Considering the properties of the golden mean inherent in God, Pacioli names it the "Divine Proportion."

1.11.2. *About Pacioli's Plagiarism*

Roger Herz-Fischler wrote in his book [40]: "Perhaps no mathematician has plagiarized as much and with so little change in details and has stirred so much controversy and such vehement reactions as Luca Pacioli." The main charge for plagiarism falls on Pacioli's *De Divina Proportione*. Some have claimed that this book is a literal translation out of the Latin of Piero della Francesca's manuscript *De Quinque Corporibis*. However, *De Divina Proportione* is not the only book of Pacioli that falls under this charge. According to Davis [40], Pacioli also derived dodecahedron and icosahedron problems from Piero's *Trattato* and included them in his 1494 book, *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*.

It is now surprising why Pacioli's books caused such vehement reaction. Euclid, for instance, could also be blamed for plagiarism in *The Elements*. According to the statement of the famous mathematician and historian Van der Waerden, a majority of the mathematical results stated in *The Elements* belonged to the Pythagoreans. The following quote from Roger Herz-Fischler's book [40] is very interesting in regard to the aforementioned:

"As a final comment on Pacioli's "literary borrowings," I mention Cardano's 16th century comments ... on Pacioli's use in his Summa of a 1202 manuscript (Fibonacci?) dealing with algebra; and Agostini (1925) who, after discussing Pacioli's word for word inclusion in his Summa of a 1481 mercantile text, says that one should not accuse Pacioli of plagiarism for having used in his book what is available to everybody and is not personal intellectual property."

Thus, following the logic of Pacioli's accusers, all modern books on geometry are plagiarized, as they could not be written without using Euclid's *Elements*! Unlike artistic creative work, a distinctive feature of scientific, in particular, mathematical creative work, is that ideas of preceding fundamental scientific results can be used by all subsequent authors in new articles and books. Therefore, it is our responsibility as scientists, who are often in the same position, to stand up for the great Italian mathematician, Luca Pacioli, against such accusations of plagiarism. As mentioned earlier, Piero della Francesca was the first teacher and very good friend of Luca Pacioli. It is more Francesca's mathematical works than his art works that influenced the young Luca. There is absolutely nothing extraordinary in Luca's use of Piero della Francesca's mathematical ideas. At the very moment Pacioli began working on his book, *De Divina Proportione*; he was already a well-known mathematician and the author of the book *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*. The latter could well be considered to be the mathematical encyclopedia of the Renaissance. Considering these facts, it is hard to believe that the book *De Divina Proportione* is literal translation of Francesca's manuscript *De Quinque Corporibis*. More likely, Luca Pacioli used it as the basis of his "second great book" *De Divina Proportione*. Leonardo da Vinci unquestionably was the illustrator of this remarkable book with its 60 geometrical figures. Perhaps somebody will argue that he borrowed them from Piero della Francesca's works. Following the logic of the Pacioli accusers, all future books in the field of the "Golden Section," including the present book, should be recognized as a "plagiarism" of Pacioli's book *De Divina Proportione*, the first book focusing on the Golden Section in history!

1.11.3. *Death and Oblivion of Luca Pacioli*

Already tired and ill, in 1510, Luca Pacioli was 65 years old. In the foreword of his unpublished book *About the Forces and Quantities* he wrote the sad phrase: "The last days of my life approach." This manuscript is stored in the library of the Bologna University. Pacioli died in 1515 and is buried in the cemetery of his native town Borgo San-Sepolcoro.

After Pacioli's death, his works fell into oblivion for almost four centuries. At the end of the 19th century, his works became internationally well-known. After the 370-year hiatus of recognition, grateful descendants in 1878 placed a memorial stone in Pacioli's house of birth in Borgo San Sepolcro with the following inscription that reflects the essential significance of Pacioli:

"For Luca Pacioli, who had da Vinci and Alberti as friends and advisers, who turned algebra into a science, and applied it to geometry, who lectured in double-enrty bookkeeping, whose work was the basis and the norm for later mathematical research; for this great fellow citizen, the people of San Sepulcro, ashamed of their 370 years of silence, have placed this stone, in 1878."

The names of three geniuses of the Renaissance are mentioned in this inscription on the stone: Leonardo da Vinci, Leon Battista Alberti and Luca Pacioli. Their works became a valuable if not priceless contribution to the development of the theory of Harmony and the Golden Section!

Pacioli's memorable work is not forgotten in 20th and 21st centuries. A beautiful Monument to Luca Pacioli was sculpted out of white Carrara

marble and placed in Sansepolcro, Italy in honor of the anniversary of the 1494 publication of the *Summa*. In recognition of Pacioli's great works, a 500 lira post stamp was issued, and numerous conferences were held around the world.

Memorial stone in Borgo San Sepolcro, 1878

In honor of 500 anniversary of Summa, 1994

1.12. A Proportional Scheme of the Golden Section in Architecture

The 1935 book *Proportionality in Architecture* [10] by Russian architect Professor Grimm is well known in the theory of architecture. The purpose of the book is set forth in its Introduction:

"In view of the exceptional significance of the golden section as the proportional division that establishes a constant connection between the whole and its parts and gives a constant ratio between them, unachievable by any other division, it is foremost the normative basis upon which we will subsequently check the proportionality of both historical and modern monuments and constructions...."

Taking into consideration this general importance of the golden section in all manifestations of architectural thought, it is necessary to recognize the importance of it, as the basis of architectural proportionality theory in general."

Grimm illustrates the golden section as a division of line segment *AB* at point *C* into two unequal parts where the larger part *CB* is called *major*, and the smaller part *AC* is called *minor*. The whole line *AB* is to the *major* (segment *CB*) as the *major* (segment *CB*) is to the *minor* (segment *AC*), both in the golden ratio. Following a detailed analysis of the properties of the golden section, and in a manner not unlike Pacioli's comparison of it to the qualities Divine, Grimm places the golden mean in the forefront of all other proportions. He writes:

"In general it is necessary to recognize the extremely outstanding property of the golden section, which cannot be reached by arithmetic mean proportions and other divisions of the whole."

Grimm further illustrates examples of linear proportionality of the golden division (see the Doryphorus statue, Fig. 1.23), through analysis of golden divisions of rectangles, triangles, circles, and golden spirals. And finally, he reviews volumetric proportionality through divisions of cubes, parallelepipeds, triangular prisms, and tetrahedral pyramids.

"This analysis of the significance of the golden section and its exclusive properties for the theoretical solution of proportional division problems of linear, planar and volumetric masses, leads us to the following conclusion: For full proportional coordination of architectural structures representing a volumetric solution, it is necessary to have a golden proportioning, not only of its linear dimensions in vertical and horizontal, but also of the planar areas, and thence, of all volumes."

Figure 1.30. Saint Peter's Basilica in Rome (architect Bramante)

Grimm validates his theoretical researches in the field of the golden proportional scheme by architectural examples from classical art (Parthenon, Jupiter's temple in Tunis), Byzantine monuments, and Italian Renaissance art and architecture (Saint Peter's Cathedral, Fig. 1.30; and the Calleoni monument).

Grimm also analyzed structures of the Baroque period as they differ from the architecture of the Classical period and the Italian Renaissance, and at first sight may have been lacking the golden section. He analyzed the Smolny Cathedral (Fig. 1.31) and its surrounding monastery in St. Petersburg, Russia, a shining example of Baroque style. The

Figure 1.31. Smolny Cathedral St.-Petersburg, Russia

complex was commissioned in 1744 by Empress Elizabeth, a daughter of Peter the Great, and was constructed in place of the Admiralty's resin yard. Construction commenced in 1749 under the leadership of the architect Rastrelli, author of the *Winter Palace,* and the basic structure was completed in 1764. The magnificent blue-white cathedral rising to a height of 93.7 meters stands at the center of the Smolny monastery complex. The cathedral is stunning in its luxury, perfection of proportions, and variety of decorative forms. It possesses a strange, mystical peculiarity and magnificence that is strengthened as one approaches it. After analyzing the cathedral, Grimm concluded that its architecture was based upon the golden section.

Although there is no generally accepted opinion amongst architects as to Grimm's harmonious views, his editor's foreword to *Proportionality in Architecture* states:

"Nevertheless, his attempt at a general formulation of the golden section principle as the basis of proportionality in a variety of architectural styles, supported by analyses of material from ancient and European architecture, deserves to be published. All the more, Grimm's book presents a historical sketch of the development and use of proportionality theory, and also a comprehensive mathematical statement of the principle of the golden section."

1.13. The Golden Section in the Art of the 19th and 20th Centuries

1.13.1. *Ivan Shishkin's Picture "The Ship Grove"*

Ivan Shishkin (1832-1898) holds a place of honor in Russian painting. The entire history of Russian landscape painting in the second half of the 19th century is associated with his name. The art works of this eminent master took on special significance and became established as classics of Russian painting. Shishkin's creative work is an exclusive phenomenon amongst the masters of painting. Like many Russian artists he possessed a tremendous native talent.

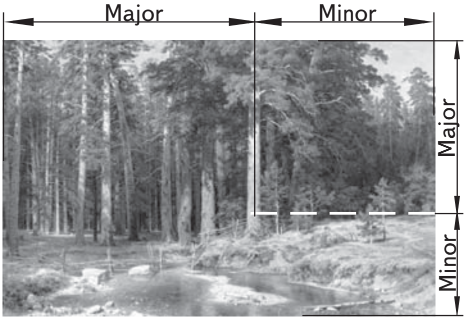


Figure 1.32. Harmonic analysis of

The picture *The Ship Grove* (Fig. 1.32), created by Shishkin in 1898, is his last work. It is considered a worthy culmination to his original and creative career. This picture has a perfect classical form, from the point of view of completeness and composition. The natural etudes created by Shishkin in his native land's forests, where he found his own idealistic synthesis of harmony and greatness, were thought to be the basis of his landscapes. In *The Ship Grove* the artist exemplified a thorough knowledge of Russian nature that was infused within him throughout his almost fifty years of creative life.

It is obvious that the golden section law can be found in this well-known picture, *The Ship Grove* (Fig. 1.32). For example, the brightly sunlit foreground pine divides the horizontal plane of the picture in the golden section. Furthermore, from the right of the pine there is the sunlit hillock. This hillock divides the vertical plane of the picture in the golden section. From the left of the foreground pine we can see many other pines. A division of the left part of the picture's horizontal plane in the golden section suggests itself. Presence of the bright vertical and horizontal lines that divide the picture in the golden section produces a balance and calmness consistent with the artist's intention.

1.13.2. *"Modulor" by Le Corbusier*

Le Corbusier (1887-1965) is an exceptional architect and theorist. His buildings became a triumph of the new architectural aesthetics with their vitality and humanism. After his book *To Architecture* was published in 1923, Le Corbusier became one of the twentieth century's leading architectural theorists.

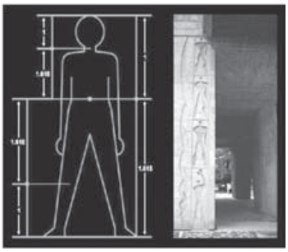


Figure 1.33. Le Corbusier's *Modulor*

During the Second World War Le Corbusier created his *Modulor*, a system of new proportional relations. He used the proportions of the human body as a principle of architectural metrics and, as a basis of his *Modulor*, he took not only the average height of a man, but also man's measurements while seated, including lengths of the hand and foot. The *Modulor* is not just a theory; it is a practical guide for the employment of human proportions in architecture. Einstein was one of the first great scientists who appreciated the *Modulor* for its true value. He recognized that Corbusier's

Figure 1.34. The Building of the United Nations Headquarters in New York

system had practical importance not only for architecture, but for other kinds of human activity as well. Harmonic analysis of the *Modulor* (Fig. 1.33) leaves no doubt of the presence of the golden section.

1.13.3. *The Building of the United Nations Headquarters in New York*

The construction of this world famous building (Fig. 1.34) is associated with the name of *Oscar Niemeyer* (1907-1989), well-known twentieth century Brazilian architect. He was the main adviser at the construction of the United Nations headquarters in New York. Its main architectural idea of this building consists of the three golden rectangles.

1.13.4. *A Picture "Near to the Window" by Konstantin Vasiliev*

Konstantin Vasiliev (1942-1976) is a modern Russian artist who unfortunately died quite young. He lived and worked near Kazan in the village Vasilevo. His popularity is owed to his mixture of traditional symbolism and modernism through the painting of folklore themes from ancient and modern Russian history. He was first introduced to the golden section at the Kazan school of art. Having studied the compositional principles of the golden section used by ancient Greeks, Konstantin decided "to investigate harmony through algebra."

He began exploring the possibility of applying the laws of harmony to the whole picture area in order to achieve maximal artistic expression of the image. He was moving the different segments expressing the golden section within the picture. Lastly, using the golden spiral, Vasiliev defined for himself how the human eye perceives the subject of the picture. Thus he would find how the picture should be constructed, and where he should set the imaginary main point of the picture so that it would absorb all the plot lines.

Vasiliev's *Near to the window* (1975) is a fine example of construction using the golden mean in all aspects of the picture (Fig. 1.35). We can only guess what the artist was trying to tell us. The main idea of this picture and all of its ramifications lies

Figure 1.35. Vasilev's picture *Near to the window* (1975)

in the face of the young woman glowing with cleanliness, dignity and quiet wisdom. The artist framed her face around the golden point of the picture that is located on the crossing of the two golden lines, horizontal and vertical. Therefore, this compositional decision is one of the reasons for the feeling of harmony of the picture that manifests within all primordial beginnings and the beauty of Russian women.

1.14. A Formula of Beauty

1.14.1. *The Golden Ratio in the External Forms of a Person*

Countless artists, poets and sculptors have admired the beauty of the human body! The French sculptor Rodin proclaimed, "*The naked body seems to me beautiful. For me it is a miracle where there can not be anything ugly*."

A human is considered the highest creation of Nature. Therefore, a human body at all times is recognized as the most perfect and worthy object of sculptural art. The problem of correct proportional representation of the human body was always one of the most important factors in art. The golden mean played a leading role in the art canons of Leonardo da Vinci, Durer and other great artists. According to these canons the golden section relates the whole body to its two unequal subdivisions cut at the waistline. This is the most simple realization of the proportional division of the human body (Fig. 1.36a). In addition, the golden proportion is often used by designers of clothes (Fig. 1.36b).

a)

b)

Figure 1.36. The golden mean in human body (a) and clothes design (b)

1.14.2. *The Beauty of a Woman's Face*

Repeated attempts have been made to analyze a woman's face using the golden mean and pentagram (Fig. 1.37).

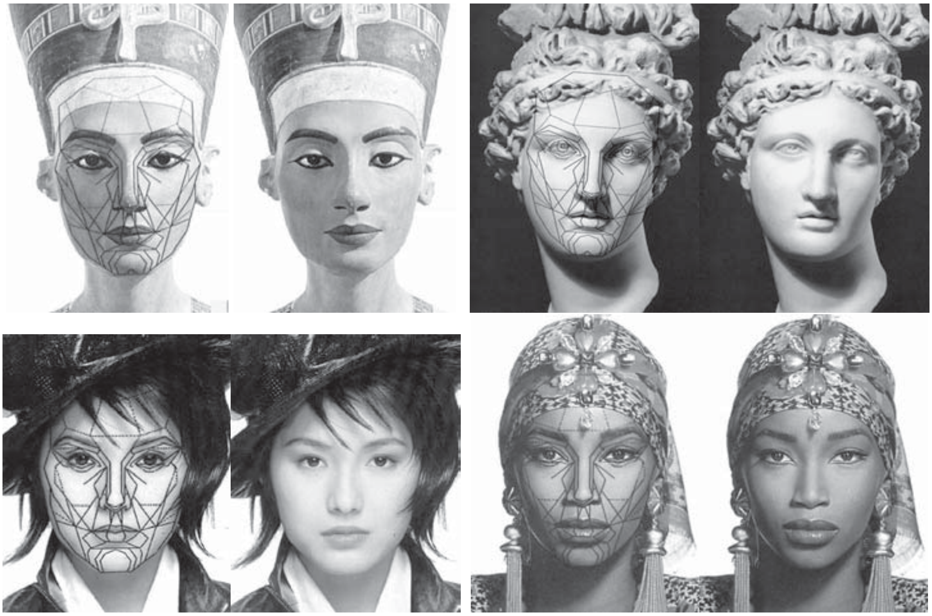


Figure 1.37. A harmonious analysis of a woman's face

Numerous researchers have come to the general conclusion that a woman's face is beautiful due to the golden section relations exemplified in it. The face of a woman displays an array of emotions that are an integrated element of her beauty. It is proven that a woman's face fits the proportions of the golden section most fully when she smiles. Any woman is perceived as more beautiful with a warm smile than with a rigid face filled with anger, arrogance and disregard. Admirers of great feminine beauty would be advised to make note of these golden section principles of aesthetics.

1.14.3. *Nefertiti*

During excavations in 1912, a German archaeological expedition digging in the deserted city of Amarna discovered a ruined house and studio complex. The building was identified as a sculptor's workshop. One of the items found had the name and job title of Thutmose, court sculptor of Egyptian Pharaoh Akhenaten. Among the many sculptures recovered was the famous head of Queen Nefertiti (in ancient Egyptian meaning the "beauty is coming"). This sculpture is recognized as a symbol of feminine beauty. After the exposition of Nefertiti in the Berlin museum, her fame rose as if she were a modern movie star rather than an ancient Egyptian queen.

With a charming head, long harmonic neck, and direct, but gently outlined nose, the Nefertiti sculpture caused increased interest in Egyptian art, including its deep mystical past of a cult of priests and esoteric wisdom. Possibly, our irrational century has selected Nefertiti as a symbol of unusual feminine beauty, feeling and as yet unrealized affinity for the grandeurs of ancient Egyptian culture. Belarusian philosopher Edward Soroko tried to determine what ideas were used by Thutmose in the sculpture of Nefertiti [138]. He thinks that Thutmose's logic was very clear and simple. In ancient Egypt, harmony was the prerogative of the Divine order that dominated the universe, and geometry was the main tool of its expression. The Queen played the role of Goddess. Hence, her image, which personified the wisdom of the world, must have been formed with geometrical perfection and irreproachable harmony, beauty and clarity. As a matter of fact, the main idea of ancient Egyptian aesthetic philosophy was to glorify the eternal, the measured, and the perfect in a constantly changing universe.

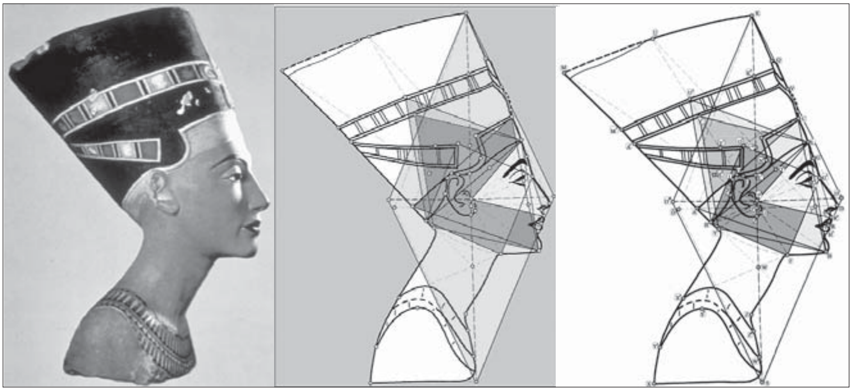


Figure 1.38. Harmonic analysis of Nefertiti's portrait

Soroko made a harmonious analysis of Nefertiti's sculpture and came to the conclusion that Thutmose had used the golden section principle as the basis of its design. During his analysis, Soroko found a harmonious system of regular geometric figures such as triangles, squares, and rhombi (see Fig. 1.38). Thus, he established that the parts of these figures, if put in order according to their sizes, are guided by one and the same ratio — the golden mean.

The metric structure of Nefertiti's statuette shows that there were ancient Egyptian sculptors consciously employing the principle of the golden section in their creative work. Soroko's analysis is one of the more compelling confirmations of the role of the golden mean in ancient Egyptian art.

1.15. Conclusion

The geometrical problem of the "Division in Extreme and Mean Ratio" (DEMR) came to us from Euclid's *Elements*. It was named later the *golden section, golden mean, golden number, golden proportion* or *divine proportion*. DEMR appears throughout Euclid's *Elements* and was used by Euclid for the geometric construction of the isosceles triangle with the angles 72°, 72° and 36° (the golden triangle), regular pentagon and dodecahedron. This fact may have given rise to the widespread belief (see Proclus, commentator of Euclid's *Elements*) that the main goal of the *Elements* was to describe the geometric construction and numerical interrelationship of the Platonic Solids. This means that the Pythagorean doctrine about the numerical Harmony of the Spheres was realized in the greatest mathematical work of ancient science, Euclid's *Elements*.

As the history of science shows, the golden section appears diffused within and throughout our cultural history. It became an aesthetic canon for the Egyptian, Greek and Renaissance cultures. During many millennia the golden section was a subject of delight for the great scientists and thinkers including Pythagoras, Plato, Euclid, Leonardo da Vinci, Luca Pacioli, Johannes Kepler, Allan Turing and countless others. A small list of some of the outstanding works of art and masterpieces which are based on the golden section could contain Khufu's Great Pyramid, the most famous of the Egyptian pyramids, Nefertiti's sculptural portrait, the majority of Greek sculptural monuments, the magnificent *Mona Lisa by Leonardo da Vinci, the works of Raphael, Shishkin and Konstantin Vasiliev, Chopin's etudes, the musical works of Beethoven, Tchaikovsky and Bella Bartok, and the Modulor* of Corbusier.

Chapter 2

Fibonacci and Lucas Numbers

2.1. Who was Fibonacci?

2.1.1. *Leonardo of Pisa Fibonacci*

Usually we associate the "Middle Ages" with the inquisition, forced conversions, burning of witches and heretics, and bloody Crusades in search of "Christ's Tomb." Science was not in the spotlight. Under these circumstances, the appearance of the 1202 mathematical book "Liber abaci" by Italian mathematician Leonardo of Pisa could not stay unnoticed by the scientific community. Who was the author of this book, and why were his mathematical works so important for Western European mathematics? To find the answers to these questions we have to return to his time and reproduce the historical epoch, in which Leonardo of Pisa, nicknamed Fibonacci, lived and worked.

Fibonacci
(c. 1170-after 1228)

It is worth noting that the period between the 11th and 12th centuries was an epoch of the brilliant flowering of Arabian culture; however, this century was the beginning of its downfall. At the end of the 11th century, before the beginning of the Crusades, the Arabs were undoubtedly the most educated people in the world and went far beyond their Christian competitors. Arabian influence had penetrated to the West long before the Crusades. After the Crusades, Arabian culture continued to influence the West and this mingling of cultures began to erode the Arabian world. Western researchers were dazzled with the Arabian world's art and scientific achievements. Their interest rapidly increased in Arabian geographical maps, algebra, astronomy tutorials, and architecture. During this time the great European mathematician of the Middle Ages, Leonardo of Pisa, nicknamed Fibonacci (son of Bonacci or son of the bull), lived and worked in Pisa. He admired Arabic science and math.

We know little about Fibonacci's life. The exact date of his birth is unknown, though he is thought to have been born around 1170. His father was a merchant and a government official who represented a new class of executives generated by the "Commercial Revolution." At that time the city of Pisa was one of the largest commercial Italian centers that actively cooperated with the Islamic East. Fibonacci's father traded in one of the trading posts founded by Italians on the northern coast of Africa. Thanks to that, he was able to give his son Fibonacci, the future mathematician, a good mathematical education in one of the Arabian educational institutions.

Moritz Cantor, well-known historian of mathematics, called Fibonacci "the brilliant meteor that flashed on the dark background of the Western European Middle Ages."

Fibonacci wrote several mathematical works, including *Liber Abaci, Liber Quadratorum,* and *Practica Geometriae*. The book *Liber Abaci* is the best known of them. This book was printed twice during the life of Fibonacci, originally in 1202 and again in 1228. The book consisted of 15 sections covering the following topics: the new Hindu numerals and representation of numbers with their help (Section 1); multiplication, addition, subtraction and division of numbers (Sections 2-5); multiplication, addition, subtraction and division of fractions (Sections 6-7); finding the prices of the goods and their exchange, the rules of mutual aid and the rule of the "double false situation" (Sections 8-13); finding the quadratic and cubic roots (Section 14); and the rules related to geometry and algebra (Section 15).

The book was intended as a manual for traders, though its significance far exceeded the bounds of trade practice. Fibonacci's book is a sort of mathematical encyclopedia of the Middle Ages epoch. Section 12 is of particular interest. It makes up nearly one third of the book and, apparently, Fibonacci paid special attention to it and demonstrated great innovation within it. The best-known Fibonacci problem is that of "rabbit reproduction." Its solution purportedly resulted in the discovery of the numerical sequence 1, 1, 2, 3, 5, 8, 13 ..., later called *Fibonacci Numbers.* This famous problem will be reviewed later.

2.1.2. *Fibonacci and Abu Kamil*

Roger Herz-Fischler [40] pointed out that Fibonacci borrowed many mathematical problems from the Arab mathematicians, in particular, Abu Kamil. However, in the summary of this comparison Herz-Fischler did not accuse Fibonacci of plagiarism. He wrote [40]: "Again when we turn to Fibonacci's presentation of the problems from Abu Kamil's, *On the Pentagon and Decagon*, we find new methods of solution that again display a deep understanding and ability. We must conclude then that either there were several, now lost, works from which Fibonacci obtained his material or he was responsible for significantly raising the level of the applications of the properties of DEMR to various computational problems."

2.1.3. *Influence of Fibonacci's Works on the Development of the European Mathematics*

Although Fibonacci was one of the great mathematical intellects in the history of Western European mathematics, his contribution in mathematics continues to be understated. The Russian mathematician Professor Vasiliev, in his book *Integer Number* (1919), noted the merits of Fibonacci's creative mathematical work:

"The works of the educated merchant from Pisa were so far above and beyond the level of mathematical knowledge of the scientists of those times, that the influence of his work on mathematical literature only became recognized two centuries after his death, at the end of the 15th century, when many of his theorems and problems were included by Luca Pacioli... and in the beginning of the 16th century, when the group of talented Italian mathematicians Ferro, Cardano, Tartalia, and Ferrari produced the beginnings of higher algebra thanks to the solution of the cubic and biquadrate equations."

It follows from this statement, that Fibonacci surpassed the Western European mathematicians of his time by almost two centuries. His historical role for Western science is similar to the role of Pythagoras who acquired his scientific knowledge from Egyptian and Babylonian sciences and then transferred them to Greek science. Fibonacci received his mathematical education in the Arabian educational institutions and transferred the Arabic knowledge of math to Western European science. Much of the knowledge acquired there, in particular, the Arabic-Hindu decimal notation, was introduced by him to Western European mathematics. Thus, he provided the fundamentals for the further development of Western European mathematics.

2.2. Fibonacci's Rabbits

2.2.1. *The "Rabbit Reproduction Problem"*

Fibonacci made a valuable contribution to science and the development of mathematics, however, the irony of his fate is that in modern mathematics he is generally only known as the author of an unusual numerical sequence, the Fibonacci numbers. He deduced that sequence when he was looking for the solution to the now famous rabbit reproduction problem. The formulation and solution of this problem is considered to be Fibonacci's main contribution to the development of combinatorial analysis. In it Fibonacci anticipated the recursive method that was recognized as one of the most powerful methods of solving combinatorial problems. The recursive relation achieved by Fibonacci is reputed to be the first recursive relation in mathematical history. Fibonacci formulated the core of the rabbit reproduction problem as follows:

"*A pair of rabbits were placed within an enclosure so as to determine how many pairs of rabbits will be born there in one year, it being assumed that every month a pair of rabbits produces another pair, provided that rabbits only begin to produce a new pair at their maturity, which is two months after their own birth.*"

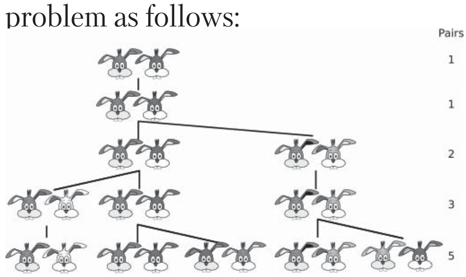


Figure 2.1. Fibonacci's rabbits

For the solution to this problem (Fig. 2.1) we define a pair of the mature rabbits by A, and a pair of the newborn rabbits by B. We display the process of reproduction with two transitions that describe the rabbits' monthly transformations:

$$A \to AB \tag{2.1}$$

$$B \to A \tag{2.2}$$

Note that the transition (2.1) simulates a monthly transformation of each pair of mature rabbits into two pairs, namely, the same pair of mature rabbits A and the newborn pair B. The transition (2.2) simulates the process of rabbit maturation when a newborn pair B is transformed into a mature pair A. Further, by beginning with a mature pair A, the process of rabbit reproduction can be represented by Table 2.1.

Table 2.1. Rabbit reproduction

Date	Pairs of rabbits	A	B	$A+B$
January, 1	A	1	0	1
February, 1	AB	1	1	2
March, 1	ABA	2	1	3
April, 1	$ABAAB$	3	2	5
May, 1	$ABAABABA$	5	3	8
June, 1	$ABAABABAABAAB$	8	5	13

Note that in the columns A, B and $A+B$ of Table 2.1 we can see the numbers, respectively, of mature (A), newborn (B) and total ($A+B$) rabbits.

Studying the A, B and $(A+B)$ sequences, we find the following regularity: each number of the sequence is equal to the sum of the previous two numbers. If we now designate the n-th number of the sequence that satisfies the rule as F_n, then the above general rule can be represented by the following mathematical formula:

$$F_n=F_{n-1}+F_{n-2}. \tag{2.3}$$

Such formula is called *recurrence* or *recursive relation*.

Note that the specific values of the numeric sequences generated by the recursive relation (2.3) depend on the initial values (the seeds) of the sequence F_1 and F_2. For example, we have $F_1 = F_2 =1$ for the A series and for this case the recursive formula (2.3) generates the following numerical sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ... \tag{2.4}$$

For the B series we have: $F_1=0$ and $F_2=1$ then the corresponding numerical sequence for this case is as follows:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

At last, for the $(A+B)$ series we have: $F_1=1$ and $F_2=2$ then the corresponding numerical sequence for this case is the following:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

The numerical sequence (2.4) is usually called Fibonacci numbers. Fibonacci numbers have a series of remarkable mathematical properties, which we will describe further.

2.2.2. *About the Rabbits*

Why do rabbits occur in the history of mathematics? Rabbits are mammals of the hare family. The countries of Spain, France, and Italy are considered to be the native land of wild rabbits; from these countries the rabbits were introduced to other countries. Wild rabbits now inhabit the southern and middle parts of Western Europe, and also Africa, Asia, Australia, New Zealand and America.

A special feature of rabbits is their surprising rate of reproduction. Female rabbits become mature at the age of 3-4 months and are able to reproduce throughout the year. A female rabbit's pregnancy lasts for 28-32 days (on the average 30 days); this means, that, by considering process of rabbit reproduction, Fibonacci employed biological facts. However, mature rabbits may produce 8 to 10 newborn rabbits monthly. Thus, the rabbits reproduce more intensively than Fibonacci suggested in his famous problem.

This exclusive ability of rabbit reproduction can explain why many countries consider the "rabbit invasion" to be a "national tragedy." Australia is one of the examples. In 1837 one Australian farmer started a rabbit farm of 24 rabbits. The rabbits that reproduced and escaped to freedom could have nearly destroyed all the greenery of the continent. The Australian government succeeded in reducing the number of rabbits by its resolute measures in the struggle with the so-called “long-eared locust.” In Australia, war was declared against the rabbits and has lasted more than 150 years. Throughout the continent, Australians had constructed a kind of “Great Wall of China” of many hundreds of kilometers, an insuperable barrier for the rabbits. The war has progressed with variable success. Wild Australian rabbits have learned to climb trees, becoming terrifyingly aggressive, capable of attacking the fields and kitchen gardens of the farmers.

The “prolific tribe of rabbits,” which had influenced the famous Italian mathematician, now took the Italian island Ustina (to the North of Sicily) under siege. 100,000 rabbits overran the 1000 inhabitants of this small island. In contrast to the Australian inhabitants the native population of Ustina yielded to the rabbits without a fight; already one fifth of the inhabitants have emigrated from the island.

It is necessary to remember the flip side of the “rabbit problem”; rabbit meat is considered useful and tasty. Italy, Fibonacci’s native land, is one of the largest producers of rabbit meat. Remembering this, the Italian (and non-Italian) historians of mathematics should work to derive the answer to the following question: what was the main reason for Fibonacci introducing rabbit reproduction into mathematics: a love for mathematics or rabbit meat?

2.2.3. *Dudeney’s Cows*

The English puzzlist Henry E. Dudeney (1857-1930) wrote several excellent books on puzzles. In one of them he adapted Fibonacci’s rabbits to cows by making Fibonacci’s problem more realistic. He replaced months by years and rabbits with bulls (males) and cows (females). Dudeney states his cow problem as follows:

If a cow produces its first she-calf at age of two and after that produces another single she-calf every year, how many she-calves are there after 12 years, assuming none die?

In Dudeney’s opinion, this simplifies the problem and makes it quite realistic.

2.2.4. *Honeybees and Family Trees*

As known, Fibonacci took a physical problem and simplified it in the way many mathematicians often do at first, namely, he simplified the problem and then observed to see what would happen. This series has many interesting and useful applications, which we will see later. Now let us consider another real-life situation, honeybee reproduction, which can also be precisely modeled on the Fibonacci series

There are over 30,000 species of bees and most of them live solitary lives. The honeybees are the best known among them. They usually live in a colony called a hive and have an unusual "Family Tree." In fact, there are many surprising features of the honeybees and, in this section, we will show how the Fibonacci numbers can be used to construct the genealogical trees of honeybees.

Before we discuss the Fibonacci series and honeybees, let us look at an unusual fact about them. Not all honeybees have two parents! In a colony of honeybees there is one special bee called the **queen bee**. There are many worker-bees, who are female too, but unlike the queen bee, they do not produce eggs. There are some male bees called **drone-bees.** Drone-bees are produced by the unfertilized eggs of the queen bee, so the male bees have only a mother, and no father. All female bees are produced when the queen bee has mated with a male bee and so female bees have two parents (Fig. 2.2). The females usually become worker-bees. However, when some of them are fed with a special substance called **royal jelly**, each grows into a queen bee ready to go off to start a new colony. When this occurs, the bees form a swarm and leave their hive in search of a place to build a new nest.

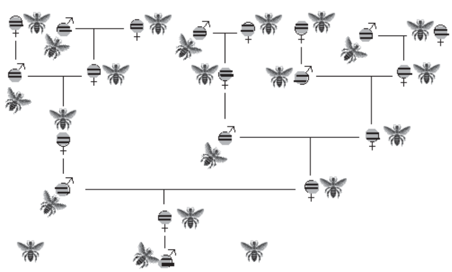


Figure 2.2. Family tree of honeybees

Here we follow the convention of "Family Trees" where *parents appear above their children*, so the latest generations are at the bottom, and the higher up we go the older the generations. Such trees place all *ancestors* of the descendent at the top of the diagram. We would get quite a different tree if we listed all the *descendants* (offspring) of an ancestor in a way similar to the rabbit problem, where we placed all descendants of the initial pair at the bottom.

Now, let us look at the family tree of the male-bee.

1. The male bee has **1** parent, the queen bee.
2. The male bee has **2** grand-parents because his mother (the queen bee) had two parents, a male and a female.
3. The male bee has **3** great-grand-parents: i.e. his grand-mother had two parents; however, his grand-father had only one parent.
4. How many great-great-grand parents did the male-bee have?

It is possible to make the following table, which sets forth a family tree of each male bee and each female bee.

It follows from Table 2.2 that the reproduction of honeybees is carried out according to "Fibonacci's principle!"

Table 2.2. Fibonacci numbers in a family tree of honeybees

	Number of parents	Number of grand parents	Number of great-grand parents	Number of great-great-grand parents	Number of great-great-great-grand parents
Number of male-bees	1	2	3	5	8
Number of female-bees	2	3	5	8	13

2.3. Numerology and Fibonacci Numbers

2.3.1. *Some Mathematical Properties of Numerological Values*

Increasingly popular today, numerology is an ancient tradition that attracted attention of civilization's greatest minds. There is evidence that numerology was in use in China, Greece, Rome, and Egypt for a long time before Pythagoras, who is generally considered to be the "father" of numerology.

Numerology, not unlike astrology and other esoteric subjects, has experienced a revival in recent years. Many modern numerologists have adopted the original Pythagorean system. It is a simple system that assigns a number value (from one to nine) to every letter of the alphabet: A is 1, B is 2, and so on. According to traditional summation, we add for example, 7+8=15. In numerology, this is not enough: we use a further method of summation known as the "Fadic" system, or "natural summation." This simply means we sum two or more digits together until we arrive at a single digit. For our example, 7 and 8 add up to 15, however, in numerology we sum 1+5, for a final answer 6.

Despite its esoteric aspects, numerology involves a deep mathematical conception. In fact, a calculation of the numerological values of one or another number from the mathematical point of view is very close to obtaining the remainder of some number through division by 9. In mathematics such operation is called a reduction by modulo 9.

Let us denote a numerological value of the integer a by $k(a)$. We say that two integers a and b are comparable by modulo m (m being a natural number) if at the division by m they result in the same remainder. In other words, a and b are comparable by modulo m if their remainder *a and b* are identical.

Example: 32 and 39 are comparable by modulo 7 because 32=7×4+4, 39=7×5+4, both having the same remainder: 4.

The statement "a and b are comparable by modulo m" is written in the form:

$a \equiv b \pmod{m}$.

Note that there is only one distinction between the numerological value of the number a and its remainder by modulo m=9. Consider, for example, the Fibonacci number of 144. As this number is perfectly divisible by 9, $144 \equiv 0 \pmod 9$ and the numerological value of the number 144 is equal to 9, i.e. $k(144)=1+4+4=9$.

A relation of comparison by the modulo m has many properties similar to the properties of traditional integers. For example, if

$a_1 \equiv b_1 \pmod{m}$ and $a_2 \equiv b_2 \pmod{m}$,

then we have

$$a_1 a_2 \equiv b_1 b_2 \pmod{m} \tag{2.5}$$

and

$$a_1 + a_2 \equiv (b_1 + b_2) \pmod{m}. \tag{2.6}$$

As $10 \equiv 1 \pmod 9$ then by using (2.5) it is easy to prove that

$$(10^i) \equiv 1 \pmod 9. \tag{2.7}$$

Now, let us represent a natural number N in the decimal system:

$$N = b_n 10^{n-1} + b_{n-1} 10^{n-2} + \ldots + b_i 10^{i-1} + \ldots + b_2 10^1 + b_1 10^0, \tag{2.8}$$

where $b_n, b_{n-1}, \ldots, b_i, \ldots, b_2, b_1$ are decimal numerals of the number N that take their values from the set of the decimal numerals {0,1,2,3,4,5,6,7,8,9}.

The abridged notation of the sum (2.8) $N= b_n, b_{n-1}, \ldots, b_i, \ldots, b_2, b_1$ is called a decimal notation of the number N.

By using (2.5) through (2.8), we can write:

$$k(N) = k(b_n + b_{n-1} + \ldots + b_i + \ldots + b_2 + b_1). \tag{2.9}$$

If we represent the sum

$$S = b_n + b_{n-1} + \ldots + b_i + \ldots + b_2 + b_1$$

in the decimal system (2.8)

$$S = d_m 10^{m-1} + d_{m-1} 10^{m-2} + \ldots + d_j 10^{j-1} + \ldots + d_2 10^1 + d_1 10^0, \tag{2.10}$$

then we can obtain a new expression for the numerological value of the number N:

$$k(N) = k(d_m + d_{m-1} + \ldots + d_j + \ldots + d_2 + d_1). \tag{2.11}$$

This procedure continues until the last sum of the kind (2.11) is less than the number 10. It is the numerological value of the number N.

From the above examination, it is easy to deduce a number of mathematical properties of the "numerological values." First of all, we can find a numerological value of the sum of two numbers, for example, N_1+N_2. For this purpose, we represent the numbers N_1 and N_2 in the decimal notation (2.8):

$$\tag{2.12}$$

$$N_2 = d_n 10^{n-1} + d_{n-1}10^{n-2} + \dots + d_i 10^{i-1} + \dots + d_2 10^1 + d_1 10^0 \tag{2.13}$$

Then, we can represent the sum N_1+N_2 as follows:

$$\begin{aligned} N_1 + N_2 &= \left[c_n 10^{n-1} + c_{n-1}10^{n-2} + \dots + c_i 10^{i-1} + \dots + c_2 10^1 + c_1 10^0 \right] \\ &+ \left[d_n 10^{n-1} + d_{n-1}10^{n-2} + \dots + d_i 10^{i} + \dots + d_2 10^{i-1} + d_1 10^0 \right]. \end{aligned} \tag{2.14}$$

It follows from (2.14) that the numerological value of the sum N_1+N_2 is calculated according to the expression:

$$\begin{aligned} k(N_1+N_2) &= k\left[(c_n + c_{n-1} + \dots + c_i + \dots + c_2 + c_1) + (d_m + d_{m-1} + \dots + d_j + \dots + d_2 + d_1) \right] \\ &= k(c_n + c_{n-1} + \dots + c_i + \dots + c_2 + c_1) + k(d_m + d_{m-1} + \dots + d_j + \dots + d_2 + d_1). \end{aligned} \tag{2.15}$$

We can write the expression (2.15) as follows:

$$k(N_1+N_2) = k(N_1) + k(N_2). \tag{2.16}$$

Thus, the numerological value of the sum N_1+N_2 is equal to the sum of the numerological values of the initial numbers N_1 and N_2. As examples, we calculate the numerological value of the sum:

17711+5702887=5720598.

It is clear that the numerological value of the numbers 17711, 5702887 and 5720598 are equal, respectively,

k(17711)=1+7+7+1+1=17, 1+7=8;

k(5702887)=5+7+0+2+8+8+7=37, 3+7=10, 1+0=1;

k(5720598)= 5+7+2+0+5+9+8=36, 3+6=9.

On the other hand, we have

k(5720598)=k(17711)+k(5702887)=8+1=9.

2.3.2. *Fibonacci Numerological Series*

Let us now examine the Fibonacci numbers from numerological points of view. For this purpose we write the first 48 Fibonacci numbers together with their numerological values $k(F_1)$ through $k(F_{48})$ (Table 2.3).

Table 2.3. Numerological values of Fibonacci numbers

n	F_n	$k(F_n)$	n	F_n	$k(F_n)$
1	1	1	25	75025	1
2	1	1	26	121393	1
3	2	2	27	196418	2
4	3	3	28	317811	3
5	5	5	29	514229	5
6	8	8	30	832040	8
7	13	4	31	1346269	4
8	21	3	32	2178309	3
9	34	7	33	3524578	7
10	55	1	34	5702887	1
11	89	8	35	9227465	8
12	144	9	36	14930352	9
13	233	8	37	24157817	8
14	377	8	38	39088169	8
15	610	7	39	63245986	7
16	987	6	40	102334155	6
17	1597	4	41	165580141	4
18	2584	1	42	267914296	1
19	4181	5	43	433494437	5
20	6765	6	44	701408733	6
21	10946	2	45	1134903170	2
22	17711	8	46	1836311903	8
23	28657	1	47	2971215073	1
24	46368	9	48	4807526976	9

Let us consider the recursive relation (2.3) for Fibonacci numbers. If we use a general identity (2.16), we can write the following recursive relation for the numerological values of the adjacent Fibonacci numbers:

$$k(F_n)=k(F_{n-1})+k(F_{n-2}). \tag{2.17}$$

We can see from Table 2.3 that this regularity is valid for all Fibonacci numbers in that table. For example, by using Table 2.3, we can write: $k(F_{41}) = 4$. We can also calculate the numerological value of the Fibonacci number F_{41} by using the recursive relation (2.17). In fact,

$$k(F_{41})=k(F_{40})+k(F_{39})=6+7, \quad 1+3=4.$$

Analysis of the numerological values $k(F_1)$ to $k(F_{48})$ lead us to an unexpected result. Beginning with the Fibonacci number F_{25}, the numerological values start to recur, that is, the numerological values $k(F_{25})$ to $k(F_{48})$ are a repetition of the numerological values $k(F_1)$ to $k(F_{48})$. We can continue Table 2.3 in order to be convinced that the numerological values $k(F_{49})$ to $k(F_{72})$ are a repetition of $k(F_1)$ to $k(F_{24})$ and $k(F_{25})$ to $k(F_{48})$. Thus, the *Fibonacci Numerological Series* $k(F_i)$ (i=1, 2, 3, ...) is a periodic sequence with the period of length 24:

1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9 (2.18)

Let us divide the period (2.18) into two parts (with 12 numbers in each part):

1	2
1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9	8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9

and then sum them in pairs. As a result, we obtain the regularity presented in Table 2.4.

It follows from Table 2.4 that the sum of the first 11 pairs of the Fibonacci numerological series is equal to the number 9. However, the sum of the last, that is, the 12th pair is equal to 9+9 = 18; note that the numerological value of this last sum is also equal to 9.

If we sum all numbers of the first period, then this sum is equal to

1+1+2+3+5+8+4+3+7+1+8+9+8+8+7+6+4+1+5+6+2+8+1+9 = 117.

The numerological value of the sum 117 is again equal to 9. This means that the numerological value of the sum of the first 24 Fibonacci numbers is equal to 9. By analogy, we can assert that the numerological sum of the next 24 Fibonacci numbers will also be equal to 9. Thus, if we begin from the first Fibonacci numbers F_1 to F_{24} and examine the subsequent periods of 24 Fibonacci Numbers, for example, F_{25} to F_{48}, F_{49} to F_{72}, and so on, then we will find that the numerological values of these sums are each equal to 9. In order to calculate the numerological value of the Fibonacci series, we should calculate the numerological value of the following infinite sum 9+9+...+9+.... It is easy to prove that the numerological value of this sum is equal to 9. This reasoning results in the following theorem.

Table 2.4. Regularity of the Fibonacci numerological series

1	+	8	=	9
1	+	8	=	9
2	+	7	=	9
3	+	6	=	9
5	+	4	=	9
8	+	1	=	9
4	+	5	=	9
3	+	6	=	9
7	+	2	=	9
1	+	8	=	9
8	+	1	=	9
9	+	9	=	18

Theorem 2.1. The Fibonacci numerological series has the period (2.18) of length 24; here the numerological value of the sum of the first 24 Fibonacci numbers and all subsequent periods of 24 Fibonacci numbers are each equal to 9.

This means that the number 9 is a "numerological essence" of the Fibonacci series, that is, this number expresses a "sacred" property of the Fibonacci series.

2.3.3. *Another Periodic Properties of Fibonacci Numbers*

It is necessary to point out that the period (2.18) that appears in the numerological study of Fibonacci numbers is not the only example of a periodic property of Fibonacci numbers. For example, we can calculate the values of the Fibonacci series by Mod 2 and Mod 3 (see Table 2.5).

Table 2.5. Periodicities of Fibonacci numbers taken by Mod 2 and Mod 3

F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
Mod 2	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1
Mod 3	1	1	2	0	2	2	1	0	1	1	2	0	2	2	1	0

We can see that the Fibonacci numbers by Mod 2 have a period 1, 1, 0 of length 3 and by Mod 3 a period 1, 1, 2, 0, 2, 2, 1, 0 of length 8. By studying the numerical sequences obtained by taking Fibonacci series by Mod k, the American scientist Jay Kappraff found the following periodicities of such sequences as given in Table 2.6.

Table 2.6. Periodicities of Fibonacci numbers by different k module

Modulo k	2	3	4	5	6	7	8	9	12	16
Length of the period	3	8	6	20	24	16	12	24	24	24

Note that mathematicians discovered many interesting number-theoretic properties of Fibonacci numbers concerning their divisibility. Consider the following remarkable properties of Fibonacci numbers proved in [13]:

1. If n is divisible by m, then F_n is divisible by F_m. For example, consider two Fibonacci numbers, F_{26}=121393 and F_{13}=233. We can see that F_{26} is divisible by F_{13} (121393:233=521) because 26 is divisible by 13 (26:13=2).

2. If $F_n \equiv 0$ (mod k) and $F_m \equiv 0$ (mod$k$) then also $F_{n=m} \equiv 0$ (mod k) and $F_{n-m} \equiv 0$ (mod k) (for $n>m$). For example, consider two Fibonacci numbers, F_{24}=46368 and F_{12}=144. We can calculate that $F_{24} \equiv 0$ (mod 9) and $F_{12} \equiv 0$ (mod 9); then for the Fibonacci numbers $F_{24+12}=F_{36}$=14930352 and $F_{24-12}=F_{12}$=144 we have: $F_{36} \equiv 0$ (mod 9) and $F_{12} \equiv 0$ (mod 9).

A surprising periodicity of the Fibonacci series taken by Mod k (see Tables 2.4 - 2.6) produces a feeling of rhythm and harmony hidden in this remarkable numerical sequence!

2.4. Variations on Fibonacci Theme

The variations on a given theme in music are known as genre. A distinctive feature of the musical works of various genres consists of the fact that they begin, in most cases, with one simple essential musical theme, which thereinafter undergoes considerable changes of tempo, mood and nature. However, no matter how extreme the variations are, the listeners are absolutely impressed that each variation is a natural development of the main theme.

If we follow the example of musical genre and select a simple mathematical subject, such as the Fibonacci series, we can consider this series together with its numerous variations.

2.4.1. *Formulas for the Sums of Fibonacci Numbers*

Fibonacci numbers have a number of delightful mathematical properties that have stimulated the imaginations of mathematicians over the centuries. Let us calculate, for example, the sum of the first n Fibonacci numbers. Begin from the simplest sums:

$$\begin{aligned} 1+1 &= 2 = \mathbf{3}-1 \\ 1+1+2 &= 4 = \mathbf{5}-1 \\ 1+1+2+3 &= 7 = \mathbf{8}-1 \\ 1+1+2+3+5 &= 12 = \mathbf{13}-1 \end{aligned} \tag{2.19}$$

If we consider in the sums (2.19) the numbers marked by bold type: **3**, **5**, **8**, **13**, ..., then it is easy to see that they are Fibonacci numbers! Then, we can write the sums (2.19) as follows:

$F_1+F_2=F_4-1;\quad F_1+F_2+F_3=F_5-1;\quad F_1+F_2+F_3+F_4=F_6-1.$

It is clear that the general formula has the following form:

$$F_1+F_2+...+F_n=F_{n+2}-1. \tag{2.20}$$

Now, let us consider the sum of the n sequential Fibonacci numbers with the odd indexes 1,3,5,...,2n-1,.... To this end we start from the simplest sums:

$$\begin{aligned} 1+2 &= \mathbf{3} \\ 1+2+5 &= \mathbf{8} \\ 1+2+5+13 &= \mathbf{21} \\ 1+2+5+13+34 &= \mathbf{55} \end{aligned} \tag{2.21}$$

By analyzing (2.21), we find the following regularity for Fibonacci numbers: **the sum of the *n* sequential Fibonacci numbers with the odd indexes is equal to Fibonacci numbers!** In general, the partial sums (2.21) can be written in the form of the following general identity:

$$F_1+F_3+F_5+...+F_{2n-1}=F_{2n}. \tag{2.22}$$

It is easy to prove the similar formulas for the sums of the *n* sequential Fibonacci numbers with even indexes:

$$F_2+F_4+F_6+...+F_{2n}=F_{2n+1}-1. \tag{2.23}$$

Now, let us find the sum of the squares of the *n* sequential Fibonacci numbers:

$$F_1^2+F_2^2+...+F_n^2. \tag{2.24}$$

Start from the analysis of the simplest sums of the kind (2.24):

$$\begin{aligned} 1^2+1^2 &= 2=\mathbf{1}\times\mathbf{2} \\ 1^2+1^2+2^2 &= 6=\mathbf{2}\times\mathbf{3} \\ 1^2+1^2+2^2+3^2 &= 15=\mathbf{3}\times\mathbf{5} \\ 1^2+1^2+2^2+3^2+5^2 &= 40=\mathbf{5}\times\mathbf{8} \end{aligned} \tag{2.25}$$

The analysis of (2.25) results in the following general formula:

$$F_1^2+F_2^2+...+F_n^2=F_nF_{n+1}, \tag{2.26}$$

that is, **the sum of squares of *n* sequential Fibonacci numbers is equal to the product of the greatest Fibonacci number used in this sum multiplied by the next Fibonacci number!**

Also we can find the sum of the squares of two adjacent Fibonacci numbers:

$$F_n^2+F_{n+1}^2. \tag{2.27}$$

Start from the analysis of the simplest sums of the kind (2.27):

$$\begin{aligned} 1^2+1^2 &= 1+1 = \mathbf{2} \\ 1^2+2^2 &= 1+4 = \mathbf{5} \\ 2^2+3^2 &= 4+9 = \mathbf{13} \\ 3^2+5^2 &= 9+25 = \mathbf{34} \end{aligned} \tag{2.28}$$

By analyzing (2.28), we can find another remarkable regularity: **the sum of squares of two adjacent Fibonacci numbers is always equal to a Fibonacci number!** The general form of this regularity can be expressed as follows:

$$F_n^2+F_{n+1}^2=F_{2n+1}. \tag{2.29}$$

2.4.2. *Connection of Fibonacci Numbers to the Golden Mean*

Now, let us show a connection of Fibonacci numbers to the golden mean. With this purpose in mind, we can examine a sequence of fractions that are built up by the ratios of adjacent Fibonacci numbers:

$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{21}{13},\ \frac{34}{21},\ \ldots \tag{2.30}$$

The first terms of the sequence (2.30) have the following values:

$$\frac{1}{1}=1;\ \frac{2}{1}=2;\ \frac{3}{2}=1.5;\ \frac{5}{3}=1.666;\ \frac{8}{5}=1.6;\ \frac{13}{8}=1.625;\ \frac{21}{13}=1.615;\ldots$$

The question is: what is the limit of the sequence (2.30) if we direct the index n to infinity? To answer this question, let us consider the representation of the golden mean in the form (1.15). It is easy to prove that the sequence (2.30) is connected directly with the representation (1.15). In fact, the fractions (2.30) are sequential approximations of the continuous fraction (1.15), namely:

$$\frac{1}{1}=1 \qquad \text{(the first approximation);}$$

$$\frac{2}{1}=1+\frac{1}{1} \qquad \text{(the second approximation);}$$

$$\frac{3}{2}=1+\cfrac{1}{1+\cfrac{1}{1}} \qquad \text{(the third approximation);}$$

$$\frac{5}{3}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}} \qquad \text{(the fourth approximation).}$$

If we continue this process to infinity, we obtain:

$$\lim_{n\to\infty}\frac{F_n}{F_{n-1}}=\tau=\frac{1+\sqrt{5}}{2}. \tag{2.31}$$

The result given by the expression (2.31) is the “key” result for our research because it shows a deep connection between the Fibonacci numbers and the golden mean. This means, that like the golden mean itself, Fibonacci numbers express harmony in the world around us!

2.4.3. *“Iron Table” by Steinhaus*

Renowned Polish mathematician Steinhaus constructed a table of random numbers by using the golden mean. For this purpose, he multiplied 10 000 integers from 1 to 10 000 by the number $\varphi=\tau-1=0.61803398$, where τ is the golden mean. As a result, he obtained the sequence of numbers multiplied by φ, that is:

1φ, 2φ 3φ, ... , 4181φ, ... , 6765φ, ... , 10000.

Steinhaus called this numerical sequence the *Golden Numbers*. Each *Golden Number* consists of integer and fractional parts. For example, the number 1000φ = 618.03398 has the integer part 618 and the fractional part 0.03398. The number 4181φ=2584.00007 has the integer part 2584 and the fractional part 0.0007, and so on. Moreover, neither a *Golden Number* with fractional part equal to 0 exists nor do two *Golden Numbers* with the equal fractional parts. This means that every *Golden Number* has a unique fractional part.

If we put the *Golden Numbers* in order in a special table according to their increasing fractional parts, we find that the number 4181φ has the least fractional part and, therefore, this *Golden Number* should start this table; also we find that the number 6765φ has the largest fractional part and, therefore, this *Golden Number* has to end the table:

4181	8362	1597	5778	9959
3194	7365	0610	4791	8972
……	……	……	……	……
8739	1974	6155	3571	7752
0987	5168	9349	**2584**	**6765**

Steinhaus named this the *Iron Table* taking into consideration some of its unique properties. The *Iron Table* demonstrates deep connections with Fibonacci numbers. The first property is that a difference between the adjacent *Golden Numbers* of the *Iron Table* by absolute value is always equal to one of three numbers: **4181**, **6765** and **2584**. In fact, we have:

$$8362-4181=4181,\ 8362-1597=6765,\ 5778-1597=4181,...$$
$$9349-5168=4181,\ 9349-2584=6765,\ 6765-2584=4181.$$

It is easy to find the numbers of 2584, 4181 and 6765 if we continue the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 897, 1597, **2584**, **4181**, **6765**,...

Hence, the numbers **2584**, **4181**, **6765** are three adjacent Fibonacci numbers:

$$F_{18}=2584, F_{19}=4181, F_{20}=6765.$$

We can see that the *Iron Table* starts with the Fibonacci number F_{18}=2584 and ends with the Fibonacci numbers F_{19}=4181 and F_{20}=6765.

It is clear that the *Iron Table* may be constructed for any arbitrary natural number *N*. Polish scientist Jan Grzedzielski in his book *Energy-Geometric Code of Nature* [26] analyzed the *Iron Tables* for the cases $N=F_n$ where F_n is Fibonacci number. He obtained an interesting regularity that appears at the conversion of the *Iron Table* with $N=F_{n-1}$ into the next *Iron Table* with $N=F_n$. This latter table is constructed from the preceding table (with $N=F_{n-1}$) by means of the disposition of new numbers $F_{n-1}+1, F_{n-2}+2,...,F_n-1,F_n$ on particular positions in the new *Iron Table*. In Grzedzielski's opinion, this method of *Iron Table* construction "resembles the functioning of all radiation spectra in Nature."

2.5. Lucas Numbers

2.5.1. *Francois-Edouard-Anatole Lucas*

Fibonacci did not continue to study the mathematical properties of the numerical series (2.4). However, the study of Fibonacci numbers was continued by other mathematicians. Since the 19th century, the mathematical works devoted to Fibonacci numbers, according to the witty expression of one mathematician, "began to be reproduced like Fibonacci's rabbits." The French mathematician Lucas became one of the leaders of this research in the 19th century.

Francois-Edouard-Anatole Lucas
(1842-1891)

What do we know about Lucas? The French mathematician Francois-Edouard-Anatole Lucas was born in 1842. He died in 1891 as a result of an accident that occurred at a banquet, when a dish was smashed and a splinter wounded his cheek. Lucas died from an infection some days later.

Lucas' major works fall in the area of number theory and indeterminate analysis. In 1878 Lucas gave the criterion for the determination of the "primality" of Mersenne numbers of the kind $M_p=2^p-1$. Applying his own method, Lucas proved that the Mersenne number

$$M_{127}=2^{127}-1=170141183460469231731687303715884105727$$

is prime. For 75 years, this number was the greatest prime number known in mathematics. He also found the 12th "perfect number" and formulated a number of interesting mathematical problems.

We can give some explanation for Lucas' scientific achievements. It is well known that the prime numbers are divisible only by 1 and itself. The first few prime numbers are: 2, 3, 5, 7, 11, 13, ... Already the Pythagoreans knew that the number of prime numbers is infinite (the proof of this fact is in *The Elements* of Euclid). A study of the prime numbers and determination of their distribution in the natural series is a rather difficult problem in number theory. Therefore, the scientific result that was obtained by Lucas in the field of prime numbers, undoubtedly, belonged to the category of outstanding mathematical achievements.

It is curious, that Lucas took a great interest in the so-called *Perfect Numbers*. What are *Perfect Numbers*? As is well known, the Pythagorean theory of numbers had a qualitative character, that is, the Pythagoreans were interested in the qualitative aspects of numbers. They attributed to numbers some unusual properties. In this connection, the so-called *Perfect Numbers* are of special interest. For example, consider the number 6. Its feature is that this number is equal to the sum of its divisors, that is, 6=1+2+3. Besides the number 6, the Pythagoreans knew another two *Perfect Numbers*, 28 and 496:

28=1+2+4+7+14; 496=1+2+4+8+16+31+62+124+248.

The fourth perfect number is 8128. It was also known by ancient mathematicians.

It is proven, that in the process of movement along natural series the *Perfect Numbers* are found less frequently. Only four *Perfect Numbers* (4, 28, 496 and 8128) are found in the first 10,000 numbers of the natural series. The search for *Perfect Numbers* became a fascinating passion for many mathematicians. The fifth *Perfect Number* $2^{12}(2^{13}-1)$ was found in the 15th century by the German mathematician Regiomontan. In the 16th century, the German scientist Sheybel found two new *Perfect Numbers*: 8589869056 and 137438691328. Lucas in the 19th century found the twelfth *Perfect Number*. Research in this area continues today where all the power of modern computers is being used. For example, the 18th *Perfect Number* that was found by means of computer modeling has 2000 decimal digits.

In honor of Lucas, it is necessary to note one more of his scientific predictions. Already in the 19th century, long before the occurrence of modern computers, Lucas paid attention to technical advantages of the binary notation for technical realization of computers and machines. This means that he anticipated by a century the outstanding American physicist and mathematician John von Neumann's preference for the binary system in the technical realization of electronic computers (John von Neumann's principles).

2.5.2. *Some Properties of Lucas Numbers*

However, it is most relevant for our book that in the 19th century Lucas attracted the attention of mathematicians to the remarkable numeric sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ..., which he named *Fibonacci Numbers*. In addition, Lucas introduced the concept of generalized Fibonacci numbers that are calculated according to the following general recursive relation:

$$G_n = G_{n-1} + G_{n-2} \tag{2.32}$$

at the initial terms (seeds) G_1 and G_2.

For example, the sequence of numbers 2, 8, 10, 18, 28, 46, ... falls into the generalized class of Fibonacci numbers satisfying the recursive relation (2.5) at the seeds G_1=2 and G_2=8.

However, the main numerical sequence of the type (2.32), introduced by Lucas in the 19th century, is a numerical sequence given by the following recursive relation:

$$L_n = L_{n-1} + L_{n-2} \tag{2.33}$$

at the seeds

$$L_1 = 1, \quad L_2 = 3. \tag{2.34}$$

Then, by using the recursive relation (2.33) at the seeds (2.34) we can calculate the following numerical sequence called *Lucas Numbers*:

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ... \tag{2.35}$$

If we make the reasonings for Lucas numbers similar to Fibonacci numbers, we can prove the following identities:

$$L_1 + L_2 + ... + L_n = L_{n+2} - 3$$

$$L_1 + L_3 + L_5 + ... + L_{2n-1} = L_{2n} - 2$$

$$L_1 + L_3 + L_5 + ... + L_{2n-1} = L_{2n} - 2$$

$$L_1^2 + L_2^2 + L_3^2 + ... + L_n^2 = L_n L_{n+1} - 2$$

$$L_n^2 + L_{n+1}^2 = 5F_{2n+1} \tag{2.36}$$

$$\lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \tau = \frac{1+\sqrt{5}}{2} \tag{2.37}$$

2.5.3. *The "Extended" Fibonacci and Lucas Numbers*

Up until now, we have studied Fibonacci and Lucas series F_n and L_n with the positive indices n, that is, n=1,2,3,.... However, they can be extended with the negative values of the indices n, that is, when the indices n take their values from the set: n=0,-1,-2,-3,....

The "extended" Fibonacci and Lucas numbers are represented in Table 2.7.

Table 2.7. "Extended" Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

It follows from Table 2.7 that the elements of the "extended" numerical sequences F_n and L_n have a number of remarkable mathematical properties. For example, for the odd indices $n=2k+1$ the elements of the sequences F_n and F_{-n} coincide, that is, $F_{2k+1}= F_{-2k-1}$, and for the even indices $n=2k$ they are opposite by a sign, that is, $F_{2k}=F_{-2k}$. For the Lucas numbers L_n all is vice versa, that is, $L_{2k}=L_{-2k}$; $L_{2k+1}= L_{-2k-1}$.

Now, let us consider the numerical sequences given in Table 2.7. Take, for example, the Lucas number L_4=7 and compare it with Fibonacci numbers. It is easy to find that L_4=7=5+2. However, the numbers 2 and 5 are the Fibonacci numbers F_3=2 and F_5=5.

Is this a mere coincidence? By continuing examination of Table 2.7, we can find the following correlations that connect Fibonacci and Lucas numbers: 1=0+1, 3=1+2, 4=1+3, 7=2+5, 11=3+8, 18=5+13, 29=8+21, and so on.

Now, let us compare the numerical sequences L_{-n} and F_{-n}. Here we find the same regularity, that is, -1=0+(-1), 3=1+2, -4=(-1)+(-3), and so on. Thus, we found the following surprising mathematical identity that connects Lucas and Fibonacci numbers:

$$L_n = F_n + F_{n+1}, \tag{2.38}$$

where the index n takes the following values: n= 0, ±1, ±2, ±3, ...

If we continue examination of Table 2.7, we can find other interesting identities for Fibonacci and Lucas numbers:

$$L_n = F_n + 2F_{n-1} \tag{2.39}$$

$$L_n + F_n = 2F_{n-1}. \tag{2.40}$$

Note that the above formulas for Fibonacci and Lucas numbers can be considered to be the "golden treasure" of mathematics! And, by comprehending these formulas, we can understand the delight of many outstanding mathematicians of the 20th century, in particular, the Russian mathematician Nikolay Vorobyov and the American mathematician Verner Hoggatt. It was Professor Nikolay Vorobyov who was the author of the remarkable brochure

Fibonacci Numbers [13] that became a mathematical bestseller of the 20th century. Professor Verner Hoggat was the founder of the Fibonacci Association, the mathematical journal *The Fibonacci Quarterly* and the author of the book *Fibonacci and Lucas Numbers* [16]. They saw in Fibonacci and Lucas numbers a "mathematical secret of nature." The desire to uncover this "secret" inspired them to study these unique mathematical sequences!

2.5.4. *Other Remarkable Identities for Fibonacci and Lucas Numbers*

The next group of formulas is based on the following identity that connects the generalized Fibonacci numbers G_n with the classical Fibonacci numbers F_n:

$$G_{n+m} = F_{m-1}G_n + F_m G_{n+1}. \tag{2.41}$$

We can prove this formula by induction on m. For the cases m=1 and m=2 this formula is valid because

$G_{n+1} = F_0 G_n + F_1 G_{n+1} = G_{n+1}$ and $G_{n+2} = F_1 G_n + F_2 G_{n+1} = G_n + G_{n+1}$.

The basis of the induction is proved.

Let us suppose that the formula (2.41) is valid for the cases m=k and m=k+1, that is,

$$G_{n+k} = F_{k-1}G_n + F_k G_{n+1},\ G_{n+k+1} = F_k G_n + F_{k+1}G_{n+1}.$$

By summarizing these formulas termwise, we obtain the following identity:

$$G_{n+k+2} = F_{k+1}G_n + F_{k+2}G_{n+1}.$$

The identity (2.41) is proved.

A number of interesting identities for Fibonacci and Lucas numbers follow from the identity (2.41). Suppose that G_i=F_i and m=n+1. Then, the identity (2.41) is reduced to the following:

$$F_{2n+1} = F_{n+1}^2 + F_n^2. \tag{2.42}$$

For the case G_i=F_i and m=n, the formula (2.41) is reduced to the following:

$$F_{2n} = F_{n-1}F_n + F_n F_{n+1} = F_n\left(F_{n-1} + F_{n+1}\right) = F_n L_n. \tag{2.43}$$

By using the formula (2.43), we can write:

$$F_{n+1}L_{n+1} - F_n L_n = F_{2n+2} - F_{2n} = F_n L_n. \tag{2.44}$$

2.5.5. *Lucas Numbers and Numerology*

As was mentioned above, the Fibonacci series generates a periodic Fibonacci numerological series with the period (2.18). There is a question: whether there is a similar periodicity for the Lucas series? Table 2.8 presents 48 Lucas numbers and their numerological values.

Table 2.8. Numerological values of Lucas numbers

n	L_n	$k(L_n)$	n	L_n	$k(L_n)$
1	1	1	25	167761	1
2	3	3	26	271443	3
3	4	4	27	439204	4
4	7	7	28	710647	7
5	11	2	29	1149851	2
6	18	9	30	1860498	9
7	29	2	31	3010349	2
8	47	2	32	4870847	2
9	76	4	33	7881196	4
10	123	6	34	12752043	6
11	199	1	35	20633239	1
12	322	7	36	33385282	7
13	521	8	37	54018521	8
14	843	6	38	87403803	6
15	1364	5	39	141422324	5
16	2207	2	40	228826127	2
17	3571	7	41	370248451	7
18	5778	9	42	599074578	9
19	9349	7	43	969323029	7
20	15127	7	44	1568397607	7
21	24476	5	45	2537720636	5
22	39603	3	46	4106118243	3
23	64079	8	47	6643838879	8
24	103682	2	48	10749957122	2

The analysis of Table 2.8 shows that the numerological Lucas series $\{k(L_n, n=1, 2, 3,...)\}$ possess mathematical properties similar to the numerological Fibonacci series, that is, the numerological Lucas series is periodic with the period of length 24 of the following kind:

$$\mathbf{1, 3, 4, 7, 2, 9, 2, 2, 4, 6, 1, 7, 8, 6, 5, 2, 7, 9, 7, 7, 5, 3, 8, 2} \qquad (2.45)$$

If we divide the period (2.45) into two parts (by 12 numbers in each part)

1	2
1, 3, 4, 7, 2, 9, 2, 2, 4, 6, 1, 7	8, 6, 5, 2, 7, 9, 7, 7, 5, 3, 8, 2

and then sum them in pairs, we find the following regularity (see Table 2.9):

That is, the sums of all 11 mutual numerological values of the period (2.45) are equal to 9 and only one to the sum 9+9=18; however, its numerological value is equal to 9. If we use the same reasoning as for Fibonacci numbers, we can prove the following theorem.

Theorem 2.2. The Lucas numerological series has the period (2.45) of length 24; here the numerological value of the sum of the first 24 Lucas numbers and all the next sequential 24 Lucas numbers are equal to 9.

This means that the number 9 is the "numerological essence" of the Lucas series, that is, it expresses some "sacred" property of Lucas numbers.

It is easy to prove that the generalized Fibonacci numbers given by (2.32) has similar general property (see Theorem 2.3).

Theorem 2.3. The numerological series of the generalized Fibonacci numbers given by the recursive relation (2.32) has the period of length 24; here the numerological value of the sum of the first 24 generalized Fibonacci numbers and all the next sequential 24 generalized Fibonacci numbers are equal to 9.

Table 2.9. Regularity of the Lucas numerological series

1	+	8	=	9
3	+	6	=	9
4	+	5	=	9
7	+	2	=	9
2	+	7	=	9
9	+	9	=	18
2	+	7	=	9
2	+	7	=	9
4	+	5	=	9
6	+	3	=	9
1	+	8	=	9
7	+	2	=	9

2.6. Cassini Formula

2.6.1. *Great Astronomer Giovanni Domenico Cassini*

Cassini is the name of a famous dynasty of French astronomers. **Giovanni Domenico Cassini** (1625-1712) is the most famous of them and the founder of this dynasty. The following facts illustrate his outstanding contribution to astronomy. The name **Cassini** was given to numerous astronomical objects, the "Cassini Crater" on the Moon, the "Cassini Crater" on Mars, "Cassini Slot" in Saturn's ring, and "Cassini Laws" of the Moon's movement.

Giovanni Domenico Cassini (1625-1712)

However, the name of Cassini is widely known not only in astronomy, but also in mathematics. Cassini developed a theory of the remarkable geometrical figures known under the name of *Cassini Ovals*. The mathematical identity that connects three adjacent Fibonacci numbers is well known under the name *Cassini Formula*. Below we will review this famous formula.

Giovanni Cassini was born on June, 8, 1625 in the Italian town of Perinaldo. He got his education in Jesuit collegiums in Genoa. During 1644-1650 he worked in the observatory located near Bologna.

In 1650 Cassini took a professorial chair in mathematics and astronomy at the University of Bologna. Cassini's basic scientific works concerned observational astronomy and he became famous, first of all, as a talented observer. Working in Bologna, for the first time in history, he executed numerous positional observations of the Sun with meridian tools. On the basis of these observations he made new Solar tables published in 1662. Owing to Cassini's researches, the *Parisian Meridian* was established. Cassini's work provided the possibility for the creation of the well-known map of France, the *Cassini Map*.

The glory of Cassini as an astronomer was so great that in 1669 he was elected a member of the Parisian Academy of Sciences. According to Pikar's recommendation, King Louis XIV invited Cassini to take a position as Director of Parisian observatory. France became his second native land up to the end of his life. In Paris, Cassini made a number of outstanding astronomical discoveries. During 1671-1684 he discovered several satellites of Saturn. In 1675 he found that the ring of Saturn consists of two parts divided by dark strip called the *Cassini Slot*. During 1671-1679 he observed details of the Lunar surface and in 1679 he made a finer map of the Moon. In 1693 Cassini formulated three empirical laws for the Moon's movement called *Cassini Laws*.

Cassini died in Paris on September 14, 1712 at the age of 87, absolutely blind but a highly honored man.

2.6.2. *Cassini Formula for Fibonacci Numbers*

The history of science is silent as to why Cassini took such a great interest in Fibonacci numbers. Most likely it was simply a hobby of the Great astronomer. At that time many serious scientists took a great interest in Fibonacci numbers and the golden mean. These mathematical objects were also a hobby of Cassini's contemporary, Kepler.

Now, let us consider the Fibonacci series: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Take the Fibonacci number 5 and square it, that is, $5^2=25$. Multiply two Fibonacci numbers 3 and 8 that encircle the Fibonacci number 5, so we get $3\times 8=24$. Then we can write:

$$5^2-3\times 8=1.$$

Note that the difference is equal to (+1).

Now, we follow the same process with the next Fibonacci number 8, that is, at first, we square it $8^2=64$ and then multiply two Fibonacci numbers 5 and 13 that encircle the Fibonacci number 8, $5\times 13=65$. After a comparison of the result $5\times 13=65$ with the square $8^2=64$ we can write:

$$8^2-5\times 13=-1.$$

Note that the difference is equal to (-1).

Further we have:

$$13^2-8\times 21=1, \quad 21^2-13\times 34=-1,$$

and so on.

We see that the square of any Fibonacci number F_n always differs from the product of two adjacent Fibonacci numbers F_{n-1} and F_{n+1}, which encircle it, by 1. However, the sign of 1 depends on the index n of the Fibonacci number F_n. If the index n is even, then the number 1 is taken with minus, and if odd, with plus. The indicated property of Fibonacci numbers can be expressed by the following mathematical formula:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \qquad (2.46)$$

This wonderful formula evokes a reverent thrill, if one recognizes that this formula is valid for any value of n (we remember that n can be some integer in limits from -∞ up to +∞). The alternation of +1 and -1 in the expression (2.45) at the successive passing of all Fibonacci numbers produces genuine aesthetic enjoyment and a feeling of rhythm and harmony.

2.7. Pythagorean Triangles and Fibonacci Numbers

2.7.1. *Pythagorean Theorem*

The Pythagorean Theorem is probably the best-known theorem in all of geometry. It is remembered by anyone who studied geometry in secondary school, though having "absolutely forgotten" all mathematics. The essence of this theorem is extremely simple. The Pythagorean Theorem asserts that the sides a, b and c of a right triangle are connected by the following formula:

$$a^2+b^2=c^2. \qquad (2.47)$$

Despite its ultimate simplicity, the Pythagorean Theorem, in the opinion of many mathematicians, refers to a category of the most outstanding mathematical theorems. Earlier we mentioned Kepler's widely known statement concerning the Pythagorean Theorem and the golden mean. Among all the vast sea of geometrical results and theorems, Kepler chose only two results that he named "treasures of geometry": the Pythagorean Theorem and the "division in extreme and mean ratio" (the golden mean).

2.7.2. *Pythagorean Triangles*

Amongst the infinite number of right-angled triangles, it is the *Pythagorean Triangles,* for which the numbers a, b, c in (2.47) are integers, which have always caused a special interest.

Pythagorean triangles also can be referred to as a category of the "treasures of geometry," and a study of such triangles is one of the most fascinating pages in the history of mathematics. The right triangle with sides 3, 4 and 5 is the most widely known Pythagorean triangle (Fig. 2.3). It is called the *Sacred* or *Egyptian Triangle* because it was widely used in Ancient Egyptian culture. We mentioned above that this triangle is the main geometrical idea of the *Chephren Pyramid* in Giza.

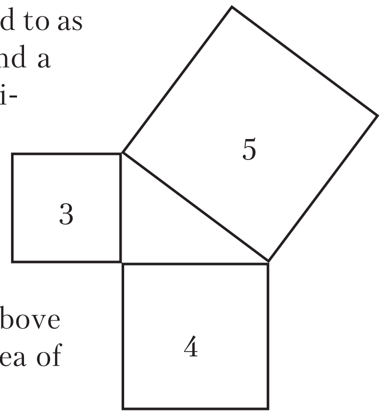


Figure 2.3. Sacred or Egyptian Triangle

For the *Egyptian Triangle* in Fig. 2.3 the Pythagorean Theorem (2.47) takes the following form:

$$4^2+3^2=5^2. \tag{2.48}$$

There is a legend that the identity (2.48) was used by the Egyptian land surveyors and builders for the definition of a right angle on the earth's surface. For this purpose, they used a rope of the length 12*m*, for example (12=3+4+5). The rope was divided by means of special loops into three parts of the lengths 3, 4, 5*m*, respectively. For the definition of a right angle, the Egyptian land surveyor pulled one of the parts of the rope, for example, of the length 3*m*, and then fixed it on the ground by using special pegs, inserted in two loops. Then, the rope was pulled by means of the third loop and this loop was fixed by using the third peg. Clearly, the angle that is formed between the two smaller sides of the triangle accurately equals 90°. Tradition has it that at the foundation of the Pyramid a ritual procedure to define the right angles at the base of the Pyramid was carried out by the Pharaoh.

2.7.3. *Fibonacci-Pythagorean Triangles*

There is a question: are there other Pythagorean triangles besides the *Egyptian Triangle*? Rasko Jovanovich [139] gives an original answer to this question.

First, we try to express the identity (2.47) by Fibonacci numbers beginning with the first four Fibonacci numbers:

$$F_1=1, F_2=1, F_3=2, F_4=3. \tag{2.49}$$

Perform the following procedure:

1. Multiply the two middle Fibonacci numbers of the series (2.49): $F_2 \times F_3=1\times2=2$.
2. Double this result: $2\times(F_2\times F_3)=2\times2=4$. The product 4 is the length of the side a of the *Egyptian Triangle* in Fig. 2.3, that is, $a=2\times(F_2\times F_3)=4$.
3. Multiply the two external numbers of the series (2.49): $F_1\times F_4=1\times3=3$.

The product 3 is the length of the side b of the *Egyptian Triangle* in Fig. 2.3, that is, $b= F_1 \times F_4 = 3$.

4. The third, the longest side c of the *Egyptian Triangle*, is determined if we add the squares of the interior numbers of the series (2.49), that is, the numbers $(F_2)^2=1^2=1$ and $(F_3)^2=2^2=4$; then their sum is equal to 1+4=5. Hence, $c=(F_2)^2+(F_3)^2=5$.

Therefore, we can represent a fundamental identity for the *Egyptian Triangle* (2.48) by Fibonacci numbers as follows:

$$\left[2\left(F_2 \times F_3\right)\right]^2 + \left(F_1 \times F_4\right)^2 = \left[\left(F_2\right)^2 + \left(F_3\right)^2\right]^2. \tag{2.50}$$

Is this result (2.50) a mere coincidence? To answer this question, we examine the next four Fibonacci numbers:

$$F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5. \tag{2.51}$$

If we repeat the above procedure for the series (2.51), we obtain:

1. 2×3=6
2. 2×6=12; hence, a=12
3. 1×5=5; hence, b=5
4. $2^2=4$, $3^2=9$, 4+9=13; hence, c=13

It is easy to be convinced, that the sides a=12, b=5 and c=13 make up a Pythagorean triangle because:

$12^2+5^2=13^2$.

These examples allow us to formulate a general rule to find Pythagorean triangles by using four adjacent Fibonacci numbers:

$$F_n, F_{n+1}, F_{n+2}, F_{n+3}. \tag{2.52}$$

Perform the following:

1. Multiply two middle Fibonacci numbers of the series (2.52): $F_{n+1} \times F_{n+2}$.
2. Double this result: $2\times\left(F_{n+1} \times F_{n+2}\right)$; hence, $a = 2\times\left(F_{n+1} \times F_{n+2}\right)$.
3. Multiply the two external Fibonacci numbers of the series (2.52): $F_n \times F_{n+3}$; hence, $b = F_n \times F_{n+3}$.
4. Square the two interior Fibonacci numbers of the series (2.52), $\left(F_{n+1}\right)^2$ and $\left(F_{n+2}\right)^2$, and then add them: $c + \left(F_{n+1}\right)^2 + \left(F_{n+2}\right)^2$.

For the general case of (2.52) the main identity for Pythagorean triangles is as follows:

$$\left(2\times F_{n+1} \times F_{n+2}\right)^2 + \left(F_n \times F_{n+3}\right)^2 = \left(F_{n+1}^2 + F_{n+2}^2\right)^2. \tag{2.53}$$

Now let us consider the Pythagorean triangle for the case n=3. For this case, the four adjacent Fibonacci numbers are the following:

$$2, 3, 5, 8. \tag{2.54}$$

Then, according to the above algorithm, the sides of the Pythagorean triangle can be found in the following manner:

$a = 2\times3\times5 = 30;\ b = 2\times8 = 16;\ c = 3^2 + 5^2 = 9 + 25 = 34.$

The Pythagorean Theorem for this case is: $30^2+16^2=34^2$.

Finally, for the case n=4 the four adjacent Fibonacci numbers are the following: 3, 5, 8, 13, and the sides of Pythagorean triangle are equal to, respectively,

$a = 2\times5\times8 = 80;\ b = 3\times13 = 39;\ c = 5^2 + 8^2 = 35 + 64 = 89.$

The Pythagorean Theorem for this case is as follows: $80^2+39^2=89^2$.

Table 2.10 represents Pythagorean triangles for the initial values n.

Table 2.10. Fibonacci-Pythagorean triangles

n	F_n	F_{n+1}	F_{n+2}	F_{n+3}	a	b	c
1	1	1	2	3	4	3	5
2	1	2	3	5	12	5	13
3	2	3	5	8	30	16	34
4	3	5	8	13	80	39	89
6	8	13	21	34	546	272	610
7	13	21	34	55	1428	715	1597
8	21	34	55	89	3740	1869	4181
9	34	55	89	144	9790	4869	10946

It is essential to note that side c of the Pytha-gorean triangles of Table 2.10 are calculated by the formula:

$$c = F_{n+1}^2 + F_{n+2}^2. \quad (2.55)$$

By using the identity (2.29), we can write:

$$c = F_{2(n+1)+1},$$

that is, the hypotenuse c of the Fibonacci-Pythagorean triangle is always equal to some Fibonacci number that is confirmed by Table 2.10.

2.7.4. *Lucas-Pythagorean Triangles*

The above procedure for the Fibonacci-Pythagorean triangles also proves to be valid for Lucas numbers. For example, the first four adjacent Lucas numbers 1, 3, 4, 7 result in the Lucas-Pythagorean triangle with the sides:

$a = 2\times3\times4 = 24;\ b = 1\times7 = 7;\ c = 3^2 + 4^2 = 9 + 16 = 25.$

For this case the Pythagorean Theorem is as follows:
$24^2+7^2=25^2$.

The next four adjacent Lucas numbers 3, 4, 7, 11 result in the next Lucas-Pythagorean triangle with the sides:

$a = 2\times4\times7 = 56;\ b = 3\times11 = 33;\ c = 4^2 + 7^2 = 16 + 49 = 65.$

For this case the Pythagorean Theorem is as follows:
$56^2+33^2=65^2$.

Table 2.11 represents the Lucas-Pythagorean triangles for the initial values of n.

Table 2.11. Lucas-Pythagorean triangles

n	L_n	L_{n+1}	L_{n+2}	L_{n+3}	a	b	c
1	1	3	4	7	24	7	25
2	3	4	7	11	56	33	65
3	4	7	11	18	154	72	170
4	7	11	18	29	396	203	445
5	11	18	29	47	1044	517	1165
6	18	29	47	76	2726	1368	3050
7	29	47	76	123	7144	3567	7985
8	47	76	123	199	18696	9353	20905
9	76	123	199	322	48954	24472	54730

2.7.5. *Fibonacci and Lucas Right Triangles*

Now, let us consider the identity $F_n^2+F_{n+1}^2=F_{2n+1}$ (2.29) as an expression of the Pythagorean Theorem (2.47) for the right triangle with the sides $a=F_n; b=F_{n+1}$ and $c=\sqrt{F_{2n+1}}$. We name these right triangles *Fibonacci Right Triangles* because they are based on the important identity (2.29) that connects the adjacent Fibonacci numbers. Note that *Fibonacci Right Triangles* are not Pythagorean. Nevertheless, they are a very interesting class of right triangles. Fibonacci right triangles are given by Table 2.12.

Table 2.12. Fibonacci right triangles

n	F_n^2	F_{n+1}^2	F_{2n+1}
1	1	1	2
2	1	4	5
3	4	9	13
4	9	25	34
5	25	64	89
6	64	169	175
7	169	441	610
8	441	1156	1597
9	1156	3025	4181

Note that similar triangles can also be constructed on the basis of Lucas numbers if we use the identity (2.36) that gives Lucas right triangle with the sides: $a=L_n, b=L_{n+1}$ and $c=\sqrt{5F_{2n+1}}$.

Table 2.13. Lucas right triangles

n	L_n^2	L_{n+1}^2	$5F_{2n+1}$
1	1	9	10
2	9	16	25
3	16	49	65
4	49	121	170
5	121	324	445
6	324	841	1165
7	841	2209	3050
8	2209	5776	7985
9	5776	15129	7305

Table 2.13 gives the Lucas right triangles.

The above connection of Fibonacci and Lucas numbers with Pythagorean triangles allows us to propose the existence of an infinite set of Fibonacci and Lucas Pythagorean triangles. This fact is additional testimony for the fundamental character of Fibonacci and Lucas numbers!

2.8. Binet Formulas

2.8.1. *Jacques Philippe Marie Binet*

In the 19th century, the interest in Fibonacci numbers and the golden mean in science and mathematics again arose. The mathematical works of the French mathematicians Lucas and Binet were a bright reflection of this interest. We have mentioned above about the French 19th century mathematician Lucas (1842-1891), who revived the interest in Fibonacci numbers of the scientific 19th century community.


Jacques Philippe Marie Binet (1776-1856)

The French mathematician *Jacques Philippe Marie Binet* was the other 19th century enthusiast of Fibonacci numbers and the golden mean. He was born on February 2, 1776 in Renje and died on May 12, 1856 in Paris. Following his graduation from the Polytechnic School in Paris in 1806 Binet worked at the Bridge and Road Department of the French government. He became a teacher at the Polytechnic school in 1807 and then Assistant Professor of applied analysis and descriptive geometry. Binet studied the fundamentals of matrix theory and his work in this direction was continued later by other researchers. He discovered in 1812 the rule for matrix multiplication, which glorified his name more than any of his other works.

Binet worked in other areas in addition to mathematics. He published many articles on mechanics, mathematics and astronomy. In mathematics, Binet introduced the notion of the "beta function"; also he considered the linear difference equations with alternating coefficients and established some metric properties of conjugate diameters, etc. Among his many honors, Binet was elected to the Parisian Academy of sciences in 1843.

Binet entered Fibonacci number theory as author of the famous mathematical formulas called *Binet Formulas*. These formulas link the Fibonacci and Lucas numbers with the golden mean and, undoubtedly, belong amongst history's most famous mathematical formulas.

2.8.2. *Deducing Binet Formulas*

In order to deduce Binet formulas, we must first consider the remarkable identity that connects the adjacent degrees of the golden mean:

$$\tau^n = \tau^{n-1} + \tau^{n-2}, \tag{2.56}$$

where τ is the golden mean and n takes its values from the set: 0, ±1, ±2, ±3, ...

Now, let us write the expressions for the zero, first and minus-first terms of the golden mean:

$$\tau^0 = 1 = \frac{2+0\times\sqrt{5}}{2},\ \tau^1 = \frac{1+\sqrt{5}}{2},\ \text{and}\ \tau^{-1} = \frac{-1+\sqrt{5}}{2}. \tag{2.57}$$

Let us remember that the degrees of τ are connected one to the other by the following identity:

$$\tau^2 = \tau + 1, \tag{2.58}$$

that is a partial case of the general identity (2.56) for the case n=2.

By employing the expressions of (2.57) and the identities (2.56) and (2.58), we can represent the second, third and fourth degrees of the golden mean as follows:

$$\tau^2 = \tau^1 + \tau^0 = \frac{3+\sqrt{5}}{2};\ \tau^3 = \tau^2 + \tau^1 = \frac{4+2\sqrt{5}}{2};\ \tau^4 = \tau^3 + \tau^2 = \frac{7+3\sqrt{5}}{2}. \tag{2.59}$$

Is it possible to see some regularity in the formulas (2.57) and (2.59)? First of all, we can see that each expression for any degree of the golden mean has the following typical form:

$$\frac{A+B\sqrt{5}}{2}.$$

What are the numerical sequences A and B in these formulas? It is easy to see that the series of the numbers A is the number sequence 2, 1, 3, 4, 7, 11, 18, ..., and the series of the numbers B is the number sequence 0, 1, 1, 2, 3, 3, 5, 8, ... Thus, the first sequence is Lucas numbers L_n and the second one is Fibonacci numbers F_n! It follows from this reasoning that the general formula that allows representation of the n-th degree of the golden mean by Fibonacci and Lucas numbers has the following form:

$$\tau^n = \frac{L_n + F_n\sqrt{5}}{2}. \tag{2.60}$$

Note that the formula (2.60) is valid for each integer n taking its values from the set 0, ±1, ±2, ±3, ...

By using the formula (2.60), it is possible to represent the "extended" Fibonacci and Lucas numbers by the golden mean. For this purpose, it is enough to write the formulas for the sum or difference of the n-th degrees of the golden mean $\tau^n + \tau^{-n}$ and $\tau^n - \tau^{-n}$ as follows:

$$\tau^n + \tau^{-n} = \frac{(L_n + L_{-n}) + (F_n + F_{-n})\sqrt{5}}{2} \tag{2.61}$$

$$\tau^n - \tau^{-n} = \frac{(L_n - L_{-n}) + (F_n - F_{-n})\sqrt{5}}{2}. \tag{2.62}$$

Now, let us consider the expressions of the formulas (2.61) and (2.62) for the even values of the index n=2k. For this purpose, recall once again the following wonderful property of Fibonacci numbers: for the even values of n the Fibonacci numbers F_{2k} and F_{-2k} are equal by absolute value and opposite by sign, that is, $F_{-2k}=-F_{2k}$ and likewise for the Lucas numbers L_{2k} and L_{-2k}, that is, $L_{-2k}=L_{2k}$. Then for the case of n=2k the (2.61) and (2.62) take the following form:

$$\tau^{2k} + \tau^{-2k} = L_{2k} \tag{2.63}$$

$$\tau^{2k} - \tau^{-2k} = F_{2k}\sqrt{5}. \tag{2.64}$$

For the odd n=2k+1 we have the following relations for the "extended" Fibonacci and Lucas numbers: $F_{-2k-1}=-F_{2k+1}$ and $L_{-2k-1}=L_{2k+1}$. Then, for this case the formulas (2.63) and (2.64) take the following form:

$$\tau^{2k+1} + \tau^{-(2k+1)} = F_{2k}\sqrt{5} \tag{2.65}$$

$$\tau^{2k+1} - \tau^{-(2k+1)} = L_{2k+1}. \tag{2.66}$$

We can now represent the formulas (2.65) and (2.66) in the following compact forms:

$$L_n = \begin{cases} \tau^n + \tau^{-n} \text{ for } n = 2k \\ \tau^n - \tau^{-n} \text{ for } n = 2k+1 \end{cases} \tag{2.67}$$

$$F_n = \begin{cases} \dfrac{\tau^n + \tau^{-n}}{\sqrt{5}} \text{ for } n = 2k+1 \\ \dfrac{\tau^n - \tau^{-n}}{\sqrt{5}} \text{ for } n = 2k \end{cases}. \tag{2.68}$$

The analysis of the formulas (2.67) and (2.68) provide us with "aesthetic pleasure" and once again we are convinced of the power of the human mind! We know that the Fibonacci and Lucas numbers are always integers. On the other hand, any degree of the golden mean is an irrational number. It follows from this that the integer numbers L_n and F_n can be represented with the help of the formulas (2.67) and (2.68) by the special irrational number, the golden mean!

For example, according to (2.67) and (2.68) we can represent the Lucas number 3(n=2) and the Fibonacci number 5(n=5) as follows:

$$3 = \left(\frac{1+\sqrt{5}}{2}\right)^2 + \left(\frac{1+\sqrt{5}}{2}\right)^{-2}, \tag{2.69}$$

$$5 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^5 + \left(\frac{1+\sqrt{5}}{2}\right)^{-5}}{\sqrt{5}}. \qquad (2.70)$$

It is easily proven that the identity (2.69) is valid because according to (2.60) we have:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{L_2 + F_2\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2} \text{ and } \left(\frac{1+\sqrt{5}}{2}\right)^{-2} = \frac{L_{-2} + F_{-2}\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}.$$

If now we substitute these expressions to the right of the formula (2.69), then the right-hand side of (2.69) can be represented as follows:

$$\frac{3+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2}. \qquad (2.71)$$

By adding the items of (2.71), we can see that all "irrationalities" in (2.71) are mutually eliminated and we obtain the number 3 from the sum, that is, the identity (2.69) is valid.

We can be convinced in the validity of the identity (2.70), if we remember that according to (2.60) we have the following representations:

$$\left(\frac{1+\sqrt{5}}{2}\right)^5 = \frac{L_5 + F_5\sqrt{5}}{2} = \frac{11+5\sqrt{5}}{2}$$

and $$\left(\frac{1+\sqrt{5}}{2}\right)^{-5} = \frac{L_{-5} + F_{-5}\sqrt{5}}{2} = \frac{-11+5\sqrt{5}}{2}.$$

If we substitute these expressions into (2.70), then we obtain the following expression for the right-hand side of (2.70):

$$\frac{10\sqrt{5}}{2\sqrt{5}} = 5,$$

whence follows the validity of the identity (2.70).

Note that this reasoning has a general character, that is, for any Lucas or Fibonacci numbers that are given by formulas (2.67) and (2.68), all "irrationals" in the right-hand parts of (2.67) and (2.68) are always mutually eliminated and we obtain integers as the outcome!

2.8.3. *A Historical Analogy*

The situation of the mutual elimination of all "irrationals" in the formulas (2.67) and (2.68) reminds one of the situations that appeared in mathematics with the introduction of complex numbers. In the 16th century, Italian mathematicians contributed significantly to the development of algebra: they solved radicals with equations of the 3rd and 4th degrees. Cardano's famous book, *The Great Art* (1545), contains the algebraic solution of the cubic equation:

$$x^3 + px + q = 0 \tag{2.72}$$

according to the formula $x = u + v$, where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}; \; v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}; \; uv = -\frac{p}{3}.$$

It was proven algebraically that there are three roots of the algebraic equation (2.72), namely:

1. For the case $\Delta = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} > 0$, Eq. (2.72) has one real root and two complex conjugate roots; for example, the equation $x^3+15x+124=0$ with $\Delta>0$ has the following roots: $x_1=-4$; $x_{2,3} = 2 \pm 3i\sqrt{3}$.
2. For the case $\Delta=0$, $p\neq 0$, $q\neq 0$, Eq. (2.72) has three real roots; for example, the roots of the equation $x^3-12x+16=0$ are: $x_1=-4$, $x_{2,3}=2$.
3. For $\Delta<0$ we have the most interesting instance, the so-called "non-reducible" case, where we need to extract the root of the 3rd degree from complex numbers and the cubic roots u and v are complex numbers. Nevertheless, in this case the equation (2.72) has real roots. For example, the equation $x^3-21x+20=0$ with

$$\Delta = -243, \; u = \sqrt[3]{-10 + \sqrt{-243}}, \; v = \sqrt[3]{-10 - \sqrt{-243}} \tag{2.73}$$

has real roots 1, 4 and -5 !

This fact seemed paradoxical to the 16th century mathematicians! Really, all factors of the equation $x^3-21x+20=0$ are real numbers, all its roots are real numbers, however, the intermediate calculations result in the "imaginary," "false," "nonexistent" numbers of the kind (2.73). Mathematicians were in a very difficult situation again (starting with the discovery of irrationals). To completely ignore the numbers of the kind (2.73) would mean to refuse the general formulas for the solution of algebraic equations of the 3rd degree, and other remarkable mathematical achievements. On the other hand, to recognize that these obtrusively appearing "monstrous" numbers such as (2.73) are equivalent with real numbers was inadmissible from the point of view of common sense. For a long time the "monstrous" numbers of the kind (2.73) were not recognized by many mathematicians. For example, Descartes considered that the complex numbers do not have any real interpretation and are doomed forever to remain only "imaginary" numbers (the name "imaginary numbers" came into mathematics in 17th century after Descartes). Many eminent mathematicians, in particular, Newton and Leibniz, adhered to the same opinion.

17th Century English mathematician Vallis in the book *Algebra: Historical and Practical Treatise* (1685) pointed out a possible geometric interpretation of complex numbers. Ultimately, the complex numbers came into use in 18th century after the works of the French mathematician Moivre (1667-1754) who introduced the following well-known formula:

$$(\cos\varphi \pm i\sin\varphi)n = \cos n\varphi \pm \sin n\varphi. \tag{2.74}$$

After the introduction of Moivre's formula (2.74), a representation of the complex numbers in trigonometric form came into use, which facilitated a solution of numerious mathematical problems. However, the famous *Euler's Formulas* became the "moment of celebration" for complex numbers. By using Moivre's formula (2.74), Euler proved the following formulas for trigonometric functions:

$$\cos x = \frac{e^{xi} + e^{-xi}}{2},\ \sin x = \frac{e^{xi} - e^{-xi}}{2i}. \tag{2.75}$$

Note that finding the connection between trigonometric and exponential functions expressed by Euler's formulas emphasize a fundamental connection between the numbers π and e, two numerical constants of mathematics, that play an important role in mathematics, in particular, in the development of the complex number concept.

By returning back to Binet formulas (2.67) and (2.68) and taking into consideration our reasoning concerning the complex numbers, we may suppose that Binet formulas touch upon some rather deep number-theoretical problems that are at the intersection of integers (Fibonacci and Lucas numbers) and irrationals (the golden mean). Further, in Chapter 9 we will try to broaden this idea by considering the number systems with irrational bases, which may overturn our ideas about number systems.

2.9. Fibonacci Rectangle and Fibonacci Spiral

2.9.1. *Fibonacci Rectangles*

Let us consider the Fibonacci series: 1, 1, 2, 3, 5, 8, 13, 21, ... Take two squares with all of the sides equal to 1 (the area of each square is equal to 1) and put them together. As a result we get the 2×1 or "double square" rectangle. Then, construct a new square of the size 2×2 on the longer side of the "double square." Here we obtain a new rectangle of size 3×2. Then, construct a new square of size 3×3 on the longer side of the preceding rectangle; as a result, we obtain a new rectangle by the size 5×3. Continuing this process, results in rectangles, with side equal to adjacent Fibonacci numbers, that is, the rectangles with the following sizes: 8×5, 13×8, 21×13, and so on (Fig. 2.4).

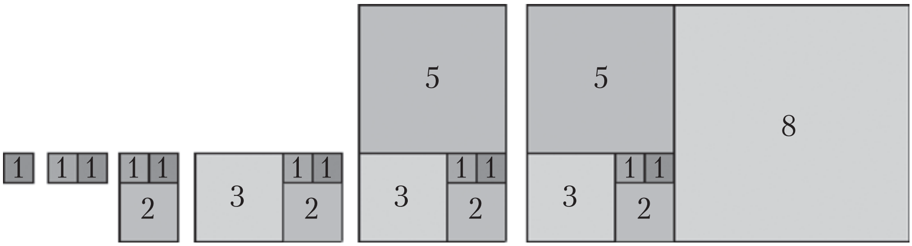


Figure 2.4. Fibonacci rectangles

Such rectangles are called *Fibonacci Rectangles*. Consider a sequence of the ratios of the Fibonacci rectangles sides: 1:1, 2:1, 3:2, 5:3, 8:5, 13:8, 21:13, 34:21, ... As these ratios aim for the golden mean, we can conclude that the Fibonacci rectangles aim for the golden rectangle, that is, the Fibonacci rectangles are sequential approximations of the golden rectangle.

2.9.2. *Fibonacci Spiral*

Fibonacci rectangles in Fig. 2.4 consist of the squares with sides 1, 2, 3, 5, 8, 13, ... Now, in each square we draw an arc that is equal to a quarter of a circle as shown in Fig. 2.5. If we connect these arcs, we obtain a curve that is suggestive of a spiral by its form (Fig. 2.5). Strictly speaking, this curve is not a spiral from the mathematical point of view. However, it is a very good approximation of the golden spirals that are widely met in nature. The curve in Fig. 2.5 we call a *Fibonacci Spiral*.

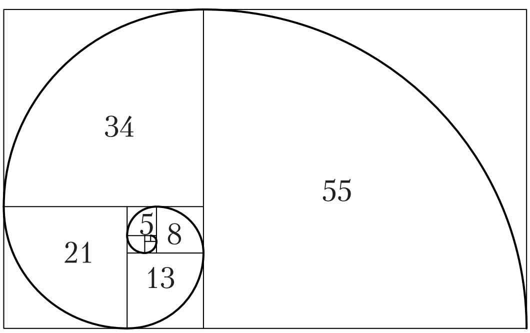
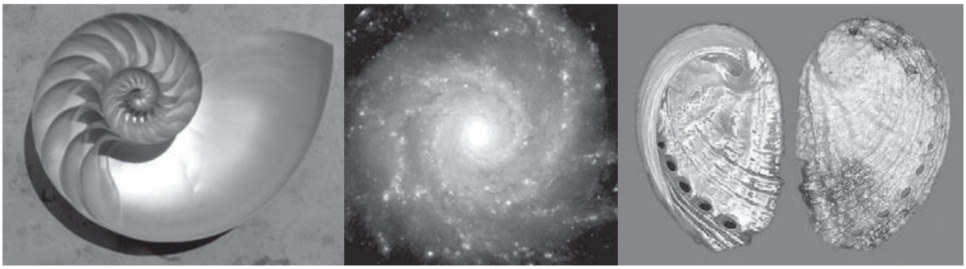


Figure 2.5. Fibonacci spiral

2.9.3. *Fibonacci Spirals in Nature*

The great poet and scientist Goethe considered a spiral form as one of the characteristic attributes of all living organisms, as a manifestation of the most secret essence of life. Short moustaches of plants and horns of rams are twisted by a spiral, a growth of tissues in trunks of trees is carried out by a spiral, sunflower seeds are disposed by a spiral, and so on. Everyone admires the forms of shells that are constructed under the spiral law. Let us begin from the spiral form of the *Nautilus* (Fig. 2.6-a). If we compare its shape with the Fibonacci spiral (Fig. 2.5), we can conclude that the nautilus shell is constructed according to the Fibonacci spiral principle. The motives of the Fibonacci spiral can be found in the shape of the Galaxy (Fig. 2.6-b) and other sea shells (Fig. 2.6-c).



a) b) c)

Figure 2.6. Fibonacci spirals in Nature

2.10. Chemistry by Fibonacci

2.10.1. *Law of Multiple Ratios*

There is an opinion that the accumulated sum of knowledge in some discipline can be called science only when it passes into a precise, quantitative analysis beyond the merely qualitative perspective. To the end of the 18th century, chemistry accumulated a significant volume of knowledge and chemists had learned to decompose many complex substances into simple ones and to build the complex substances from the simple ones. During this process the problem of quantitatively describing various chemical compounds composing complex compounds appeared. It was necessary for the creation of a graceful theory of chemical structures, which would meet the requirements for the production practice of various chemical products.

The *Law of Constant Proportions of Chemical Compounds* is one of the fundamental chemical laws. This law came into chemical science after research by French scientist Prust (1754-1826). By studying chemical compounds, in particular the oxides of metals, he came to the conclusion that the chemical compounds have a strictly constant structure that is not dependent on the conditions of their formation. Thanks to the works of the English scientist Dalton (1766-1844), the atomic doctrine came into chemistry and the *Law of Simple Multiple Ratios* was formulated. According to this law, simple integer-valued ratios between atoms in chemical structures exist. Now students know that the structure of water is described by the formula H_2O, of common salt $NaCl$, and zinc oxide ZnO. Chemistry became a precise science. A new branch of chemistry arose that studies the ratios of atoms in chemical structures called *Stoichiometry*.

The discovery of the law of multiple ratios is one of the remarkable achievements of chemistry as a science: a beautiful simply-ordered system of chemical compounds appeared from the chaos of atomic representations. Atoms of different elements can form different combinations that are connected by the forces of chemical connection. However, only some of them are stable; the other chemical combinations perish by disintegrating into more stable compounds. Only those combinations of atoms of different elements will be steady, if they correspond to simple integer-valued ratios of their components. This idea is surprisingly simple and clear and corresponds completely to the Pythagorean doctrine about the dominating role of integer numbers in the organization of the Universe!

However, such formulation of the main chemical law evokes some bewilderment. It is not clear as to what the "simple integer-valued ratios" of atoms in the formulas of chemical compounds really means. While we studied rather simple chemical compounds, atomic ratios in them usually corresponded to "small" numbers, for example, H_2O, Al_2O_3, Fe_3O_4, As_2O_5. However, the range of the studied chemical compounds started to broaden rapidly. Formulas of chemical compounds appeared there with the stoichiometric factors of 7, 9, 15, 21, etc. When the chemists started studying the structure of the organic compounds, it became inconvenient to speak about the "simple integer-valued ratios." The chemical structure of the bacteriophage is a peculiar champion in stoichiometry because it has the following formula: $C_{5750}H_{7227}N_{2215}O_{4131}S_{590}$. What do the ratios of the "small" integers mean in this formula? Here we see four-digit numbers!

Thus, not all is so simple in stoichiometric laws: simplicity here is combined with complexity, and the question about the possible ratios of atoms in compounds remains open.

2.10.2. *Research by the Ukrainian Scientist Nikolai Vasyutinsky*

We will not delve too deeply into the chemistry of different compounds. We are interested in only one problem — whether Fibonacci numbers are found in the formulas of chemical compounds. The Ukrainian chemist Nikolai Vasyutinsky attempted to give the answer to this question in his book [31].

By analyzing oxides of uranium and chromium, Vasyutinsky found that these chemical compounds are based on Fibonacci numbers. With the oxides of uranium, the structure of the generating oxides is changing not continuously, but spasmodically, from one steady compound with a certain integer-valued ratio of atoms into another one. Between the uranium oxides UO_2 and UO_3 a lot of the intermediate compounds are forming; their structures are described by the formulas U_2O_5, U_3O_8, U_5O_{13}, U_8O_{21}, $U_{13}O_{34}$. We can see that the atomic ratios in these compounds are equal to the ratios of Fibonacci numbers: 2:5, 3:8, 5:13, 8:21, 13:34. It is easy to prove that such ratios aim in the limit for the quadrate of the golden mean! Each of the described uranium oxides can be represented as the sum of two oxides UO_2 and UO_3 taken in the different proportions, for example: $U_5O_{13}=3UO_3+2UO_2$, $U_8O_{21}=5UO_3+3UO_2$. Here the factors of the oxides UO_2 and UO_3 correspond to the adjacent Fibonacci numbers! This means that the structures of the above-considered uranium oxides are subordinated completely to the Fibonacci number regularity. Note that according to Vasyutinsky's research the chromium oxides Cr_2O_5, Cr_3O_8, Cr_5O_{13} have structures that are described by Fibonacci numbers.

By considering the equalities of the type $U_5O_{13}=3UO_3+2UO_2$, we can find their similarity to the algebraic golden mean equation of the 4th degree $x^4=3x+2$ that is used for the description of the butadiene structure. By comparing the equality $U_8O_{21}=5UO_3+3UO_2$ with the algebraic golden mean equation of the 5th degree $x^5=5x+3$, we can see that they also have a similar mathematical structure. Perhaps these analogies can be the beginning of rather interesting research into the field of stoichiometry.

It is generally accepted procedure to determine the structures of the chemical compounds by the ratio of atoms of the elements that is included in this compound. However, it is possible to estimate the chemical compounds by the ratios of the atoms (ions) of different elements to the mobile valence electrons that are responsible for the formation of chemical connections between atoms. So, for example, in the chromium oxide Cr_2O_5 the 10 valence electrons correspond to the 7 atoms of the chromium and oxygen. Making similar calculations for all the above oxides, we obtain the following ratios of the sums of the atoms to the sums of the valence electrons: 10:7, 16:11, 26:18, 42:29, 68:47. Note that the numerators of these fractions are double Fibonacci numbers ($10=2\times5$, $16=2\times8$, $42=2\times21$, $68=2\times34$) and the denominators are the Lucas numbers 7, 11, 18, 29, 47. If we decreased sequentially the numerators and the denominators of these fractions on the Fibonacci numbers that correspond to the metal atom quantity in compounds, that is, on 2, 3, 5, 8, 13, we obtain the series of ratios of adjacent Fibonacci numbers: 8:5, 13:8, 21:13, 34:21, 55:34; in the limit these ratios aim for the golden mean.

Thus, the Ukrainian chemist Vasyutinsky convincingly demonstrated that chemical compounds organized by Fibonacci Numbers do exist.

2.11. Symmetry of Nature and the Nature of Symmetry

2.11.1. *Basic Concepts of Symmetry*

Symmetry is one of the most fundamental scientific concepts that among the concept of Harmony has a relation to practically all branches of Nature, Science and Art. The outstanding mathematician Hermann Weil evaluated the role of symmetry in modern science in the following words:

"Symmetry, as though wide or narrow we did not understand this word, there is an idea, with the help of which a person attempts to explain and create an order, beauty and perfection."

What is symmetry? When we look at a mirror we can see us in its reflection; this is an example of mirror symmetry. The mirror reflection is an example of the so-called *Orthogonal Transformation* that changes an orientation. In the general case *Symmetry* in mathematics is perceived as a transformation of space (plane), when each point M of space (plane) turns into the other point M' with respect to some plane (or a straight line) a; here, the line segment MM is perpendicular to a plane (or a straight line) a and is divided by it in half. The plane (or the straight line) a is called *Plane (or Axis) of Symmetry*.

Plane of Symmetry, *Symmetry Axis* and *Center of Symmetry* are fundamental concepts of symmetry. A plane of symmetry P is a plane that divides the figure into two mirror-symmetrical parts that are disposed one to another as some subject and its mirror reflection. For example, the isosceles triangle ABC shown in Fig. 2.7 at

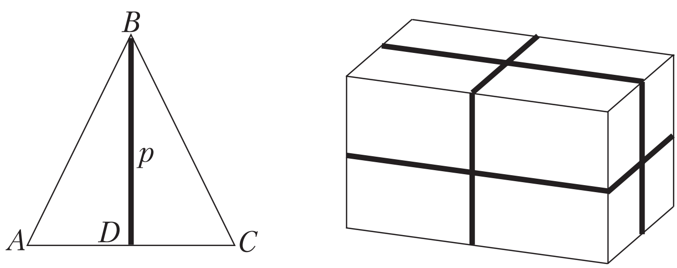


Figure 2.7. A symmetry of triangle and rectangular parallelepiped

the left is divided by the altitude BD into two mirror-symmetrical halves ABD and BCD; thus, the altitude BD is the "track" of the plane of symmetry P that is perpendicular to the plane of the triangle. We say that the isosceles triangle ABC has only the plane of symmetry P. In Fig. 2.7 at the right a rectangular parallelepiped (matchbox) is shown; it has three orthogonal planes of symmetry $3P$. It is easy to show, that a cube has nine planes of symmetry — $9P$. These examples could be continued.

A symmetry axis L is called a straight line, around which a symmetrical figure can be turned several times in such a manner that each time the figure coincides with itself in space. For example, the equilateral triangle has the symmetry axis L_3 that is perpendicular to the triangle plane in the center of the triangle. It is clear that there are three ways of turning a triangle around the symmetry axis L_3 when the triangle coincides with itself. It is clear, that a square has the symmetry axis L_4 and the pentagon L_5. The cone also has a symmetry axis; there are an infinite number of the turns of the cone around the symmetry axis. This means that the cone has the symmetry axis of the type L_∞.

Finally, a symmetry center C of a figure is constructed inside the figure when any straight line, drawn through the point C, meets the identical points on the figure at equal distances from the center C. A sphere is a perfect example of a figure with center symmetry.

2.11.2. *Symmetry of Crystals*

For many centuries, the geometry of crystals seemed a mysterious and unsolvable riddle. In 1619, the Great German mathematician and astronomer *Johannes Kepler* (1571-1630) paid close attention to the six-fold symmetry of snowflakes. He tried to explain it by the fact that their crystals are constructed from small identical balls that are connected one to another. Kepler's idea was developed by the English polymath *Robert Hooke* (1635-1703) and the Russian polymath *Mikhail Lomonosov* (1711-1765), who made important contributions to Russian literature, education and science. They also assumed that it is possible to liken the elementary particles inside crystals to densely packed balls. Presently the *Principle of Dense Spherical Packing* underlies structural crystallography, only the spherical particles of the ancient authors are replaced now by atoms and ions.

Fifty years after Kepler, the Danish geologist and crystallographer *Nicolas Stenon* (1638-1686) for the first time formulated the main idea of crystal growth: "Crystal growth is implemented not from within, similar to plants, but by means of the superposition on the external planes of the crystal's smallest particles, which are brought from outside by certain liquids." This idea about crystal growth as the outcome of the sediment forming new stratums of substance on crystal faces preserves its significance until now. It was Nicolas Stenon who discovered the main law of geometric crystallography — the law of *Constancy of Interface Angles.*

The law of Constancy of Interface Angles became a reliable basis for the development of geometric crystallography and provided the richest material for verification of the true symmetry of crystallographic structures. The French researcher *Rene Just Hauy* assumed that the crystals consist not of the small balls (according to the assumption of Kepler, Hooke and Lomonosov), but of the molecules of parallelepiped shape, and these molecules are the extreme small splinters of the same shape. In other words, the crystals are peculiar "masonries" forming molecular "bricks." Despite all the naivety of this theory from the modern point of view, this idea did play an important role in the history of crystallography and gave an impetus to the origin of the theory of crystal lattice structures.

The French crystallographer Brave was Hauy's direct follower. He replaced the molecular "bricks" by points, the centers of molecular weight. As a result of this approach, the spatial lattices of crystals were obtained. By developing a hypothesis about the lattice structure of all crystals, Brave created the fundamentals of modern structural crystallography long before the experimental research of crystal structures with the help of X-rays.

We can see in Fig. 2.8 the spatial crystal's lattices of the common salt *NaCl* (*a*) and the calcium oxide *CaO* (*b*). In all crystals, we can see the set of the identical atoms located like the points of a spatial lattice. The straight lines, on which atoms in the lattice are located, are named *rows*, and the planes, filled by atoms, are named *planar lattices*.

A presentation about the lattice structure of crystals allowed us to give a general definition of crystals. It is accepted that crystals are solid bodies, which consist of particles (atoms, ions, molecules), located the strictly regular knot-like spatial lattice. This general definition can be applied to all types of Nature's crystals.

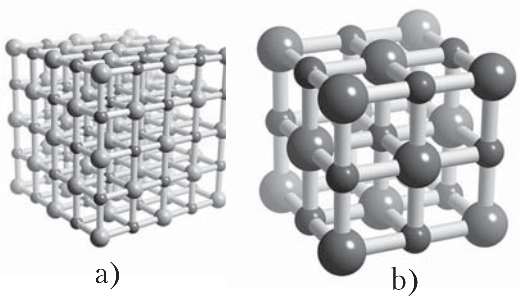


Figure 2.8. Spatial crystal lattices of *NaCl* (*a*) and *CaO* (*b*)

However, already in 1830, long before Brave, the German professor *Johannes Fridrich Hessel* (1796-1872) published the long article *Crystallometry*, where he developed an approach to crystal geometry based on the symmetry concept. In this article, Hessel gave a full research report of the set of symmetry elements of crystals and spatial geometric figures. Only in 1890, that is, in 60 years after publication of this article, and in 18 years after the death of its author, crystallographers could evaluate the significance of Hessel's outstanding discovery.

In 1890, the eminent Russian crystallographer Fedorov proved the existence of the 230 sets of symmetry elements for all existing types of crystals. Fedorov's symmetry groups corresponding to the geometric laws of atom location in crystal structure underlies modern structural crystallography.

A presentation about the lattice structure of crystals, in particular, a concept of "planar lattice," on which the knots of the crystal lattice are located, resulted in the discovery of the *Major Law of Crystallographic Symmetry*. According to this law of crystal, symmetry axes the first, second, third, fourth and sixth orders are allowed. However, it is impossible for crystals to have the fifth order symmetry or anything greater than the sixth order of symmetry.

According to the main crystallography law, there is a fundamental difference between the symmetry of the mineral world and the symmetry of the living world, where five-fold symmetry is widely used. For crystals, the five-fold axis of symmetry and the symmetry axes greater than the 6th order are prohibited: this strict rule of classical crystallography existed until 1982, when the Israeli physicist Dan Shechtman proclaimed the startling discovery of quasi-crystals.

2.11.3. *Symmetry Laws in Nature*

The concept of symmetry is used widely in physics. If the laws that determine relations between physical magnitudes and a change of these magnitudes in the course of time do not vary at the definite operations (transformations), they say, that these laws have symmetry (or they are invariant) with respect to the given transformations. For example, the law of gravitation is valid for any points of space, that is, this law is invariant with respect to the system of coordinates.

In the opinion of outstanding Russian scientist and academician Vernadsky, "symmetry encompasses properties of all fields of physics and chemistry."

The Pythagoreans paid very close attention to the phenomenon of symmetry in Nature that was connected with their Harmony doctrine. The two kinds of symmetry, *Mirror* and *Radial*, are widespread throughout Nature. A butterfly, a leaf, and a beetle (Fig. 2.9-a) have "mirror" symmetry

Figure 2.9. Natural forms with "bilateral" (a) and "radial" (b) symmetries

and often such type of symmetry is called *Leaf Symmetry* or *Bilateral Symmetry*. A mushroom, a chamomile, and a pine tree (Fig. 2.9-b) have a "radial" symmetry, and often such type of symmetry is called a *Chamomile-Mushroom Symmetry*.

Already in the 19th century the researchers in this area came to the conclusion that symmetry of natural forms largely depend upon the influence of the Earth's gravitational forces that have the symmetry of a cone in each point. In the outcome, the following law was found:

"Everything that grows or moves in a vertical direction, that is, upwards or down relative to the Earth's surface is subordinated to the "radial" ("chamomile-mushroom") symmetry. Everything that grows and moves horizontally or with an inclination relative to the Earth's surface is subordinated to the "bilateral" or "leaf" symmetry."

2.11.4. *Symmetry Laws in Art*

The principle of symmetry is used widely in Art. The curbs in architectural and sculptural works, the ornamental designs in the applied art are examples of the application of the laws of symmetry.

Symmetry together with the golden mean principle is often used in works of art. Raphael's picture *The Engagement of Virgin Mary* (Fig. 2.10) is just such an example.

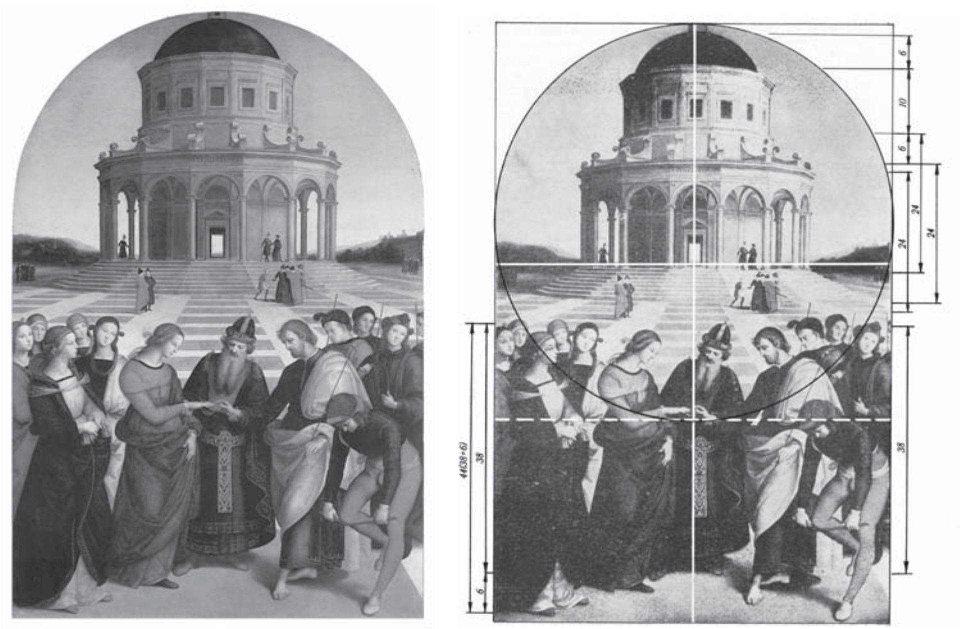


Figure 2.10. Raphael's picture "*The Engagement of Virgin Mary*" and its harmonic analysis

2.12. Omnipresent Phyllotaxis

2.12.1. *A Helical Symmetry*

Everything in Nature appears subordinate to stringent mathematical laws. For example, leaf disposition on plant stems also has stringent mathematical regularity, a phenomenon called *Phyllotaxis* in botany. The essence of phyllotaxis consists of a spiral disposition of leaves on plant stems, trees, petals in flower baskets, seeds in a pinecone and those of a sunflower head, etc. This phenomenon was already known to Kepler and was a subject of discussion of many scientists throughout the years, including Leonardo da Vinci, Turing, Veil, and so on. Much more composite concepts of symmetry are used in the phenomenon of phyllotaxis, in particular, the concept of *Helical Symmetry*. Let us examine, for example, a disposition of leaves on the plant stem (Fig. 2.11). We can see in Fig. 2.11 that the leaves are on different heights on the stem along the helical curve that encircles the stem. To move up from the lower leaf to the higher one, it is necessary virtually to turn the leaf at some angle around the vertical axis and then to raise the leaf up a definite distance. This transformation is the essence of helical symmetry.

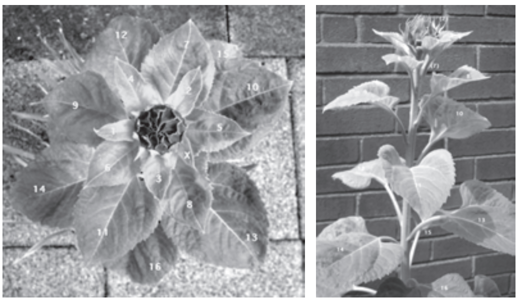


Figure 2.11. A helical symmetry

Let us examine typical helical axes that can appear along the plant stems (Fig. 2.12). In Fig. 2.12-a we can see the stem of a plant with helical symmetry axis of the 3rd order. We can unite all leaves of the plant by any virtual line. We start from leaf 1. To move up from leaf 1 to leaf 2, it is necessary to turn leaf 1 around the stem axis 120° counterclockwise (if looking from below) and then move up leaf 1 along the stem in the vertical direction until it coincides with leaf 2. If

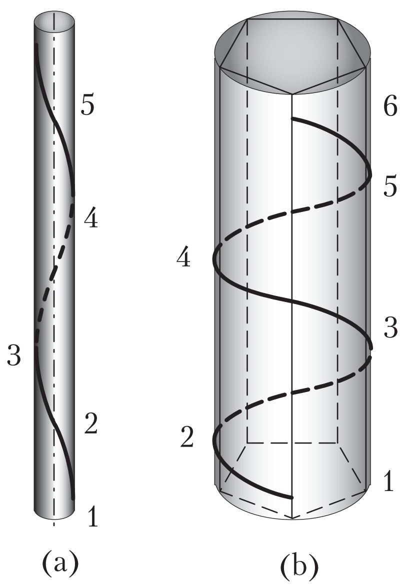


Figure 2.12. The helical axis on stems of plants

we repeat a similar operation, we can move up from leaf 2 to leaf 3 and then on to leaf 4. It is necessary to note that leaf 4 is on the same line with respect to leaf 1 (as though it repeats on the higher level). It is clear if we want to move up from leaf 1 to leaf 4 we have to make three turns of 120° i.e. we have to carry out a full rotation around the stem axis (120°×3=360°).

Botanists called the rotation angle of the helical axis the *Leaf's Divergence Angle*. A vertical straight line that is parallel to the stem of the plant connecting two leaves is called an *Ortho-line*. The line segment of leaf 1 to leaf 4 of the ortho-line corresponds to a full transformation of the helical axis. We will see further that the number of turns around the stem axis to move up from the bottom leaf to the next leaf directly above it along the ortho-line, can be equal not only to 1, but also to 2, 3, and so. This number of turns is called the *Leaf's Cycle*. In botany, it is acceptable to characterize helical leaf location by the ratio m/n where the numerator m is a number of turns in the leaf's cycle, and the denominator n is the number of leaves in this cycle. In the above case, we have a helical axis of 1/3.

In Fig. 2.12-*b* the helical axis of five-fold symmetry with a leaf cycle 2 is presented; this means that for the transition from leaf 1 to leaf 6 it is necessary to make two full turns. The fraction 2/5 characterizes the given helical axis; the leaf divergence angle is equal to 144° that is, we have: 360/5=72°; 72°×2=144°. Note that there are also more complex helical axes, for example, of the kind 3/8, 5/13, and so forth.

There is a question as to what values can take the numbers m and n that describe the helical axis of the kind m/n. And here Nature gives us the following surprise called the *Law of Phyllotaxis*.

Botanists assert that the fractions that describe the helical axes of plants build up a strict mathematical sequence of the following kind, for example:

$$1/2, 1/3, 2/5, 3/8, 5/13, 8/21, 13/34, ... \tag{2.76}$$

Note that the fractions in the sequence (2.76) are the ratios of two adjacent Fibonacci numbers beginning with number one. It is easy to prove, that the sequence (2.76) aims for the number τ^{-2}=0.382, that is, to the inverse of the square of the golden mean.

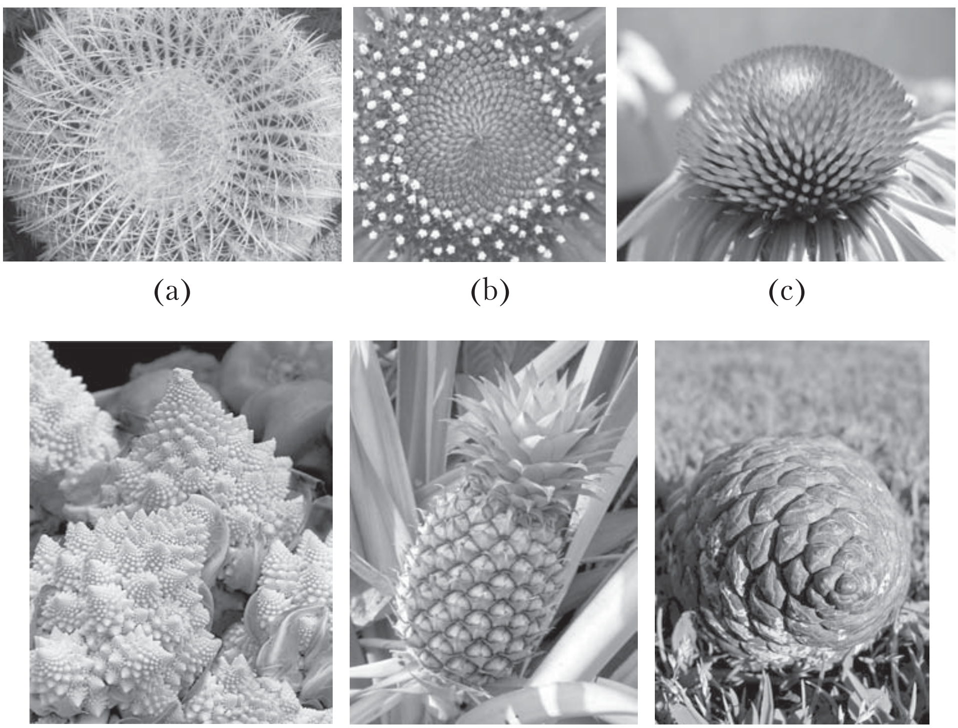
Botanists found that different plants have different ratios of the above sequence (2.76). For example, the fraction 1/2 is peculiar to cereals, birch and grapes; 1/3 to sedge, tulip, and alder; 2/5 to pear, currant, and plum; 3/8 to cabbage, radish, and flax; 5/13 to fir-tree and jasmine, etc.

What is the "physical" cause that underlies the phyllotaxis law (2.76)? The answer is very simple. It proves to be the disposition of leaves that allow for the maximum inflow of solar energy to the plant.

2.12.2. *Densely Packed Phyllotaxis Structures*

The phyllotaxis phenomenon shows itself in inflorescences and densely packed botanical structures, such as, pinecones, pineapples, cacti, sunflowers, cauliflowers and many other structures.

Figure 2.13 gives examples of phyllotaxis objects (cactus, sunflower, coneflower, Romanescue cauliflower, pineapple, pinecone), in which the phyllotaxis law is based on the numerical sequence (2.30) that consists of the ratios of adjacent Fibonacci numbers. This means that the seeds or small parts on the surface of such botanical objects are located at the crossings of left-hand and right-hand spirals; here the ratio of the numbers of left-hand and right-hand spirals is always equal to the ratio of adjacent Fibonacci numbers (2.30). The same regularity is observed in baskets of flowers.



(a) (b) (c)

(d) (e) (f)

Figure 2.13. Phyllotaxis structures: (a) cactus; (b) head of sunflower; (c) coneflower; (d) Romanescue cauliflower; (e) pineapple; (f) pine cone

Geometric models of phyllotaxis structures in Fig. 2.14 provide a clearer representation of this unique botanical phenomenon.

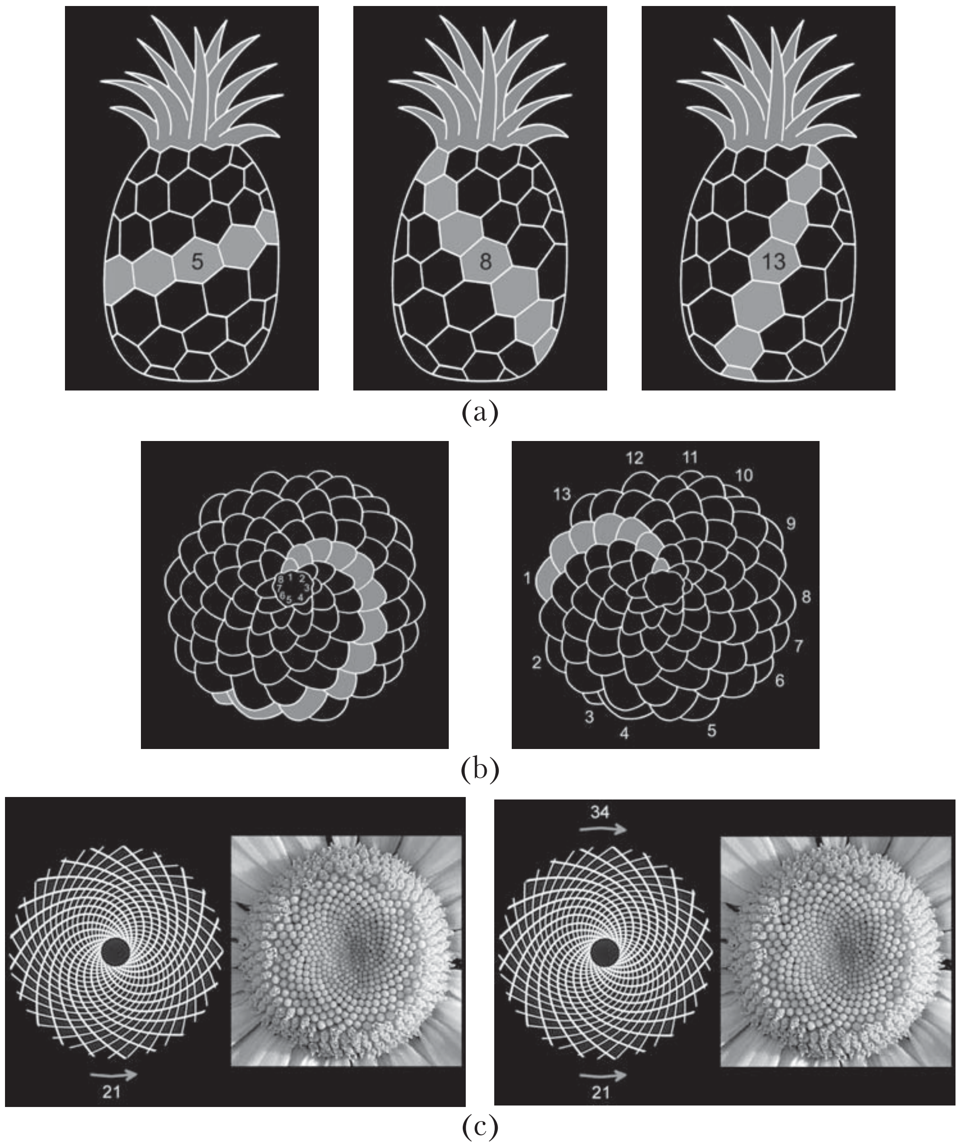


Figure 2.14. Geometric models of phyllotaxis structures:
(a) pineapple; (b) pinecone; (c) shasta daisy

Thus, we find strict mathematics in the dispositions of leaves on plant stalks, flower petals, in the cross-section of an apple (pentacle), in the spiral dispositions of seeds of a pinecone, pineapple, cactus and the head of a sunflower. And this mathematical law is expressed by Fibonacci numbers and therefore the golden mean! Once again, it appears as though Nature is subordinate to a coherent plan, the uniform *Law of the Golden Section*! To discover and explain this fundamental law of Nature in all its expressions is one of the most important problems of science and philosophy.

2.12.3. *Use of Phyllotaxis Lattices in Painting*

In the above we mentioned a wide-ranging use of the golden rectangle and other golden geometrical figures in artwork. One art method used by artists of the Renaissance was the use of phyllotaxis grids in painting. This is not unlike the botanical phyllotaxis phenomenon in which Nature appears to design pinecones, pineapples, cacti, sunflower heads, and many other botanical structures.

According to phyllotaxis law, cactus spines are located along Fibonacci spirals; here the adjacent Fibonacci numbers 21 and 34 are the numbers of the left-hand and right-hand spirals. If we unroll on a plane the spines of a cactus, we can obtain a raster grid (Fig. 2.15). In the raster grid in Fig. 2.15 the inclined lines with right-hand and left-hand inclination represent the principle of disposition of the spines on the cactus surface. This raster grid has 21 lines with right-hand inclination and 34 lines with left-hand inclination. The net of the lines in Fig. 2.15, *Phyllotaxis Raster Grid*, from an aesthetic point of view looks as if it is a golden rectangle. Many artists of the Renaissance used the phyllotaxis raster grid in their art work.

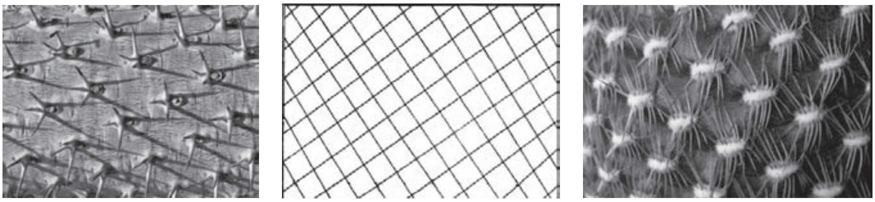


Figure 2.15. Phyllotaxis raster grid

Paturi, the Austrian researcher and author of the remarkable book, Plants as Ingenious Engineers of Nature [140], made an analysis of the use of phyllotaxis raster grids in the artworks of great artists. For this purpose, he put together the phyllotaxis raster grid with Titian's picture Bacchus and Ariadne (Fig. 2.16).

After an analysis of Titian's painting Paturi concluded:

"All basic lines of the perspective coincide with the raster grid. Even the set of the minor parts and forms were placed by the artist in that field of internal constraints, on which the picture is constructed. Pay attention to the line along the small hill seen on the horizon near to the church campanile, to the branches of the large tree, to the outline of the cloud lying under the constellation, to the hind paws and the belly line of the large wild cat,

Figure 2.16.
Titian's *Bacchus and Ariadne*

along the direction of the axis of the overturned vase, to the raised right hand of the satyr in the garland of grapevines in the right corner of the canvas and, at last, to the raised leg of the horse."

Paturi further concluded:

"At all times the artists consciously or unconsciously comprehended the laws of aesthetic perception by watching nature. Artists were always enchanted by the simple and simultaneously rational geometry of the growth of biological forms."

2.13. "Fibonacci Resonances" of the Genetic Code

2.13.1. *The Initial Data about the Genetic Code*

Among the biological concepts [141] that are well formalized and have a level of general scientific significance, the genetic code takes special precedence. Discovery of the striking simplicity of the basic principles of the genetic code places it amongst the major modern discoveries of mankind. This simplicity consists of the fact that inheritable information is encoded in the texts from three-lettered words — triplets or codonums compounded on the basis of the alphabet that consists of the four characters or nitrogen bases: *A* (Adenine), *C* (Cytosine), *G* (Guanine), *T* (Thiamine). The given system of the genetic information represents a unique and boundless set of diverse living organisms and is called the *Genetic Code*.

2.13.2. *DNA SUPRA-code (Jean-Claude Perez's Discovery)*

In 1990 Jean-Claude Perez, an employee of IBM, made a rather unexpected discovery in the field of the genetic code. He discovered the mathematical law that controls the self-organization of bases *A, C, G* and *T* inside the DNA. He found that the consecutive sets of the DNA nucleotides are organized in frames of remote order called "RESONANCES." Here, the resonance means a special proportion that divides the DNA sequence according to Fibonacci numbers (1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 ...).

The key idea of Perez's discovery, called the DNA SUPRA-code, consists of the following. Let us consider some fragment of the genetic code that consists of the *A, C, G* and *T* bases. Suppose that the length of this fragment is equal to some Fibonacci number, for example, 144. If a number of the *T*-bases in the DNA fragment is equal to 55 (Fibonacci number), and a total number of the *C, A* and *G* bases is equal to 89 (Fibonacci number), then this fragment of the genetic code forms is a resonance — that is, a proportion between three adjacent Fibonacci numbers (55:89:144). Here it is permissible to consider any combinations of the bases, that is, *C* against *AGT*, *A* against *TCG*, or *G* against *TCA*. The discovery consists of the fact that the arbitrary DNA-chain forms some set of the resonances. As a rule, the fragments of the genetic code of the length equal to the Fibonacci number F_n are divided into the subset of the *T*-bases, and the subset of the remaining *A, C, G* bases; here the number of *T*-bases is equal to the Fibonacci number F_{n-2} and the total number of the remaining *A, C, G* bases is equal to the Fibonacci number F_{n-1}, where $F_n=F_{n-1}+F_{n-2}$. If we make a systematic study of all the Fibonacci fragments of the genetic code, we can obtain a set of the resonances that is called the *SUPRA-code of DNA*.

2.13.3. *A Verification of Jean-Claude Perez's Law*

In the Petoukhov book [141] the sequences of triplets for the α- and β-chains of insulin are given. For the β-chain this sequence has the following form:

A**T**G-**TT**G-G**T**C-AA**T**-CAG-CAC-C**TT**-**T**G**T**-GG**T**-**T**C**T**-CAC-C**T**C-G**TT**-GAA-GC**T**-**TT**G-**T**AC-C**TT**-G**TT**-**T**GC-GG**T**-GAA-CG**T**-GG**T**-**TT**C-**TT**C-**T**AC-AC**T**-CC**T**-AAG-AC**T**

Note that all the *T*-bases in the indicated sequence are marked by bold type.

A verification of Jean-Claude Perez's Law using the β-chain of the insulin molecule as an example (see above) results in the following outcome. The total number of the triplets in the β-chain is 30, that is, the molecule contains 90 bases (the nearest Fibonacci number is 89). If we count the number of *T*-bases in the above β-chain, we find that there are 34 (a Fibonacci number). Then the number of the remaining *A, C, G* bases is equal to 90-34=56 (the nearest Fibonacci number being 55). Thus, there is the following proportion between the *T*-bases and the rest, i.e. the *A, C, G* bases in the β-chain are 90:56:34. This proportion is very close to the Fibonacci resonance of 89:55:34. It follows from this analysis that Jean-Claude Perez's Law for the insulin β-chain is fulfilled with great accuracy. If now we take the initial segment of the above β-chain that consists of the first 18 triplets, that is, of the 54 bases (the nearest Fibonacci number is 55), and count the number of *T*-bases in this fragment we find that it is 22 (the nearest Fibonacci number being 21). This means that we have the following proportion in the first fragment of the β-chain, namely, 54:32:22 that is close to a Fibonacci resonance of 54:34:21.Thus, the Perez's Law is also fulfilled for the first fragment. If we take the segment that consists of the rest of the 12 triplets (36 bases) then the number of the T-bases in this segment is 12 (the nearest Fibonacci number being 13). Thus, for this case we have a proportion of 34:24:12 that is close to the Fibonacci resonance of 34:21:13. Therefore, both for the β-chain of the insulin molecule as a whole, and for its separate fragments, the Perez Law is fulfilled in practice with sufficient accuracy. In addition, it is possible to see that practically in any segment of the β-chain there is a tendency towards the golden mean.

This surprising discovery of Jean-Claude Perez has a number of interesting applications in the so-called plastic arts and market analysis. Below we can see application of this law in, for example, music, poetry, cinema and market processes (Elliott Waves).

2.14. The Golden Section and Fibonacci Numbers in Music and Cinema

2.14.1. *Pythagorean Theory of Musical Harmony*

Music is a form of art reflecting reality and influencing a person by means of organized sound sequences. Preserving some similarity to natural sounds, musical sounds principally differ from the latter by constrained pitch and rhythmic organization (musical harmony). Since antiquity, the search for the "laws of musical harmony" has been one of the important pursuits of science. Greek musicology does not correspond exactly with the character of modern musicology. Greek musicology was not directed toward analyzing musical works. Rather, the Greeks saw musicology as a study of the acoustic aspects of sound. An essential feature of Greek musicology was the aspiration for a mathematical description of musical harmony.

Pythagoras is credited with the discovery of two fundamental harmonic musical laws:

1. If the ratio of the oscillation frequencies of two tones is describable in small integers, a harmonic sound may result;
2. To get a harmonic triad, it is necessary to add a third tone to the chord, which consists of two consonant tones. The oscillation frequency of the third tone should be in harmonic proportional connection to the two tones of the chord.

Pythagoras' work on the scientific explanation of musical harmony was the first well-grounded scientific theory of musical harmony! Pythagoras attempted to apply his musical theory to cosmology; according to his ideas, the planets of the Solar System are arranged pursuant to the musical octave ("harmony of the spheres").

There are many legends surrounding the acoustic experiments of Pythagoras. The most popular of them describes how Pythagoras, passing by a black smith, heard the sounds of hammer blows on an anvil and found in them an octave, a fifth and a fourth. In his excitement, Pythagoras hastened to the smithy and after a series of experiments with hammers, found that the difference in sounds depended on the weight of the hammers.

The Pythagorean discovery of the deep connection between music and mathematics caused harmonics to be included amongst the mathematical sciences influencing the future development of musical theory.

2.14.2. *Chopin's Etudes in the Lighting of the Golden Section*

Any piece of music has a temporal duration and is divided into separate parts by marks that attract our attention ("expression marks") and simplify our perception of the musical work. These marks are dynamic culmination points within the musical work. The question is whether or not there is certain regularity in the placing of "expression marks" in the musical work. An attempt to answer this question was made by Russian musicologist Sabaneev. He showed [142] that the separate time intervals of the musical works of Chopin are connected, as a rule, by "culmination events" dividing the musical work in the golden ratio. Sabaneev writes:

"All such events are referred by the author's instinct to such points of the musical work that divide the temporal durations into separate parts being in the ratio of the "Golden Section." Observation shows that quite often similar "expression marks" are found at golden points. It is quite surprising, because frequently the knowledge about these things is absent for many composers, and these facts are a consequence of internal feelings of rhythm."

Sabaneev's analysis of a large number of musical works allowed him to conclude that the organization of music is often done so that its cardinal parts, divided by the "marks," are based upon the golden mean series. Such organization of music corresponds to the most economical perception and consequently creates an impression of the best "regularity" within the musical structure. In Sabaneev's opinion, quantity and frequency of golden mean appearances in music depends upon a "composer's expertise." The musical work of great composers is distinguished by the highest percentage of golden mean appearances, that is, "the intuition of the form and regularity, as would be expected, is highest for the first class geniuses."

According to Sabaneev, the golden mean is repeatedly met in the musical works of different composers. Every such golden point reflects the overall musical event, the qualitative jump in the development of the musical theme. In the 1,770 musical works of 42 composers that were studied by him, the golden mean is met 3,275 times. The greatest frequency of musical works based upon the golden mean is found in Beethoven (97%), Haydn (97%), Arensky (95 %), Chopin (92%), Mozart (91%), Schubert (91%), and Scriabin (90 %).

Chopin's 27 etudes were studied by Sabaneev in particular detail. Golden sections were found 154 times in them; the golden section being absent in only three etudes. In some cases, the principles of symmetry and the golden mean are combined simultaneously in the structure of a musical work; in these cases, the musical work is divided into some symmetrical parts; however, the golden section is met within every separate part. For example, Beethoven's music is often divided into two symmetrical parts; however, the golden section is always met inside each part.

2.14.3. *Rosenov's Research*

The Russian art critic Rosenov paid particular attention to research into the harmonic laws of music. He asserted that the stringent proportional relations in music and poetry that are present with the golden mean should play an outstanding role.

Rosenov chose for his analysis a number of typical musical works of outstanding composers: Bach, Beethoven, Chopin and Wagner. For example, by studying Bach's Chromatic fantasy and fugue, he used the quarter duration as the unit of measure. There are 330 such units of measure in this musical work. The 204th quarter from the beginning corresponds to the golden section here. This golden point coincides precisely with the fermata (in musical pieces the fermata sign augments a pause's duration, usually in 1.5-2 times). Furthermore, the golden point separates the first part of the musical work (the prelude) from the second part. The fugue, which follows the fantasy, also demonstrates an astonishing harmony of parts. Looking at the scheme of the fugue's harmonic analysis, "one experiences a sacred thrill [and] comes into contact with the greatness of the composer, who embodies the innermost laws of creative musical work with such accuracy."

Rosenov analyzed the final of the sonata of *Cis-moll* by Beethoven, the *Fantasia-Impromptu* by Chopin, the introduction to *Tristan and Isolda* by Wagner and so on. The golden section occurs very often in all of these musical works. He paid special attention to Chopin's *Fantasia*, which was an impromptu creation and not subject to any editing. Since this was impromptu, there was no conscious application of the golden section, which nevertheless is present throughout this musical piece even in the smallest musical parts.

Thus, we should recognize the golden section as a harmonic criterion in many musical works appearing frequently. Every part can be divided into two parts by the golden section and each new part can be divided by the golden section. Here we can see an analogy with Fibonacci resonances of the genetic code where each segment of the genetic code is divided in the golden ratio!

2.14.4. *Bella Bartok's Music*

The Hungarian Bella Bartok is one of the composers who consciously used the golden section in his music. The deep harmonic analysis of Bartok's music was given by Jay Kappraff [47] who wrote: "Bartok based his music on the deepest layer of folk music. He believed that all folk music of the world can ultimately be traced to a few primeval sources." According to the Hungarian musicologist Erno Lendai, "[Bartok] discovered and drew into his art the laws governing the depths of the human soul which have been untouched by civilization." As Kappraff noted, "Bartok based the entire structure of his music on the golden mean and Fibonacci series — from the largest elements of the whole piece, whether symphony or sonata, to the movement, principal, and secondary themes and down to the smallest phrase."

2.14.5. *The Golden Mean and Fibonacci Numbers in Cinema*

Sergey Eisenstein (1898-1948**)** is the great Soviet film director and art theorist. Having seen inexhaustible new opportunities for art in cinema, Eisenstein soon created his first film, *Strike*. In 1925 Eisenstein created the film *The Battleship Potemkin* that caused a triumph in the world. The American Film Academy recognized *The Battleship Potemkin* as the best film of 1926. At the Parisian Exhibition of Arts this film received the highest award: "Super Grand-Prix." It became a classic of new cinema art. Creating the films *October*, *Old and New*, etc., Eisenstein developed a theory of "intellectual cinema" that gave the spectator an opportunity to get not only the artistic, but also scientific knowledge and concepts.

Eisenstein was a pioneer in the use of the golden mean in cinema. It is most interesting how that Eisenstein used the "Golden Section Principle" in his film *The Battleship Potemkin*. First of all this film consists of 5 acts. The partitioning of the film into two separate parts corresponds to the proportion 2:3, that is, the first Fibonacci approximation to the golden mean. This 2:3 watershed occurs between the end of the second and the beginning of the third act of the five-act film, the basic caesura of the film occurs as a *Zero Point* of the action stop. However, perhaps, the most curious fact in all of this is that the “Golden Section Law” in *The Battleship Potemkin* is observed not only for the zero point of the movement, but also is true for the *apogee point*: a *Red Flag* on the battleship mast. Finally, the moment of the appearance of the red flag is the golden section of the film’s duration.

2.15. The Music of Poetry

In many respects, poetic works are close to music. A distinctive rhythm, regular alternation of stressed and unstressed syllables, a certain dimensionality, and their emotional saturation make poetry a native sister to music. Each verse has its musical form, rhythm and melody. It is possible to expect some musical features in poems, for example, regularity of musical harmony, and consequently, the golden section can often be present.

2.15.1. *Pushkin’s Poetry*


Alexander Pushkin

Alexander Pushkin was a Romantic author considered to be the greatest Russian poet and the founder of modern Russian literature. Pushkin’s father came from a distinguished family of the Russian nobility. His mother’s grandfather was Abram Petrovich Gannibal, an Ethiopian, who as a child was abducted by the Turks during their governing of Ethiopia, and he became a great military leader, engineer and nobleman in Russia under the auspices of his adoptive father, Peter the Great.

Born in Moscow, Pushkin published his first poem at the age of fifteen. By the time he finished the first grade of the prestigious Imperial Lyceum in Tsarskoe Selo near St. Petersburg, his talent was widely recognized by the Russian literary community. In 1820 he published his first epic poem, *Ruslan and Lyudmila*. Because of his 1820 poem, *Ode to Liberty,* he was exiled by the Czar Alexander I to the south of Russia. First coming to Kishinev in 1820, he there became a Freemason. He was in Kishinev until 1823. After a summer trip to the Caucasus and to the Crimea, he wrote two Romantic poems which brought him wide popularity, *The Captive of the Caucasus* and *The Fountain of Bakhchisaray*.

When Alexander’s brother, Nicholas I, came to power in 1825, he invited Pushkin back to the capital and gave him a government post. However, Nicholas acted as his personal censor to make sure that Pushkin did not publish anything that would hurt the government. The Czar ordered his spies to follow him, cutting out whole stanzas from his manuscripts. In the autumn of 1830 Pushkin left the capital to visit a small village left to him by his father. There Pushkin wrote some of his best poems, including completing his most famous poem *Eugene Onegin.*

On January 19, 1831, when he was almost 30 years old, Pushkin married the beautiful young Nathalie Goncharova. Although they had three children, they were not a happy couple. Nathalie was very beautiful and a favorite at the court, often encouraging the attention of other men. In 1837, influenced by rumors that his wife had entered into a scandalous liaison, Pushkin challenged her alleged lover, Georges d'Anthиs, to a duel. Pushkin was mortally wounded in the duel and died on January 29, 1837.

Pushkin's poetic works were analyzed by many researchers from the golden mean point of view. Let us begin with the poem's size, the number of lines. At first, it seemed that this parameter of the poem could change somewhat arbitrarily. However, this proved not to be so. For example, the analysis of Pushkin's poems by Vasjutinsky [31] showed that the sizes of Pushkin's poems are not distributed uniformly; Pushkin appearantly preferred poems of 5, 8, 13, 21 and 34 lines (Fibonacci numbers!).

It was noted by many researchers that his poetical verses and poems are similar to musical pieces; there are also culmination points in them dividing the poems at the golden section. By studying Pushkin's poem *Shoemaker*, Vasjutinsky noted that the poem consists of 13 lines. Two semantic parts can be singled out in the verse: the first part consists of 8 lines and the second one (the parable moral) consists of 5 lines. Note of course that the numbers 13, 8, 5 are Fibonacci numbers and their ratios are thus an approximation to the golden mean proportion!

One of Pushkin's last poems, *Not Dearly I Appreciate the High-Sounding Rights …*, consists of 21 lines in which two semantic parts are singled out: the first part is 13 lines and the second part 8 lines (21, 13 and 8 are of course Fibonacci numbers).

The analysis of the famous novel, *Eugene Onegin,* by Vasyutinsky [47] is very intriguing. This novel consists of 8 chapters; each chapter is, on the average, about 50 verses. The eighth chapter is the most perfect and the most emotional. This chapter consists of 51 verses. Together with the letter of Eugene to Tatjana, the size of this chapter corresponds precisely to the Fibonacci number 55.

Vasyutinsky writes: "Eugeny's declaration of love to Tatjana is the chapter's culmination (the line 'To turn pale and to die away... it is bliss!" This line divides the 8th chapter into two parts, the first part consists of 477 lines and the second part consists of 295 lines. Their ratio is equal to 1.617! We can see the finest conformity to the golden mean! This fact is the great miracle of harmony created by Pushkin's genius!"

2.15.2. *Lermontov's Poetry*

Rosenov analyzed many poetic works of Lermontov, Schiller and Tolstoy, also finding in them the golden mean.

Michael Lermontov

Michael Lermontov was born on the 15th of October, 1814 into a family of nobility. The poet spent his youth at Tarkhany. He enrolled in Moscow University, but very soon had to leave the University. Later he enrolled in and graduated from St. Petersburg School of Cavalry Cadets. In 1837, he was exiled to the Caucasus because of his poem on Pushkin's death, in which he blamed the ruling circles of Russia and the Czar Nicolas I. In 1841, Lermontov was sent into exile in the Caucasus once again. As a result of intrigues between the officers, he was provoked into a personal quarrel with an old schoolfellow. This led him to a duel on July 15, 1841 in which the poet was killed.

Lermontov began writing his poems when he was very young. However, he became famous because of his poem on Pushkin's death. Lermontov's poems, including *The Demon* and *Mtsyri*, and his innumerable lyrical poems such as, *A Hero of Our Time* and *Masquerade,* are masterpieces of Russian literature.

Lermontov's famous poem, *Borodino* (see English translation by Eugene M. Kayden at www.lermontov.net/content/view/39/2/)*,* is divided into two parts: the short introduction and the main part. The introduction consists of only one stanza:

"But tell me, uncle, why our men
Let Moscow burn, yet fought again
To drive the French away?
I hear it was a dreadful fight,
A bitter war, by day and night;
That's why we celebrate the might
Of Borodino today."

The main part of the poem consists of 13 verses, and it is divided into two parts: first consist of 8 verses, and second of 5 verses. In the first part, a battle of Borodino is described with rising tension an expectation of action; in the second part, the battle itself is described with gradual reduction of tension to the end of the poem. Each verse consists of 7 lines, that is, in total the main part consists of 91 lines. If we divide the main part in the golden section (91/1.618 = 56.238), we find, that the division point corresponds to the beginning of the 57th line, that begins from the short phrase: "O what a day!" This phrase represents the culmination point of the tense anticipation, which completes the first part of the poem - the expectation of the action, and opens its second part- a description of the battle:

O what a day! The Frenchmen came,
A solid mass, like clouds aflame,
Straight for our redoubt.
Their lancers rode with pennons bright;
Dragoons came on in all their might
Against our walls, and in the fight
They scattered in a rout.

2.15.3. *Shota Rustaveli's Poem*

Shota Rustaveli

Shota Rustaveli was a Georgian poet of the 12th century. He is one of the greatest representatives of medieval literature. There is not much biographical data about Rustaveli: the exact dates of his birth and death, and historical data about the main events of his life are unknown. Many poetic works written by him were lost. He is the author of the literary work *The Knight in the Panther's Skin*, a Georgian national epic poem. Rustaveli was a Georgian noble and treasurer to the Queen of Georgia Tamara. He also restored and painted frescoes in the Georgian monastery of the Holy Cross in Jerusalem. One of the pillars of this monastery bears a portrait, which is believed to be that of the poet. According to the legend, he fell hopelessly in love with Tsarina Tamara. It was here in the cell of this monastery that Rustaveli terminated his life.

A grandiose poem, *The Knight in the Panther's Skin,* brought to Rustaveli world glory. Translated into many languages this poem is rightfully considered one of the world's greatest pieces of literature.

Many researchers of Shota Rustaveli's poem note its harmony and melody. In the opinion of Georgian scientist and academician Zereteli, these properties of Rusataveli's poem arise, thanks to his conscientious use of the golden mean in both its poetic forms and the construction of its verses. Rustaveli's poem consists of 1 637 stanzas; each consisting of four lines. Each line consists of 16 syllables divided into two equal parts, the half-lines, which therefore consist of 8 syllables. All half-lines are divided into two segments of two kinds: the **A-**kind segment is the half-line with equal segments and with an even number of syllables (4+4), the **B**-kind segment is the half-line with asymmetrical division into two unequal parts (5+3 or 3+5). Thus, we can see the following 3:5:8 ratio in the half-line of the **B-**kind; that is, Fibonacci numbers, whose ratios are, of course, close to the golden mean. Zereteli proved that 863 of the 1 637 stanzas in Rustaveli's poem are constructed according to the golden mean principle!

2.16. The Problem of Choice: Will Buridan's Donkey Die?

The Russian academician Zeldovich wrote in one of his works: "Do not treat with contempt "simple" reasoning. Those researchers, who obtain deep results out of the simple, but firmly established facts, deserve the highest praise." The works of American psychologist Vladimir Lefevre have an utter "simplicity" of initial reasoning giving them aesthetic charm.

2.16.1. *A Parable about Buridan's Donkey*

Often times we have to choose the best amongst various possibilities. This leads to a rather unexpected consequence — the pleasure of choice suddenly turns into a complex problem, and this problem can be extremely serious. This *problem of choice* is a perennial problem. The Greek philosopher Aristotle wrote of the difficulty for a person experiencing both hunger and thirst who is equally distant from food and drink. He remains unmoved at the same place because he cannot make a choice. This problem of a person not being able to make a choice between food and drink, was formulated more precisely by the French philosopher Jean Buridan, who demonstrated an apparent absence of the freedom of will. He gave the example of the donkey who, being situated between two equally sized haystacks each equally distant, should by all means die when the donkey cannot choose which haystack to go and eat. Since then a popular expression, *Buridan's donkey,* has arisen. A person, irresolute in a choice or vacillating between two equivalent situations, is called "Buridan's donkey."

2.16.2. *A Psychological Experiment*

We are quite confident that repeatedly throwing a coin inevitably results in 50% heads and 50% tails. American psychologist Vladimir Lefevr began to reflect upon whether one can apply this result to psychology. For this reason, he carried out the following experiment. He asked a person to divide a pile of string beans into two piles; the good string beans are in one of them and the bad string beans in the other. String beans in the initial pile are all very similar to one another, and it is difficult to give any objective criterion for their division into the "good" and "bad" string beans. It is clear that this problem is similar to throwing a coin and we can expect the result: 50% by 50%. However, the actual result overturned the expectations: the number of the good string beans steadily built up to 62% (0.62) from the initial number of string beans. What a surpise! As we know, 62% is the golden mean of 100%! The result of the psychological experiment proved very surprising: a person divides the string beans into two piles in the golden ratio! Vladimir Lefevre offered an original explanatory theory for this experiment. However, to understand Lefevre's theory, it is necessary to know some basic concepts of mathematical logic and psychology.

2.16.3. *What is an Implication?*

The name of the English mathematician *George Boole* (1816-1864) is widely known in modern science. In 1854 he published *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*. His work produced the beginning of a new algebra, the algebra of logic, or *Boolean Algebra*. Boole showed for the first time, that there is an analogy between algebraic and logical operations, if we assume that logical variables take only two values — true or false (symbolized as 1 or 0). He invented a system of designations and rules, which allows one to code any statements and then to operate on them as traditional numbers. Boolean algebra has three basic operations — AND, OR, NOT, which allow one to perform logical operations of conjunction, disjunction and negation on statements. In his *Laws of Thought* (1854) Boole finally formulated the basis of mathematical logic. However, as often happens with mathematicians, Boole wrote that he did not see the practical applications of his algebra. Fortunately, regarding practical applications of his algebra the great mathematician was mistaken. Modern science, in particular computer science, is impossible without Boolean logic, which is widely used in the analysis and synthesis of digital automatons, as well as in the solutions of logic problems on computers.

Propositions can be 1 (true) or 0 (false) and *Boolean Functions* connecting them are the basis of Boolean algebra. A proposition is any statement that can be evaluated from the point of view of its truth or falsity. As an example, consider the two propositions:

A = "*George Boole is the creator of Boolean algebra*"

B = "2×2=5"

The first proposition is true, that is, A=1, the second proposition is false, that is, B=0.

Propositions can be *Simple* or *Complex*. We gave examples above of simple propositions. Their basic property is the fact that they can be either true or false. The complex proposition consists of several simple propositions, thus, the complex proposition is also either true or false (1 or 0). However, the truth value of the complex proposition depends on the truth values of the simple propositions that make up the complex proposition. Thus, the complex proposition Y is a logic function (Boolean function) of the simple propositions contained within it (for example, A and B), that is, $Y=f(A,B)$.

The following logic functions – *Negation, Conjunction* (the logic function AND), *Disjunction* (the logic function OR), *Addition by the Module* 2, etc. are the most widespread elementary Boolean functions. Boolean functions are employed in computers. Boolean functions are also used in other fields of knowledge including implication in psychology. Implication is a logical operation that builds up the complex proposition corresponding to the logical connective *"if ... then."* Implication consists of two simple propositions, the *Antecedent* which follows "if" and the *Consequent* which follows "then." If A and B are the antecedent and the consequent, respectively, then the implication is the Boolean function $Y=A \to B$ that is false ($Y=0$) only in the case where the antecedent is true ($A=1$) and the consequent is simultaneously false ($B=0$); for the rest of the cases the implication is true ($Y=1$).

The implication is represented by Table 2.14.

Table 2.14. A truth table for implication

A	0	0	1	1
B	0	1	0	1
Y	1	1	0	1

What would be a physical application of the implication? How can the implication connect two simple propositions? We will show this in the example of the statements: A = "the given quadrangle is a square" and B = "around a given quadrangle we can describe a circle." Let us examine the complex proposition $A \to B$ = "if the given quadrangle is a square, then around the given quadrangle we can describe a circle." There are **three variants**, when the proposition $A \to B$ is true (1):

1. A is true (1) and B is true (1), that is, a given quadrangle is a square and it is possible to describe a circle around it.
2. A is false (0) and B is true (1), that is, a given quadrangle is not a square, however, it is possible to describe a circle around it (it is clear that the proposition B is not true for all quadrangles).
3. A is false (0) and B is false (0), that is, a given quadrangle is not a square and it is not possible to describe a circle around it.

Note that only one variant for the implication is false (0), when A is true and B is false, that is, when a given quadrangle is a square, and at the same time, it is not possible to describe a circle around it.

In usual speech a connective “*if ... then*” describes a relationship of cause and effect between propositions. However, in terms of logical operations the meanings of the propositions are not considered. We only consider the truth or falsity of the propositions. Therefore, we should not be confused by the seeming “nonsense” of the implications formed by the propositions that cannot be connected by their contents. For example: *“If the President Bush is a democrat, then giraffes are in Africa”, “if a watermelon is a berry, then gasoline is sold at the gas station.”*

2.16.4. *What is a Reflection?*

In the most general sense reflection maybe though of as the human soul pondering itself. There is variety of definitions for this concept. More often, a reflection is determined as the analysis of one’s own ideas and experiences; a reflection is full of doubts and uncertainties. For the first time the concept of *Reflection* in a contemporary sense was used by the British philosopher John Locke (1632-1704). From his point of view, a reflection is a special operation by a person upon his own consciousness. As a result, this operation generates ideas about one’s own consciousness. Further down, in relation to a person words “his” and “himself” stands for both genders.

A person is reflecting, when one says: “I think that I think,” and so forth. According to Lefevre’s opinion, a description of *reflection* signifies that a person a_0 can be represented symbolically by two elements: a person and his “map of consciousness,” in which the information about how the person thinks about himself is placed. This means that the “map of consciousness” contains some image Z of a person regarding himself (Fig. 2.17).

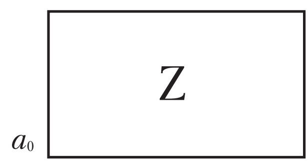


Figure 2.17. A person with his “map of consciousness” *Z*

There therefore appears to be a reflective structure, in which the person possesses a “map of consciousness,” where the person’s image about himself is represented. Symbolically such a structure can be expressed by the formula $A = a_0^{a_1}$, where a_0 is the person himself (how the “absolute observer” can see a person from outside), and a_1 is the image of a person about himself. On the other hand, it is possible to consider the reflection of the second order $A = a_0^{a_1^{a_2}}$, where a_1 is the image of a person thinking about himself, and a_2 is the image of a person about his own thoughts (Fig. 2.18).

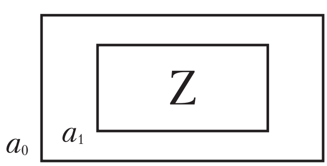


Figure 2.18. A reflective structure of the second order

Now, let us introduce one more intriguing word, *Intention* (from the Latin word *intentio* —

aspiration). This word means an aspiration, a purpose, a direction or an orientation of consciousness, wish, and feelings towards any subject. In the expression $a_0^{a_1}$ the symbol a_0 means the intention of the person and the symbol a_1 means the intention of his image. We can estimate the intentions and the acts of the person as positive (1) or negative (0) as done in Boolean algebra. A negative estimation of oneself means the presence of guilt. A negative estimation of the partner means the feeling of the partner's guilt and self-condemnation.

Let us consider the expression:

$$A = a_0^{a_1^{a_2}}. \tag{2.77}$$

Here, the power $A_1 = a_1^{a_2}$ means the image of the person in his representation. The appraisal A_1=0 means that a person estimates himself negatively, vice versa, A_1=1 means that a person estimates himself positively. The power a_2=1 is a representation of what a person thinks about how he estimates himself. The appraisal a_2=1 means that a person has a positive self-appraisal. The appraisal a_2=0 means that a person has a negative self-appraisal.

Lefevre started to study this kind of self-reflection (2.77), that is, the reflection of the second order, where a_1 is a representation (a thought) of a person about himself, a_2 is a representation (a thought) of a person about his own thought.

Lefevre's important idea consists of the fact that the appraisal of a one's self and negative or positive determination of this appraisal is fulfilled by a person without any efforts from the person's consciousness, that is, each person seems to have a "reflective computer" that automatically makes these self-appraisals according to the formula (2.77).

Now consider the values taken by the function (2.77) with dependence on the variables a_0, a_1, a_2. With this purpose in mind, we first take into account the expression. We can write the numerical values of this expression with dependence on the values of a_1 and a_2 as follows:

$$1^1=1;\ 0^1=0;\ 1^0=1;\ 0^0=1. \tag{2.78}$$

These expressions do not raise any objections from the mathematical point of view. In mathematics, the expression $0^0=1$ is accepted as "correct" by definition.

Now compare the expressions (2.78) with the truth table for implication (Table 2.13). It follows from the comparison that the expression $A_1 = a_1^{a_2}$ sets the implication $A_1=a_2\to a_1$. If we consider the values of the function (2.77), we can conclude that this function creates the implication of a more complicated kind, namely:

$$A=(a_2\to a_1)\to a_0. \tag{2.79}$$

Note that the expression (2.79) is Lefevre's key idea! Let us consider a truth table for this logical function (2.79).

Table 2.15. A truth table for Lefevre's function

a_0	1	1	1	1	0	0	0	0
a_1	1	0	1	0	1	0	1	0
a_2	1	1	0	0	1	1	0	0
A	1	1	1	1	0	1	0	0

Analysis of Table 2.15 results in very interesting conclusion. The total number of possible values for Lefevre's function (2.79) is equal to 8. Five of them take the value 1 (a positive estimation) and the remaining 3 values take the value 0 (a negative estimation). Note that 8, 5 and 3 are adjacent Fibonacci numbers! This means that according to Table 2.15, our "reflective computer" automatically (that is, without participation of our consciousness) gives a Fibonacci ratio between positive (5) and negative (3) estimations. This 5:3 ratio is a Fibonacci approximation to the golden mean of 1.618! That is, Lefevre's theory explains an occurrence of the golden mean in the above psychological experiment with the string beans, that is, a person automatically, without the participation of his consciousness, tends to make a choice according to the golden mean principle! This paradoxical conclusion can have very unexpected applications to the so-called "behavioral" sciences. The theory of Elliott's wave that has actively developed in American science is one such theory.

2.17. Elliott Waves

However, if Lefevre's theory is true, that is, all processes of the behavior of a person and his decisions are based on (or related to) the "Golden Mean Principle," then perhaps Fibonacci numbers and the golden mean can be discovered in such "behavioral systems," such as in an economy that is a function of human behavior through the decisions made by many people. Here we may be in for a big surprise! In the first half of the 20th century the American bookkeeper and economist Ralph Elliott developed an original theory about stock market price fluctuations. This research resulted in the modern science of *Elliott Waves.*

2.17.1. *Ralph Nelson Elliott*

Let us start from who was Elliott and how did he come by his discovery? Ralph Nelson Elliott (1871-1948) was one of the brightest representatives of the "American Renaissance." Being an accountant by education, he specialized in reorganizing and revitalizing large companies such as export-import houses and railroads in the U.S. and Central America. In 1924 the U.S. State Department appointed Elliott to serve as Chief Accountant for Nicaragua where the U.S. had large economic ties.

Later, Elliott wrote a foreign policy proposal based upon his long-term experience in Central and South America and submitted it to the State Department. The essential ideas of his proposal were reflected in the program "Good Neighbor" as elaborated by Roosevelt's administration, and used later in the more recent programs of the World Bank.

In the 1930s Elliott turned his attention toward studying the stock market. In 1938 at the age of 67, he published his first monograph about market models, *The Wave Principle*. In the next year, he published a series of articles in Financial World magazine that detailed his discovery. In 1946 he completed a large book, *Nature's Law*, in which he expanded his 1938 essay on the connection between the Wave Principle, the golden mean and the stock market.

2.17.2. *Rhythm in Nature*

The proposition that the Universe is subordinated to some general Laws returns us to Pythagoras and Plato, and the origin of this great universal recognition in modern science. It is obvious that without the Laws of Nature we would have chaos. A strict ordering and constancy of Nature follows. This is confirmed by a surprising periodicity and repetition in all of Nature's processes.

A person is a natural object similar to the Sun or the Moon. One's activity can be expressed by the language of numbers and is subject to scientific analysis. Human activity, for example the heart's activity, could be considered from the point of view of rhythmic processes. A person's heart beats uniformly (about 60 beats per minute). The heart acts as a cylinder piston by drawing in and then pushing out the blood. Blood pressure changes during this cardiac cycle. It reaches its greatest value in the left ventricle at the moment of compression (systole). In arteries (during systole) the blood pressure is reaching its maximum value, equal to 115-125 mm Hg. At the moment of the cardiac reduction (diastole), the pressure is decreasing until there is 70-80 mm Hg. The ratio of the maximum (systolic) pressure to the minimum (diastolic) pressure is equal, on the average, to 1.6, which is a close approximation to the golden mean.

Is this a mere random coincidence, or does it reflect some objective regularity of the organized cardiac activity? The heart beats continuously from a person's birth up to his death. And its activity should be optimal and be subordinate to the self-organizational laws of biological systems. Because the golden mean is one of the criteria of self-organizing systems, one might naturally expect the cardiac cycle to be subordinate to the golden mean law. This hypothesis underlies the mammal cardiac research activity of Russian biologist Tscvetkov [39].

Judgments about heart activity are done through an electrocardiogram, displaying a curve that reflects cardiac performance (Fig. 2.19).

In a cardiogram, a comparison is made of two time intervals of different duration corresponding to the heart's systolic (t_1) and diastolic (t_2) activity. Tscvetkov found that there exists the optimal (golden) frequency for people and other mammals; here, the durations of systole, diastole and the full cardiac cycle (T) are in golden mean proportion, that is, T: t_2= t_2: t_1. For example, for a person this golden frequency is equal to 63 heart beats per minute, and 94 beats per minute for dogs.

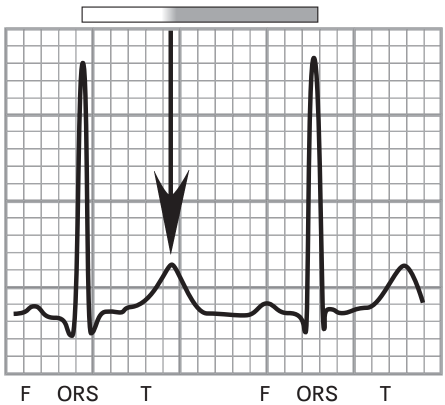


Figure 2.19. Human electrocardiogram

Tscvetkov found that if we take the middle blood pressure in the aorta as the measurement unit, then the systolic blood pressure is 0.382, and the diastolic pressure is 0.618, that is, their ratio corresponds to the golden ratio (0.618:0.382=1.618). It means that cardiac performance in the timing cycles and blood pressure variations are optimized according to the same principle, the law of the golden mean.

If we move on from the investigation of the person as a biological unit to the consideration of our social and economic activity, we find that human activity follows certain laws, which force the social and economic processes to repeat in the form of a certain set of waves or impulses. The best way this idea can be shown is through the example of stock exchange processes.

2.17.3. *Elliott's Wave Principle*

Let us now return to a consideration of the main ideas concerning the stock market that underlie the Elliott wave theory:

1. Natural Laws embrace the most important element of all, namely, timing. They are not themselves some simple system or method for playing the market. However, as expressed in market phenomena they appear to mark the expression of all human activities. The application of these laws to forecasting may have a revolutionary character.

2. This law can only be discerned when the market is viewed and analyzed as a product of human decisions. Simply put, the stock market is a human creation and therefore reflects human idiosyncrasy.

3. All human activities have three distinctive features — pattern, time and ratio — each of which appears to be based upon the Fibonacci series.

According to the Elliott Wave Theory, stock prices can be simulated by using a specific wave that is connected with Fibonacci numbers. Elliott's main wave is shown in Fig. 2.20. More specifically, Elliott believed that the main wave is simulated by distinct upward and downward movements. For example, the ascending part of the wave consists of 5 movements, three movements up and 2 movements down (3 and 2 are of course adjacent Fibonacci numbers); the descending part of the wave consists of 3 movements, 2 movements down and 1 movement up (again adjacent Fibonacci numbers).

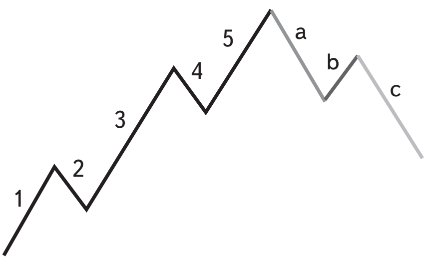


Figure 2.20. Elliott Waves

The movements 1, 3 and 5 represent the *Impulses* in a *Major Bull Movement*. The movements 2 and 4 represent the *Corrective Movements* in a major bull movement. The movements A, B and C define a *Minor Bear Wave*; here, the movements A and C represent the descending movements of the minor bear wave, while B represents one ascending movement of the minor bear wave. **The major waves determine the major trends of the market, and the minor waves determine the minor trends.** Elliott carefully examined the market's waves, and found that the golden mean and Fibonacci numbers play an important role in the pressures and trends of the stock market.

Elliott proposed that the main wave exists at many levels; this means that new sub-waves could appear within the primary wave. To clarify, this means that the chart above (Fig. 2.20) represents only the primary wave pattern. However, the same kind of wave occurs, for example, between the points 2 and 4. The diagram below (Fig. 2.21) shows how primary waves could be broken down into smaller waves. We can see in Fig. 2.21 that the smaller waves are subordinate to one and the same principle, the golden mean principle.

Trading which is based upon the Elliott Wave patterns is quite simple. A trader identifies the main wave as a super-cycle and then acts accord-

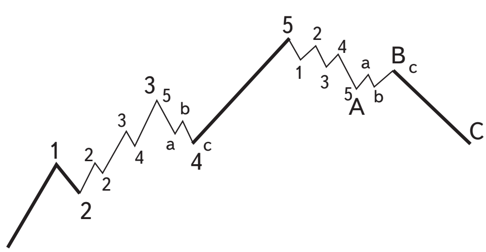


Figure 2.21. The smaller Elliott Waves

ing to the Elliott Wave model, that is, increasing or decreasing the sales because market behavior is subordinated to the golden mean principle. This goes on in progressively shorter cycles until the cycle is complete.

Of course, real market processes are more complex than the Elliott model. Even if we do not agree with some of Elliott's findings, the idea is of great methodological interest. Many scientists, especially in the U.S., are convinced Elliott supporters. The American scientist Robert Prechter is the most famous follower of Elliott's ideas. In 1999, he published a book [143], which is dedicated to the development of the Elliott Wave Principle, and organized Elliott Wave International to promote Elliott's ideas.

Prechter made the following very ambitious statement: "(R.N. Elliott's) Wave Principle is to sociology what Newton's laws were to physics" [143].

Time will tell whether Prechter's comparison of the Elliot Wave Principle to Newton's Laws has merit. However, one thing is doubtless: thanks to the activities of Elliott and his followers, a theory of modern sociology and market economics was added with deep scientific significance. According to this concept, Fibonacci numbers and the golden mean are not only behind the growth of the pinecone or sea-shell, but also determine the laws of human behavior, and through them the laws of the stock market!

2.18. The Outstanding Fibonacci Mathematicians of the 20th Century

The 20th century is characterized by an increasing interest in Fibonacci numbers and the golden section. Two scientific works, the book *Aesthetics of Proportions in Nature and Art* [11] by the French scientist Matila Ghyka and the book *Proportionality in Architecture* [10] by Russian architect Professor G.D. Grimm were very important achievements in the "golden" area in the first half of the 20th century.

In addition, considerable results in this area were obtained in the 20th Century by Danish mathematician Willem Abraham Wythoff and Belgian amateur mathematician Edouard Zeckendorf.

2.18.1. *Willem Abraham Wythoff*

Many specialists in combinatorial analysis and number theory know about the Wythoff game. However, few know anything about the man for whom this game was named. W. A. Wythoff was born in Amsterdam in 1865, the son of a sugar refinery operator, and obtained his Ph.D. in mathematics in 1898 from the University of Amsterdam. Wythoff's game was described in his article *A Modification of the Game of Nim* [*Nieuw Archief voor wiskunde*, 2 (1905-07), 199-202].

Dr. Wythoff described his famous game as follows:

"The game is played by two persons. Two piles of counters are placed on the table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an equal number. The player who takes up the last counter, or counters, wins."

The solutions to Wythoff's game that are based on Fibonacci numbers can be found in numerous articles in *The Fibonacci Quarterly*, and also in the famous article *The Golden Section, Phyllotaxis, and Wythoff's Game* by Coxeter (*Scripta Mathematica,* 19 (1953) 135-143).

2.18.2. *Edouard Zeckendorf*

Edouard Zeckendorf (1901-1983)

Many number theorists know about *Zeckendorf sums*, however, here again, few know anything about the man for whom these sums are named. Edouard Zeckendorf was born in Liиge, Denmark. In 1925, he qualified as a medical doctor at the University of Liиge and then became a Belgian army officer. He also obtained a license for dental surgery some time prior to 1930.

Mathematics nevertheless was Zeckendorf's main passion. In 1939, he published the article devoted to Zeckendorf sums. According to his theorem, each positive integer has a unique representation as a sum of Fibonacci numbers, though two adjacent Fibonacci numbers are actually never employed. Research on Zeckendorf sums became the subject of many articles published in *The Fibonacci Quarterly*.

2.18.3. *Verner Emil Hoggatt*

Verner Emil Hoggatt (1921-1981)

The mathematical organization of the Fibonacci Association was founded in 1963 by a group of American mathematicians making it one of the most outstanding events in the history of Fibonacci number theory. Beginning of 1963, the Fibonacci Association began to issue *The Fibonacci Quarterly*. The American mathematicians Verner Emil Hoggatt (1921-1981) and Alfred Brousseau (1907-1988) were founders of the Fibonacci Association. Verner Hoggatt, along with Brother Alfred Brousseau, published the first volume of *The Fibonacci Quarterly* in 1963, thereby founding the Fibonacci Association. The April 4, 1969 issue of *TIME* magazine reported the phenomenal growth of the Fibonacci Association. In the same year, Houghton Mifflin published Verner Hoggatt's book, *Fibonacci and Lucas Numbers* [16], perhaps the world's best introduction to Fibonacci number theory. Howard Eves wrote that "during his long and outstanding tenure at San Jose State University, Vern directed an enormous number of master's theses, and put out an amazing number of attractive papers. He became the authority on Fibonacci and related numbers."

2.18.4. *Alfred Brousseau (1907-1988)*

Alfred Brousseau was another outstanding person involved in the organization of the Fibonacci Association. Brother Alfred belonged to the religious order *Fratres Scholarum Christianarum* that translates as *Brothers of the Christian Schools*, or simply, *The Christian Brothers*.

Brother Alfred was an avid photographer. He made a collection of some 20,000 slides of California wildflowers. Images of more than half of these and other collections by Brother Alfred are preserved in the form of widely used websites at the University of California, Berkeley.

Alfred Brousseau
(1907-1988)

In 1969, *Time* magazine featured two founders of the Fibonacci Association in an article titled *The Fibonacci Numbers*. Brother Alfred was pictured in the *Time* article holding a pineapple, one of the best known representatives of phyllotaxis. The article referred to the many natural applications of Fibonacci numbers: for example, male bees reproduce "fibonaccically," and Fibonacci numbers occur in the formations visible in many sunflowers, pine cones, and leaf-positions on branches of trees. Brother Alfred recommended that people who learn about Fibonacci numbers should focus their attention on the aesthetic pleasure involved in it.

2.18.5. *Steven Vaida*

In 1989, twenty years after Verner Hoggat's book, the publishing house of *Ellis Horwood Limited* published the book *Fibonacci & Lucas Numbers and the Golden Section* [28] by Prof. S. Vaida. The book attracted wide attention from Fibonacci mathematicians, as it is considered to be one of the best mathematical books on the subject. Who is the author of this book? From his brief scientific biography, we learn that the mathematician Steven Vaida was Professor of Mathematics of University Sussex (England) at the time of the book's publication. He obtained a Doctorate in Philosophy at Vienna University. In 1965, he began to work as Professor of operational researches at Birmingham University. It is also noteworthy that he was an honorary member of many mathematical organizations, in particular, the Mathematics and Statistics Institute of the Society of Operational Researches (England).

2.18.6. *Herta Taussig Freitag*

Among the modern Fibonacci mathematicians, it is necessary to name Professor Herta Taussig Freitag, another member of the Fibonacci Association. She was born on December 6th, 1908 in Vienna, Austria and died January 25th, 2000 in Roanoke, Virginia.

Herta Taussig Freitag
(1908-2000)

Herta Freitag graduated with a Ph.D. from Columbia University in 1953. Her colleagues at the Fibonacci Association named Herta Freitag the Queen of the Fibonacci Association. Over the years, she attended and presented a paper at every International Conference of Fibonacci numbers starting with the first Conference in 1984. It is curious to note that Professor Freitag delivered the lecture *Elements of Zeckendorf Arithmetic* (coauthor G.M. Phillips) at the 7th International Conference on Fibonacci Numbers and Their Applications. The appearance of this lecture title is rather distinctive. It testifies to the fact that Fibonacci mathematicians came close to the creation of the new computer arithmetic based on Fibonacci numbers and Zeckendorf sums. This became a central subject of the Soviet scientific and engineering developments in the 1970s and 1980s.

2.19. Slavic "Golden" Group

The outstanding Hungarian geometer Janos Bolyai, one of the creators of Non-Euclidean geometry, once wrote the following remarkable words:

"For ideas, as well as for plants, the time comes, when they mature in their various locales, just as in the spring the violets blossom wherever the Sun shines."

The history of science should tell us why the 80s and 90s became the historical period in which interest in Fibonacci numbers and the Golden Section began to peak. Particularly in this period, scientists of different disciplines put forward hypotheses concerning the applications of the golden mean and made discoveries that have fundamental significance for the development of science and its many branches.

The small brochure *Fibonacci Numbers* [13] published in 1961 by the Russian mathematician Nikolay Vorobyov resulted in an expanding mathematical interest in Fibonacci numbers. The brochure was re-issued repeatedly and translated into many languages.

First of all, it is necessary to mention the primary direction that arose in the 1970s in Soviet science. It is well known that Fibonacci's problem of "rabbit reproduction" became a source for research into Fibonacci numbers. However, the "rabbit reproduction problem" is not the only problem suggested by Fibonacci. He also proposed the well-known problem of *Choosing the Best System of Standard Weights*. This problem is named the *Weighing Problem* or *Bashet-Mendeleev Problem* in Russian historic-mathematical literature. In 1977, the author of this book published the book *Introduction into Algorithmic Measurement Theory* [20] and in 1979 the brochure *Algorithmic Measurement Theory* [21]. These works generalize the Bashet-Mendeleev problem and provide a new measurement theory based on Fibonacci numbers, namely *Algorithmic Measurement Theory*. Published at the end of the 70s they helped determine the applied nature of research in the Fibonacci field (Fibonacci codes, Fibonacci arithmetic, Fibonacci computers) that began to develop in Soviet science. The practical direction of Fibonacci research by the Soviet scientific school is distinctively different from the direction of the American Fibonacci Association.

In 1984, two books were published dedicated to the golden section. The Byelorussian philosopher Eduard Soroko in his book *Structural Harmony of Systems* [25] made a courageous attempt to revive for modern science the Pythagorean idea of the numeric harmony of the Universe, proposing the so-called *Law of Structural Harmony of Systems* that is expressed with the help of the generalized golden sections, namely, the golden p-sections [20]. In the same year Stakhov's book *Codes of the Golden Proportion* [24] was published. This book developed number systems with irrational bases or codes of the golden proportion that are a generalization of Bergman's number system [86].

One year later in 1985, the book *Aesthetic Foundations of Ancient Egyptian Art* [145] by Russian art critic Natalia Pomerantseva was published; the book demonstrates quite convincingly the role of the golden section in Ancient Egyptian culture.

The year 1986 was marked by the publication of two books on the Golden Section. In Poland the *Energy-Geometric Code of Nature* [26] by Polish scientist and journalist Jan Grzedzielski was published. This book, probably for the first time, uncovered the physical sense of the golden mean as the main code of the Universe, as the proportion of thermodynamic equilibrium in self-organizing systems. In the same year a teacher at the Kiev State Art Institute, Kovalev, published a manual for artists called *The Golden Section in Painting* [29].

The beginning of the 1990s was characterized by the publication of two more books on the Golden Section. The first was *The Golden Section: Three Approaches to the Nature of Harmony* (1990) [146]. The book was authored by three representatives of Russian art, the architect Shevelev, the composer Marutaev and the architect Shmelev. The abstract states: “This book is dedicated to the theoretical substantiation of the phenomenon of the golden section, one of the universal laws of Nature.”

The popular book *The Golden Proportion* [31] by Nikolay Vasyutinsky, a Ukrainian researcher, chemist, geologist, metallurgist and mechanic, was of considerable importance of 1990. Written with great proficiency, it states: “the manifestation of the Golden Proportion laws in architecture, music, poetry, chemistry, biology, botany, geology, astronomy, and engineering sciences are herein described.”

At the beginning of the 1990s, it became clear that Slavic science (Ukraine, Russia, Belarus, and Poland) had developed an immensely powerful group of researchers formed of representatives of various sciences and arts, and authors of very original books on the golden section. The idea arose to unite these researchers together and create a Slavic golden scientific society. The First International Seminar of *The Golden Proportion and Problems of System Harmony* was held in Kiev in 1992 under the scientific supervision of Professor Stakhov, the author of this book. The Belarussian philosopher Eduard Soroko (Minsk), the Ukrainian architect and art critic Oleg Bodnar (Lvov), the Ukrainian mathematician and economist Ivan Tkachenko, the Russian mechanical engineer Victor Korobko (Stavropol), the Ukrainian physician and anatomist Pavel Shaparenko (Vinnitsa), the Ukrainian chemist Nikolay Vasjutinsky (Zaporozhye), and the Polish journalist Jan Grzedzielski (Warsaw) participated as members of the group’s organizing committee. This group of highly respected scientists became the skeleton of the Slavic scientists’ association called the *Slavic Golden Group*.

The Second International Seminar of *The Golden Proportion and Problems of System Harmony* was held in 1993 in Kiev. Then on the initiative of Professor Korobko, the Seminar continued its work in 1994, 1995 and 1996 in Stavropol, Russia.

The Seminars stimulated research in the golden section in a variety of fields. During the 1990s the Slavic Golden Group published a number of very interesting books on the subject.

In 1994, the book *The Golden Section and Non-Euclidean Geometry in Nature and Art* [37] was published by Oleg Bodnar. A new geometrical theory of phyllotaxis (*the Law of Spiral Bio-symmetry Transformation*) discovered by Bodnar became the primary outcome of the book.

The end of the 20th century was very successful for the Slavic Golden Group. In 1997 *Heart, Golden Section and Symmetry* [39] by Russian biologist Tsvetkov was published. The book is a summary of the author's long-term research in the area of Golden Section applications to cardiac activity in mammals. In this book the set of Golden Sections in the different structures of the cardiac cycle is found. Tsvetkov also demonstrated the role of the Golden Section and Fibonacci numbers in optimization of the heart activity (minimization of energy consumption, in the blood, muscle and vascular matter) of mammals.

In 1998, the book *The Golden Proportion and Problems of System Harmony* [43] was published by Professor Korobko (Russia), an active member of the Slavic Golden Group. The book contains considerable information concerning the application of the golden section throughout Nature, Science and Art. The great strength of the book is that it can serve as a manual for university and college teachers of post-graduate students in both the engineering sciences and the liberal arts. In fact, the Russian Association for Building Universities recommended it as a manual for all students in colleges and universities.

In 1999, Alexey Stakhov with assistance of Vinancio Massingue and Anna Sluchenkova published *Introduction into Fibonacci Coding and Cryptography* [44]. This book presented a new coding theory based on Fibonacci matrices. In 2000, the famous Russian architect Shevelev [46] published the book *The Meta-language of Living Nature*.

The beginning of the 21st century has been characterized by increasing scientific activity by the Slavic Golden Group. In 2001, Alexey Stakhov and Anna Sluchenkova created a website ***Museum of Harmony and the Golden Section*** www.goldenmuseum.com. The uniqueness of this site in its bilingualism (Russian/English) and big concentration of exclusive information about Golden Section and its applications.

In 2003, the Slavic Golden Group organized the international conference *Problems of Harmony, Symmetry and the Golden Section in Nature, Science and Art* in the Ukrainian city of Vinnitsa. Here the great contribution of the Slavic scientists to the development of the theory of the golden section and its applications was recognized in the 38 books written on the golden section. According to the Conference resolution, the Slavic Golden Group was transformed into the *International Club of the Golden Section.* This group stimulated Slavic researchers to further develop their creative work. Included amongst the many interesting books written by this prestigious group of leading Slavic scientists, were the following very popular ones:

1. Bodnar, O.J. *The Golden Section and Non-Euclidean Ggeometry in Science and Art* (2005) (in Ukrainian) [52]. This book is the second edition of Bodnar's preceding book [37].

2. Petrunenko, V.V. *The Golden Section in Quantum States and its Astronomical and Physical Manifestations* (2005) (in Russian) [53].
3. Soroko, E. M. *The Golden Section, Processes of Self-organization and Evolution of System. Introduction into General Theory of System Harmony* (2006) (in Russian) [[56]. This book is the second edition of Soroko's preceding book [25].
4. Stakhov, A.P., Sluchenkova, A.A., and Scherbakov, I.G. *The da Vinci Code and Fibonacci Series* (2006) (in Russian) [55].

Together with the American Fibonacci Association, the Slavic Golden Group greatly influenced the development of contemporary research in the field of the golden section and Fibonacci numbers, and their applications.

2.20. Conclusion

The Fibonacci and Lucas numbers are two remarkable numerical sequences, which are becoming widely known in modern science. Fibonacci numbers were introduced into mathematics in the 13th century by the famous Italian mathematician Leonardo of Pisa (Fibonacci) as the solution to the Fibonacci rabbit reproduction problem. Fibonacci and Lucas numbers have many interesting mathematical properties. If we take the ratio of two adjacent Fibonacci numbers F_n/F_{n-1} and direct n towards infinity, the ratio approaches the golden mean in the limit. The discovery of this mathematical property is attributed to Johannes Kepler. In the 17th century Kepler's contemporary, the great astronomer Giovanni Cassini, proved the remarkable Cassini formula connecting three neighboring Fibonacci numbers. If we calculate the numerological values of all the terms of the Fibonacci series, then we will find that there is an intriguing periodicity, equal to 24, in this series of numerological values. The French 19th century mathematicians Lucas and Binet made great contributions to the development of Fibonacci number theory. Lucas introduced Lucas numbers into mathematics and Binet derived the famous mathematical formulas (Binet formulas) which connect Fibonacci and Lucas numbers with the golden mean. The Fibonacci and Lucas numbers show themselves throughout Nature's structures. The botanic phenomenon of phyllotaxis is the best recognized amongst them. However, during the last few decades, Fibonacci numbers have been revealed in the genetic code (Fibonacci Resonances), in psychology (Lefevre's experiments), and the extension of market processes (Elliott Waves), and numerous other areas and disciplines.

Chapter 3

Regular Polyhedrons

3.1. Platonic Solids

3.1.1. *Regular Polygons*

People show an interest in regular polygons and polyhedrons (i.e. polyhedra) throughout life – from the two-year-old child, playing with wooden cubes, to the mature mathematician. Some regular and semi-regular solids appear in Nature in the form of crystals, others as viruses seen only by using an electron microscope.

What is a polygon? Recall that geometry may be defined as a science of space and spatial figures – two-dimensional and three-dimensional. A two-dimensional geometric figure can be defined as a set of line segments limiting some part of a plane. Such a planar figure is called a *Polygon*.

Scientists have been interested for some time in the *Ideal* or *Regular* polygons, that is, the polygons having equal sides and equal angles. The idea of "regularity" and "ideal geometric figures" is very old in geometry. It dates back to the ancient Greek mathematicians and philosophers.

The equilateral triangle is considered to be the simplest regular polygon because it has the least number of sides necessary to limit part of a plane. The square (four sides), pentagon (five sides), hexagon (six sides), octagon (eight sides), decagon (ten sides) and so on together with the equilateral triangle (Fig. 3.1) provide a general picture of the regular polygons. It is obvious that there are no theoretical restrictions on the number of sides of a regular polygon, meaning there are an infinite number of polygons.

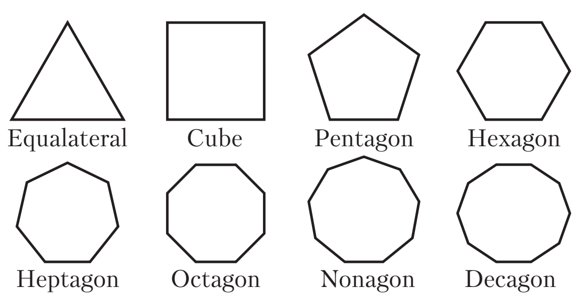


Figure 3.1. Regular convex polygons

Note that all regular polygons in Fig. 3.1 are *Convex*. What is a convex geometric figure? The *Convex* geometric figure is the opposite of a *Concave* geometric figure. To understand the distinction between convex and concave geometric figures, we examine two famous figures, the pentagon (Fig. 3.2-*a*) and pentagram (Fig. 3.2-*b*).

The regular pentagon in Fig. 3.2-*a* is a planar convex figure, while the pentagram in Fig. 3.2-*b* is a planar concave figure.

The notion of convex and concave geometric figures should be intuitively clear. Below we clarify these geometric concepts in greater detail when we begin the study of regular polyhedra.

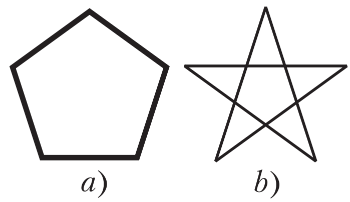


Figure 3.2. Regular pentagon (*a*) is a convex figure, pentagram (*b*) is a concave figure

The first textbook of geometry, Euclid's *Elements*, assumed convexity without providing a precise definition. Convex polygons have an interesting property connected with the interior angles of polygons. We know that the three angles in any triangle always total 180°. As the equilateral triangles have equal angles, each of its angles must be equal to 60° (60+60+60=180). The angles of any quadrangular polygon add to 360°. As the angles of a regular polygon are equal, each angle of the regular quadrangular polygon (square) is equal to 90° (90+90+90+90=360). The angles of any five-sided polygon add to 540°. It is clear that each angle of the regular pentagonal polygon (pentagon) is equal to 108° (108+108+108+108+108=540).

How can we calculate the interior angle of any regular *n*-gon? Take any vertex of a regular *n*-gon and draw from this vertex all possible diagonals within the polygon. This process divides the regular *n*-gon into $n-2$ (*n* minus 2) nonintersecting triangles. As three angles of a triangle total 180°, then the interior angles of any regular *n*-gon total [180° (*n*-2)], a characteristic property of the convex polygon. Table 3.1 lists the numerical values for the interior angles of regular *n*-gons.

Table 3.1. Interior angle measures in regular polygons

Name	Number of sides	Sum of interior angles	Interior angle
Triangle	3	180	60
Square	4	360	90
Pentagon	5	540	108
Hexagon	6	720	120
Octagon	8	1080	135
Nonagon	9	1260	140
Decagon	10	1440	144
Dodecagon	12	1800	150
...	...	...	...
n-gon	*n*	180(*n*-2)	180(*n*-2) /*n*

Note that regular polygons have many lines of symmetry. This characteristic is very important in their ability to build up different tessellations. We can see from Fig. 3.3 that the regular octagon has eight axes of symmetry: one between each pair of opposite vertices and one between each pair of opposite sides. The regular pentagon has five axes of symmetry: one between each vertex and its opposite side. A regular polygon with *n* sides has *n* axes of symmetry.

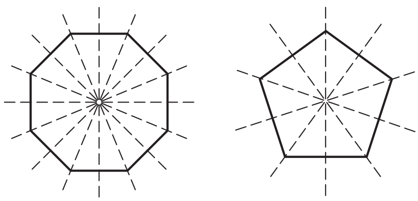
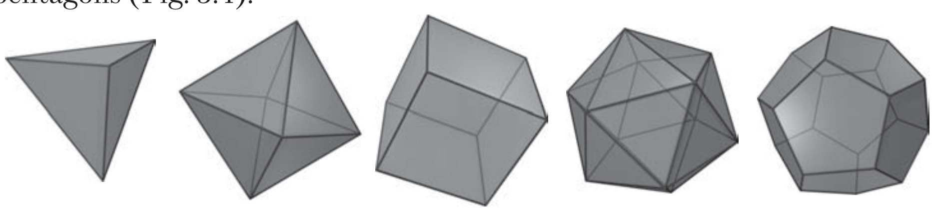


Figure 3.3. Symmetry lines of the regular octagon and the regular pentagon

3.1.2. *Regular Polyhedra*

A polyhedron is a "solid" three-dimensional figure similar to two-dimensional polygons discussed above. Polyhedra have vertices, edges and faces. If a polyhedron has faces that are regular polygons and if at each vertex exactly the same number of faces meets, such a polyhedron is called a *Regular Polyhedron*. How many regular polyhedra exist? At first sight, the answer to this question is very simple: as many as there are regular polygons that are faces of regular polyhedra. However, this is not the case. To find the correct answer to this question we must divide all regular polyhedra into convex and concave types.

Let us start with the convex polyhedra. Euclid's *Elements* give a strict proof of the fact that only five convex regular polyhedra exist, and their faces must be one of only three types of the regular polygons: triangles, squares and pentagons (Fig. 3.4).



a) *b*) *c*) *d*) *e*)

Figure 3.4. Platonic Solids: (*a*) tetrahedron (Fire), (*b*) octahedron (Air), (*c*) hexahedron or cube (Earth), (*d*) icosahedron (Water), (*e*) dodecahedron (Universal Mind)

Many books are devoted to the theory of polyhedra. *Polyhedron Models* by English mathematician M. Wenninger is the best known amongst them. The Russian translation of this book was published in 1974 [147]. Bertrand Russell's statement was chosen as an epigraph to this book: "Mathematics possesses not only truth, but also supreme beauty ... sublimely pure, and capably of a stern perfection such as only the greatest art can show."

The book [147] begins with a description of the convex regular polyhedra, that is, the polyhedra that are regular polygons of one type. These polyhedra are named the *Platonic Solids* in honor of the ancient Greek philosopher Plato, who used regular polyhedra in his cosmology. We begin our consideration with the regular polyhedra possessing equilateral triangles as faces. The *Tetrahedron* is the first and simplest of them (Fig. 3.4-*a*). The key observation is that the sum of the interior angles of the polygons that are meeting at every vertex is always less than 360 degrees. In the tetrahedron, three equilateral triangles (the sum of their interior angles is equal to $180°=3\times60°$) meet at each vertex; thus, their bases create a new equilateral triangle. The tetrahedron has the least number of faces among the Platonic Solids and therefore it is the three-dimensional analog to the equilateral triangle (which of course has the least number of sides among the regular polygons).

The *Octahedron* (Fig. 3.4-*b*) is the next spatial geometric figure based on equilateral triangles. In the octahedron, four equilateral triangles (the sum of their interior angles is equal to $240°=4\times60°$) come together at one vertex; as a result, a pyramid with a quadrangular base arises. If one connects two such pyramids by their bases, then the symmetric figure with eight triangular faces, called the *Octahedron*, appears.

Now, we can try to connect 5 equilateral triangles at one vertex (the sum of the interior angles is equal to $300°=5\times60°$). As a result, we obtain a spatial geometric figure with 20 triangular faces called the *Icosahedron* (Fig. 3.4-*d*).

A square is the next regular polygon (with interior angle of 90°). If we unite 3 squares at one vertex (the sum of their interior angles equaling $270°=3\times90°$) and then add to this figure three new squares, we obtain a perfect geometric figure with 6 faces called a *Hexahedron* or *Cube* (Fig. 3.4-*c*).

Finally, there is one more regular polyhedron to construct based on the use of a pentagon with the interior angle 108°. If we collect 12 pentagons so that 3 regular pentagons come together at each vertex (the sum of their interior angles is equal to $324°=3\times108°$), then we obtain the next Platonic Solid with 12 pentagonal faces called the *Dodecahedron* (Fig. 3.4-*e*).

The hexagon is the next regular polygon after the pentagon. It has an interior angle 120°. If we connect 3 hexagons at one vertex, we obtain a surface because the sum of their interior angles is equal to $360°=3\times120°$. This means that it is impossible to construct a three-dimensional geometric figure from hexagons alone. Other regular polygons after the hexagon have interior angles greater than 120°. This means that we cannot construct spatial geometric figures from them. It follows from this examination that **there are only 5 convex regular polyhedra, the faces of which are limited to equilateral triangles, squares and pentagons**.

There are surprising geometrical connections between all regular polyhedra. For example, the cube (Fig. 3.4-*c*) and the octahedron (Fig. 3.4-b) are dual to one another, that is, they can be obtained from each other, if the centroids of the faces of the first figure are taken as the vertices of the other and conversely. Similarly, the icosahedron (Fig. 3.4-*d*) is dual to the dodecahedron (Fig. 3.4-*e*). The tetrahedron (Fig. 3.4-*a*) is dual to itself. The fact of the existence of only five convex regular polyhedra is surprising because the number of regular polygons is infinite! This fact was proven in Euclid's *Elements*.

3.1.3. *Numerical Characteristics of Platonic Solids*

Before continuing, let us collect some data about the regular polyhedra. Let

- ***m*** be the number of polygons meeting at one vertex,
- ***n*** be the number of vertices of each polygon,
- ***f*** be the number of faces of a polyhedron,
- ***e*** be the number of edges of a polyhedron, and
- ***v*** be the number of vertices of a polyhedron.

The values of these numbers for each of the regular polyhedra are listed in Table 3.2.

Table 3.2. Numerical characteristics of regular polyhedra

	n	m	f	e	v
Tetrahedron	3	3	4	6	4
Octahedron	3	4	8	12	6
Icosahedron	3	5	20	30	12
Hexahedron	4	3	6	12	8
Dodecahedron	5	3	12	30	20

Further, our aim is to show that for any pair of numbers n and m the values of the other parameters f, e, and v are uniquely determined. As two faces come together at one edge, we can write:

$$e=nf/2. \qquad (3.1)$$

Next, as m faces come together at each vertex, we can write:

$$v=nf/m. \qquad (3.2)$$

It is apparent from Table 3.2 that for all five regular polyhedra we have:

$$f=2+e-v. \qquad (3.3)$$

The result (3.3) is known as Euler's Polyhedron Theorem.

3.1.4. *The Golden Section in the Dodecahedron and Icosahedron*

The dodecahedron and its dual the icosahedron (Fig. 3.4-*d, e*) take up a special place among the Platonic Solids. First of all, it is necessary to emphasize that the geometry of the dodecahedron and icosahedron is directly connected with the golden section. The faces of the dodecahedron (Fig. 3.4-*e*) are pentagons based on the golden section. If we look carefully at the icosahedron (Fig. 3.4-*d*), we can see that five triangles come together at each vertex of the icosahedron; here, their external sides produce a pentagon based on the golden section. Already these facts demonstrate that the golden section plays an essential role in these two Platonic Solids.

A duality of the dodecahedron and the icosahedron is expressed in the fact that the number of the dodecahedron faces (f=12) is equal to the number of the icosahedron vertices (v=12) and the number of the icosahedron faces (f=20) is equal to the number of the dodecahedron vertices (v=20), but they both have the same number of edges (e=30). There is another numerical characteristic that unites the dodecahedron and the icosahedron: the number of planar angles on the surface of both spatial figures is equal to 60. As the number of vertices (and sides) n of the regular polygons (triangle and pentagon) that are the faces of the icosahedron and dodecahedron are equal to 3 and 5, respectively, and the number of faces of the icosahedron and dodecahedron are equal to 20 and 12, respectively, it follows that the following formula for the number of the planar angles on the surface of the icosahedron and the dodecahedron are valid:

$$60=3\times 20=5\times 12. \tag{3.4}$$

There are also deeper confirmations of the fundamental role of the golden section in the icosahedron and the dodecahedron. It is recognized that these Platonic Solids have three unique spheres. The first sphere (the inscribed sphere or insphere) is a sphere that is inserted into the Platonic Solid and touches the centroids of its faces. Define the radius of this insphere by R_i. The second or middle sphere (midsphere or intersphere) touches the centroids of its edges. Define the radius of the midsphere by R_m. The third (external) sphere or circumsphere is circumscribed around the Platonic Solid and passes through its vertices. Define this radius by R_c. It was proven in geometry that the radius lengths of the indicated spheres for the dodecahedron and the icosahedron with sides equal to 1 are expressed by the golden mean τ (see Table 3.3).

Table 3.3. Connection of the icosahedron and dodecahedron with the golden mean τ

	R_c	R_m	R_i
Icosahedron	$\frac{\tau\sqrt{3-\tau}}{2}$	$\frac{\tau}{2}$	$\frac{\tau^2}{2\sqrt{3}}$
Dodecahedron	$\frac{\tau\sqrt{3}}{2}$	$\frac{\tau^2}{2}$	$\frac{\tau^2}{2\sqrt{3-\tau}}$

Note that the ratios of the radii $\frac{R_c}{R_i} = \frac{\sqrt{3(3-\tau)}}{\tau}$ are equal for both the icosahedron and the dodecahedron. Thus, if the dodecahedron and the icosahedron have identical inserted spheres, then their described spheres are also identical. A proof of this mathematical theorem is given in Euclid's *Elements*.

Table 3.4. The golden mean in the areas and volumes of the icosahedron and dodecahedron

	Icosahedron	Dodecahedron
Outer area	$5\sqrt{3}$	$\frac{15\tau}{\sqrt{3-\tau}}$
Volume	$\frac{5\tau^5}{6}$	$\frac{5\tau^3}{2(3-\tau)}$

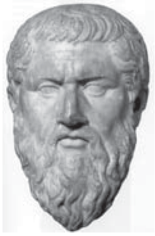
There are other well-known geometric relations for the dodecahedron and the icosahedron verifying their connection with the golden mean. For example, if we take the icosahedron and the dodecahedron with edge length equal to 1 and compute their external areas and volumes, then they will be expressed in terms of the golden mean (see Table 3.4).

Thus, there are a large number of relations obtained by the mathematicians of antiquity, which verify the remarkable fact, that the golden mean is the main proportion of the dodecahedron and icosahedron. A connection of the golden mean with dodecahedron and icosahedron is especially interesting from the point of view of the so-called "dodecahedron-icosahedron doctrine" considered below.

3.1.5. *Plato's Cosmology*

We mentioned above that the regular polyhedra were named *Platonic Solids* because they played a very important part in Plato's cosmology.

In Plato's cosmology, the first four polyhedra personified four "essences" or "elements." The *Tetrahedron* symbolized *Fire* with its top pointed upwards; the *Icosahedron* symbolized *Water* as the most "fluid" polyhedron; the *Cube* symbolized *Earth*, as the "steadiest" polyhedron; the *Octahedron* symbolized *Air* presumably the most "aerial" polyhedron. The fifth polyhedron, the *Dodecahedron*, symbolized *"the real World,"* "*Universal Reason,"* and was considered the main geometrical figure of the Universe or entire Cosmos.



Plato
(427 - 347 BC)

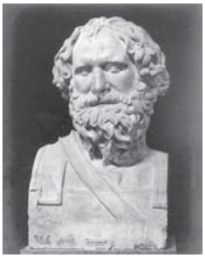
The ancient Greeks considered harmonious relations as a basis of the Universe, therefore the first four "elements" were connected by the following proportion: Earth/Water = Air/Fire. According to Plato, the atoms of the "elements" are in perfect consonances similar to four strings of lyre. It is pertinent to tell that such system of "elements" that includes the four elements, Earth, Water, Air and Fire, was canonized also by Aristotle. These four "elements" remained the four fundamental stones of the Universe within many centuries. Here we can use an analogy with today's well-known four states of substance: rigid, liquid, gaseous and plasma.

Thus, the ancient Greek idea about the Harmony of the Universe was bound together with its embodiment in the Platonic Solids. Plato's idea about the role of the regular polyhedra in the structure of the Universe influenced Euclid in his *Elements*. In this famous book, which over the centuries was the unique textbook of geometry, a description of "ideal" lines and figures is given. A straight line is the most "ideal" line, and also the regular polygons and polyhedra are the most ideal geometric figures. It is interesting that Euclid's *Elements* begins with the description of an equilateral triangle that is the simplest regular polygon and ends with a study of the five Platonic Solids. We mentioned above that the theory of the Platonic Solids was stated in the 13th, that is, final book of Euclid's *Elements*. That is why, the ancient Greek mathematician Proclus, a commentator on Euclid, put forward the interesting hypothesis about the true purpose for which Euclid wrote the *Elements*. In Proclus' opinion, Euclid wrote his *Elements* in order to provide a complete theory of the construction of the "ideal" geometric figures, in particular, the five Platonic Solids. In passing he gave in the *Elements* some advanced achievements of the Greek mathematics necessary to give a complete theory of the "ideal" geometric figures! Thus, we can consider Euclid's *Elements* as the first historically geometric theory of the Harmony of the Universe, based on the "Golden Section" (division in extreme and mean ratio) and the Platonic Solids!

3.2. Archimedean Solids and Star-shaped Regular Polyhedra

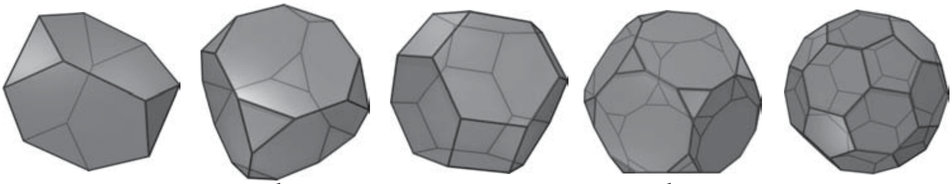
3.2.1. *Archimedean Solids*

There are 13 semi-regular convex polyhedra attributed to Archimedes. This well-known set of the perfect geometric figures is named *Archimedean* or *Semi-regular Polyhedra*.



Archimedes
(287 – 212 BC)

The Archimedean Solids can be divided into several groups. The first group consists of the five polyhedra that are formed from the Platonic Solids by means of the truncation of their vertices. For the Platonic Solids the truncation can be made in such a manner that the new faces and the remaining parts of the old faces are regular polygons. For example, the tetrahedron (Fig. 3.4-a) can be truncated so that its four triangular faces are converted into hexagonal faces and the four new regular triangular faces are added to them. In this manner the five Archimedean Solids are obtained: *Truncated Tetrahedron, Truncated Hexahedron* (*Cube*), *Truncated Octahedron*, *Truncated Icosahedron,* and *Truncated Dodecahedron* (Fig. 3.5).



a) b) c) d) e)

Figure 3.5. Archimedean solids: (*a*) truncated tetrahedron, (*b*) truncated cube, (*c*) truncated octahedron, (*d*) truncated dodecahedron, (*e*) truncated icosahedron

In his Nobel lecture (1996) the American chemist Richard E. Smalley, one of the authors of the experimental discovery of fullerenes, spoke about Archimedes (287 - 212 BC) as the first researcher of truncated polyhedra, in particular, the *Truncated Icosahedron.* It is his opinion, however, that Archimedes may have ascribed to himself this discovery, even though the icosahedron was truncated long before him. All these solids were described by Archimedes, although, his original works on this topic were lost and were known only from second-hand sources. Various scientists gradually rediscovered all these polyhedra during the Renaissance, and Kepler finally reconstructed the entire set of Archimedes' polyhedra in his 1619 book *The World Harmony* ("Harmonice Mundi").

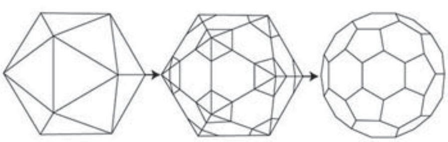
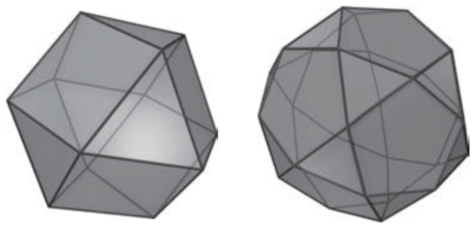


Figure 3.6. Construction of the Archimedean truncated icosahedron from the Platonic icosahedron

How we can construct the Archimedean truncated icosahedron from the Platonic icosahedron? The answer to this question is given in Fig. 3.6. We can see from Table 3.2 that five faces converge in each of the 12 vertices of the Platonic icosahedron. If we truncate 12 vertices of the icosahedron by a plane, then 12 new pentagonal faces will be formed. The old triangular faces are converted into hexagonal faces. 12 new pentagonal faces together with 20 hexagonal faces compose a truncated icosahedron with 32 faces. Here the edges and vertices are equal to 90 and 60, respectively.

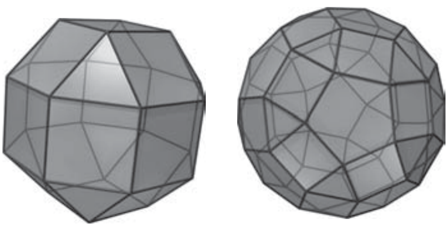
Another group of Archimedean solids consists of two quasi-regular polyhedra. A polyhedron is called *Quasi-regular* if it consists of two sets of regular polygons, let's say, m-sided and n-sided, respectively, and it is constructed so that each polygon of the first set is completely surrounded by polygons of the second set. There are two quasi-regular Archimedean solids named *Cuboctahedron* and *Icosidodecahedron* (Fig. 3.7). For the cuboctahedron (Fig. 3.7-*a*) $m = 3$, $n = 4$, and for the icosidodecahedron (Fig. 3.7-*b*) $m = 3$, $n = 5$.



a) b)

Figure 3.7. Cuboctahedron (a) and icosidodecahedron (b)

There are two Archimedean solids called the *Rhombicuboctahedron* (Fig. 3.8-*a*) and the *Rhombicosidodecahedron* (Fig. 3.8-*b*). The rhombicuboctahedron (Fig. 3.8-*a*) consists of two kinds of faces, squares and triangles. Each square is surrounded by four squares and each triangle is surrounded by three squares.

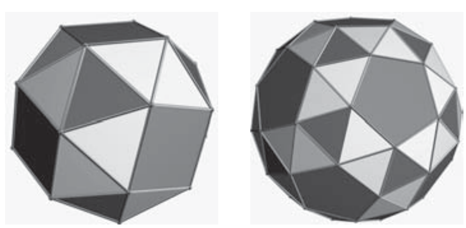


a) b)

Figure 3.8. Rhombicuboctahedron (*a*) and rhombicosidodecahedron (*b*)

Some believe that the rhombicosidodecahedron (Fig. 3.8-*b*) is the most attractive among the Archimedean solids. The rhombicosidodecahedron consists of pentagons, squares and triangles. Each pentagon is surrounded by five squares and there is a triangle surrounded by three squares.

Finally, there are two so-called "snub-nosed" versions of the Archimedean solids – one for the hexahedron (cube), *Snub Hexahedron* (Fig. 3.9-*a*), another – for the dodecahedron, *Snub Dodecahedron* (Fig. 3.9-*b*). The *s*nub hexahedron (Fig. 3.9-*a*) consists of six squares surrounded by triangles. Each square is surrounded by four triangles. Each triangle adjoined to a square is surrounded by two triangles. The same idea underlies a snub dodecahedron (Fig. 3.9-*b*). It consists of 12 pentagons where each is surrounded by five squares. The gap between squares is filled by triangles.



a) b)

Figure 3.9. Snub hexahedron (*a*) and snub dodecahedron (*b*)

3.2.2. *Star-shaped Regular Polyhedra*

In *Polyhedron Models* by M. Wenninger [147] we find 75 various models of regular and semi-regular polyhedra. The Russian mathematician Ljusternak, who made much of this area, wrote: "A theory of polyhedra, in particular, the convex polyhedra, is one of the most fascinating chapters of geometry." The development of this theory is connected with the names of several outstanding scientists. We mentioned Kepler's contribution to the development of the theory of polyhedra. He wrote the etude *About snowflakes* where he noted the following: "Among the regular polyhedra the cube is the very first one, the beginning and the primogenitor of others, if it is permissible so to say, its spouse is the octahedron because it has as many vertices as the cube has faces." Kepler published a full list of the 13 Archimedean solids and gave them the names under which they are now known.

Also Kepler started to study the so-called *Star-shaped Polyhedra* that unlike Platonic and Archimedean Solids are regular concave polyhedra. In the beginning of the 19th century the French mathematician and physicist Poinsot (1777 - 1859), whose geometric works relate to star-shaped regular polyhedra, followed Kepler's work and discovered two new star-shaped regular polyhedra. So, thanks to Kepler and Poinsot's works, four types of the star-shaped regular polyhedra (Fig. 3.10) became known. In 1812, the French mathematician Cauchy (1789 - 1857) proved that other star-shaped regular polyhedra do not exist.

Many readers may ask a question: "For what purpose is it necessary to study regular polyhedra? What benefit can we have from them?" We could answer this question by another question: "What benefit can we derive from music or poetry? Is all that is beautiful merely useful?" The models of polyhedra presented in Figs. 3.4-3.10, first of all, make an aesthetic impression upon us and can be used as decorative ornaments. However, it will be shown that a wide manifestation of regular polyhedra in natural structures caused a great deal of interest in modern science.

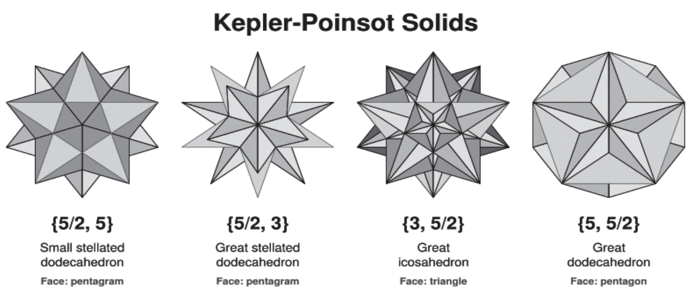


Figure 3.10. Star-shaped regular polyhedra (Kepler-Poinsot solids)

3.3. A Mystery of the Egyptian Calendar

3.3.1. *What is a Calendar?*

A Russian proverb states: "Time is the eye of history." Everything that exists in the Universe: the Sun, the Earth, stars, planets, known and unknown worlds – they all have spatial-temporal measurement. Time is measured by observation of the periodically repeating processes of definite duration.

In remote antiquity people noted that the day is always followed by night, the seasons of the year alternating on a regular basis: after the winter the spring comes, after the spring the summer, after the summer the autumn...

By searching for the cause of these phenomena, a person paid attention to the celestial heavenly bodies, the Sun, the Moon, stars, and the strict periodicity of their movement in the firmament. It was the first celestial observations that preceded the origin of astronomy, one of the most ancient sciences. As the basis of time measurement, the astronomers had used three important astronomical phenomena: rotation of the Earth around the axis, motion of the Moon around the Earth and motion of the Earth around the Sun. The various concepts of time often depended on the physical phenomenon used for time measurement. Astronomy knows of a stellar time, solar time, local time, zone time, nuclear time, etc. The Sun, as well as all other heavenly bodies, participates in the motion of the firmament. Except for diurnal motion, the Sun has so-called annual motion, and the totality of the annual motion of the Sun on the firmament is an *Ecliptic*. If, for example, we fix the location of the constellations at some definite moment of time and then repeat this observation each month, we can see different pictures of the palate. A view of the starry sky changes continuously: each season has its own picture of vesper constellations recurring yearly. Therefore, on the expiration of one year the Sun returns to its initial location in the starry sky.

To be oriented conveniently in the stellar world, astronomers divided the entire firmament into 88 constellations. Each of them has its own name. Among the 88 constellations, those that are on the ecliptic, play a special role. These constellations have a generalized title – "the zodiac" (from the Greek word "zoop" – animal), and are widely known throughout the world as symbols (signs) with a variety of allegorical meanings and calendar systems.

During its apparent movement along the ecliptic, the Sun intersects 13 constellations. However, astronomers found it necessary to divide the Sun's path into 12 parts (not 13), by uniting the constellations of Scorpions and the Dragon in one unified constellation under the common name Scorpio.

The special science of *Chronology* studies the problems of time measurement. This science underlies all calendar systems constructed by mankind. In antiquity the creation of calendars was one of the major problems of astronomy.

What are "calendars" and "calendar systems?" The word "calendar" originated from the Latin word "calendarium" which meant a "debt book"; in Ancient Rome interest was tracked in such books and was to be paid on the first day of each month, called "calendas."

The calendar makers paid great attention to periodicities in the motion of the Sun, the Moon, as well as, Jupiter and Saturn, the two gigantic planets of the Solar system. The idea of Jupiter's calendar with the celestial symbolism of a 12-year animal cycle may be connected to the rotation of Jupiter, which makes its full rotation around the Sun in approximately 12 years (11.862 years). On the other hand, Saturn, the second gigantic planet of the Solar system, makes its full rotation around the Sun in approximately 30 years (29.458 years). By wishing to coordinate the cyclic movements of the gigantic planets, the ancient Chinese introduced the idea of a 60-year cycle of the Solar system. During this cycle, Saturn makes 2 full rotations around the Sun, and Jupiter makes 5 rotations.

From antiquity the calendar makers of Eastern and South-East Asia made use of the following astronomical phenomena: alternation of day and night, change of lunar phases and alternation of seasons. The use of different astronomical phenomena resulted in the creation of three kinds of calendars: the lunar one based on the motion of the Moon, the solar one based on the motion of the Sun, and the combination lunar-solar one.

3.3.2. *Structure of the Egyptian Calendar*

The Egyptian calendar that was produced in the 4th millennium BC was one of the first solar calendars. One year in this calendar consisted of 365 days. One *Year* was divided into 12 months, each *Month* consisted of 30 days; at the end of the year the 5 holidays that were not part of the month structure were added. Thus, the Egyptian calendar year had the following structure: $365 = 12\times30 + 5$. It is important to note that the Egyptian calendar is a predecessor of the contemporary calendar.

Why the Egyptians divided the calendar year into 12 months? We know that there were calendars with other numbers of months in the year. For example, in the Maya calendar one year consisted of 18 months, each month being 20 days. Furthermore, why did each month in the Egyptian calendar have exactly 30 days? One may ask questions concerning the Egyptian system of time measurement, in particular, regarding the choice of such units of time, such as *Hour*, *Minute* and *Second*. In particular, the question is: why was unit of hour chosen so that 1 day=24(2×12) hours? Further, why 1 hour=60 minutes, and 1 minute=60 seconds? The same questions concerning the choice of the measurement units of angular magnitudes, in particular, why is a circumference divided into 360° that is, why $2\pi=360°=12\times30°$? We can ask other questions, for example: why do astronomers find it expedient to introduce 12 “zodiacal” constellations, instead of the 13 the Sun appears to intersect during its motion along the ecliptic? And why does the Babylonian number system have a rather esoteric radix, the number 60?

3.3.3. *Connection of the Egyptian Calendar with the Numerical Characteristics of the Dodecahedron*

Analyzing the above questions we discover with surprising consistency the following four numbers are repeated: 12, 30, 60 and the derivative number 360 (360=12×30). Is there some scientific fact that could give a simple and logical explanation of the use of these numbers in the Egyptian calendar, and their system of time and angle measurement? To answer this question, we return once again to the regular dodecahedron based on the golden section (Fig. 3.4-*e*).

Did the Egyptians know the dodecahedron? Historians of mathematics recognize that the ancient Egyptians had information about the regular polyhedra. The ancient Greek mathematician Proclus attributes to Pythagoras the construction of all 5 regular polyhedra. However, we know that Pythagoras borrowed from the ancient Egyptians many mathematical theorems and discoveries, in particular, the Pythagorean Theorem. Some accounts claim Pythagoras spent 22 years in Egypt and 12 years in Babylon. Therefore, it is possible that Pythagoras could have also borrowed the knowledge about the regular polyhedra from the ancient Egyptians.

However, there is more substantial proof that the Egyptians possessed information about all 5 regular polyhedra. In particular, the British Museum has preserved the dice from Ptolemy’s epoch that have the form of an icosahedron, the Platonic dual to the dodecahedron. These facts allow us to put forward the hypothesis that the dodecahedron was known to the Egyptians. The very unusual theory of the origin of the Egyptian calendar, as well as the Egyptian measurement system of the time and geometric angles follows from this hypothesis.

Earlier we found that the dodecahedron has 12 faces, 30 edges and 60 planar angles on its surface. If we accept the hypothesis that the ancient Egyptians knew the dodecahedron and its numerical parameters 12, 30 and 60, then the scientists of antiquity should not have been surprised, when they discovered that the cycles of the Solar system are expressed by the same numbers (12-year cycle of Jupiter, 30-year cycle of Saturn, and 60-year cycle of the Solar system). Thus, there is a deep mathematical connection between the Solar system and this perfect spatial figure, the dodecahedron. Scientist of antiquity apparently came to this conclusion. This may explain why the Egyptians (and Plato) chose the dodecahedron as the "Main Geometric Figure," that symbolizes the "Harmony of the Universe." It appears that the Egyptians made all their main systems (calendar system, systems of time and angle measurement) correspond to the numerical parameters of the dodecahedron! According to ancient thought, the motion of the Sun on the ecliptic was strictly circular. By then choosing the 12 Zodiac constellations with the distance of 30°, the Egyptians were able to coordinate the yearly motion of the Sun on the ecliptic with the structure of their calendar year: **one month corresponded to the apparent motion of the Sun along the ecliptic between two adjacent Zodiacal constellations**! Moreover, **the movement of the Sun one degree along the ecliptic corresponded to one day in the Egyptian calendar**! Thus, the ecliptic was divided automatically into 360°. By dividing one day into two parts, the Egyptians thereby automatically divided each half of one day into 12 parts (12 faces of the dodecahedron) and introduced the *Hour*, a major unit of time. By dividing one hour into 60 minutes (60 planar angles on the surface of the dodecahedron), the Egyptians introduced the *Minute*, the next important unit of a time. And of course this allowed them to introduce the *Second* (1 minute = 60 seconds), the smallest unit of time in that period.

Thus, by choosing the dodecahedron as the **Main Harmonic Figure of the Universe** and by following strictly to its numerical characteristics (12, 30 and 60), the Egyptians designed a perfect calendar together with the systems of time and angle measurement that have stood the test over several millennia. These systems of course correspond to the golden mean "Theory of Harmony," the underlying proportional basis of the dodecahedron.

These surprising conclusions follow from a simple comparison of the dodecahedron with the Solar system. And if our hypothesis is correct (let somebody attempt to deny it), it follows that for several millennia mankind has lived under the standard of the golden section! And each time, when we look at the index dial of our watch based on the numerical parameters of the dodecahedron 12, 30 and 60, we touch the "Main Secret of the Universe," the "Golden Section!"

3.3.4. *The Mayan Calendar*

The Maya were a very advanced race that made great achievements in the fields of mathematics and astronomy. Their calendar was, at the time, the most accurate calendar in the world, surpassed later only by the current Gregorian calendar (introduced in the 16th century). Their calendar was the product of the obsession with measuring long periods of time, which they tracked by means of highly accurate observations of the stars and the planets. The Mayan calendar had the following structure: 1 year=360+5=20×18+5. Unlike the Egyptian calendar, the Mayan calendar year was divided into 18 months of 20 days. It is clear that the structure of the Mayan calendar was similar to the structure of the Egyptian solar calendar: 1 year=360+5=12×30+5. Remember that the numbers 12 and 30 are numerical parameters of the dodecahedron. However, what does the number 20 refer to in the Maya calendar? Let us address again the icosahedron and the dodecahedron. In these "sacred" figures, there is one more "sacred" numerical parameter: the number of the icosahedron faces is 20 and the number of the dodecahedron vertices is 20! Thus, the Maya undoubtedly used these numerical characteristics of the icosahedron and dodecahedron in their calendar by means of the division of one year into 20 months.

3.4. A Dodecahedron-Icosahedron Doctrine

3.4.1. *Sources of the Doctrine*

Plato's cosmology became a source of the so-called *Dodecahedron-Icosahedron Doctrine*, which by a golden thread passes through all human science. The essence of this doctrine consists in the fact that the dodecahedron and the icosahedron are typical forms of Nature in all its manifestations from the macrocosm down to the microcosm.

According to the remark of one commentator on Plato's works, for Plato "all cosmic proportionality is based on the principle of the "golden" or harmonic proportion." Plato's cosmology is based on the Platonic Solids. Each Platonic Solid symbolized one of the five "beginnings" or "elements": the *Tetrahedron* – the body of *Fire*, the *Octahedron* – the body of *Air*, the *Hexahedron* (*Cube*) – the body of *Earth*, the *Icosahedron* – the body of *Water*, the *Dodecahedron* – the body of the *Universe*. A representation of the general harmony of the Universe was invariably associated with its embodiment in these regular polyhedra. The fact that the dodecahedron, the "Main Cosmic Figure," was based on the golden section gave to the latter a deep sense in its qualifying to be the "Main Proportion of the Universe."

We mentioned above that, according to Proclus, the commentator on Euclid's *Elements*, Euclid did consider a theory of geometric construction of the Platonic Solids, the "Main Geometric Figures of the Universe," as the main goal of Euclid's *Elements*. Therefore, Euclid placed this major mathematical information in the final, that is, 13th book of his *Elements*.

3.4.2. *A Shape of the Earth*

The question about Earth's shape has perennially attracted the attention of scientists since antiquity. When the hypothesis about the spherical shape of the Earth received scientific substantiation, the idea arose that the Earth is a dodecahedron in shape. Socrates wrote: "The Earth, if to look at it from outside, is similar to the ball consisting of 12 pieces of skin." Socrates' hypothesis found further scientific development in the works of physicists, mathematicians and geologists. So, the French geologist de Bimon and the French mathematician Poincaré proposed the hypothesis that the Earth is a deformed dodecahedron in shape.

In the first half of the 20th century, the Russian geologist Kislitsin proposed the hypothesis that 400-500 million years ago the geo-sphere's dodecahedral shape was converted into the geo-icosahedron. However, such transformation appeared incomplete. As a result, the geo-dodecahedron appeared as if to be inserted into the frame of the icosahedron.

Recently, the Moscow researchers Makarov and Morozov have proposed another interesting hypothesis that concerns the shape of the Earth. They supposed that the kernel of the Earth has the shape and properties of a growing crystal that is either a dodecahedron or icosahedron. This crystal influences the development of all natural processes that occur on our planet. Its energetic field is the cause of this dodecahedral-icosahedral structure of the Earth. The influence of this energetic field reveals itself in the fact that we find evidence on the Earth's surface of the dodecahedron and the icosahedron as if they were inserted into the globe.

In recent years, the hypothesis about the icosahedron-dodecahedron shape of the Earth was subjected to verification. For this purpose, scientists combined the axis of the dodecahedron with the axis of the Globe; then they started to rotate the dodecahedron around this axis. They saw that the edges of the dodecahedron coincide with the gigantic disturbances of Earth's crust. Then they started to rotate the icosahedron around the Globe. They saw that its edges coincide with the smaller-sized partitioning of Earth's crust (mountain ranges, breaks etc.). These observations support the hypothesis of the closeness of the tectonic framework of Earth's crust with the shapes of the dodecahedron and the icosahedron.

It is as if the vertices of the hypothetical geo-crystal are the centers of certain anomalies on the planet: in them all global centers of extreme atmospheric pressure including the regions of hurricane origin are located. In one of the hubs of the geo-icosahedron (in Gabon) the natural nuclear reactor, which acted 1.7 million years ago, was found. The gigantic mineral fields (for example, Tyumen's oil field), the anomalies of the animal world (the lake of Baikal), the centers of the development of mankind's civilizations (Ancient Egypt, Northern Mongolia, etc.) coincide with many of the vertices of the two polyhedra. All of these examples tend to confirm Plato's surprising intuition.

3.5. Johannes Kepler: from "Mysterium" to "Harmony"

3.5.1. *"Mysterium Cosmographicum"*

Among the fathers of the new European science there was not a person more mysterious than Johannes Kepler: it seems that he connected two scientific epochs not only by his "elliptic" astronomical works, but also his unique personality. On the one hand, Kepler was a professional astrologer, a dreamer and visionary, whose style of thinking was unacceptable to the creators of classical science, including Galileo and Newton. On the other hand, this astrologer, almost medieval in his style of thinking, introduced basic concepts into science. The modern, that is, mechanistic concept of *Force* was introduced by Kepler. He introduced the concept of *Inertia* that distinguishes modern physics from all former physics. At the same time he introduced the concept of *Energy*. However, the discovery of *New Quantitative Laws of Astronomy* was his main scientific achievement. Kepler is the founder of physics of the heavens. This remarkable word-combination is a subtitle of his basic work: "New Astronomy based on the Causes, or Physics of the Heavens."

Johannes Kepler is known by all of educated mankind as the author of three famous astronomical laws that overturned astronomical ideas that had existed from antiquity. However, it is less well-known that these laws were obtained by Kepler as insights resulting from his ambitious research program into Universal Harmony that he pursued at a young age.

Johannes Kepler was born in 1571 into a poor Protestant family. In 1591, he enrolled in the Tubingen Academy where he received quite good mathematical education. In the Tubingen Academy the future great astronomer became acquainted with the heliocentric system of Nicolaus Copernicus. After graduation from the Academy, Kepler obtained a Masters degree and then was appointed mathematics professor at the Graz secondary school (Austria). The small book with the intriguing title *Mysterium Cosmographicum,* his first astronomical work, was published by Kepler in 1596 at the age of 25.

Johannes Kepler
(1571-1630)

Reading the *Mysterium Cosmographicum*, it is impossible not to be surprised by his visionary insights. A deep belief in the existence of the Harmony of the Universe was Kepler's main idea.

Kepler formulated the purpose of his research in the Foreword as follows:

"To you kind reader! In this book I intend to demonstrate that our almighty God at the creation of our moving world and at the disposition of the celestial orbits used the five regular polyhedra that are from Pythagoras' and Plato's times and up to now have received great honor. He chose the number and proportions of the celestial orbits and also the relations between the planetary motions pursuant to the nature of the regular polyhedra. I am especially interested in the nature of three things: why are the planets arranged this way and not otherwise, namely, the number, sizes and motions of the celestial orbits."

So, already in the Foreword of his first book the 25-year-old Kepler put forward the main problem of contemporary physics, the problem of the natural causes of physical phenomena. Though natural today, this problem in Kepler's times sounded unusual. In Ptolemy's and even in Copernicus' astronomy, this problem had not been formulated. According to that old tradition, the astronomers considered a problem of their science only in terms of a precise description of planetary motion and the possible prediction of celestial phenomena.

3.5.2. *Kepler's Cosmic Cup*

How did Kepler answer the surprising questions raised by him in *Mysterium Cosmographicum*? After the verification of the numerous hypotheses connected with the arrangement of planets, Kepler found the following geometrical model of the Solar system based on the "Platonic solids" (see Fig. 3.11):

"Earth's orbit is the measure of all orbits. We place the dodecahedron around this orbit. The orbit around the dodecahedron is Mars' orbit. We place the tetrahedron around Mars' orbit. The orbit around the tetrahedron is Jupiter's orbit. We place the cube around Jupiter's orbit. The orbit around the cube is Saturn's orbit. Then we insert the icosahedron inside Earth's orbit. The orbit inside the icosahedron is Venus' orbit. We insert the octahedron inside Venus' orbit. The orbit inside the octahedron is Mercury's orbit."

This model gave Kepler an opportunity to collect together all Platonic Solids so that they could unite the then "known" planetary spheres. There are only 5 Platonic Solids and only 5 interplanetary spaces, and all of them appear to allow the regular polyhedra to be placed in them. Could this be mere coincidence?

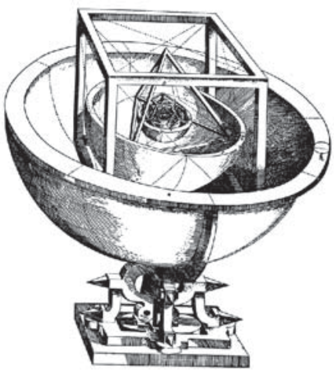


Figure 3.11. Kepler's *Cosmic Cup* as a model of the Solar system

Kepler's *Cosmic Cup* (Fig. 3.11) that inserts the Platonic Solids into the crystalline spheres embodies this model of reality. The most precious treasure of ancient geometry, the Platonic Solids, were used in Kepler's astronomy. After that Kepler had the right to say that he comprehended the Universe as if he created it with his own hands.

Kepler sent his *Mysterium Cosmographicum* to Galileo and Brahe and received reassuring responses from them. Armed with this support, Kepler addressed the court of Wurttemberg's duke Frederick in hopes of receiving the means for the creation of the new model of the Universe, the Cosmic Cup, in silver. On his application the duke recommended that he first make it in copper. The astronomer began to glue a paper model and then threw out the result after one week of hard work.

Certainly, the creation of the *Cosmic Cup* was a great success for the young astronomer. This scientific result had brought scientific renown to Kepler. However, this success was incomplete and even somewhat doubtful, because Kepler's basic scientific purpose had not been achieved. In his first scientific work, Kepler merely anticipated the secret of the world. The *Cosmic Cup* provided him with access into the invaluable observational data collected in the *Heavenly Castle* of Tycho Brahe. Kepler assisted the great astronomer in the study of Mars. By using Brahe's data, the most exact in the world, he intended to polish his new Cosmos, faceted by Platonic Solids, into a cosmic brilliance. It was necessary only to determine how the planetary orbits could be placed into the *Cosmic Cup*.

The *Cosmic Cup* resulted in an important conclusion that disclosed the main secret of the Universe: the Universe is created on the basis of a general geometric principle, the regular polyhedra! Unfortunately, Kepler's joy was premature. In spite of his over-fervid character, Kepler had all the characteristics of a serious scientist. He believed that the theory should fit the observational data. By restraining his delight, Kepler undertook to verify his model.

The universal geometric principle allowed Kepler to give answers to two of the three problems that had been raised by him: (1) to explain the number of the then known planets (with the help of the five Platonic Solids, it is possible to construct 6 orbits; the conclusion follows regarding the existence of 6 planets known in that period); (2) to give an answer about the distances between planets. The answer to the third problem (about the motion of planets) turned out to be the most difficult and Kepler only attained this answer many years later.

3.5.3. *Discovery of Kepler's First Two Astronomical Laws*

Kepler's model in Fig. 3.11 was based upon the supposition that nature's planetary motion is spherical. By obtaining data from the perennial observations of the famous astronomer Brahe, and then carrying out his own observations, Kepler decided to reject the astronomical models of his forerunners, Ptolemy and Copernicus, and even his own models. After a thorough study of the planetary orbits, he came to the following conclusion:

"The fact that planetary motions are circular is confirmed by their incessant recurrence. The human intellect, by extracting this truth from experience, at once concludes from here, that the planets are rotated on the 'ideal' circles, because among the planar figures a circle and among the spatial figures a sphere can be considered as the most perfect geometric figures. However, after closer examination we can conclude that experience gives another result, namely, the planetary orbits differ from simple circles."

The results of this work are presented in Kepler's main book *A New Astronomy Based on the Causes, or Physics of the Heavens* published in 1606. The importance of this book consists, first of all, of the fact that Kepler therein formulates the first two of three astronomical laws named after him. According to *Kepler's First Law*, the orbit of each planet is an *Ellipse* with the Sun as one of its foci. *Kepler's Second Law* asserts that the areas, covered by the radii-vectors of the planets, are constant.

Kepler's Laws are the first quantitative laws that are of great importance for the development of astronomy. By testing the *Cosmic Cup*, Kepler found that this model based upon circular planetary orbits did not fit the experimental data. Therefore, Kepler decided to reject the idea of circular planetary motion. After numerous and difficult calculations Kepler found that the orbits of planets were not circles but *Ellipses*. This result dramatically altered much of the basis of previous cosmology.

3.5.4. *"Harmonices Mundi"*

Kepler's third astronomical law, formulated by him in *Harmonices Mundi* (*The Harmony of the World*) completed the creation of a new astronomy.

Kepler's first two laws could not answer one major question of astronomy. How is it that the planetary distances from the Sun change? Kepler tried to grasp a new principle for the solution to this challenge. He used his philosophical views going back to Pythagoras and Plato. According to Kepler's deep belief, Nature was created by God on the basis of not only mathematical, but also harmonious principles. Kepler believed in the "music of the spheres" that generate bewitching melodies embodied not in sounds, but in the movements of the planets capable of generating harmonious chords. Based on this idea, Kepler combined mathematical and musical arguments to discover the third law of planetary motion that asserts the following:

"If T is the cycle of time of a planet's orbit around the Sun, and D is its middle distance from the Sun, then they are connected by the following correlation: $T^2=kD^3$, where k is a constant value equal for all planets."

Kepler arrived at the following conclusion from this discovery: "Thus, the heavenly movements are a never ending polyphonic music that is perceived by the human intellect, but not the ear."

The Russian scientist Predtechensky (1860-1904), Kepler's biographer, wrote about this Kepler's discovery as follows: "The wonderful harmony, reigning in the world, was perceived by Kepler not only in an abstract sense of the organization of the Universe. The harmony was sounding in its poetic soul by a true form of music, which could be understood by us if we could enter into the circle of his ideas and would be imbued with his mighty enthusiasm about the marvelous construction of the world and the Pythagorean belief in numerical relations. Really, it is surprising that what makes sounds 'beautiful' for hearing depends upon a strict numerical ratio, for example, between the lengths of the strings, which produce the sounds as discovered by Pythagoras. But in Kepler, undoubtedly, part of Pythagoras' soul lived on, and it is not accidental that he saw the numerical ratios in the planetary cosmos."

Kepler's third Law is the outstanding scientific result, at the summit of his career. The result obtained filled Kepler's soul with great joy and gratitude for the Creator. He expressed this gratitude in the following words:

"The wisdom of the Creator is endless and his glory and power are boundless. You, the heavens, glorify and praise be to Him! The Sun, the Moon and the planets, glorify God with their ineffable language! Celestial harmonies, comprehended by His wonderful creations, do glorify and praise to Him! Let my soul eulogize the Creator! Everything is built by Him and everything is embodied within Him. Everything the best which is known to us is created by Him in our bustling life. Praise, Honor and Glory to Him for all of eternity!"

The dramatic period in astronomical development was completed. This period ended with the discovery of three major astronomical laws of planetary motion, Kepler's Laws. This history started with an original model of the Solar system based on the Platonic Solids. Though Kepler's model, in the end, appeared somewhat erroneous, Kepler did not reject from this model what he described in his *Mysterium Cosmographicum*. He always considered this model to be one of his greatest scientific achievements. By submitting to the requests of his friends, Kepler in the twilight of his life decided to undertake the second main issue of his first book "for benefit not only to booksellers, but also scientists." Addressing the new readers, Kepler wrote in the dedication with pride:

"Almost 25 years have passed since I issued a small book *Mysterium Cosmographicum*. Though in that time I was very young and this publication was my first astronomical work, nevertheless the success accompanying this book in subsequent years, testifies eloquently that no one could write a first book with a more substantive, successful and valuable treatment of the subject. It is as if an oracle from the heavens dictated through me the chapters of this book, because all of them, as is generally accepted, were excellent and corresponded to the truth. During the 25 years, the chapters of this book illuminated my way in astronomy many times. Almost all my astronomical works, which I published by this time, had their beginning in one or another chapter of my first book, and therefore these later books can be considered to be a more in-depth or more complete presentation of these original chapters."

3.5.5. *Life through the Centuries*

Kepler's life is an example of scientific selflessness based upon the timeless belief in the Harmony of the Universe. His entire life was a struggle on two fronts. On the one hand, the "mathematician of the famous province Stiria" struggled against poverty and the almost intolerable "life" of the poor having so many children. His life was accompanied by the depression of his wife, the untimely deaths of his children, the charges of sorcery against his mother, and the dullness of coreligionists. On the other hand, Kepler's scientific life was a world of calculations, improbable in their complexity. "The intense, incessant and vigorous thoughts" were the basis of Kepler's unique scientific life.

However, old age was coming. Kepler's death (in 1630) interrupted his work on the last book *Somnium* (*Dreams*), a science fiction novel about flight to the Moon. Unfortunately, the Harmony was not written in this book. There were no captious verifications; there were no new hypotheses. Kepler was tired: "My brain got tired when I attempted to understand what I wrote, and it is already difficult for me to re-establish a connection between figures and text found by me at an earlier time ..."

So, the drama had been finished. With Kepler's death, his discoveries were forgotten. Even the wise Descartes did not know about Kepler's works. Galileo had not found necessary to read his books. Only in Newton's works Kepler's Laws find a new life. However, the Harmony is not of interest to Newton. He is interested in Equations. A new era had arrived.

Kepler's life terminated the epoch of "scientific romanticism," the epoch of the Harmony of the Golden Section that is characteristic of the Renaissance. On the other hand, his scientific works became the beginning of the new science that started with the works of Descartes, Galileo and Newton.

And in conclusion we once again remember Kepler's well-known statement about the Golden Section:

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."

For those, who treat skeptically Kepler's comparison of the Pythagorean Theorem with the Golden Section, we should remind them that Kepler was not only a great astronomer, but also a great mathematician! Predictions of great scientists can sometimes dramatically advance the development of a science. The development of modern science confirms that Kepler was right: the Golden Section becomes one of the major ideas of modern science!

Unfortunately, after Kepler's death, the Golden Section, that was considered by him to be one of the great "treasures of geometry," was forgotten. This unfortunate ignorance was continued for almost two centuries. With few exceptions interest in the Golden Section was revived only in the 19th century!

3.6. The Regular Icosahedron as the Main Geometrical Object of Mathematics

3.6.1. *Felix Klein*

Among the five Platonic Solids, the icosahedron and the dodecahedron have a special place. In Plato's cosmology the icosahedron symbolizes Water, and the dodecahedron - the Harmony of the Universe. These two Platonic

Solids are connected directly with the pentagon and through it the Golden Section. The dodecahedron and icosahedron give rise to the so-called "icosahedron-dodecahedron doctrine" that permeates the history of human culture from Pythagoras and Plato up to the present. It is not an accident that this doctrine received a unique boost or development in the works of the German mathematician Felix Klein (1849-1925).

Felix Klein
(1849-1925)

Felix Klein, a graduate of the University of Bonn, was born in 1849 and died in 1925. Beginning in 1875 he worked as a Professor of the Higher Technical School in Munich, then from 1880 as a Professor of the University of Leipzig. In 1886 he arrived in Gettingen, where he headed the Mathematical Institute of the University of Gettingen. During the first quarter of the 20th century this Mathematical Institute was recognized as the World's leading center of mathematics.

Klein's main works were dedicated to Non-Euclidean geometry, and the theories of continuous groups, algebraic equations, and elliptic and automorphic functions. Klein presented his ideas in the field of geometry in a *Comparative Consideration of the New Geometrical Researches* (1872) known under the title *Erlangen's Program*.

According to Klein, each geometry is some invariant theory of a special group of transformations. By dilating or narrowing down the group, it is possible to pass from one type of geometry to another. The Euclidean geometry is a science about invariants of the metric group, projective geometry - about invariants of the projective group, etc. The classification of transformation groups gives us the classification of the geometries. A proof for the existence of different Non-Euclidean geometries is considered to be Klein's most essential achievement.

3.6.2. *Elementary Mathematics from the Point of View of Higher Mathematics*

Felix Klein was not only a well-known mathematician-theorist, but also a reformer of mathematical education in the schools. Prior to World War I he organized the commission on reorganization of mathematical teaching. The book *Elementary Mathematics from the Point of View of Higher Mathematics* was devoted to the development of a solution to this problem.

Mathematics in the 19th century produced a number of remarkable ideas that influenced all branches of knowledge and engineering. The main idea behind the reformers headed by Klein was to increase the role of mathematics and natural science in the general education. The study of natural sciences and mathematics, a deepening of the connection between theoretical and applied mathematics, the introduction of functional analysis and calculus into mathematical teaching, and a broad use of graphic methods are the main principles suggested by Klein and his followers as a basis for mathematical education. It is important to recognize that Klein's ideas are rather topical for modern mathematical education.

3.6.3. *Role of the Icosahedron in Mathematical Progress*

Besides the *Erlangen Program* and other outstanding mathematical achievements, Klein's greatness consists of the fact that 100 years ago he predicted an outstanding role for the Platonic Solids, in particular, the icosahedron in the future development of science. In 1884, Klein published the book *Lectures on the Icosahedron and Solution of the 5th Degree Equations* [58] dedicated to a geometric theory of the icosahedron.

The icosahedron (and its dual, the dodecahedron) play a special role in "living" nature; many viruses and other living things have the shape of the icosahedron, that is, the icosahedron shape and pentagonal symmetry play a fundamental role in the organization of living substance.

According to Klein, the tissue of mathematics extends widely and freely by the sheets of the different mathematical theories. However, there are geometric objects, in which many mathematical theories converge. Their geometry unites these mathematical theories and allows us to embrace the general mathematical sense of the various theories. The icosahedron, in Klein's opinion, is just such a mathematical object. Klein treats the regular icosahedron as the central mathematical object, from which the branches of the five mathematical theories follow, namely: *Geometry, Galois' Theory, Group Theory, Theory of Invariants* and *Differential Equations*.

Thus, the great mathematician Felix Klein following after Pythagoras, Plato, Euclid, Johannes Kepler could realize the fundamental role of the Platonic Solids, in particular the icosahedron, for the development of science and mathematics. Klein's main idea is extremely simple: "Each unique geometrical object is somehow or other connected to the properties of the regular icosahedron."

Unfortunately, Klein's contemporaries could not understand and appreciate the revolutionary importance of Klein's idea proposed by him in the 19th century. However, its significance was appreciated one century later, when the Israeli scientist Dan Shechtman discovered in 1982 a special alloy with an icosahedral phase called *Quasi-crystals*. And the famous researchers

Robert F. Curl, Harold W. Kroto and Richard E. Smalley discovered in 1985 a special kind of carbon called *Fullerenes*. In 1996, they became Nobel Prize winners for their discovery. It is important to emphasize that the quasi-crystals are based on the Platonic icosahedron and the fullerenes on the Archimedean truncated icosahedron.

3.7. Regular Polyhedra in Nature and Science

3.7.1. *Symmetry Groups of Regular Polyhedra*

The influence of Klein's *Erlangen Program* on school geometry is especially important. The influence of Klein's "group approach" can be traced in all themes of school geometry. Each geometric figure *F* determines some group of movements; this group contains all those movements that convert the figure *F* in a manner similar to itself. This is called the *Symmetry Group* of the figure *F*. In many respects, knowledge of the symmetry group of the figure *F* determines the geometric properties of that figure.

Let us consider in greater detail some important concepts of symmetry theory. We start with the *Symmetry Plane P* that is familiar to us from the previous Chapter. We offer to the reader to be convinced that a square has four planes of symmetry (4*P*), however, a rectangle has only two planes of symmetry (2*P*). It can be easily proven that a cube has 9 planes of symmetry (Fig. 3.12), that is, 9*P*.

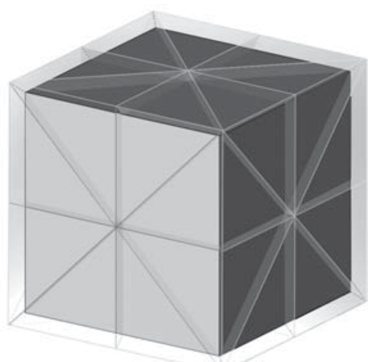
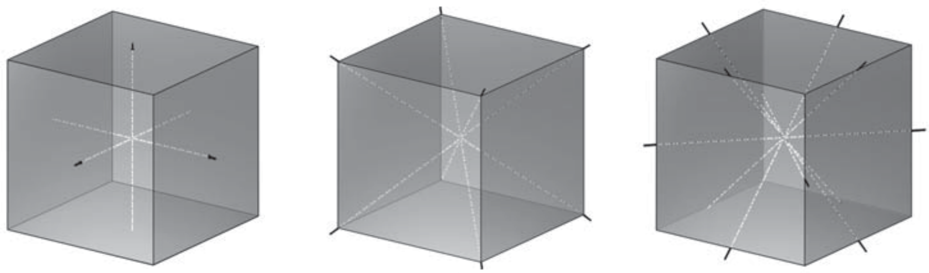


Figure 3.12. The 9 planes of symmetry of the cube

Let us now consider the second type of symmetry elements, the *Axis of Symmetry*. The axis of symmetry is based upon a straight line, around which the identical parts of a symmetric figure can be repeated a particular number of times. These identical parts are located so that after a turn around, the symmetry axis on the certain angle the figure occupies the same position as before the turn. As a result, the figure comes to "self-coincidence." A faceted glass in a glass holder is the best visual example of "self-coincidence." Taking the glass out of the glass holder and then inserting it back in the changed position, we execute the operation of "self-coincidence." The number of all possible "self-coincidences" that can be carried out around a given axis is named its *Symmetry Order*. Usually the axis of symmetry is denoted by the capital letter L, but its order is denoted by a small subscript number that stands after this capital letter. Thus, for example, the symmetry axis of the 3rd order is denoted by L_3. It is clear, that the equilateral triangle has the symmetry axis L_3, the square – L_4, the pentagon – L_5, and the circle has the symmetry axis of the infinite order L_∞.

As an example, for demonstration of symmetry axes, we examine a cube (Fig. 3.13).



a) *b*) *c*)

Figure 3.13. The symmetry axes of the cube: $3L_4(a)$, $4L_3(b)$, $6L_2(c)$

We can draw three axes of symmetry of the 4th order through the center of a cube perpendicularly to each pair of opposite faces (Fig. 3.13-*a*). This means, that a cube has 3 axes of symmetry of the fourth order, that is, $3L_4$. A cube has 8 vertices. We can draw a triple symmetry axis that coincides with the corresponding diagonal of the cube through each pair of the opposite vertices of the cube (Fig. 3.13-*b*). This means, that the cube has 4 symmetry axes of the 3rd order, that is, $4L_3$. The cube has 12 edges. We can draw double symmetry axes through the middles of each pair of the opposite edges, in parallel to the diagonals of the faces (Fig. 3.13-*c*). This means, that the cube has 6 symmetry axes of the 2nd order, that is, $6L_2$. Therefore, the full set of the cube axes of symmetry is the following: $3L_4 4L_3 6L_2$.

There are geometric figures that have symmetry axes of the infinite order L_∞. The so-called "figures of rotation," cylinder, cone, etc. have similar axes. Any diameter of a full-sphere is the axis of the type L_∞. This means, that a full-sphere has an infinite set of symmetry axes of infinite order, that is, ∞L_∞.

Now, let us consider one more element of symmetry, the *Center of Symmetry*. The center of symmetry is a special point inside the figure, when any straight line, drawn through this point, meets an identical point of the figure at equal distances from this point. The cube, for example, has a center of symmetry.

Usually for the description of symmetry of some geometric figure a full set of symmetries is used. For example, the symmetry group of a snowflake is $L_6 6P$. This means that a snowflake has one symmetry axis of the 6th order L_6, that is, the snowflake “self-coincides” 6 times at its rotation around the symmetry axis, and 6 planes of symmetry. The symmetry group of camomile having 24 petals is $L_{24}24P$, that is, the camomile has one symmetry axis of the 24th order and 24 planes of symmetry. By uniting all symmetry elements of the cube, we can write the following symmetry group of the cube: $3L_4 4L_3 6L_2 9PC$.

The cube is, of course, one of 5 Platonic Solids. Table 3.5 presents the symmetry groups of all Platonic Solids.

Table 3.5. Symmetry groups of the Platonic Solids

Polyhedron	**A shape of faces**	**Symmetry group**
Tetrahedron	Equilateral triangles	$4L_3 3L_2 6P$
Cube	Squares	$3L_4 4L_3 6L_2 9PC$
Octahedron	Equilateral triangles	$3L_4 4L_3 6L_2 9PC$
Dodecahedron	Pentagons	$6L_5 10L_3 15L_2 15PC$
Icosahedron	Equilateral triangles	$6L_5 10L_3 15L_2 15PC$

Analysis of the symmetry groups of Platonic Solids, given in Table 3.5, shows that the symmetry groups of the cube and the octahedron, on the one hand, and the dodecahedron and the icosahedron, on the other hand, coincide. Thus the dodecahedron is dual to the icosahedron, and the cube is dual to the octahedron.

3.7.2. *Applications of Regular Polyhedra in the Living Nature*

Ernst Heinrich Philipp August Haeckel (1834-1919) was the eminent German biologist and philosopher. He became famous following the publication of his book *Kunstformen der Nature* (*Art-shapes of Nature*)

Haeckel wrote in his book: “Nature has an inexhaustible number of surprising creations, which by beauty and variety far surpass all works created by human art.”

Nature’s creations presented in Haeckel’s book are beautiful and symmetric (see Fig. 3.14). This fact is an inseparable property of natural harmony. In his book he gave the examples of the single-cell organisms similar to the icosahedron in shape. The icosahedron attracted the attention of biologists in their disputes concerning the shape of viruses. The virus cannot be absolutely round as was considered earlier. To establish its form, the biologists took the various polyhedra and directed a light on them under the same angle, such as a stream of atoms on the virus. It is proved that only the icosahedron gives a precisely identical shadow. It is considered that the geometrical properties of the icosahedron allow it to preserve genetic information in the best possible manner.

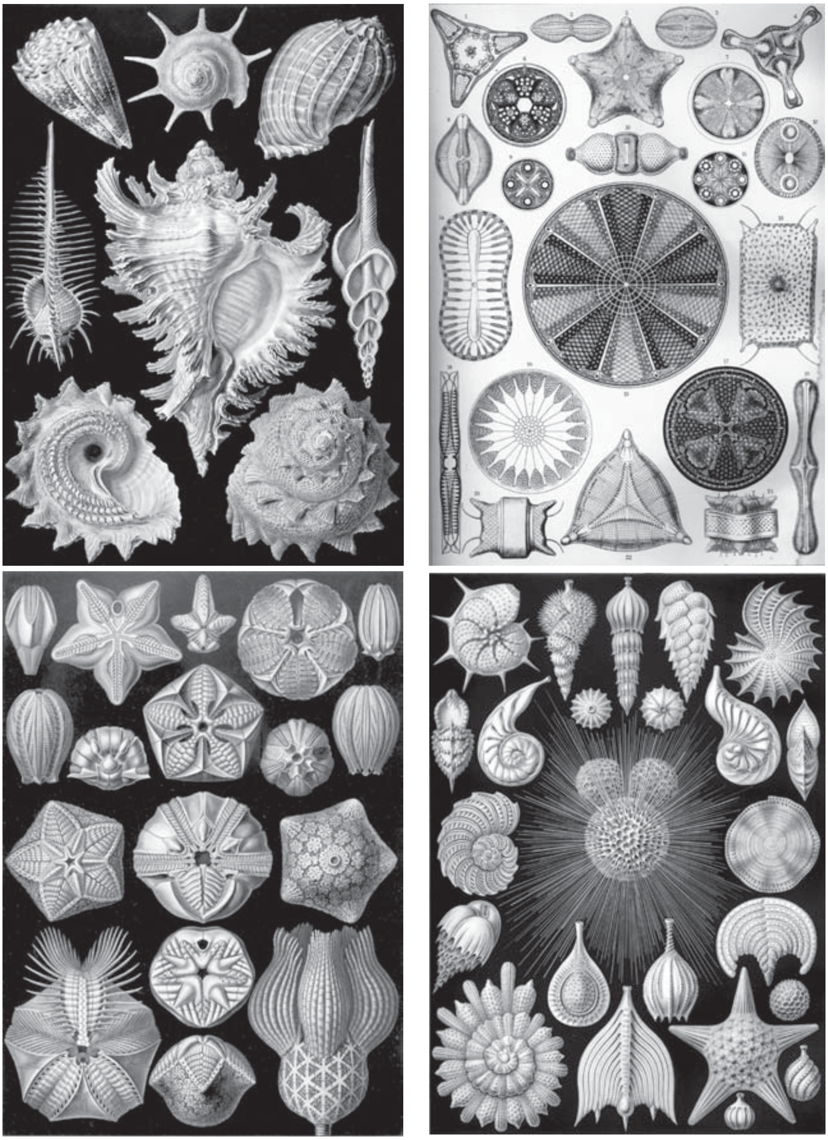


Figure 3.14. Art-shapes of Nature from Haeckel's book *Kunstformen der Nature* (1904)

The regular polyhedra are the most "preferred" and "symmetric" figures. Nature makes wide use of this property of the polyhedra. The crystals of many chemical substances have the shape of regular polyhedra. For example, a crystal of table salt *NaCl* has the shape of a cube, a mono-crystal of the potassium alum has the shape of an octahedron, a crystal of the chemical substance *FeS* has the shape of a dodecahedron, a sodium sulphite – a tetrahedron, a boron *Br* – an icosahedron and so on.

Today it has been proven that the process of the formation of a human embryo from the ovum is carried out by its divisions according to the "binary" law, that is, at first the ovum is transformed into two cells, these new cells are transformed in turn into four cells, etc. Earlier we considered that at the second stage of the division the four cells form a square. In fact, it occurs in another manner: at the second stage of the division, the four cells form the tetrahedron (Fig. 3.4-a). The cells are further divided into eight; together forming a geometric figure that consists of one tetrahedron pointing upward and the other tetrahedron pointing downwards, connected together. As a result we obtain the *Star Tetrahedron* that reminds one of the *Egg of Life* in esoteric sciences (Fig. 3.15).

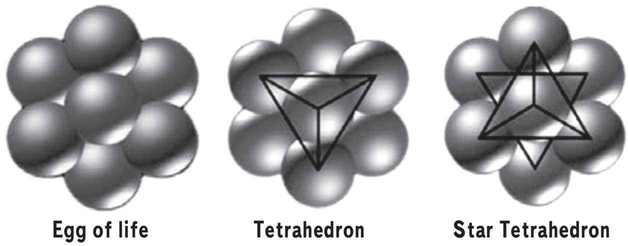


Figure 3.15. A star tetrahedron

3.7.3. *"Parquet Problem" and Penrose Tiling*

Since ancient times, "parquet's problem" in geometry has been the problem of how to fill a plane with regular polygons. Already the Pythagoreans proved that only regular (equilateral) triangles (symmetry axis of the 3rd order), squares (symmetry axis of the 4th order) and hexagons (symmetry axis of the 6th order) can be solutions to this problem. "Parquet's problem" has a direct relation to the main law of crystallography, according to which only the symmetry axes of the 3rd, 4th and 6th order are allowed in crystals. The symmetry axes of the 5th order and more than 6 are prohibited in crystals.

The English mathematician Sir Roger Penrose was one of the first scientists, who found another solution to "parquet's problem." In 1972, he covered a plane in a non-periodic manner, by using only two simple polygons. In the simplest form, the *Penrose's Tiles* are a non-random set of rhombi of two types, the first one (Fig. 3.16-a) has the internal angle 36°, and the second one (Fig. 3.16-b) has the internal angle 72°.

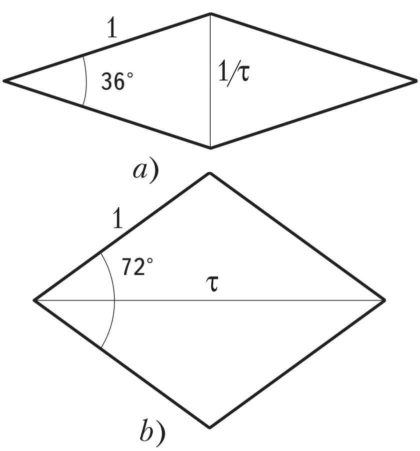


Figure 3.16. Penrose tiles: (*a*) "thin" rhombus; (*b*) "thick" rhombus

To understand the mathematical essence of the *Penrose Tiles*, we return to the *Pentagon* and *Pentagram* (Fig. 3.17).

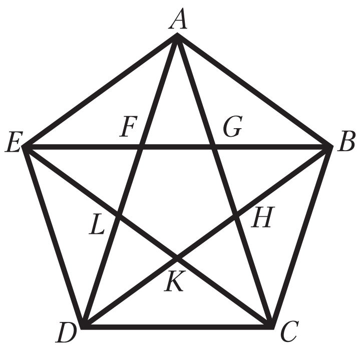


Figure 3.17. Pentagram

The pentagram contains a number of characteristic isosceles triangles. The triangle of the type *ADC* is the first of them. The acute angle at the vertex of *A* is 36°, and the ratio of the side *AC*=*AD* to the base *DC* is equal to the golden mean, that is, the given triangle is the golden isosceles triangle. Note that the pentagram consists of 5 identical golden isosceles triangles *ADC, BED, CAE, DBA,* and *ECB* (the 5 crossing triangles). Besides, we have in the pentagram 5 smaller "golden" isosceles triangles *AGF, BHG, CKH, DLK,* and *EFL* similar to *ACD*. If we now take two such triangles and connect them together by their bases, we obtain the Penrose's tile displayed in Fig. 3.16-*a* and named "*Thin*" *Rhombus*. The "thin" rhombus has four vertices with the following angles: 36°, 36°, 144°, 144°.

Now, let us consider one more type of isosceles triangle presented in the pentagram, for example, *ABE*. In such triangle the acute angles at the vertices *E* and *B* are each 36°, and the obtuse angle at the vertex *A* is 108°. Note that the ratio of the base *EB* of the triangle *ABE* to its sides *AE*=*AB* is equal to the golden mean, that is, this triangle is also the golden isosceles triangle. Note also, the pentagram has 5 identical golden isosceles triangles of this kind, namely, *ABE, BCA, CDB, DEC,* and *EAD*. If we connect two such triangles together at their bases, we obtain the second *Penrose tile* represented in Fig. 3.16-*b* and named *"Thick" Rhombus*. The thick rhombus has the four vertices with the angles: 72°, 72°, 108°, 108°.

Below in Fig. 3.18 we can see a process of sequential construction of the *Penrose Tiling*. Take 5 "thick" rhombi and connect them together, as shown in Fig. 3.18-*a*. Then, we add to the figure in Fig. 3.18-*a* the "thick" and "thin" rhombi, as shown in Fig. 3.18-*b*. Figure 3.18-*c* is a further development of the *Penrose tiling*.

It is proved that the ratio of the number of "thick" rhombi (Fig. 3.16-*b*) to the number of "thin" rhombi (Fig. 3.16-*a*) in the Penrose tiling aims in the limit for the golden mean.

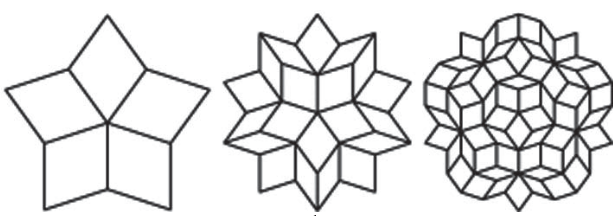
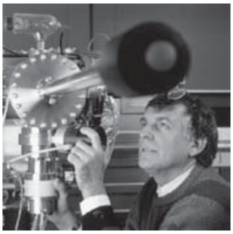


Figure 3.18. Penrose tiling

3.7.4. *Quasi-crystals*



The Israel physicist
Dan Shechtman

On November 12, 1984 in a small article, published in the authoritative journal *Physical Review Letters*, the experimental proof of the existence of a metal alloy with exclusive physical properties was presented. The Israeli physicist Dan Shechtman was the author of this article.

Dan Shechtman is Philip Tobias Professor of Materials Science at the Israel University Technion. A special alloy discovered by Professor Shechtman in 1982 and called *Quasi-crystal* is the focus of his research. By using methods of electronic diffraction, Shechtman found new metallic alloys having all the symptoms of crystals. Their diffraction pictures were composed from the bright and regularly located points similar to crystals. However, this picture is characterized by the so-called "icosahedral" or "pentagonal" symmetry, strictly prohibited according to geometric reasons. Such unusual alloys are called "quasi-crystals."

The concept of quasi-crystals generalizes and completes the definition of a crystal. Gratia wrote in the article [148]: "A concept of the quasi-crystals is of fundamental interest, because it extends and completes the definition of the crystal. A theory, based on this concept, replaces the traditional idea about the 'structural unit,' repeated periodically, with the key concept of the distant order. This concept resulted in a widening of crystallography and we are only beginning to study the newly uncovered wealth. Its significance in the world of crystals can be put at the same level with the introduction of the irrational to the rational numbers in mathematics."

What are quasi-crystals? What are their properties and how we can describe them? We mentioned above that according to the *Main Law of Crystallography* some strict restrictions are imposed on the structure of a crystal. According to classical ideas, the crystal is constructed from one single cell. The identical cells should cover a plane densely without any gaps.

As we know, the dense filling of a plane can be carried out by means of *Equilateral Triangles* (Fig. 3.19-*a*), *Squares* (Fig. 3.19-*b*) and *Hexagons* (Fig. 3.19-*d*). A dense filling of the plane by means of *Pentagons* is impossible (Fig. 3.19-*c*).

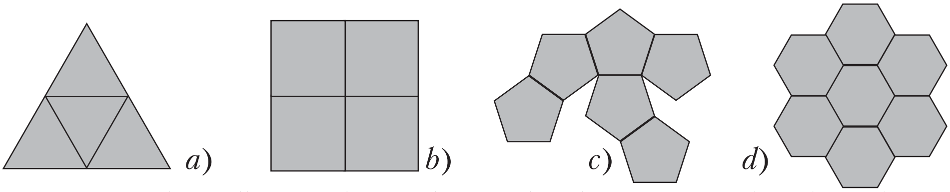


Figure 3.19. A dense filling of a plane can be carried out by means of *equilateral triangles* (*a*), *squares* (*b*) and *hexagons* (*d*)

Penrose tiling (Fig. 3.18) that is a "planar analogy" of quasi-crystals was used for the theoretical explanation of the quasi-crystal phenomenon. In the spatial model, the "regular icosahedrons" (Fig. 3.20) played the role of "Penrose rhombi" in the planar model. By using regular icosahedrons, we can fulfill a dense filling of three-dimensional space.

What is the practical significance of the discovery of quasi-crystals? Gratia wrote in [148] that "the mechanical strength of the quasi-crystals increased sharply; here the absence of periodicity resulted in slowing down the distribution of dislocations in comparison to the traditional metals .... This property is of great practical significance: the use of the "icosahedral" phase allows for light and very stable alloys by means of the inclusion of small-sized fragments of quasi-crystals into the aluminum matrix."

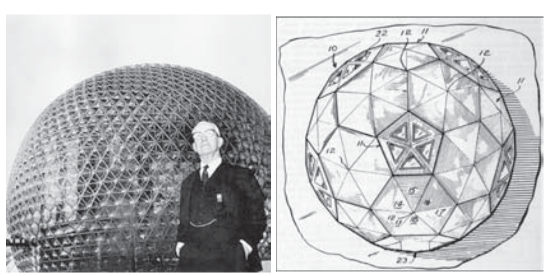
Figure 3.20.
A regular icosahedron

What is the significance of the discovery of quasi-crystals from the point of view of the main idea of our book, the golden mean? First of all, this discovery is a great triumph for the "icosahedron-dodecahedron doctrine," which passes throughout the history of natural sciences and is a source of profoundly practical scientific ideas. Secondly, the quasi-crystals shattered the conventional picture of an insuperable barrier between the mineral world where the "pentagonal" symmetry was prohibited, and the living world, where the "pentagonal" symmetry is widespread.

Note that Dan Shechtman published his first article about the quasi-crystals in 1984, that is, exactly 100 years after the publication of Felix Klein's *Lectures on the Icosahedron in* 1884. This means that this discovery is a worthy gift to the centennial anniversary of Klein's book, in which the famous German mathematician predicted an outstanding role for the icosahedron in future scientific development.

3.7.5. *Fullerenes*

Fullerenes were an important modern discovery in chemistry. This discovery was made in 1985, several years after the quasi-crystal discovery. The "fullerene" is named after *Buckminster Fuller* (1895 - 1983), the American designer, architect, poet, and inven-



a) b)

Figure 3.21. Fuller's inventions:
(*a*) Buckminster Fuller; (*b*) Fuller's geodesic dome

tor. Fuller created a large number of inventions, primarily in the fields of design and architecture. The best-known invention was the *Geodesic Dome* based on the truncated regular icosahedron (Fig. 3.21-*b*).

The geodesic dome, *Montreal Biosphere*, designed by Buckminster Fuller for the American Pavilion at Expo 67 (Fig. 3.22), is Fuller's best-known architectural construction.

Figure 3.22. The Montreal Biosphere, formerly the American Pavilion of Expo 67

Fuller was the author of many inventions in designing and architecture. In the USA, a postage stamp was produced immortalizing Buckminster Fuller's contributions to architecture and science (Fig. 3.23). Fuller's head on this postage stamp depicts the structure of his geodesic dome.

Figure 3.23. The U.S. postage stamp immortalizing Buckminster Fuller

After the discovery of *Fullerenes,* the name of Buckminster Fuller became famous worldwide. The title "fullerenes" refers to the carbon molecules C_{60}, C_{70}, C_{76}, and C_{84} in which all the atoms are on a spherical or spheroid surface. In these molecules the atoms of carbon are located at the vertices of regular hexagons or pentagons that cover the surface of the sphere or spheroid. We start from a brief history of the C_{60} molecule. This molecule plays a special role among the fullerenes. It is characterized by the greatest symmetry and as a consequence is highly stable. By its shape, the C_{60} molecule reminds one of a typical white and black soccer ball (Fig. 3.24) that has the structure of a truncated regular icosahedron (Fig. 3.5-*e*, Fig. 3.6).

Figure 3.24. The Soccer Ball

The atoms of carbon in this molecule are located on the spherical surface at the vertices of 20 regular hexagons and 12 regular pentagons; here each hexagon is surrounded by three hexagons and three pentagons, and each pentagon is surrounded by five hexagons (Fig. 3.25).

Figure 3.25. The structure of the C_{60} molecule

The most striking property of the C_{60} molecule is its high degree of symmetry. There are 120 symmetry operations that convert the molecule into itself making it the most symmetric molecule.

It is not surprising that the shape of the C_{60} molecule has attracted the attention of many artists and mathematicians over the centuries. As mentioned earlier, the truncated icosahedron was already known to Archimedes. The oldest known image of the truncated icosahedron was found in the Vatican library. This picture was from a book by the painter and mathematician Piero della Francesca. We can find the truncated icosahedron in Luca Pacioli's *Divina Proportione* (1509). Also Johannes Kepler studied the Platonic and Archimedean Solids actually introducing the name "truncated icosahedron" for this shape.

The fullerenes, in essence, are "man-made" structures following from fundamental physical research. They were discovered in 1985 by Robert F. Curl, Harold W. Kroto and Richard E. Smalley. The researchers named the newly-discovered chemical structure of carbon C_{60} the *Buckminsterfullerene* in honor of Buckminster Fuller. In 1996 they won the Nobel Prize in chemistry for this discovery.

Fullerenes possess unusual chemical and physical properties. At high pressure the carbon C_{60} becomes firm, like diamond. Its molecules form a crystal structure as though consisting of ideally smooth spheres, freely rotating in a cubic lattice. Owing to this property, C_{60} can be used as firm greasing (dry lubricant). The fullerenes also possess unique magnetic and superconducting properties.

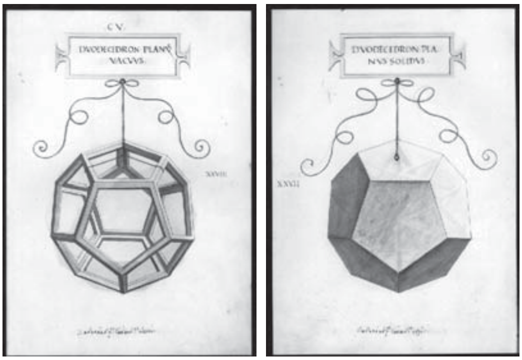
3.8. Applications of Regular Polyhedra in Art

3.8.1. *Leonardo da Vinci's Methods of Regular Polyhedra Representation*

Many authors pay particular attention to Leonardo da Vinci's original methods of the spatial representation of icosahedron, dodecahedron and truncated icosahedron, for the book *Divina Proportione* (1509) by his contemporary, the Franciscan monk and mathematician Luca Pacioli (1445 - 1514). It is probably impossible to consider Leonardo's participation in studying such perfect geometrical figures as Platonic and Archimedean Solids as a mere coincidence. And what is more, this fact is deeply symbolic. A true titan of the Renaissance, artist, sculptor, scientist, engineer and inventor, Leonardo da Vinci (1452 - 1519) is a symbol of the inseparable bond between Art and Science. His interest in such fine and highly symmetrical figures as regular and semi-regular polyhedra was natural.

Below (Fig. 3.26) we examine the different representations of the dodecahedron used by Leonardo da Vinci in Pacioli's *Divina Proportione*. Leonardo used two methods, a method of *Rigid Edges* (Fig. 3.26-*a*) and a method of *Continuous Faces* (Fig. 3.26-*b*). Comparison of these methods with the example of the dodecahedron convincingly shows the advantage of the method of rigid edges.

The essence of the method of rigid edges consists in the fact that the faces of the polyhedron are represented as "empty," not continuous. Strictly speaking, the faces are not represented at all; they exist only in our imagination. However, the edges of the polyhedron are represented not by geometrical lines (which have neither width nor thickness), but by the rigid three-dimensional segments. Such techniques of polyhedron representation allow the spectator to accurately determine, first of all, which edges belong to the front and which belong to the back faces of the polyhedron (which is practically impossible when the edges are represented by geometrical lines). On the other hand, we can look through a geometrical body to see the body in perspective and depth. It is clear that this is impossible if we use the method of continuous faces.



a) b)

Figure 3.26. Leonardo's representations of the dodecahedron by the methods of (a) rigid edges and (b) continuous faces in Pacioli's Divina Proportione

Below (Fig. 3.27) we see Leonardo's representation of the truncated icosahedron by the method of rigid edges. At the top of the picture we see the Latin inscription *Ycocedron Abscisus* (truncated icosahedron) *Vacuus.* The Latin word *Vacuus* meant, that the faces of the polyhedron are represented as "empty," not continuous.

It is necessary to note that the representation of polyhedra by the method of rigid edges began to be used widely in science and works of art following Leonardo. As an example, Johannes Kepler used the method of rigid edges for the representation of the polyhedra from which he constructs his *Cosmic Cup* (Fig. 3.11).

Figure 3.27. Leonardo's representation of the truncated icosahedron by the method of rigid edges in Pacioli's Divina Proportione

3.8.2. *Pacioli's Polyhedron*

As we saw previously, Pacioli was one of the greatest mathematicians of 15th century Europe. He also invented the principle of the so-called double record used now in all modern systems of book-keeping. That is why, he can be called "the father of modern book-keeping." However, the creative works of Luca Pacioli, who was a very mysterious person in his time, have up to now been the source of fierce disputes amongst historians of science. It is known that Luca Pacioli was born in 1445 in the Italian city Borgo San Sepulcro. In his childhood he was a pupil of the artist and mathematician Piero della Francesca. Later he was a student at the University of Bologna, which during the 15th and 16th centuries was one of the best educational institutions in Europe (at various periods Copernicus and Durer were its students). In 1472, Pacioli came back to his native city Borgo San Sepulcro and started to write his best known book *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*. This book was not published in Venice until 1494. In 1496, he was invited to Milan to give the lectures on geometry and mathematics. Here is when he met Leonardo da Vinci. After reading Pacioli's *Summa,* Leonardo stopped writing his own book on geometry and started to illustrate Pacioli's new book *Divina Proportione*.

Figure 3.28. Jacopo de Barbari's painting of *Luca Pacioli*

Some researchers accuse the author of *Divina Proportione* of plagiarism of the unpublished manuscript belonging to Pacioli's teacher Piero Della Francesca. However, it is rather doubtful, if we take into consideration that in the period when Pacioli was writing this book, he was already a well-known mathematician, and his glory reverberated throughout all of Italy. Pacioli is well known to us today owing to the portrait (Fig. 3.28) painted of him by the Italian artist Jacopo de Barbari (1440 - 1515). Barbari's picture is revealing in several respects, first of all, in respect of the presentation of Luca Pacioli's personality. Numerous compositional details in Barbari's picture are full of deep scientific sense. The artist demonstrates an understanding of the interrelation between Art and Science which was peculiar to Renaissance experts. Pacioli in the robe of the Franciscan monk is represented by standing behind the table of geometrical tools and books (in the lower right-hand corner of the picture we can see a model of the dodecahedron). The attention of Pacioli and the handsome young man, who stands on the right and somewhat behind Pacioli, is directed to the glass model of the polyhedron that hangs in the upper left-hand corner of the picture. The choice of the polyhedron is not accidental: it is Pacioli's rhombic cube-octahedron (Fig. 3.29).

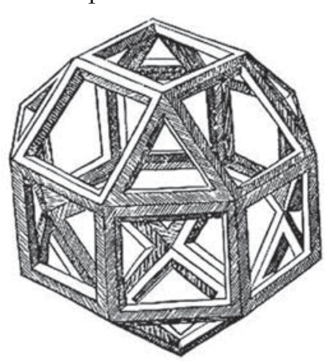


Figure 3.29. Pacioli's rhombic cube-octahedron

3.8.3. *Albrecht Durer*

The identity of the young man, who is near to Pacioli in Jacopo de Barbari's picture (Fig. 3.28), has often caused disputes amongst historians of art. Some of them assert that this is Barbari's self-portrait, others identifying the individual with Albrecht Durer (1471 - 1528), German painter, draughtsman and art theorist who is generally known as the greatest German Renaissance artist. However, and this is very important in our context, Durer was amazed by the art method of Barbari, who later created his pictures on the basis of a deep study of the system of proportions, that is, strict use of particular ratios of parts of the represented objects among themselves.

Durer was one of the first artists to start studying the laws of perspective. He dreamed of meeting with the glorified Italian artists to study their art works and to compete with them. With this purpose in mind Durer traveled to Italy in 1505. We do not know his teachers at the school of perspective, but we do know that throughout his life Durer continued to teach at this school. He wrote several books on the theory of arts. One of them, *The Painter's Manual*, was dedicated to geometry and perspective and was published in Nuremberg in 1525. His last work on human proportion was published in 1528 following his death.

Durer's books are a serious scientific contribution to the theory of perspective and stereometry of polyhedra. He described several Archimedean Solids unknown at that time, and also developed and published for the first time books that included spatial models of the planar unfolding of various polyhedra, including the unfolding of the truncated icosahedron. Today a similar unfolding of volumetric models of polyhedra is widely used to study the elementary forms of crystals, the structures of molecules (fullerenes, for example), and viruses and so on.

In 1512 in the rough draft of his first treatise about proportions, Durer wrote: "All needs of the person are satiated by the fleeting things in the case of their surplus, so that they cause in him disgust, except for a thirst for knowledge.... A desire to know a lot and through this to grasp the essence of all things is inherent within us from nature." These words became a prologue to Durer's theoretical works. Art of that epoch is often penetrated by a thirst for knowledge and

Figure 3.30. Albrecht Durer's Melancholia

many representatives of that epoch became scientists and researchers. The idea of the unity of artistic inspiration and mathematical theory is also reflected in Durer's well-known picture *Melancholia* created in 1514. This picture embodies the image of a person, who is near to God. The individual is surrounded by various geometric tools (Fig. 3.30). The presence in the engraving of a polyhedron (most likely, truncated rhombohedra), is certainly not accidental.

3.8.4. *Piero Della Francesca*

We should start a list of the greatest experts of the Renaissance, who made a concerted study of the geometry of polyhedra beginning with Piero Della Francesca (1420 - 1492). We know a little about the life of Piero Della Francesca, who was a great Italian artist, art theorist and mathematician. It is known, that he was born into the family of a craftsman in the small city of Borgo San Sepulcro. He studied in Florence, then worked in some Italian cities, including Rome. Francesca's creativity went far beyond the local art schools and helped define the whole of art of the Italian Renaissance. Fortunately, at the beginning of the 20th century the originals of three mathematical manuscripts by Piero Della Francesca were discovered. They are now housed in the Vatican Library. After five centuries of obscurity, the glory of this great mathematician has been returned to Piero Della Francesca. Now it is known for certain that Francesca was the Renaissance's first expert, who rediscovered and described in detail all Archimedean Solids, in particular, the five truncated Platonic Solids, the truncated tetrahedron, hexahedron, octahedron, dodecahedron, and, what is especially important here, the truncated icosahedron.

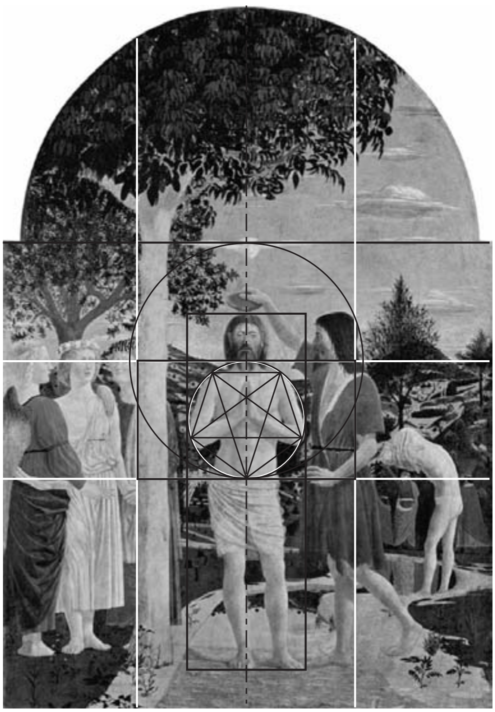


Figure 3.31. Piero Della Francesca's *The Baptism of Christ* and its harmonious analysis based on the golden section

Piero Della Francesca was both a talented mathematician and great artist. His art works express majestic solemnity, nobleness and harmony of images, reasonableness of proportions, and clarity of perspective in their construction. His painting of *The Baptism of Christ* is one of the highlights of his artistic style. Christ is placed in the center of the picture (Fig. 3.31). His feet are being washed by the river water; his hands are combined in a gesture of Catholic prayer. Near to Christ we see John the Baptist, who pours water from a dish on Christ's head. The dove, representing the Holy Spirit, descends from above the head of Christ.

3.8.5. *Art of Intarsia*

At the end of the 15th and beginning of the 16th centuries the *art of Intarsia*, a special kind of inlay, a mosaic, constructed from thousands of fine slices of various species of trees, was very popular in Northern Italy. The mosaic created by Fra Giovanni da Verona (1457 - 1525) for the church Santa Maria in Verona in roughly 1520, is an example of intarsia (Fig. 3.32). The image of half-opened shutters creates on the planar mosaic a spatial effect, which increases through the representation of various polyhedra (including the truncated icosahedron) by using Leonardo's technique of rigid edges.

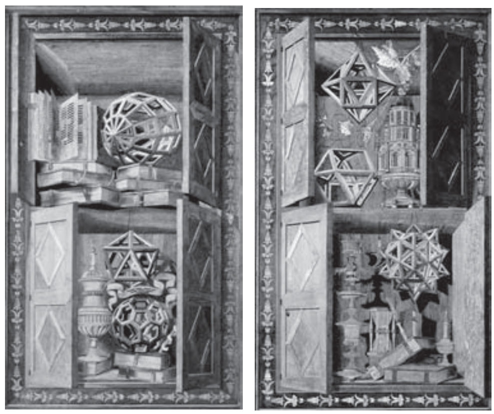


Figure 3.32. Fra Giovanni da Verona's intarsia created for the church Santa Maria in Verona

3.8.6. *Salvador Dali's Last Supper*

Now, let us consider an example of the representation of polyhedra by the well-known 20th century artist Salvador Dali (1904 - 1989). This great Spanish artist is one of the best known representatives of *surrealism*. An excellent artist, Dali created images similar to dreadful visions, called by him "drawing pictures of dreams." Some of the most common repeat images, for example hours, which lose their forms under the sun beams, became Dali's logo. Dali's creativity continues to cause disputes (e.g. some critics even suggest that after 1930 he did not create anything of real worth). However, in 1955 Dali created one of his best known pictures, *Last Supper* (Fig. 3.33).

This big picture is an original masterpiece of painting. Geometrical rationalism testifies to the invincible belief in the sacred force of number. In the center of the big horizontal picture (167×288) we see Christ. Together with his pupils he sits at the table as God's son descended from the Heavens (Dodecahedron) upon the Earth. The apostles are represented with heads held low inclined to the table. They appeared as if in deep worship of Christ. The dodecahedron in this picture plays a central role because it personifies the spiritual harmony, moral cleanliness and greatness of character.

Figure 3.33. Salvador Dali's *Last Supper*

3.8.7. *Escher's Creative Work*

Maurits Cornelis Escher (1898 - 1972) is one of the world's most famous graphic artists. His art continues to amaze millions of people all over the world. In his works we recognize his keen observation of the world around us and the expressions of his own fantasies. Escher shows us that reality is wonderful and fascinating. He is most famous for his so-called impossible structures, such as *Ascending and Descending, Relativity*, and his *Transformation Prints* such as *Metamorphosis I*, *Metamorphosis II* and *Metamorphosis III, Sky & Water and Reptiles* (see www.mcescher.com).

Escher's creative work was highly esteemed by many scientists, in particular, by mathematicians and crystallographers. At the International Crystallographic Congress in Cambridge (1960) the exhibition of Escher's pictures became a sensation attracting the special attention of the crystallographers. What do art and crystallography hold in common? The question arises as to whether Maurits Cornelis Escher intuitively discovered the laws of symmetry, those laws which dominate over crystals and define their external shape, nuclear structure and physical properties, and then illustrated these laws in his pictures. Escher took great interest in periodic figures and drew up the mosaic patterns of the repeating figures in his pictures. He inserted one image in another so that identical figures are periodically repeated without leaving any empty space between them. In fact, it is based on the same law, according to which the particles in crystal structures are placed, namely, *the law of the densest packing:* the periodic recurrence of identical groups of particles without intervals and infringements.

3.9. Application of the Golden Mean in Contemporary Art

3.9.1 *Abstract Art by Astrid Fitzgerald*

Astrid Fitzgerald is an internationally acclaimed artist, born and educated in Switzerland and now residing and working in New York. Her works are represented in major public museums and private collections in the United States, Europe and Asia. Fitzgerald's installation, *Amish Quilts,* was chosen by the judges to represent the United States at the Artcanal Exposition (2002) in Switzerland. She is also the author of *An Artist's Book of Inspiration – A Collection of Thoughts on Art, Artists and Creativity* (1996) and *Being Consciousness Bliss – A Seeker's Guide* (2002), both published by Lindisfarne Books. Fitzgerald refers to her recent work as *Cosmic Measures* – a phrase that expresses her continuing search for the true nature of things. Here she began to work with the fundamental laws of geometric forms that include the Golden Mean – the universal principle that underlies Nature from the spiral of our DNA to that of our galaxy. These harmonious proportions have fascinated philosophers, architects and artists for millennia. Fitzgerald embodies the golden mean within her abstract pictures.

Astrid Fitzgerald

In Figs. 3.34 we can see some pictures of Astrid Fitzgerald based on the Golden Mean.

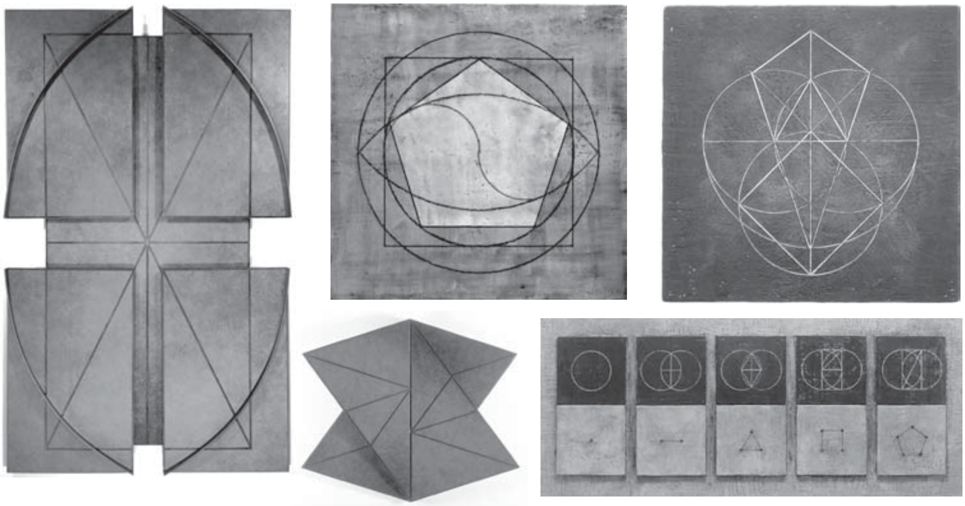


Figure 3.34. Abstract art of Astrid Fitzgerald

3.9.2. *The World of Matjuska Teja Krasek*

Matjuska Teja Krasek obtained her B.A. degree in painting from the Arthose–College for visual arts in Ljubljana, Slovenia. Her theoretical and practical work focuses primarily on symmetry as a connecting concept between art and science. Through her art, Krasek wishes to convey her experience and feelings that are connected with her research results from various disciplines. She wants to convey this to all who are interested in exploring the way and functioning of our universe and nature. In this manner she wishes to contribute to the awareness of certain characteristics such as the various kinds of symmetry, the golden mean, and the Fibonacci sequence's connection to nature, natural science and art. She also explores in her works how the use of various formal elements of artistic expression (lines, colors, structures, etc.) can influence the stability of art work. She uses contemporary computer technology as well as a classical painting technique. And her artworks have been represented at many international exhibitions and published in international journals (*Leonardo Journal*, *Leonardo on-line* and so on).

Matjuska Teja Krasek

In Fig. 3.35 we can see some pictures of Matjuska Teja Krasek.

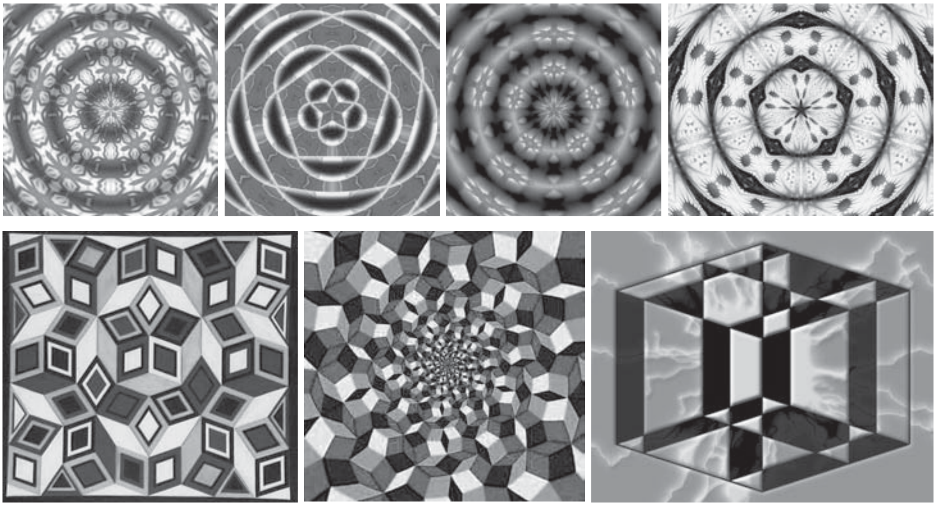


Figure 3.35. The art works of Matjuska Teja Krasek

3.9.3. *The Geometric Art of John Michell*

John Michell, born in 1933, was educated at Eton and Cambridge and published his first book in 1967. The author of 12 books, he is a specialist in

John Michell

sacred geometry and the geometry of reconciliation. With *The View over Atlantis* (1969) and *City of Revelation* (1972) Michell helped to change the world-views of a whole generation by illuminating the science, culture and wisdom of past civilizations. Michell's *City of Revelation* was summarized and extended with further research in *The Dimensions of Paradise*, in which the proportions and symbolic numbers of ancient cosmology were explained.

Michell's books, *Twelve Tribe Nations* and *Science of Enchanting the Landscape* (1991), the latter with Christine Rhone, explain that throughout the history of civilization an ideal social order harmonically related to nature and the zodiac was imposed upon the landscapes of the world. This is depicted, for example, by the 12 tribe divisions of people and the careful alignment of various holy places.

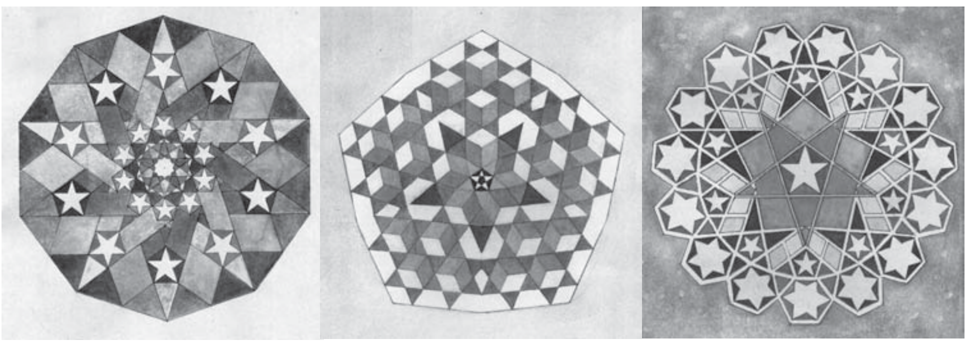


Fig. 3.36. The Geometric Art of John Michell

3.9.4. *Quantum Connections by Marion Drennen*

Marion Drennen is a Louisiana Artist, who received her Bachelor in Fine Arts from LSU, working in Acrylic on Board. These paintings incorporate the concepts of Number and Quantum Physics in an effort to evoke a sense of connectedness across time and space, within ourselves and in our relationships.

Marion Drennen

In her artistic statement she describes her artistic process – "Being an idea person, an avid reader and researcher, my paintings manifest from contemplations on a variety of subjects – Mathematics, The Golden Ratio, Quantum Physics, and Spiritual. I glimpse the connections and begin to draw thumbnail sketches around the edges of my notes. The internal dialogue is about the concept, then words gradually disappear and visual elements expand until only sketches are coming off the end of the pencil. Later, within the painting, words may reappear." She compares her creation process with a dance – "I used to dance. There's something exquisite about losing yourself in movement. Sometimes, when I'm painting, I'll start to dance. I almost always paint to music. After working with the Golden Ratio for several years, I now have custom painting surfaces made that allow me to paint within that format. It sets the stage. It is the architecture, the structure on which I begin to build. I break up the space and then insert my one or two shapes, the initial idea, and then the dance begins."

In Fig. 3.37 we represent some pictures of Marion Drennen from her Quantum Connections exhibit that was showing at the Brunner Gallery in the Shaw Center for the Arts.

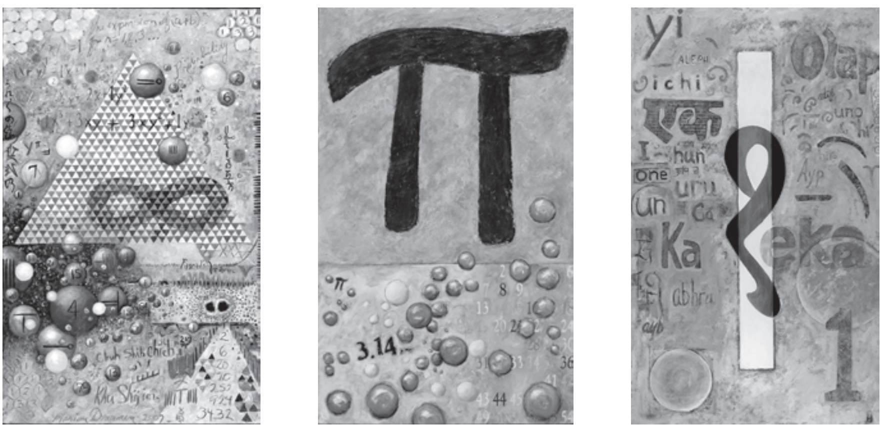


Fig. 3.37. Quantum Connections by Marion Drennen.

3.10. Conclusion

The regular and semi-regular polyhedra have been known from antiquity. The regular polyhedra got the name *Platonic Solids,* because they played such an important role in Plato's cosmology. According to Plato, the atoms of the Universe's "Basic Elements" have the form of the Platonic Solids (Fire – a Tetrahedron, Earth – a Hexahedron or Cube, Air – an Octahedron, Water –an Icosahedron). The Dodecahedron was considered to be the primary figure of the Universe, expressing the Universal Intellect and the Harmony of the Universe. The semi-regular polyhedra are named *Archimedean Solids.* A geometric theory of the Platonic Solids was presented in the 13th or final Book of Euclid's

Elements. This is the reason why the ancient Greek mathematician Proclus, a commentator on Euclid, put forward the hypothesis about the true purpose of Euclid writing *The Elements*. In Proclus' opinion, Euclid wrote his *Elements* to give a full and systematized theory of geometric construction of the "ideal" geometric figures, in particular, the five Platonic Solids. Thus, we can consider *The Elements* of Euclid to be the first historical geometrical theory of the Harmony of the Universe, based upon the golden section (division in the extreme and mean ratio) and the Platonic Solids! Since antiquity the Platonic and Archimedean Solids have been a source of many scientific hypotheses, theories and discoveries. The surprising coincidence of the numerical characteristics of the dodecahedron (12 faces, 30 edges and 60 planar angles on its surface) with the main cycles of the Solar System (12-year cycle of Jupiter, 30-year cycle of Saturn, and 60-year basic cycle of the Solar System) apparently became one reason why the numbers 12, 30, 60 and $360=12\times30$ were used by the Ancient Egyptians in their calendar and their systems of time and angle measurement. In the 19th century the prominent mathematician Felix Klein began to consider the regular Icosahedron as the main geometric object, from which the branches of the five mathematical theories follow, namely, geometry, Galois' theory, group theory, theory of invariants and differential equations. Shechtman's quasi-crystals were based on the icosahedron and fullerenes (1996 Nobel Prize in chemistry). These are brilliant confirmations of the role of the Platonic and Archimedean Solids, and therefore the golden section, in modern physics.

Chapter 4

Generalizations of Fibonacci Numbers and the Golden Mean

4.1. A Combinatorial Approach to the Harmony of Mathematics

4.1.1. *Mathematical, Aesthetic and Artistic Understanding of Harmony*

In the Introduction we mentioned that the *Harmony Problem* is one of the "key" problems of mathematics entering the scene at its very origin. But what does a concept of "harmony" mean?

The Russian philosopher Shestakov, one of the best researchers in the field, pointed out three basic understandings of "harmony" that have been developed in science and aesthetics since antiquity [7]:

1. **Mathematical Understanding of Harmony or Mathematical Harmony**. In this sense, harmony is understood as equality or proportionality of parts one to another and the parts to the whole.
2. **Aesthetic Harmony**. In contrast to the mathematical harmony, the aesthetic harmony is not quantitative, but qualitative notion and expresses the internal nature of things. The aesthetic harmony is connected with aesthetic excitements and estimations. Most precisely this type of harmony is shown at perception of beauty of Nature.
3. **Artistic Harmony**. This type of harmony is connected with art. Artistic harmony is an actualization of the harmony principle in the realm of art.

In the present book, our attention is concentrated on **Mathematical Harmony**. It is clear that mathematical harmony is expressed in the form of certain numerical proportions. Shestakov emphasizes that mathematical harmony "attracts attention to its quantitative side and is indifferent to qualitative originality of the parts forming conformity.... The mathematical understanding of the harmony fixes, first of all, quantitative definiteness of the harmony, but it does not express aesthetic quality of the harmony, its expressive connection with beauty."

Thus, in this book we consciously restrict the area of our research to the mathematical understanding of harmony. This approach is fruitful in mathematics and allows us to create a *Mathematical Theory of Harmony* or *Mathematics of Harmony* that extends the area of the mathematical models of the harmonic processes of Nature.

4.1.2. *Concept of the Mathematical Theory of Harmony*

We can ask a question: how does one create a **Mathematical Theory of Harmony**? As is known, mathematics studies the quantitative aspect of this or that phenomenon. Starting with a mathematical analysis of the concept of harmony, we should concentrate our attention on the quantitative aspects of harmony. What is the quantitative aspect of this concept?

There are many definitions of the harmony concept. However, the majority of them are reduced to the following definition that is given in *The Great Soviet Encyclopedia*:

"Harmony is proportionality of parts and the whole, a combination of the various components of the object in the uniform organic whole. It is the internal order and measure obtained in the harmony external expression [of the harmony]."

In the article *Harmony in Nature and Art* [149], the Russian crystallographer Shubnikov compared harmony with the *Order* studied by science to discover Nature's laws. He writes: "Law, Harmony, Order underlie not only scientific work but also any work of art."

Now let us analyze a question of the origin and meaning of the word "harmony." "Harmony" has a Greek origin. The Greek word *αρμονια* has the following meaning: *connection, consent*. The analysis of the word of "harmony" and its definitions demonstrate that the most important, key notions, which underlie this concept, are the following: *Connection, Consent, Combination, Order*.

We can ask a question: what branch of mathematics studies these concepts? A search for the answer leads us to *Combinatorial Analysis*. "Combinatorial analysis studies the various kinds of combinations and connections, which can be formed from the elements of some finite set. The term 'combinatorial' is derived from the Latin word *combinary* which means to combine or connect." [150]

It follows from this consideration that both the Latin word *combinary* and the Greek word *αρμονια*, in essence, have the same meaning, namely, *Combination* and *Connection*. It allows us to put forward the following hypothesis: the "Laws of combinatorial analysis" can be used for the analysis of the Harmony concept from the quantitative point of view.

Therefore, the main concepts underlying the Harmony Mathematics and the combinatorial analysis, in essence, coincide. And this allows us to use combinatorial analysis as the main mathematical theory for the analysis of the harmony concept from the quantitative point of view. This is the main idea behind the research, in which the author made an attempt to create the *Fundamentals of the Harmony Mathematics* based upon combinatorial analysis.

Thus, Harmony Mathematics is a mathematical theory studying the notion of harmony from a quantitative point of view. Its main goal is to study mathematical laws, mathematical proportions, which underlie the Harmony of the Universe.

4.1.3. *An Analogy between the Theory of Information and Mathematics of Harmony*

Are there similar theories in modern science? The *Mathematical Theory of Information* developed by Claude Shannon [151] is possibly a brilliant example of a similar theory. Information is a complex and interdisciplinary concept similar to *Harmony*. Both information and harmony are non-material and omnipresent. In spite of its non-material character, the Mathematical Theory of Harmony may be considered to be an original mathematical theory similar to the Theory of Information.

By developing a *Theory of Information*, Claude Shannon used the concept of *Probability* as the starting point of the theory. The concept of *Entropy*, based on the notion of probability, became the underlying basic concept of the Theory of Information. It is necessary to emphasize that Shannon's Theory of Information is a mathematical theory and can be effectively used, first of all, for the quantitative analysis of any informational system. In his well-known article *Bandwagon* [151] Shannon warned that one must be cautious applying this theory to other areas of human activity.

Shannon's Theory of Information is sometimes considered to be a branch of probability theory. By continuing the analogy between Shannon's Theory of Information [151] and Harmony Mathematics, it is possible to consider the Harmony Mathematics as a special branch of combinatorial analysis.

From this point of view, it becomes imperative to answer the question about practical application of the Harmony Mathematics. What are the areas of the effective application of this theory? By answering this question, it is important to emphasize that the most effective areas are those where the quantitative aspects of the Harmony are most important, such as theoretical physics, computer science, biology, botany, economics, and so on.

4.2. Binomial Coefficients and Pascal Triangle

4.2.1. *The Main Concepts of Combinatorial Analysis*

Combinatorial analysis studies different kinds of combinations that can be built up from the elements of some set. The term "combinatorial analysis" originates from the Latin word *combinare* that has the meaning to combine or connect.

Some elements of combinatorial analysis were known in India already in the 2nd century AD. The Indian mathematicians knew how to calculate numbers C_n^m called *Combinations* from n elements by m, and they knew the following formula:

$$C_n^0 + C_n^1 + \dots + C_n^n = 2^n. \tag{4.1}$$

A theory of binary codes, the basis of modern computers, is an example of the effective application of the formula (4.1). We can consider a set of the n-digit binary words starting from the code combination 00 ... 0 and ending by the code combination 11 ... 1. As is known, the number of elements of this set is equal to 2^n. We can divide this set into the $(n+1)$ disjoint subsets. Then we can refer all binary words consisting only of 0's to the first subset. It is clear that the only code combination 00 ... 0 satisfies this condition, that is, the number of the elements of this subset is equal to $C_n^0 = 1$. Then, we can refer to the second subset all the binary words containing only 1 and the $(n$-$1)$ 0's. It is clear that the number of elements of this subset is equal to C_n^1. We can refer to the $(m+1)$-th subset all the n-digit binary words containing m 1's and $(n$-$m)$ 0's; the number of code combinations of this subset is equal to C_n^m. At last, we can refer to the $(n+1)$-th subset all the n-digit binary words which contain only 1's. It is clear that the only code combination 11 ... 1 satisfies this condition, that is, the number of elements of this subset is equal to $C_n^n = 1$. From this reasoning the validity of the formula (4.1) follows.

The term "combinatorial analysis" began to be used after publication in 1666 of Leibniz's *Reasoning about Combinatorial Art*; in this book he gave for the first time scientific substantiation of the theory of combinations and permutations. Bernoulli introduced for the first time a notion of *Distribution* in the second part of his famous book *Art of Guessing* published in 1713. He introduced and used in our sense the term *Permutation*. The term *Combination* was introduced by Pascal in his *Treatise about the Arithmetical Triangle* (1665).

Thus we have the well-known formula

$$C_n^m = \frac{n!}{m!(n-m)!}, \tag{4.2}$$

where $n!=1\times2\times3\times\ldots\times n$ is a *Factorial* of n.

4.2.2. *Binomial Formula*

Now, write the following formulas:

$$\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= 1a+1b\\
(a+b)^2 &= 1a^2+2ab+1b^2\\
(a+b)^3 &= 1a^3+3a^2b+3ab^2+1b^3.
\end{aligned} \tag{4.3}$$

Note that the first two formulas from (4.3) are trivial; the other two formulas are well-known from secondary school.

We can ask the question: how do we calculate the binom $(a+b)^n$? The well-known mathematical formula called the *Binomial Formula* gives the answer to this question:

$$(a+b)^n = a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}b^2 + \ldots + C_n^k a^{n-k}b^k + \ldots + C_n^{n-1}ab^{n-1} + b^n. \tag{4.4}$$

Here the numbers C_n^k are named *Binomial Coefficients* or *Binomial Factors*.

Sometimes the discovery of formula (4.4) was attributed to Newton. However, long before Newton, the mathematicians of many countries, in particular, the Arab mathematician Al Kashi, the Italian mathematician Tartalja, the French mathematicians Fermat and Pascal, knew this formula. Newton's merit consists in the fact that he derived this formula for the case of any real number n, that is, he proved that the formula (4.4) is true when n is rational or irrational, positive or negative.

The formulas (4.3) are special cases of a general formula (4.4). In particular, for the case $n=1$ the formula (4.4) is reduced to the following:

$$(a+b)^1 = C_1^0 a + C_1^1 b = 1a+1b,$$

whence it appears that $C_1^0=1$ and $C_1^1=1$.

For the case $n=2$, the formula (4.4) takes the following form:

$$(a+b)^2 = C_2^0 a^2 + C_2^1 ab + C_2^2 = 1a^2+2ab+1b^2,$$

whence it appears that $C_2^0=1, C_2^1=2, C_2^2=1$.

Thus, we can get the decomposition (4.4) which can be very easy if we know how to calculate the binomial coefficients C_n^k.

4.2.3. *Pascal Triangle*

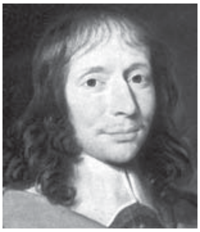
The French 17th century mathematician *Blaise Pascal* (1623-1662) suggested an original method for the calculation of the coefficients C_n^k for any arbitrary non-negative integers n and k.

In addition to the properties (4.1) and (4.2), the binomial coefficients C_n^k have a number of remarkable properties that are given here without a proof:

$$C_n^0 = C_n^n = 1 \tag{4.5}$$

$$C_n^k = C_n^{n-k} \tag{4.6}$$

$$C_{n+1}^k = C_n^{n-1} + C_n^k. \tag{4.7}$$



Blaise Pascal
(1623-1662)

The last property (4.7) is also named the *Pascal Law*. Using the recursive relation (4.7), Pascal had offered an original method for the calculation of binomial factors that are based on their disposition in the form of a special numerical table called *Pascal's Triangle*.

Let us examine an infinite table of numbers constructed according to the Pascal Law (4.7).

The top of the indicated table (Fig. 4.1) that is named *Zero-row*, consists of the only binomial coefficient $C_0^0 = 1$. The next row – the 1st row - consists of two binomial coefficients $C_1^0 = C_1^1 = 1.$ Each succeeding row can be constructed from the preceding row according to the rules (4.5)-(4.7). It is easy to prove the following properties of Pascal triangle:

1. The sum of the binomial coefficients of the n-th row of Pascal triangle is equal to 2^n what corresponds to the identity (4.1).
2. All rows of Pascal triangle are symmetric relative to the binomial coefficient $C_0^0 = 1$ of the zero-row that corresponds to the property (4.6).

```
                1
             1     1
          1     2     1
       1     3     3     1
     1    4     6     4     1
   1    5    10    10    5    1
  1   6   15    20    15   6   1
 1  7   21   35    35   21   7  1
1  8  28   56   70   56   28  8  1
1 9 36  84  126   126  84  36  9 1
```

Figure 4.1. Pascal Triangle

The above Pascal Triangle appeared for the first time in Pascal's *Treatise about Arithmetical Triangles* written in 1665. However, one century prior to the publication of Pascal's *Treatise,* this numerical table (but in rectangular form rather than triangular) was described in the *General Treatise about Number and Measure* written by the Italian mathematician Nikola Tartalja (1500-1557). Tartalja's *Treatise* was published immediately after his death.

Here the first or top row and the first column on the left consist only of 1's; each "internal" number of other rows is equal to the sum of two numbers, the first one that stands in the same row on the left to it and the second one that stands in the preceding row above it. This table of binomial factors is called *Tartalja's Rectangle* (Table 4.1).

Table 4.1. Tartalja's Rectangle

1	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
1	5	15	35	70	126
1	6	21	56	126	252

4.3. The Generalized Fibonacci *p*-Numbers

4.3.1. *Rectangular Pascal Triangle*

There are many forms of the representation of the Pascal Triangle, for example, in the form of an isosceles triangle, in the form of a rectangular table (*Tartalja's Rectangle*), etc. We will examine the so-called *Rectangular Pascal Triangle* that can be represented by the following table of binomial coefficients (Fig. 4.2).

$$\begin{array}{cccccccc}
C_0^0 & C_1^0 & C_2^0 & C_3^0 & C_4^0 & C_5^0 & \ldots & C_n^0 \\
 & C_1^1 & C_2^1 & C_3^1 & C_4^1 & C_5^1 & \ldots & C_n^1 \\
 & & C_2^2 & C_3^2 & C_4^2 & C_5^2 & \ldots & C_n^2 \\
 & & & C_3^3 & C_4^3 & C_5^3 & \ldots & C_n^3 \\
 & & & & C_4^4 & C_5^4 & \ldots & C_n^4 \\
 & & & & & C_5^5 & \ldots & C_n^5 \\
 & & & & & & \ddots & \vdots \\
 & & & & & & & C_n^n
\end{array}$$

Figure 4.2. Rectangular Pascal Triangle

The rows of the Pascal Triangle are numbered from top to bottom. The binomial coefficients $C_0^0 = C_1^0 = C_2^0 = ... = C_n^0 = 1$ make up a "zero" row. Every n-th row starts with the binomial coefficient of the kind $C_n^n = 1\left(n = 0, 1, 2, 3, ...\right)$.

The columns of the Pascal Triangle are numbered from left to right; the first left-hand column that consists of the only binomial coefficient $\left(C_0^0 = 1\right)$ is called the *Zero-column*. The n-th column (n=0,1,2,3,...) includes the following binomial coefficients:

$C_n^0, C_n^1, C_n^2, ..., C_n^k, ..., C_n^{n-k}, ..., C_n^n,$
where $C_n^k = C_n^{n-k}$.

As mentioned above, Pascal Triangle is based on the recursive relation (4.7).

What is the correlation between Pascal Triangle and Fibonacci numbers? In the second half of the 20th century many Great mathematicians (Martin

Gardner [12], George Polya [17], Alred Renyi [23] and others) independently one after another discovered the connection of Fibonacci numbers with Pascal's Triangle and binomial coefficients. This discovery demonstrates a fundamental connection of the Harmony Mathematics based on Fibonacci numbers and the golden mean with combinatorial analysis and outlines a way for the future generalization of Fibonacci numbers. Below we will demonstrate a surprisingly simple mathematical regularity that connects Pascal Triangle and Fibonacci numbers. A generalization of this regularity resulted in the mathematical discovery called the Generalized Fibonacci p-Numbers [19, 20].

4.3.2. *Pascal p-Triangles and Fibonacci p-Numbers*

Let us examine the Rectangular Pascal Triangle represented in numerical form (Fig. 4.3).

1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9
		1	3	6	10	15	21	28	36
			1	4	10	20	35	56	84
				1	5	15	35	70	126
					1	6	21	56	126
						1	7	28	84
							1	8	36
								1	9
									1
1	2	4	8	16	32	64	128	256	512

Figure 4.3. Pascal 0-Triangle

We can name the given table of binomial coefficients *Pascal 0-Triangle* (the meaning of this definition will become clear below). If we sum the binomial coefficients of the Pascal 0-Triangle by columns starting from the 0-column, then according to (4.1) we obtain the binary sequence:

$$1, 2, 4, 8, 16, ..., 2^n, ... \tag{4.8}$$

Now, we do some "manipulations" around the Pascal 0-Triangle. We move each row of the Pascal 0-Triangle one column to the right with respect to the previous row. As a result of such move, we obtain a table called *Pascal 1-Triangle* (Fig. 4.4).

Now, sum the binomial coefficients of the Pascal 1-Triangle in each column. To our amazement, we find that this summation results in the Fibonacci numbers:

$$1,1,2,3,5,8,13,21,...,F_{n+1},.... \tag{4.9}$$

where F_{n+1} is the $(n+1)$-th Fibonacci number given by the following recursive relation:

$$F_{n+1}= F_n+F_{n-1}. \tag{4.10}$$

The numerical sequence (4.9) is generated by the recursive relation (4.10) at the seeds:

$$F_1=F_2=1. \tag{4.11}$$

If we move in the initial Pascal 0-Triangle (Fig. 4.1) the binomial coefficients of each row by p columns to the right with respect to the previous row

(p=0,1,2,3,...), we get the numerical table named *Pascal p-Triangle*. It is clear that the Pascal 0-Triangle that corresponds to the case p=0 is the initial Pascal Triangle (Fig. 4.3). The Pascal 1-Triangle is represented in Fig. 4.4. The Pascal p-Triangles for the cases p=2 and p=3 have the forms shown in Fig. 4.5.

1	1	1	1	1	1	1	1	1	1	1	1
		1	2	3	4	5	6	7	8	9	10
				1	3	6	10	15	21	28	36
						1	4	10	20	35	56
								1	5	15	35
										1	6
1	1	2	3	5	8	13	21	34	55	89	144

Figure 4.4. Pascal 1-Triangle

Now, sum the binomial coefficients in each column of the Pascal 2- and 3-Triangles. As a result, we obtain two new numerical sequences:

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, \ldots \tag{4.12}$$

$$1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, \ldots \tag{4.13}$$

Pascal 2-Triangle

1	1	1	1	1	1	1	1	1	1	1	1	1
			1	2	3	4	5	6	7	8	9	10
						1	3	6	10	15	21	28
									1	4	10	20
												1
1	1	1	2	3	4	6	9	13	19	28	41	60

Pascal 3-Triangle

1	1	1	1	1	1	1	1	1	1	1	1	1
				1	2	3	4	5	6	7	8	9
								1	3	6	10	15
												1
1	1	1	1	2	3	4	5	7	10	14	19	26

Figure 4.5. Pascal 2-Triangle and 3-Tirangle

Denote by $F_2(n)$ and $F_3(n)$ the n-th elements of the sequences (4.12) and (4.13), respectively. It is easy to see the following regularities in the numerical sequences (4.12) and (4.13) that can be expressed by the following recursive relations

$$F_2(n)= F_2(n-1)+F_2(n-3) \quad \text{for } n\geq 4 \tag{4.14}$$

at the seeds

$$F_2(1)=F_2(2)=F_2(3)=1 \tag{4.15}$$

and by the recursive relation

$$F_3(n)=F_3(n-1)+F_3(n-4) \quad \text{for } n\geq 5 \tag{4.16}$$

at the seeds

$$F_3(1)=F_3(2)=F_3(3)=F_3(4)=1. \tag{4.17}$$

Thus, as a result of this examination we have found two new numerical sequences. The first of them that is given by the recursive relation (4.14) at the seeds (4.15) is named the *Fibonacci 2-numbers* and the second one that is given by (4.16) at the seeds (4.17) is named the *Fibonacci 3-numbers*.

In the general case, for arbitrary p if we sum the binomial coefficients of each column of the Pascal p-Triangle, we obtain the numerical sequences given by the following recursive relation:

$F_p(n)=F_p(n\text{-}1)+F_p(n\text{-}p\text{-}1)$ for $n\geq p+1$ (4.18)

at the seeds

$F_p(1)=F_p(2)=...=F_p(p+1)=1.$ (4.19)

The numerical sequences that correspond to the recursive relation (4.18) and (4.19) are named [20] the *Fibonacci p-numbers.*

4.3.3. *Partial Cases of the Fibonacci p-Numbers*

It is clear that for the case p=0 the recursive relation (4.18) and the seeds (4.19) takes the following form:

$F_0(n)=F_0(n\text{-}1)+F_0(n\text{-}1)$ for $n\geq 2$ (4.20)

$F_0(1)=1.$ (4.21)

It is easy to guess that the recursive relation (4.20) at the seed (4.21) generates the binary sequence (4.8) that is a special case of the Fibonacci p-numbers for p=0.

Let us examine the case p=1. For this case the recursive relation (4.18) and the seeds (4.19) are reduced to the following:

$F_1(n)=F_1(n\text{-}1)+F_1(n\text{-}2)$ for $n\geq 3$ (4.22)

$F_1(1)=F_1(2)=1.$ (4.23)

Comparing these formulas with the recursive relation for the classical Fibonacci numbers (4.10) and (4.11), we can conclude that the Fibonacci 1-numbers coincide with the classical Fibonacci numbers, that is, $F_1(n)=F_n$.

At last, we find that for the case $p=\infty$ the sequence of the Fibonacci p-numbers consists only of the unities: {1,1,1,...}.

4.3.4. *A Representation of the Fibonacci p-Numbers by the Binomial Coefficients*

In the above we have examined the formula (4.1) that allows us to represent binary numbers by the binomial coefficients. Analyzing the Pascal 1-Triangle (Fig. 4.4), it is easy to derive the mathematical formula that allows us to represent the Fibonacci 1-numbers by the binomial coefficients:

$$F_1(n+1)=C_n^0+C_{n-1}^1+C_{n-2}^2+C_{n-3}^3+C_{n-4}^4+.... \quad (4.24)$$

This means that there are two ways to calculate the Fibonacci numbers, namely, by using the recursive relation (4.10) at the seeds (4.11) or by using the formula (4.24).

For example, using the formula (4.24), we can represent the Fibonacci number $F_1(7)$=13 as follows:

$$F_1(7)=C_6^0+C_5^1+C_4^2+C_3^3+C_2^4+.... \quad (4.25)$$

Note that the binomial coefficient C_2^4 in the sum (4.25) as well as all of the binomial coefficients following C_2^4 are equal to 0 identically. This means that the expression (4.25) is the following finite sum of the binomial coefficients:

$$F_1(7)=C_6^0+C_5^1+C_4^2+C_3^3=1+5+6+1=13. \quad (4.26)$$

Studying the Pascal p-Triangle, we can represent the Fibonacci p-number $F_p(n+1)$ by the binomial coefficients as follows:

$$F_p(n+1)=C_n^0+C_{n-p}^1+C_{n-2p}^2+C_{n-3p}^3+C_{n-4p}^4+.... \quad (4.27)$$

Note that the known formula (4.1) is a partial case of (4.27) for p=0 and the formula (4.24) is a partial case of (4.27) for p=1.

4.3.5. *The "Extended" Fibonacci p-Numbers*

Until now we have studied the Fibonacci p-numbers $F_p(n)$ given for the positive values of n. In Chapter 2 we extended the classical Fibonacci numbers into the side of the negative values of n. By analogy, we can find the extended Fibonacci p-numbers, if we extend the Fibonacci p-numbers to the side of the negative values of n. With this purpose, we will find some general properties of such extended sequences. For the calculation of the Fibonacci p-numbers $F_p(n)$ corresponding to the non-negative values of n=0,-1,-2,-3,... we use the recursive relation (4.18) and the seeds (4.19). Let us represent the Fibonacci p-number $F_p(p+1)$ in the form (4.18) as follows:

$$F_p(p+1)=F_p(p)+F_p(0). \quad (4.28)$$

According to (4.19), we have: $F_p(p+1)=F_p(p)=1$. This means that $F_p(0)=0$.

Continuing this process, that is, representing the Fibonacci p-numbers $F_p(p)$, $F_p(p-1)$,..., $F_p(2)$ in the form (4.18), we have:

$$F_p(0)=F_p(-1)=F_p(-2)=...=F_p(-p+1)=0. \quad (4.29)$$

Now, represent the Fibonacci p-number $F_p(1)$ in the form (4.18):

$$F_p(1)=F_p(0)+F_p(-p). \quad (4.30)$$

As $F_p(1)=1$ and $F_p(0)=0$, we get from (4.30):

$$F_p(-p)=1. \quad (4.31)$$

Representing the Fibonacci p-numbers $F_p(0)$, $F_p(-1)$, ..., $F_p(-p+1)$ in the form of (4.18) , we get:

$$F_p(-p-1)=F_p(-p-2)=...=F_p(-2p+1)=0. \quad (4.32)$$

Table 4.2. The "Extended" Fibonacci p-Numbers

N	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7
$F_1(n)$	8	5	3	2	1	1	0	1	-1	2	-3	5	-8	13
$F_2(n)$	4	3	2	1	1	1	0	0	1	0	-1	1	1	-2
$F_3(n)$	3	2	1	1	1	1	0	0	0	1	0	0	-1	1
$F_4(n)$	2	1	1	1	1	1	0	0	0	0	1	0	0	0
$F_5(n)$	1	1	1	1	1	1	0	0	0	0	0	1	0	0

By continuing this process, we can obtain all values of the Fibonacci p-numbers $F_p(n)$ for the negative values of n. Table 4.2 gives some values of the "extended" Fibonacci p-numbers for the cases p=1, 2, 3, 4, 5.

4.3.6. *Some Identities for the Sums of the Fibonacci p-Numbers*

Once again, let us examine the recursive relation (4.14). Decomposing the Fibonacci 2-number $F_2(n\text{-}1)$ in (4.14) according to the same recursive relation (4.14), that is, representing $F_2(n\text{-}1)$ in the form $F_2(n\text{-}1)=F_2(n\text{-}2)+F_2(n\text{-}4)$, we can represent the recursive relation (4.14) as follows:

$$F_2(n)=F_2(n-2)+F_2(n-3)+F_2(n-4). \tag{4.33}$$

This means that the sum of the three successive Fibonacci 2-numbers is always equal to the Fibonacci 2-number, which is two positions from the senior Fibonacci 2-number of the sum.

Now, consider the recursive relation for the Fibonacci 3-numbers given by (4.16). Decomposing the Fibonacci 3-number $F_3(n\text{-}1)$ in (4.16) according to the same recursive relation (4.16), we can represent the recursive relation (4.16) as follows:

$$F_3(n)=F_3(n-2)+F_3(n-4)+F_3(n-5). \tag{4.34}$$

Decomposing the number $F_3(n\text{-}2)$ in (4.34) according to the recursive relation (4.16), we can represent (4.34) as follows:

$$F_3(n)=F_3(n-3)+F_3(n-4)+F_3(n-5)+F_3(n-6), \tag{4.35}$$

that is, the sum of the four sequential Fibonacci 3-numbers is always equal to the Fibonacci 3-number, which is three positions from the senior Fibonacci 3-number of the sum.

If we use a similar approach to the Fibonacci p-numbers in the general case, we obtain the following general identity:

$$F_p(n)=F_p(n-p)+F_p(n-p-1)+F_p(n-p-2)+...+F_p(n-2p). \tag{4.36}$$

Note that the identity (4.36) is valid for all "extended" Fibonacci p-numbers $F_p(n)$ when n takes the values from the set: $\{0,\pm1,\pm2,\pm3,...\}$.

Now, let us consider the sum of the first n Fibonacci p-numbers:

$$F_p(1)+F_p(2)+F_p(3)+...+F_p(n). \tag{4.37}$$

To get the required result, we write the basic recursive relation (4.18) for the Fibonacci p-numbers as follows:

$$F_p(n)=F_p(n+p+1)-F_p(n+p). \tag{4.38}$$

By using (4.38), we can write the following equalities:

$$F_p(1)=F_p(2+p)-F_p(1+p)$$
$$F_p(2)=F_p(3+p)-F_p(2+p)$$
$$F_p(3)=F_p(4+p)-F_p(3+p)$$
$$\ldots$$
$$F_p(n-1)=F_p(n+p)-F_p(n+p-1)$$
$$F_p(n)=F_p(n+p+1)-F_p(n+p).$$

Summing term by term the left-hand and right-hand parts of these equalities and taking into consideration that $F_p(1+p)=1$, we obtain the following expression for the sum (4.37):

$$F_p(1)+F_p(2)+F_p(3)+\ldots+F_p(n)=F_p(n+p+1)-1. \tag{4.39}$$

The formula (4.39) includes a further number of remarkable formulas of discrete mathematics. In fact, for the case $p=0$ this formula is reduced to the following well-known formula for binary numbers:

$$2^0+2^1+2^2+\ldots+2^{n-1}=2^n-1.$$

For the case $p=1$, the Fibonacci p-numbers coincide with the classical Fibonacci numbers, that is, $F_1(n)=F_n$. And then the formula (4.39) is reduced to the following formula:

$$F_1+F_2+F_3+\ldots+F_n=F_{n+2}-1,$$

that is well known from Fibonacci number theory [13, 16].

Thus, our “manipulations” with Pascal’s Triangle resulted in a small mathematical discovery! We found an infinite number of the new numerical sequences named the Fibonacci p-numbers ($p=0,1,2,3,\ldots$). These numerical sequences include the binary numbers (4.8) ($p=0$) and the classical Fibonacci numbers (4.9) ($p=1$) as partial cases. These numerical sequences possess a number of interesting mathematical properties, and their study can result in widening the Fibonacci number theory.

4.3.7. *The Ratio of Adjacent Fibonacci p-Numbers*

In Chapter 2 we found that the classical Fibonacci numbers are closely connected with the golden mean. In particular, the limit of the ratio F_n/F_{n-1} aims for the golden mean. There is a question: what is a limit of the ratio of the two adjacent Fibonacci p-numbers? Introduce the following definition:

$$\lim_{n\to\infty}\frac{F_p(n)}{F_p(n-1)}=x. \tag{4.40}$$

By using the recursive relation (4.18), we can represent the ratio of the two adjacent Fibonacci p-numbers as follows:

$$\frac{F_p(n)}{F_p(n-1)}=\frac{F_p(n-1)+F_p(n-p-1)}{F_p(n-1)}$$
$$=1+\cfrac{1}{\cfrac{F_p(n-1)}{F_p(n-p-1)}}=1+\cfrac{1}{\cfrac{F_p(n-1)\cdot F_p(n-2)\cdots F_p(n-p)}{F_p(n-2)\cdot F_p(n-3)\ \cdots F_p(n-p-1)}}. \tag{4.41}$$

Taking into consideration the definition (4.40), for the case $n\to\infty$ we can replace the expression (4.41) by the following algebraic equation:

$$x^{p+1}=x^p+1. \tag{4.42}$$

Note that the equation (4.42) is called the *Characteristic Equation for the Recursive Relation* (4.18).

Denote by τ_p a positive root of the characteristic equation (4.42). Let us examine Eq. (4.42) for the different values of p**.** For p=0, Eq. (4.42) is reduced to the trivial case: x=2. For p=1, Eq. (4.42) is reduced to the classical golden algebraic equation:

$$x^2=x+1 \tag{4.43}$$

with a positive root $\tau=\left(1+\sqrt{5}\right)/2$.

Thus, Eq. (4.42) can be considered to be a very broad generalization of the golden equation (4.43).

4.4. The Generalized Golden p-Sections

4.4.1. *A Generalization of the Division in Extreme and Mean Ratio (DEMR)*

Equation (4.42) has the following geometric interpretation. Let us give the integer p a non-negative value (p=0,1,2,3,...) and divide the line AB at the point C in the following proportion (Fig. 4.6):

$$\frac{CB}{AC}=\left(\frac{AB}{CB}\right)^p. \tag{4.44}$$

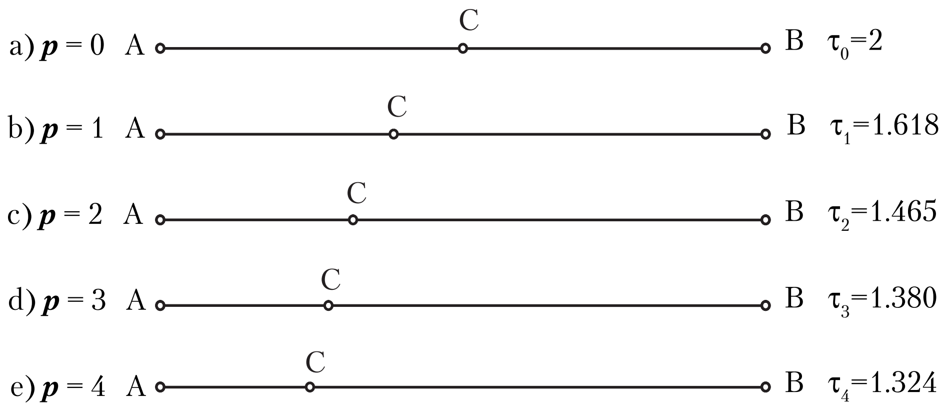


Figure 4.6. The Generalized Golden p-Sections

Note that the proportion (4.44) is reduced to the "dichotomy" for the case p=0 (Fig. 4.6-a) and to the classical "division in the extreme and mean ratio" (the golden section) for the case p=1 (Fig. 4.6-b). Taking this fact into consideration, we will name the division of the line segment AB at the point C in the proportion (4.44) a *Golden p-Section* and the positive root of Eq. (4.42) a *Golden p-Proportion* [20].

4.4.2. *Algebraic Properties of the Golden p-Proportions*

If we substitute the golden p-proportion τ_p for x in Eq. (4.42), we get the following identity for the golden p-proportion:

$$\tau_p^{p+1} = \tau_p^p + 1. \tag{4.45}$$

If we divide all terms of the identity (4.45) by τ_p^p, we get the following remarkable property of the golden p-proportion:

$$\tau_p = 1 + \frac{1}{\tau_p^p} \tag{4.46}$$

or

$$\tau_p - 1 = \frac{1}{\tau_p^p}. \tag{4.47}$$

Note that for the case p=0 (τ_p=2) the identities (4.46) and (4.47) are reduced to the following trivial cases:

$$2 = 1 + \frac{1}{1} \text{ or } 2 - 1 = \frac{1}{1}.$$

For the case p=1 we have $\tau_1 = \tau = \left(1+\sqrt{5}\right)/2$ and the identities (4.46), (4.47) are reduced to the identities (1.11) and (1.12).

If we multiply and divide repeatedly all terms of the identity (4.45) by τ_p, we obtain the following remarkable identity connecting the powers of the golden p-proportion:

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}. \tag{4.48}$$

Note that for the case p=0 the identity (4.48) is reduced to the following trivial identity for the binary numbers:

$$2^n=2^{n-1}+2^{n-1}=2\times 2^{n-1}.$$

For the case p=1 the identity (4.48) is reduced to the following well-known identity for the classical golden mean:

$$\tau^n=\tau^{n-1}+\tau^{n-2}=\tau\times\tau^{n-1}. \tag{4.49}$$

4.4.3. *Geometric Progressions Based on the Golden p-Proportions*

Now, consider a geometric progression based on the golden p-proportion:

$$\left\{\tau_p^n, \tau_p^{n-1}, ..., \tau_p^3, \tau_p^2, \tau_p^1, \tau_p^0 = 1, \tau_p^{-1}, \tau_p^{-2}, \tau_p^{-3}, ...\right\}. \tag{4.50}$$

The geometric progression (4.50) possesses the remarkable property: for the case p>0 according to (4.48) each term of the geometric progression (4.50), for example, τ_p^n can be obtained from the preceding terms in two ways: (1) by the multiplication of the preceding term by τ_p $\left(\tau_p^n = \tau_p \times \tau_p^{n-1}\right)$; (2) by the summation of the (n-1)-th and the (n-p-1)-th terms $\left(\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1}\right)$.

Note that till now we believed that only the golden geometric progression based on the classical golden mean possesses similar "additive" property. It follows from this consideration that a number of similar geometric progressions are infinite and all of them are based on the golden p-proportions.

By decomposing τ_p^n and all the cases arising from such decomposition terms $\tau_p^{n-1}, \tau_p^{n-2}, \tau_p^{n-3}, ...$ according to the recursive relation (4.48), we obtain the following identities:

$$\begin{aligned}
&\tau_p^n = \tau_p^{n-p-1} + \tau_p^{n-1}\\
&\tau_p^n = \tau_p^{n-p-1} + \tau_p^{n-p-2} + \tau_p^{n-2}\\
&\tau_p^n = \tau_p^{n-p-1} + \tau_p^{n-p-2} + \tau_p^{n-p-3} + \tau_p^{-3}\\
&\dots\\
&\tau_p^n = \left(\sum_{j=1}^{k} \tau_p^{n-p-j}\right) + \tau_p^{n-k}.
\end{aligned} \tag{4.51}$$

In particular, for the case k=p the identity (4.51) takes the following form:

$$\tau_p^n = \left(\sum_{j=1}^{p} \tau_p^{n-p-j}\right) + \tau_p^{n-p} = \sum_{j=0}^{p} \tau_p^{n-p-j}. \tag{4.52}$$

By decomposing τ_p^n and terms $\tau_p^{n-(p+1)}, \tau_p^{n-2(p+1)}, \tau_p^{n-(k-1)(p+1)},...$ arising from such decomposition according to the recursive relation (4.48), we obtain the following identities:

$$\begin{aligned}
&\tau_p^n = \tau_p^{n-1} + \tau_p^{n-(p+1)}\\
&\tau_p^n = \tau_p^{n-1} + \tau_p^{n-(p+1)-1} + \tau_p^{n-2(p+1)}\\
&\tau_p^n = \tau_p^{n-1} + \tau_p^{n-(p+1)-1} + \tau_p^{n-2(p+1)-1} + \tau_p^{n-3(p+1)}\\
&\ldots\\
&\tau_p^n = \left(\sum_{j=1}^{k} \tau_p^{n-(j-1)(p+1)-1}\right) + \tau_p^{n-k(p+1)}.
\end{aligned} \tag{4.53}$$

In particular, for the case $p=0$ ($\tau_p=2$) the identities (4.51) and (4.53) coincide and they are reduced to the following remarkable identity for the "binary" numbers:

$$2^n = \sum_{j=1}^{k} 2^{n-j} + 2^{n-k}. \tag{4.54}$$

For the case $p=1$ we have: $\tau_1 = \tau = \left(1+\sqrt{5}\right)/2$; then the identities (4.51) and (4.53) take the following forms, respectively:

$$\tau^n = \sum_{j=1}^{k} \tau^{n-j-1} + \tau^{n-k} \tag{4.55}$$

$$\tau^n = \sum_{j=1}^{k} \tau^{n-2j+1} + \tau^{n-2k}. \tag{4.56}$$

4.5. The Generalized Principle of the Golden Section

4.5.1. *Dichotomy Principle*

The remarkable book [46] by Russian architect Shevelev is devoted to a study of the most general principles that underlie Nature. The *Dichotomy Principle* and the *Golden Section Principle* are the most important of them. The *Dichotomy Principle* is based on the following trivial property of the binary numbers:

$$2^n=2^{n-1}+2^{n-1}, \tag{4.57}$$

where $n=0,\pm1,\pm2,\pm3,...$.

For the case n=0, we have:

$$1=2^0=2^{-1}+2^{-1}. \tag{4.58}$$

In the book [46] the following "dynamic" model of the *Dichotomy Principle* is given in the form of the infinite division of the "Unit" ("The Whole") according to the "dichotomy" relations (4.57) and (4.58):

$$\begin{aligned} &1=2^0=2^{-1}+2^{-1} \\ &\qquad\qquad 2^{-1}=2^{-2}+2^{-2} \\ &\qquad\qquad\qquad\qquad 2^{-2}=2^{-3}+2^{-3} \\ &1=2^0=2^{-1}+2^{-2}+2^{-3}+2^{-4}=...=\sum_{i=1}^{\infty}2^{-i}. \end{aligned} \tag{4.59}$$

4.5.2. *Classical Golden Section Principle*

The *Golden Section Principle* that came to us from Pythagoras, Plato, and Euclid is based on the following fundamental property that connects the adjacent powers of the golden mean $\tau=\left(1+\sqrt{5}\right)/2$:

$$\tau^n=\tau^{n-1}+\tau^{n-2}, \tag{4.60}$$

where $n=0,\pm1,\pm2,\pm3,...$.

For the case n=1, the identity (4.60) takes the following form:

$$1=\tau^0=\tau^{-1}+\tau^{-2}. \tag{4.61}$$

Using the golden identities (4.60) and (4.81), Shevelev developed [46] the following "dynamic" model of the *Golden Section Principle*:

$$\begin{aligned} &1=\tau^0=\tau^{-1}+\tau^{-2} \\ &\qquad\qquad \tau^{-2}=\tau^{-3}+\tau^{-4} \\ &\qquad\qquad\qquad\qquad \tau^{-4}=\tau^{-5}+\tau^{-6} \\ &1=\tau^0=\tau^{-1}+\tau^{-3}+\tau^{-5}+\tau^{-7}+...=\sum_{i=1}^{\infty}\tau^{-(2i-1)}. \end{aligned} \tag{4.62}$$

Note that the *Dichotomy Principle* (4.59) and the *Golden Section Principle* (4.62) have a great number of applications in nature, science and mathematics (binary number system, numerical methods of the algebraic equation solutions, self division and so on).

4.5.3. *The Generalized Principle of the Golden Section*

By dividing all terms of the identity (4.48) by τ_p^n, we obtain the following identity:

$$1=\tau_p^0=\tau_p^{-1}+\tau_p^{-p-1}. \tag{4.63}$$

Using (4.48) and (4.63), we can construct the following "dynamic" model of the "Unit" decomposition according to the *Golden p-Proportion*:

$$\begin{aligned}
&1=\tau_p^0=\tau_p^{-1}+\tau_p^{-(p+1)}\\
&\qquad\qquad\tau_p^{-(p+1)}=\tau_p^{-(p+1)-1}+\tau_p^{-2(p+1)}\\
&\qquad\qquad\qquad\qquad\tau_p^{-2(p+1)}=\tau_p^{-2(p+1)-1}+\tau_p^{-3(p+1)}\\
&1=\tau_p^0=\tau_p^{-1}+\tau_p^{-(p+1)-1}+\tau_p^{-2(p+1)-1}+\tau_p^{-3(p+1)-1}+\ldots=\sum_{i=1}^{\infty}\tau_p^{-(i-1)(p+1)-1}.
\end{aligned}\tag{4.64}$$

The main result of the above consideration is to find more general principle of the "Unit" division that is given by the following identity:

$$1=\tau_p^{-1}+\tau_p^{-(p+1)}=\sum_{i=1}^{\infty}\tau_p^{-(i-1)(p+1)-1},\tag{4.65}$$

where τ_p is the golden p-proportion, p=0,1,2,3,... .

It is clear that this general principle - *Generalized Principle of the Golden Section* - includes the Dichotomy Principle (4.59) and the classical *Golden Section Principle* (4.62) as special cases for p=0 and p=1, respectively.

4.6. A Generalization of Euclid's Theorem II. 11

4.6.1. *A Generalization of Euclid's Theorem II.11 for the Case p=2*

As we mentioned above, the golden p-sections that are given by the proportion (4.44) is a generalization of the classical golden section that is given by (1.3). However, in Euclid's Theorem II.11 the DEMR is formulated in the form (1.2). We can try to represent the proportion (4.44) in the form (1.2). We start from the partial case p=2. For this case the proportion (4.44) takes the following form:

$$\frac{CB}{AC}=\left(\frac{AB}{CB}\right)^2.\tag{4.66}$$

Let us denote the lengths of the line segments *AB, AC* and *CB* in (4.66) as follows: *AB=a, CB=b, AC=c*. Then, it can be represented in the form:

$$a^2\times c=b^3.\tag{4.67}$$

We can give the following geometric interpretation of the equality (4.67). The right-hand part of the equality (4.67) can be interpreted as the volume of a cube with the side equal to *b,* that is, to the length of the larger segment *CB* that arises at the division of a line segment AB in the golden 2-proportion (4.66). The left-hand part of the equality (4.67) can be interpreted as the volume of a rectangular parallelepiped. This parallelepiped has a square at its base with sides equal to a, that is, to the length of the initial segment AB. The height of the rectangular parallelepiped is equal to c, that is, to the length of the smaller segment AC in the proportion (4.66).

Then, taking into consideration (4.66) and (4.67), we can formulate a new geometric problem of the division of a line in the golden 2-section that is a generalization of Euclid's DEMR.

Generalization of DEMR (a Division in the Golden 2-Proportion). Divide the given line AB at point C into two segments, the smaller segment AC and the larger segment CB, so that a volume of the cube with the side equal to the larger segment CB is equal to the volume of a rectangular parallelepiped with a base, which is a square with sides equal to the initial line AB, and with the height equal to the smaller segment AC.

4.6.2. *Euclid's Rectangular Parallelepiped*

The rectangular parallelepiped appearing in the above problem consists of 6 faces (Fig. 4.7). The top and bottom faces are squares with sides equal to the length of the initial segment a; the lateral faces are the rectangles with sides equal to a and c. These rectangles are similar to Euclid's rectangle in Fig. 4.7 where the ratio of its sides a:c for the given case is equal to the square of the golden 2-proportion τ_2, that is,

$$\frac{a}{c} = \tau_2^2. \qquad (4.68)$$

We will name this geometric figure *Euclid's Rectangular Parallelepiped.* Thus, according to (4.68) the ratio of the side of its base to its height in Euclid's Rectangular Parallelepiped is equal to the square of the golden 2-proportion τ_2; here, according to (4.67) its volume is a cube of the length of the larger segment in the proportion (4.66).

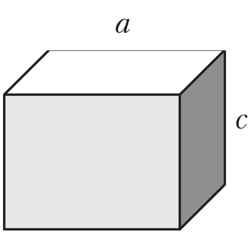


Figure 4.7. Euclid's rectangular parallelepiped

If the initial segment AB is a unit segment (AB=1), then the equality (4.67) takes the following form:

$$c=b^3. \qquad (4.69)$$

Then, we can formulate the following geometric problem of the division of the unit segment in the golden 2-proportion.

A Problem of the Division of the Unit Segment in the Golden 2-Proportion. Divide a unit segment into two unequal segments such that the smaller segment's length is equal to the cube of the larger segment's length.

Note that the formulated problem expresses the following property of the golden 2-proportion:

$$1=\tau_2^{-1}+\tau_2^{-3}=0.6823+0.3177. \tag{4.70}$$

4.6.3. *The General Case of p*

For the general case of p, we represent (4.44) in the following form:

$$a^p \times c = b^{p+1}. \tag{4.71}$$

By using geometric language, we can interpret the equality (4.71) as follows. The right-hand part of the equality (4.71) is a volume of a hypercube in the $(p+1)$-dimensional space with the side equal to the length b of the larger segment of the division of a line in the golden p-proportion. The left-hand part of the equality (4.71) is a volume of *Euclid's Hyper-Rectangular Parallelepiped* in the $(p+1)$-dimensional space; here, the p sides are equal to the length a of the initial segment at the division of a line in the golden p-proportion, and the $(p+1)$-th side (its "height") is equal to the length c of the smaller segment at the division of a line in the golden p-proportion.

It is clear that (4.71) expresses a generalized Euclidean problem of the division in extreme and mean ratio. We can consider this problem for the case of the unit segment (a=1). Then, the equality (4.71) takes the following form:

$$c=b^{p+1}. \tag{4.72}$$

Taking into consideration (4.72), we can formulate the following problem.

A Problem of the Division of the Unit Segment in the Golden *p*-Proportion. For a given p=0,1,2,3,... divide a unit segment into two unequal segments in such proportion that the smaller segment's length is equal to the $(p+1)$-th degree of the larger segment's length.

4.7. The Roots of the Generalized Golden Algebraic Equations

4.7.1. *Algebraic Equations*

As is well known, algebraic equations have the following general properties, which can be used by us before finding their roots:

1. **Algebraic equations of the *n*-th degree have *n* roots.** In special cases it can appear, that some roots are repeated some times (multiple roots); hence, the number of various roots can be less than n.
2. **Descartes' "rule of signs": the algebraic equation has no more positive roots than the number of sign changes in a series of its factors.** Often, it is not so important to calculate the roots, it is more important to understand the character of these roots. "The rule of signs" helps to solve this problem. For example, the equation x^5-$4x$-2=0 has only one positive root because the series of their factors 1,-4,-2 have only one change of a sign.
3. **In the equations with real factors, the complex roots can appear only by pairs**: alongside with the root $a+bi$ the complex number $a-bi$ is always a root of the same equation.
4. If x_i (i=1,2,3,...,n) is one of the roots of the algebraic equation

$$a_0x^n+ a_1x^{n-1}+...+a_n=0, \tag{4.73}$$

then it is easy to prove, that a polynomial that stands in the left-hand part of Eq. (4.73), is divided by the binom (x-x_i) without remainder. It is easy to prove that any polynomial of n-th degree can be represented as a product of the n multipliers of the 1-st degree of the kind (x-x_i), that is,

$$a_0x^n+ a_1x^{n-1}+...+a_n=(x-x_1)(x-x_2)\ ...\ (x-x_n). \tag{4.74}$$

This theorem (4.74) is sometimes named the *Basic Theorem of Algebra.*

4.7.2. *Properties of the Roots of the Generalized Golden Algebraic Equations*

Once again, consider the characteristic equation (4.42). By using the above rules 1-4, we can prove the following properties of the roots of Eq. (4.42):
1. According to Descartes' "Rule of Signs," Eq. (4.42) has the only positive root τ_p. This root expresses some important property of Pascal Triangle.
2. As Eq. (4.42) has the degree (p+1), this means that Eq. (4.42) has (p+1) roots $x_1, x_2, ..., x_p, x_{p+1}$. Further, without loss of generality, we suppose that the root x_1 always coincides with the golden p-proportion τ_p, that is, $x_1=\tau_p$.
3. As Eq. (4.42) has only the real factors 1, -1 and -1, this means that all complex roots of Eq. (4.42) appear in pairs, that is, each complex root $a+bi$ appears always together with the root $a-bi$, which is the complex conjugate to the root $a+bi$.
4. By using (4.74), we can represent Eq. (4.42) by its roots as follows:

$$x^{p+1}-x^p-1 =(x-x_1)(x-x_2)\ ...\ (x-x_p)(x-x_{p+1}). \tag{4.75}$$

The general identity below for the roots $x_1, x_2, ..., x_p, x_{p+1}$ comes from (4.42):

$$x_k^n = x_k^{n-1} + x_k^{n-p-1} = x_k \times x_k^{n-1}, \tag{4.76}$$

where n=0,±1,±2,±3,... and x_k (k=1,2,3,..., p+1) is a root of Eq. (4.42). Note that the identity (4.48) is a partial case of the identity (4.76) for the case k=1.

Theorem 4.1. For a given integer p>0, the following correlations for the roots of the characteristic equation x^{p+1}-x^p-1=0 are valid:

$$x_1 + x_2 + x_3 + ... + x_p + x_{p+1} = 1 \tag{4.77}$$

$$\begin{gathered}\left(x_1x_2 + x_1x_3 + ... + x_1x_{p+1}\right) + \left(x_2x_3 + ... + x_2x_{p+1}\right) + ... + x_px_{p+1} = 0 \\ \left(x_1x_2x_3 + x_1x_3x_4 + ... + x_1x_px_{p+1}\right) + \left(x_2x_3x_4 + ... + x_2x_px_{p+1}\right) + ... + x_{p-1}x_px_{p+1} = 0 \\ ... \\ x_1x_2 \cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4 \cdots x_{p-1}x_px_{p+1} + ... + x_2x_3x_4 \cdots x_{p-1}x_px_{p+1} = 0\end{gathered} \tag{4.78}$$

$$x_1x_2x_3 \cdots x_{p-1}x_px_{p+1} = (-1)^p. \tag{4.79}$$

Proof. Consider the representation of Eq. (4.42) in the form (4.75). If we remove the parentheses in (4.75), then, for the even p=2k we can write:

$$\begin{gathered}x^{p+1} - x^p - 1 = (x - x_1)(x - x_2)\cdots(x - x_{p+1}) = x^{p+1} - \left(x_1 + x_2 + ... + x_{p+1}\right)x^p \\ +\left(x_1x_2 + ... + x_1x_{p+1} + x_2x_3 + ... + x_2x_{p+1} + ... + x_{p-1}x_{p+1} + x_px_{p+1}\right)x^{p-1} \\ -\left(x_1x_2x_3 + ... + x_1x_px_{p+1} + x_2x_3x_4 + ... + x_2x_px_{p+1} + ... + x_{p-1}x_px_{p+1}\right)x^{p-2} \\ +\left(x_1x_2x_3x_4 + x_1x_2x_3x_5 + ... + x_{p-2}x_{p-1}x_px_{p+1}\right)x^{p-3} \\ ... \\ +\left(x_1x_2 \cdots x_{p-1}x_p + x_1x_3 \cdots x_px_{p+1} + ... + x_2x_3 \cdots x_px_{p+1}\right)x - x_1x_2 \cdots x_px_{p+1} = 0.\end{gathered} \tag{4.80}$$

The following outcomes follow from the comparison of (4.42) and (4.80):

$$\begin{gathered}x_1 + x_2 + ... + x_p + x_{p+1} = 1 \\ \left(x_1x_2 + ... + x_1x_{p+1}\right) + \left(x_2x_3 + ... + x_2x_{p+1}\right) + ... + \left(x_{p-1}x_p + x_{p-1}x_{p+1}\right) + x_px_{p+1} = 0 \\ \left(x_1x_2x_3 + ... + x_1x_px_{p+1}\right) + \left(x_2x_3x_4 + ... + x_2x_px_{p+1}\right) + ... + x_{p-1}x_px_{p+1} = 0 \\ ... \\ x_1x_2 \cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4 \cdots x_{p-1}x_px_{p+1} + ... + x_2x_3x_4 \cdots x_{p-1}x_px_{p+1} = 0 \\ x_1x_2 \cdots x_px_{p+1} = 1.\end{gathered} \tag{4.81}$$

For the odd p=2k+1, we have:

$$\begin{gathered}x^{p+1} - x^p - 1 = (x - x_1)(x - x_2)\cdots(x - x_{p+1}) = x^{p+1} - \left(x_1 + x_2 + \ldots + x_{p+1}\right)x^p \\ +\left(x_1x_2 + \ldots + x_1x_{p+1} + x_2x_3 + \ldots + x_2x_{p+1} + \ldots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1}\right)x^{p-1} \\ -\left(x_1x_2x_3 + \ldots + x_1x_px_{p+1} + x_2x_3x_4 + \ldots + x_2x_px_{p+1} + \ldots + x_{p-1}x_px_{p+1}\right)x^{p-2} \\ +\left(x_1x_2x_3x_4 + x_1x_2x_3x_5 + \ldots + x_{p-2}x_{p-1}x_px_{p+1}\right)x^{p-3} \\ ... \\ -\left(x_1x_2x_3 \cdots x_{p-2}x_{p-1}x_p + \ldots + x_2x_3x_4 \cdots x_{p-1}x_px_{p+1}\right)x + x_1x_2x_3 \cdots x_{p-1}x_px_{p+1} = 0.\end{gathered} \tag{4.82}$$

The following outcomes follow from the comparison of (4.42) and (4.82):

$$\begin{gathered}x_1 + x_2 + ... + x_p + x_{p+1} = 1 \\ \left(x_1x_2 + ... + x_1x_{p+1}\right) + \left(x_2x_3 + ... + x_2x_{p+1}\right) + ... + \left(x_{p-1}x_p + x_{p-1}x_{p+1}\right) + x_px_{p+1} = 0 \\ \left(x_1x_2x_3 + ... + x_1x_px_{p+1}\right) + \left(x_2x_3x_4 + ... + x_2x_px_{p+1}\right) + ... + x_{p-1}x_px_{p+1} = 0 \\ ... \\ x_1x_2 \cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4 \cdots x_{p-1}x_px_{p+1} + ... + x_2x_3x_4 \cdots x_{p-1}x_px_{p+1} = 0 \\ x_1x_2 \cdots x_px_{p+1} = -1.\end{gathered} \tag{4.83}$$

The outcomes (4.81) and (4.83) prove Theorem 4.1.

It is evident from (4.77) that the sum of the roots of Eq. (4.42) is identically equal to 1. The expression (4.78) gives the values for every possible sum of the roots of Eq. (4.42) taken by two, three, ..., or p roots from the $(p+1)$ roots of Eq. (4.42). According to (4.78), each of these sums is identically equal to zero! At last, the expression (4.79) gives the value of the product of all roots of Eq. (4.42). According to (4.79) this product is equal to 1 (for the even p) or -1 (for the odd p).

Theorem 4.2. For a given integer p=1,2,3,... and for the condition when k takes its values from the set $\{1,2,3,...,p\}$, the following identity is valid for the roots of the characteristic equation $x^{p+1}-x^p-1=0$:

$$\left(x_1+x_2+x_3+...+x_{p+1}\right)^k = x_1^k+x_2^k+x_3^k+...+x_{p+1}^k=1. \tag{4.84}$$

Proof. Consider the expression:

$$(x_1+x_2+...+x_p+x_{p+1})^k, \tag{4.85}$$

where k takes its values from (4.86)

$$k\in\{1,2,...,p\}. \tag{4.86}$$

Taking into consideration the identity (4.77), we can write:

$$(x_1+x_2+...+x_p+x_{p+1})^k=1^k=1. \tag{4.87}$$

Consider a partial case of the expression (4.87) for the case p=1. Taking into consideration the condition (4.86) for the case p=1 the expression (4.85) can take the only form:

$$x_1+x_2. \tag{4.88}$$

According to the property (4.87), we can write:

$$x_1+x_2 = 1 \tag{4.89}$$

that satisfies the expression (4.84).

Now, consider the case p=2. Taking into consideration the condition (4.86), for the case p=2 the expression (4.85) can take only two different forms:

$$x_1+x_2+x_3 \tag{4.90}$$

and

$$(x_1+x_2+x_3)^2. \tag{4.91}$$

Taking into consideration the identity (4.87), we can write for the case (4.90) the following identity:

$$x_1+x_2+x_2 = 1. \tag{4.92}$$

Now, consider the case (4.91). Representing (4.91) in the form

$$[(x_1+x_2)+x_3]^2, \tag{4.93}$$

we can write:

$$\left(x_1+x_2+x_3\right)^2=\left[\left(x_1+x_2\right)+x_3\right]^2=x_1^2+x_2^2+x_3^2+2\left(x_1x_2+x_1x_3+x_2x_3\right). \tag{4.94}$$

If we take into consideration the properties (4.78) and (4.84), we can rewrite the expression (4.94) as follows:

$$\left(x_1+x_2+x_3\right)^2=x_1^2+x_2^2+x_3^2=1. \tag{4.95}$$

By analogy, we can write 3 possible expressions like (4.85) for the case p=3:

$$x_1+x_2+x_3+x_4, \tag{4.96}$$

$$(x_1+x_2+x_3+x_4)^2, \tag{4.97}$$

$$(x_1+x_2+x_3+x_4)^3. \tag{4.98}$$

If we fulfill simple transformations of the expressions (4.96)-(4.98) and take into consideration the properties (4.77) and (4.84), we can write 3 identities for the case p=3:

$$x_1+x_2+x_3+x_4=1 \tag{4.99}$$

$$\left(x_1+x_2+x_3+x_4\right)^2=x_1^2+x_2^2+x_3^2+x_4^2=1 \tag{4.100}$$

$$\left(x_1+x_2+x_3+x_4\right)^3=x_1^3+x_2^3+x_3^3+x_4^3=1. \tag{4.101}$$

In the generalized case, for the proof of Theorem 4.2 for arbitrary p we can use the so-called *Multinomial Theorem* [35], which is a generalization of the binomial theorem (4.4). For any positive integer m and any non-negative integer n, the multinomial formula is the following:

$$(x_1+x_2+x_3+\ldots+x_m)^n=\sum_{k_1,k_2,k_3,\ldots,k_m}\binom{n}{k_1,k_2,k_3,\ldots,k_m}x_1^{k_1}x_2^{k_2}x_3^{k_3}\ldots x_m^{k_m}. \tag{4.102}$$

The summation is taken over all sequences of the nonnegative integer indexes k_1 through k_m such that $\sum_{i=1}^{m}k_i=n$. The numbers

$$\binom{n}{k_1,k_2,k_3,\ldots,k_m}=\frac{n!}{k_1!k_2!k_3!\ldots k_m!} \tag{4.103}$$

are called *Multinomial Coefficients.*

Note that the *Binomial Theorem* (4.4) and the *Binomial Coefficients* (4.2) are special cases (m=2) of the *Multinomial Formula* (4.102) and the *Multinomial Coefficients* (4.103), respectively.

In the general case of p, the expression (4.85) can be factorized if we use the *Multinomial Formula* (4.102). As is known [35], for a given k, the *Multinomial Formula* (4.102) will include in itself the sum of all k-th powers of Eq. (4.42) that are taken with the coefficient of 1, that is,

$$x_1^k+x_2^k+x_3^k+\ldots+x_p^k+x_{p+1}^k, \tag{4.104}$$

and the sum of the products of every possible combination of two (k=2), three (k=3) or the k roots of Eq. (4.42) that are taken with the factors known as *Multinomial Coefficients* (4.103) [35]. According to Theorem 4.1 all these sums are identically equal to zero. And then taking into consideration (4.86), we can write the general identity (4.84).

4.7.3. *Some Corollaries of Theorems 4.1 and 4.2*

Now, consider some corollaries of Theorems 4.1 and 4.2 for the different values of p. It is well known that for the case p=1 the golden algebraic equation (4.43) has two real roots:

$$x_1 = \tau = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = -\frac{1}{\tau} = \frac{1-\sqrt{5}}{2}.$$

Hence, from the above we can obtain the following identities - corresponding to Theorem 4.1 - for the roots x_1 and x_2:

$$x_1+x_2=1;\ x_1\times x_2=-1. \tag{4.105}$$

For the case p=2, the golden algebraic equation (4.42) takes the following form:

$$x^3=x^2+1. \tag{4.106}$$

Equation (4.106) has three roots - one real root x_1 and two complex conjugate roots x_2 and x_3 that are given below:

$$x_1 = \frac{h}{6} + \frac{2}{3h} + \frac{1}{3} = 1.4655712319... \tag{4.107}$$

$$x_2 = -\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} - i\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right) = -0.233... - (0.793...)i \tag{4.108}$$

$$x_3 = -\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} + i\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right) = -0.233... + (0.793...)i, \tag{4.109}$$

where

$$h = \sqrt[3]{116 + 12\sqrt{93}}. \tag{4.110}$$

Using direct substitution for the roots x_1, x_2 and x_3, it is easy to prove the following identities corresponding to Theorems 4.1 and 4.2 for the case p=2:

$$x_1 + x_2 + x_3 = 1,$$

$$x_1x_2 + x_1x_3 + x_2x_3 = 0,$$

$$x_1 \times x_2 \times x_3 = 1,$$

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

4.8. The Generalized Golden Algebraic Equations of Higher Degrees

4.8.1. *The Case of p=1*

In Chapter 1 we developed a theory of the golden algebraic equations of higher degrees. We proved that for the simplest golden algebraic equation (4.43) – which is a partial case (p=1) of the characteristic equation (4.42) - there is an infinite number of the characteristic equations with the degree greater than 2 that have the golden mean $\tau=\left(1+\sqrt{5}\right)/2$ as their general root. In their general form these golden equations are given by the expression:

$$x_n=F_n x^2-F_{n-2}=F_n x-F_{n-1}, \tag{4.111}$$

where F_n, F_{n-1}, F_{n-2} are Fibonacci numbers.

As we mentioned in Chapter 1, the equation $x^4=3x+2$ - which is a partial case of the general equation (4.111) corresponding to $n=4$ - describes the energy state of the butadiene molecule, a valuable chemical substance, which is used in the production of rubber. This fact at once places our interest in the golden equations (4.111), because they, probably, in general, can describe the energy conditions of the molecules of other chemical substances.

In this connection it is of great interest to study the characteristic equations given by (4.42) and the algebraic equations following from them with degrees more than p+1 - which have the golden p-proportions τ_p as their general roots. Let us demonstrate our approach for the partial cases p=2,3.

4.8.2. *The Case of p=2*

For the case of p=2 Eq. (4.42) is reduced to the algebraic equation of the third degree: $x^3=x^2+1$. Multiplying repeatedly all terms of the equation $x^3=x^2+1$ by x, we can find the following equality for the case of p=2:

$$x^n=x^{n-1}+x^{n-3}, \tag{4.112}$$

where n=3, 4, 5,... .

Using the equality (4.112) and (4.42), we derive the following algebraic equations that have the golden 2-proportion τ_2 as a root:

$$\begin{aligned}
&x^4=x^3+x=x^2+x+1\\
&x^5=x^4+x^2=2x^2+x+1\\
&x^6=x^5+x^3=3x^2+x+2\\
&\dots\\
&x^n=x^{n-1}+x^{n-3}=F_2(n-1)x^2+F_2(n-3)x+F_2(n-2),
\end{aligned} \tag{4.113}$$

where $F_2(n\text{-}1)$, $F_2(n\text{-}2)$, $F_2(n\text{-}3)$ are the Fibonacci 2-numbers that are given by the recursive relation (4.14) at the seeds (4.15).

As the golden 2-proportion τ_2 is the root of each of the equations that are given by (4.113), the following general identity, which connects the golden 2-proportion τ_2 with the Fibonacci 2-numbers $F_2(n)$, follows from (4.113):

$$\tau_2^n = \tau_2^{n-1} + \tau_{n-3}^{n-3} = F_2(n-1)\tau_2^2 + F_2(n-3)\tau_2 + F_2(n-2). \tag{4.114}$$

4.8.3. *The Case of p=3*

Consider the characteristic equation (4.42) for the case of p=3:

$$x^4=x^3+1. \tag{4.115}$$

If we use the reasoning similar to the case of p=2, we can obtain the following algebraic equations that have the golden 3-proportion τ_3 as their general root:

$$\begin{aligned}
&x^5 = x^4 + x \ \ = x^3 + x + 1\\
&x^6 = x^5 + x^2 = x^3 + x^2 + x + 1\\
&x^7 = x^6 + x^3 = 2x^3 + x^2 + x + 1\\
&\ldots\\
&x^n = x^{n-1} + x^{n-4} = F_3(n-2)x^3 + F_3(n-5)x^2 + F_3(n-4)x + F_3(n-3),
\end{aligned} \tag{4.116}$$

where $F_3(n\text{-}2)$, $F_3(n\text{-}3)$, $F_3(n\text{-}4)$, $F_3(n\text{-}5)$ are the Fibonacci 3-numbers given by the recursive relation $F_3(n)=F_3(n-1)+F_3(n-4)$ at the seeds:

$F_3(1)=F_3(2)=F_3(3)=F_3(4)=1.$

As the golden 3-proportion τ_3 is the root of any of the equations (4.116), the following identity, which connects the golden 3-proportion τ_3 with the Fibonacci 3-numbers $F_3(n)$, follows from (4.116):

$$\tau_3^n = \tau_3^{n-1} + \tau_3^{n-4} = F_3(n-2)\tau_3^3 + F_3(n-5)\tau_3^2 + F_3(n-4)\tau_3 + F_3(n-3). \tag{4.117}$$

4.8.4. *The General Case*

For the general case of p, we can write the following equality, which can be obtained from the algebraic equation (4.42):

$$x^n=x^{n-1}+ x^{n-p-1}, \tag{4.118}$$

where $n=p+1, p+2, p+3, \ldots$.

Using Eqs. (4.42) and (4.118), we can obtain the following formula for the generalized characteristic equations that have the golden p-proportion τ_p as their root:

$$x^n = F_p(n-p+1)x^p + \sum_{t=0}^{p-1} F_p(n-p-t)x^t, \tag{4.119}$$

where $n=p+1, p+2, p+3, \ldots$; $F_p(n-p+1)$ and $F_p(n-p-t)$ are the Fibonacci p-numbers given by the recursive relation (4.18) at the seeds (4.19).

It is clear that, for a given $p>0$, the formula (4.119) sets an infinite number of generalized characteristic equations with the general root τ_p. The following general identity that comes from (4.119) connects the degrees of the golden p-proportion τ_p with the Fibonacci p-numbers $F_p(n)$:

$$\tau_p^n = F_p(n-p+1)\tau_p^p + \sum_{t=0}^{p-1} F_p(n-p-t)\tau_p^t, \tag{4.120}$$

where $n=p+1, p+2, p+3, \ldots$.

Concluding this Section we may note that the above algebraic equations that are given by (4.111), (4.113), (4.116), and (4.119) are unusual algebraic equations. First of all, they follow from Pascal Triangle and express some important mathematical properties. Besides, they describe some harmonious chemical and physical structures and we may expect their application for simulation of many physical and chemical processes and structures.

4.9. The Generalized Binet Formula for the Fibonacci *p*-Numbers

4.9.1. *A General Approach to the Synthesis of the Generalized Binet Formulas*

In Chapter 2 we established the Binet formulas (2.62) and (2.63). These formulas are the representations of the "extended" Fibonacci and Lucas numbers by the golden mean τ. In this Section we try to develop a general approach to the synthesis of Binet formulas based on the representation of the "extended" Fibonacci p-numbers by the roots of the characteristic equation (4.42).

Our main hypothesis is the following. For a given $p>0$, we can represent the Binet formula for the Fibonacci p-numbers as follows:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \ldots + k_{p+1}(x_{p+1})^n, \tag{4.121}$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of Eq. (4.42), and $k_1, k_2, \ldots, k_{p+1}$ are constant coefficients that depend on the initial elements of the Fibonacci p-series.

It follows from (4.29) that $F_p(0)=0$ for any $p>0$. Therefore, we can calculate the Fibonacci p-numbers according to the recursive relation (4.18) at the seeds:

$$F_p(0)=0, F_p(1)=F_p(2)=F_p(p)=1. \tag{4.122}$$

Taking into consideration (4.121) and (4.122), we can write the following system of algebraic equations:

$$\begin{aligned}
&F_p(0)=k_1+k_2+\ldots+k_{p+1}=0\\
&F_p(1)=k_1x_1+k_2x_2+\ldots+k_{p+1}=1\\
&F_p(2)=k_1(x_1)^2+k_2(x_2)^2+\ldots+k_{p+1}(x_{p+1})^2=1\\
&\ldots\\
&F_p(p)=k_1(x_1)^p+k_2(x_2)^p+\ldots+k_{p+1}(x_{p+1})^p=1.
\end{aligned} \tag{4.123}$$

Solving the system of the equations (4.123), we obtain the numerical values of the coefficients k_1, k_2, ..., k_{p+1} for the different values of p.

4.9.2. *A Derivation of Binet Formulas for the Classical Fibonacci and Lucas Numbers*

We can use the general formula (4.121) to obtain Binet formulas for the case p=1. For this case, the characteristic equation (4.42) is reduced to Eq. (4.43), which has two roots $x_1=\tau$ and $x_2=-1/\tau$, where $\tau=\left(1+\sqrt{5}\right)/2$.

Therefore, formula (4.121), for the case of p=1, takes the following form:

$$F_1(n)=k_1(\tau)^n+k_2\left(-\frac{1}{\tau}\right)^n. \tag{4.124}$$

It is clear that for the case of p=1 the system of algebraic equations (4.123) is

$$\begin{cases}F_1(0)=k_1+k_2=0\\ F_1(1)=k_1\tau+k_2(-1/\tau)=1.\end{cases} \tag{4.125}$$

By solving the system (4.125), we obtain: $k_1=1/\sqrt{5}$ and $k_1=-1/\sqrt{5}$. If we substitute k_1 and k_2 into (4.124), we obtain the well-known Binet formula for the classical Fibonacci numbers in the form:

$$F_1(n)=\frac{\tau^n-(-1/\tau)^n}{\sqrt{5}}. \tag{4.126}$$

If we assume $k_1=k_2=1$ in (4.124), we obtain the Binet formula for the classical Lucas numbers:

$$L_1(n)=\tau^n+(-1/\tau)^n. \tag{4.127}$$

This formula generates the Lucas series at the seeds $L_1(0)=2$ and $L_1(1)=1$:

$$2, 1, 3, 4, 7, 11, 18, 29, \ldots . \tag{4.128}$$

Let us note one important fact regarding the seed $F_1(0)=0$. It follows directly from the Binet formula (4.124). In fact, according to (4.124) we have the following result for the case of n=0:

$$F_1(0)=k_1\tau^0+k_2(-1/\tau)^0=k_1+k_2=1/\sqrt{5}-1/\sqrt{5}=0. \tag{4.129}$$

These simple calculations show that the seed $F_1(0)=0$ comes from the fact that the sum of the coefficients k_1 and k_2 in the expression (4.125) is also equal to zero.

4.9.3. *Binet Formula for the Fibonacci 2-Numbers*

Let us give p=2 and use the above approach for obtaining Binet formula for the Fibonacci 2-numbers. For the case p=2, the recursive relation for the Fibonacci 2-numbers and the characteristic equation (4.42) take the following forms, respectively:

$$F_2(n)=F_2(n-1)+F_2(n-3) \tag{4.130}$$

$$F_2(0)=0, F_2(1)=F_2(2)=1 \tag{4.131}$$

$$x^3=x^2+1. \tag{4.132}$$

Equation (4.132) has three roots - real (positive) root $x_1=\tau_2$ given by (4.107) and two complex-conjugate roots x_2 and x_3 (4.108) and (4.109).

We recall that the real root x_1 of the algebraic equation (4.132) is an irrational number that is equal to the golden 2-proportion ($x_1=\tau_2$). The number h given by (4.110) is also irrational; hence, the roots x_2 and x_3 are complex numbers with irrational real parts.

For the case p=2, the formula (4.121) and the system (4.123) take the following forms, respectively:

$$F_2(n)=k_1(x_1)^n+k_2(x_2)^n+k_3(x_3)^n \tag{4.133}$$

$$\begin{aligned} F_2(0)&=k_1+k_2+k_3=0\\ F_2(1)&=k_1x_1+k_2x_2+k_3x_3=1\\ F_2(2)&=k_1(x_1)^2+k_2(x_2)^2+k_3(x_3)^2=1. \end{aligned} \tag{4.134}$$

Solving the system (4.134), we obtain:

$$k_1=\frac{2h(h+2)}{(h^3+8)} \tag{4.135}$$

$$k_2=\frac{\left[-(h+2)+i\sqrt{3}(h-2)\right]h}{(h^3+8)} \tag{4.136}$$

$$k_3 = \frac{\left[-(h+2) - i\sqrt{3}(h-2)\right]h}{\left(h^3+8\right)}, \tag{4.137}$$

where h is given by (4.110).

Note that the coefficient (4.135) is an irrational number because the number h given by (4.110) is irrational. Also the coefficients (4.136) and (4.137) are complex-conjugate numbers with irrational real parts.

Substituting (4.107)-(4.110) and (4.135)-(4.137) into the expression (4.133), we can write the following Binet formula for the Fibonacci 2-numbers, which is a generalization of the Binet formula (4.126) for the classical Fibonacci numbers :

$$\begin{aligned} F_2(n) &= \frac{2h(h+2)}{h^3+8}\left(\frac{h}{6}+\frac{2}{3h}+\frac{1}{3}\right)^n \\ &+ \frac{\left[-(h+2)+i\sqrt{3}(h-2)\right]h}{h^3+8}\left[-\frac{h}{12}-\frac{1}{3h}+\frac{1}{3}-i\frac{\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{3h}\right)\right]^n \\ &+ \frac{\left[-(h+2)-i\sqrt{3}(h-2)\right]h}{h^3+8}\left(-\frac{h}{12}-\frac{1}{3h}+\frac{1}{3}+i\frac{\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{3h}\right)\right)^n . \end{aligned} \tag{4.138}$$

It seems incredible at first sight, that the formula (4.138) - that is very complicated combination of complex numbers with irrational coefficients - represents the integer Fibonacci 2-numbers $F_2(n)$ for any integer n=0, ±1, ±2, ±3, ...

4.9.4. *Binet Formula for the Fibonacci 3-Numbers*

For the case of p=3, the recursive relation for the Fibonacci 3-numbers and the characteristic equation (4.42) take the following forms, respectively:

$$F_3(n)=F_3(n-1)+F_3(n-4) \tag{4.139}$$

$$F_3(0)=0,\ F_3(1)=F_3(2)=F_3(3)=1 \tag{4.140}$$

$$x^4=x^3+1. \tag{4.141}$$

Equation (4.141) has four roots - two real roots, x_1 and x_2, and two complex-conjugate roots, x_3 and x_4. The roots of Eq. (4.141) are irrational and complex numbers, which have a very complex symbolic representation; therefore, we can use their approximate numerical values:

$$x_1=1.380;\ x_2=-0.819;\ x_3=0.219+0.914i;\ x_4=0.219-0.914i. \tag{4.142}$$

It is important to note that the root x_1 of Eq. (4.141) is the golden 3-proportion ($x_1=\tau_3=1.380$).

The formulas (4.122) for the Fibonacci 3-numbers and the system of the algebraic equations (4.123) take the following forms, respectively:

$$F_3(n)=k_1(x_1)^n+k_2(x_2)^n+k_3(x_3)^n+k_4(x_4)^n \tag{4.143}$$

$$\begin{aligned}
&F_3(0)=k_1+k_2+k_3+k_4=0\\
&F_3(1)=k_1x_1+k_2x_2+k_3x_3+k_4x_4=1\\
&F_3(2)=k_1(x_1)^2+k_2(x_2)^2+k_3(x_3)^2+k_4(x_4)^2=1\\
&F_3(3)=k_1(x_1)^3+k_2(x_2)^3+k_3(x_3)^3+k_4(x_4)^3=1.
\end{aligned} \tag{4.144}$$

Solving the system (4.144), we obtain the following numerical coefficients:

$$k_1=0.3969;\ k_2=-0.1592;\ k_3=-0.1188-0.2045i;\ k_4=-0.1188+0.2045i. \tag{4.145}$$

Hence, using (4.142) and (4.145), we can represent Binet formula (4.143) for the Fibonacci 3-numbers in the following numerical form:

$$\begin{aligned}
F_3(n)&=0.3969(1.38)^n-0.1592(-0.819)^n\\
&+(-0.1188-0.2045i)(0.219+0.914i)^n\\
&+(-0.1188+0.2045i)(0.219-0.914i)^n.
\end{aligned} \tag{4.146}$$

4.9.5. *Binet Formulas for the Fibonacci 4-Numbers*

For the case of $p=4$, the recursive relation for the Fibonacci 4-numbers and the characteristic equation (4.42) take the following forms, respectively:

$$F_4(n)=F_4(n-1)+F_4(n-5) \tag{4.147}$$

$$F_4(0)=0,\ F_4(1)=F_4(2)=F_4(3)=F_4(4)=1 \tag{4.148}$$

$$x^5=x^4+1. \tag{4.149}$$

Equation (4.141) has five roots - one real root x_1 that coincides with the golden 4-proportion τ_4 and two pairs of complex-conjugate roots x_2, x_3 and x_4, x_5. They all can be represented in the following analytical and numerical form:

$$\begin{aligned}
&x_1=\frac{h}{6}+\frac{2}{h}=1.3247,\ x_2=\frac{1}{2}-\frac{i\sqrt{3}}{2}=0.5-0.866i,\ x_3=\frac{1}{2}+\frac{i\sqrt{3}}{2}=0.5+0.866i\\
&x_4=-\left(\frac{h}{12}+\frac{1}{h}\right)-\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{h}\right)=-0.6623-0.5623i\\
&x_5=-\left(\frac{h}{12}+\frac{1}{h}\right)+\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{h}\right)=-0.6623+0.5623i,
\end{aligned} \tag{4.150}$$

where $h=\sqrt[3]{108+12\sqrt{69}}$.

For this case, the formula (4.121) for the Fibonacci 4-numbers and the system of the algebraic equations (4.123) take the following forms:

$$F_4(n)=k_1(x_1)^n+k_2(x_2)^n+k_3(x_3)^n+k_4(x_4)^n+k_5(x_5)^n \tag{4.151}$$

$$\begin{aligned}
&F_4(0)=k_1+k_2+k_3+k_4+k_5=0\\
&F_4(1)=k_1x_1+k_2x_2+k_3x_3+k_4x_4+k_5x_5=1\\
&F_4(2)=k_1(x_1)^2+k_2(x_2)^2+k_3(x_3)^2+k_4(x_4)^2+k_5(x_5)^2=1\\
&F_4(3)=k_1(x_1)^3+k_2(x_2)^3+k_3(x_3)^3+k_4(x_4)^3+k_5(x_5)^3=1\\
&F_4(4)=k_1(x_1)^4+k_2(x_2)^4+k_3(x_3)^4+k_4(x_4)^4+k_5(x_5)^4=1.
\end{aligned} \tag{4.152}$$

Solving the system (4.152), we obtain the numerical values of the coefficients:

$$\begin{aligned}
&k_1=0.380;\ k_2=-0.171+0.206i;\ k_3=-0.071-0.206i;\\
&k_4=-0.119+0.046i;\ k_5=-0.119-0.046i.
\end{aligned} \tag{4.153}$$

Using (4.150) and (4.153), we can represent the Binet formula for the Fibonacci 4-numbers (4.151) in numerical form.

4.9.6. *A General Case of p*

In the general case, Binet formula for the Fibonacci p-numbers has the form (4.121). The coefficients $k_1, k_2, ..., k_{p+1}$ in the formula (4.121) are the solutions of the system (4.123). This outcome can be formulated as the following theorem.

Theorem 4.3 (Generalized Binet Formula for the Fibonacci p-Numbers). For a given integer $p>0$, any Fibonacci p-number $F_p(n)$ ($n=0, \pm1, \pm2, \pm3, ...$) given by the recursive relation $F_p(n)=F_p(n-1)+F_p(n-p-1)$ at the seeds $F_p(0)=0$, $F_p(1)=F_p(2)= ... =F_p(p)=1$ can be represented in the following analytical form:

$$F_p(n)=k_1(x_1)^n+k_2(x_2)^n+...+k_{p+1}(x_{p+1})^n,$$

where $x_1, x_2, ..., x_{p+1}$ are the roots of the characteristic equation $x^{p+1}=x^p+1$ and $k_1, k_2, ..., k_{p+1}$ are constant coefficients that are the solutions of the system of the algebraic equations:

$$\begin{aligned}
&F_p(0)=k_1+k_2+...+k_{p+1}=0\\
&F_p(1)=k_1x_1+k_2x_2+...+k_{p+1}x_{p+1}=1\\
&F_p(2)=k_1(x_1)^2+k_2(x_2)^2+...+k_{p+1}(x_{p+1})^2=1\\
&...\\
&F_p(p)=k_1(x_1)^p+k_2(x_2)^p+...+k_{p+1}(x_{p+1})^p=1.
\end{aligned}$$

Proof. Represent the Fibonacci p-number $F_p(p+1)$ in the analytical form according to (4.121):

$$F_p(p+1)=k_1(x_1)^{p+1}+k_2(x_2)^{p+1}+\ldots+k_{p+1}(x_{p+1})^{p+1}. \tag{4.154}$$

Using the identity (4.76), we can represent the expression (4.154) in the following form:

$$F_p(p+1)=\left[k_1(x_1)^{p}+k_2(x_2)^{p}+\ldots+k_{p+1}(x_{p+1})^{p}\right] \\ +\left[k_1(x_1)^{0}+k_2(x_2)^{0}+\ldots+k_{p+1}(x_{p+1})^{0}\right]. \tag{4.155}$$

Then, using the definition (4.121), we can write:

$$F_p(p+1)=F_p(p)+F_p(0). \tag{4.156}$$

This means that the recursive relation (4.18) is valid for the Fibonacci p-number $F_p(p+1)$, that is, the analytical formula (4.154) represents the Fibonacci p-number $F_p(p+1)$.

Furthermore, applying the formula (4.121) for the Fibonacci p-numbers $F_p(p+2)$, $F_p(p+3)$, ..., $F_p(n)$, ... and using the identity (4.76), it is easy to prove that the analytical formula (4.121) represents all Fibonacci p-numbers for the positive values of n.

Let us prove that Eq. (4.121) is valid for the negative values of n=-1, -2, -3, ... In order to do this, we consider the formula (4.121) for the case n=-1:

$$F_p(-1)=k_1(x_1)^{-1}+k_2(x_2)^{-1}+\ldots+k_{p+1}(x_{p+1})^{-1}. \tag{4.157}$$

We can rewrite the identity (4.76) in the following form:

$$x_k^{n-p-1}=x_k^p-x_k^{n-1}. \tag{4.158}$$

For the case n=p, the identity (4.158) takes the following form:

$$x_k^{-1}=x_k^p-x_k^{p-1}. \tag{4.159}$$

Using the identity (4.159), we can rewrite (4.157) as follows:

$$F_p(-1)=\left[k_1(x_1)^{p}+k_2(x_2)^{p}+\ldots+k_{p+1}(x_{p+1})^{p}\right] \\ -\left[k_1(x_1)^{p-1}+k_2(x_2)^{p-1}+\ldots+k_{p+1}(x_{p+1})^{p-1}\right]. \tag{4.160}$$

Hence, using the general formula (4.121), we can see that the formula (4.160) is equivalent to

$F_p(-1)=F_p(p)+F_p(p-1)=1-1=0$,

that is, the formula (4.160) represents the Fibonacci p-number $F_p(-1)=0$.

Furthermore, considering the formula (4.121) for the negative values of n=-2, -3, -4, ... and using (4.158), it is easy to prove that the formula (4.121) is valid for all negative values of n.

4.10. The Generalized Lucas *p*-Numbers

4.10.1. *Binet Formula for the Lucas p-Numbers*

Next let us generalize the Binet formula for the classical Lucas numbers that are given by (4.127). We recall that the formula (4.127) can be obtained from the Binet formula for Fibonacci numbers that are given by (4.126), provided we assume in it that $k_1=k_2$. We can use this approach for the introduction of a new class of the recursive sequences. If we assume that $k_1=k_2=...=k_{p+1}=1$ in the formula (4.121), we can obtain the following formula:

$$L_p(n)=(x_1)^n+(x_2)^n+...+(x_{p+1})^n. \qquad (4.161)$$

Note that for the case $p=1$ this formula is reduced to the Binet formula for the classical Lucas numbers that are given by (4.127). Let us prove that this formula represents a new class of recursive numerical sequences called *Lucas p-numbers*. They are given by the following recursive relation:

$$L_p(n)= L_p(n-1)+L_p(n-p-1) \qquad (4.162)$$

at the seeds:

$$L_p(0)=p+1 \qquad (4.163)$$

$$L_p(1)=L_p(2)=...=L_p(p)=1. \qquad (4.164)$$

In fact, for the case $n=0$, we can write the formula (4.161) as follows:

$$L_p(0)=(x_1)^0+(x_2)^0+...+(x_{p+1})^0=p+1.$$

This proves that for the case $n=0$ the formula (4.161) gives the seed (4.163).

Now, consider the formula (4.161) for the cases of $n=1, 2, 3, ..., p$:

$$\begin{aligned}
L_p(1)&=x_1+x_2+...+x_{p+1}\\
L_p(2)&=(x_1)^2+(x_2)^2+...+(x_{p+1})^2\\
L_p(3)&=(x_1)^3+(x_2)^3+...+(x_{p+1})^3\\
&...\\
L_p(p)&=(x_1)^p+(x_2)^p+...+(x_{p+1})^p.
\end{aligned} \qquad (4.165)$$

Then, according to Theorem 4.2, all expressions (4.165) are equal to 1. This proves that the expression (4.161) is valid for the cases $n=1, 2, 3, ..., p$.

To prove the validity of the recursive relation (4.162) for the general case of n, we can use the identity (4.76) and represent the formula (4.161) as follows:

$$L_p(n)=\left[(x_1)^{n-1}+(x_2)^{n-1}+\ldots+(x_{p+1})^{n-1}\right]$$
$$+\left[(x_1)^{n-p-1}+(x_2)^{n-p-1}+\ldots+(x_{p+1})^{n-p-1}\right]. \quad (4.166)$$

Using the definition (4.161), we can rewrite the expression (4.166) in the form of the recursive relation (4.162).

Our reasoning resulted in the discovery of new recursive numerical sequences - *Lucas p-numbers* given by the recursive relation (4.162) at the seeds (4.163) and (4.164). It is clear that for the case p=1 this recursive relation is reduced to the recursive relation for the classical Lucas numbers.

Let us study the partial cases of the Lucas p-numbers for the cases p=2, 3, 4.

4.10.2. *Binet Formula for the Lucas 2-Numbers*

For the case p=2, the formula (4.161) can be presented in the form below:

$$L_2(n)=(x_1)^n+(x_2)^n+(x_3)^n. \quad (4.167)$$

This formula defines the *Lucas 2-numbers* $L_2(n)$.

If we substitute the expressions for the roots x_1, x_2, x_3, given by (4.107)-(4.110) into (4.167) we can rewrite the formula (4.167) as follows:

$$L_2(n)=\left(\frac{h}{6}+\frac{2}{3h}+\frac{1}{3}\right)^n+\left[-\frac{h}{12}-\frac{1}{3h}+\frac{1}{3}+\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{3h}\right)\right]^n$$
$$+\left[-\frac{h}{12}-\frac{1}{3h}+\frac{1}{3}-\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{3h}\right)\right]^n. \quad (4.168)$$

For the case p=2, the recursive relation (4.162) and the seeds (4.163) and (4.164) are reduced to the following:

$L_2(n)=L_2(n\text{-}1)+L_2(n\text{-}3)$ (4.169)

$L_2(0)=3$ (4.170)

$L_2(1)=L_2(2)=1.$ (4.171)

Then, using the recursive relation (4.169) at the seeds (4.163) and (4.164), we can calculate all elements of the Lucas 2-series:

3, 1, 1, 4, 5, 6, 10, 15, 21, 31, 46, 67, 98, 144, ... (4.172)

4.10.3. *Binet Formula for the Lucas 3-Numbers*

For the case p=3, the formula (4.161) takes the following form:

$$L_3(n)=(x_1)^n+(x_2)^n+(x_3)^n+(x_4)^n. \tag{4.173}$$

This formula is an analytical expression for the *Lucas 3-numbers* $L_3(n)$.

Using the numerical values for the roots x_1, x_2, x_3, x_4 given by (4.142), we can represent the formula (4.173) in the following numerical form:

$$L_3(n)=(1.380)^n+(-0.819)^n+(0.219+0.914i)^n+(0.219-0.914i)^n. \tag{4.174}$$

For the case p=3 the recursive relation (4.162) and the seeds (4.163) and (4.164) are reduced to the following:

$$L_3(n)= L_3(n-1)+ L_3(n-4) \tag{4.175}$$

$$L_3(0)=4 \tag{4.176}$$

$$L_3(1)= L_3(2)=L_3(3)=1. \tag{4.177}$$

The recursive relation (4.175) at the seeds (4.176) and (4.177) generates the following Lucas 3-series:

4,1,1,1,5,6,7,8,13,19,26,34,47,66,... . (4.178)

4.10.4. *Binet Formula for the Lucas 4-Numbers*

For the case p=4, the formula (4.161) takes the following form:

$$L_4(n)=(x_1)^n+(x_2)^n+(x_3)^n+(x_4)^n+(x_5)^n. \tag{4.179}$$

This formula sets the *Lucas 4-numbers* $L_4(n)$ in analytical form.

Substituting the analytical representations of the roots x_1, x_2, x_3, x_4, x_5 given by (4.150) into (4.179), we obtain the following formula:

$$\begin{aligned}L_4(n)=&\left(\frac{h}{6}+\frac{2}{h}\right)^n+\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^n+\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^n\\&+\left[-\frac{h}{12}-\frac{1}{6}-\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{6}\right)\right]^n+\left[-\frac{h}{12}-\frac{1}{6}+\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{6}\right)\right]^n.\end{aligned} \tag{4.180}$$

For the case p=4, the recursive relation (4.162) and the seeds (4.163) and (4.164) are reduced to the following:

$$L_4(n)= L_4(n-1)+ L_4(n-5) \tag{4.181}$$

$$L_4(0)=5 \tag{4.182}$$

$$L_4(1)= L_4(2)=L_4(3)=L_4(4)=1. \tag{4.183}$$

The recursive relation (4.181) at the seeds (4.182) and (4.183) generates the following Lucas 4-series:

5, 1, 1, 1, 1, 6, 7, 8, 9, 10, 16, 23, 31, 40, ... (4.184)

4.10.5. *The "Extended" Lucas p-Numbers*

Above we introduced the so-called "extended" Fibonacci p-numbers that are given for the negative values of n (see Table 4.2). By analogy, we can introduce the "extended" Lucas p-numbers, if we extend the Lucas p-numbers to the side of the negative values of n. With this purpose in mind, we shall find some general properties of such "extended" sequences. For the calculation of the Lucas p-numbers $L_p(0)$, $L_p(-1)$, $L_p(-2)$,..., $L_p(-p)$,..., $L_p(-2p+1)$, ..., which correspond to the non-negative values of n=-1,-2,-3,..., we represent the recursive relation (4.162) as follows:

$$L_p(n-p-1)= L_p(n)-L_p(n-1). \tag{4.185}$$

In particular, for the case $n=p$ the formula (4.185) is

$$L_p(-1)= L_p(p)-L_p(p-1). \tag{4.186}$$

Now, consider the formula (4.185) for the different values of p. For p=1, the formulas (4.185) and (4.186) take the following forms, respectively:

$$L_1(n-2)=L_1(n)-L_1(n-1). \tag{4.187}$$

$$L_1(-1)=L_1(1)-L_1(0). \tag{4.188}$$

As $L_1(1)=1$ and $L_1(0)=2$, it comes from (4.188) that

$$L_1(-1)=-1. \tag{4.189}$$

Using the recursive relation (4.186), we can calculate all values of the Lucas numbers $L_1(n)$ for the negative values of n=-1, -2, -3, ... and then represent the classical Lucas numbers $L_1(n)$ as is shown in Table 4.3.

Table 4.3. The "extended" classical Lucas numbers

n	5	4	3	2	1	0	-1	-2	-3	-4	-5
$L_1(n)$	11	7	4	3	1	2	-1	3	-4	7	-11

For the case p=2, the formulas (4.185) and (4.186) take the following form:

$$L_2(n-3)=L_2(n)-L_2(n-1) \tag{4.190}$$

$$L_2(-1)=L_2(2)-L_2(1). \tag{4.191}$$

Taking into consideration (4.171), we can write:

$$L_2(-1)=0. \tag{4.192}$$

Using (4.170) and calculating numerical values of the Lucas 2-numbers $L_2(n)$ for the non-negative n=0, -1, -2, -3, -4, ... , we can get the following numerical sequence:

$$L_2(n)\ (n\le 0)\text{: } 3, 0, -2, 3, 2, -5, 1, 7, -6, 6, \ldots . \tag{4.193}$$

For the case p=3, the formulas (4.185) and (4.186) take the following form:

$$L_3(n\text{-}4)=L_3(n)\text{-}L_3(n\text{-}1) \tag{4.194}$$

$$L_3(\text{-}1)=L_3(3)\text{-}L_3(2). \tag{4.195}$$

Using (4.170) and (4.194) and (4.195), we can calculate numerical values of the Lucas 3-numbers $L_3(n)$ for the non-negative n=0, -1, -2, -3, -4, ..., we can get the following numerical sequence $L_3(n)$ ($n\leq 0$): 4, 0, 0, -3, 4, 0, 3, -7, 4, -3, ...

Of course, by using the general recursive relation (4.185), we can calculate the rest of the values of $L_p(n)$ for the non-negative values of n.

In Table 4.4 we can see the "extended" Lucas numbers $L_p(n)$ for the cases p=1, 2, 3, 4.

Table 4.4. The "extended" Lucas p-numbers

n	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6
$L_1(n)$	18	11	7	4	3	1	2	-1	3	-4	7	-11	18
$L_2(n)$	10	6	5	4	1	1	3	0	-2	3	2	-5	1
$L_3(n)$	7	6	5	1	1	1	4	0	0	-3	4	0	3
$L_4(n)$	7	6	1	1	1	1	5	0	0	0	-4	5	0

4.10.6. *Identities for the Sums of the Lucas p-Numbers*

Once again, consider the recursive relation (4.169). Decomposing the Lucas 2-number $L_2(n\text{-}1)$ in (4.169) according to the same recursive relation (4.169), that is, representing $L_2(n\text{-}1)$ in the form

$$L_2(n\text{-}1)=L_2(n\text{-}2)+L_2(n\text{-}4),$$

we can represent the formula (4.169) in the following form:

$$L_2(n)=L_2(n\text{-}2)+ L_2(n\text{-}3)+L_2(n\text{-}4). \tag{4.196}$$

This means that the sum of the three sequential Lucas 2-numbers is always equal to the Lucas 2-number that is two positions from the senior Lucas 2-number of the sum.

If we use a similar approach for the Lucas p-numbers in the general case, we obtain the following general identity:

$$L_p(n)=L_p(n\text{-}p)+L_p(n\text{-}p\text{-}1)+L_p(n\text{-}p\text{-}2)+...+L_p(n\text{-}2p). \tag{4.197}$$

Note that the identity (4.197) is valid for all "extended" Lucas p-numbers.

Now, let us examine the sum of the first n Lucas p-numbers:

$L_p(1)+L_p(2)+L_p(3)+...+L_p(n).$ (4.198)

In order to get the required result, we write down the basic recursive relation (4.162) for the Lucas p-numbers in the following form:

$L_p(n)=L_p(n+p+1)-L_p(n+p).$ (4.199)

Using (4.199), we can write the following equalities:

$$\begin{aligned}
&L_p(1)=L_p(2+p)-L_p(1+p)\\
&L_p(2)=L_p(3+p)-L_p(2+p)\\
&L_p(3)=L_p(4+p)-L_p(3+p)\\
&\ldots\\
&L_p(n-1)=L_p(n+p)-L_p(n+p-1)\\
&L_p(n)=L_p(n+p+1)-L_p(n+p).
\end{aligned}$$

Summing term by term the left-hand and right-hand parts of these equalities, we obtain the following formula:

$L_p(1)+L_p(2)+L_p(3)+...+L_p(n)= L_p(n+p+1)-L_p(1+p).$ (4.200)

It follows from the recursive relation (4.162) and the seeds (4.163) and (4.164) that

$L_p(1+p)=L_p(p)+L_p(0)=1+p+1=p+2.$ (4.201)

The following formula for the sum (4.200) follows from (4.201):

$L_p(1)+L_p(2)+L_p(3)+...+L_p(n)= L_p(n+p+1)- p-2.$ (4.202)

4.10.7. *The Ratio of Adjacent Lucas p-Numbers*

Above we found that the Fibonacci p-numbers are closely connected with the golden p-proportion. In particular, the limit of the ratio $F_p(n)/F_p(n-1)$ aims for the golden p-proportion. There is a question: what is the limit of the ratios of adjacent Lucas p-numbers? Introduce the following definition:

$$\lim_{n\to\infty}\frac{L_p(n)}{L_p(n-1)}=x. \tag{4.203}$$

Using the recursive relation (4.162), we can represent the ratio of the adjacent Lucas p-numbers as follows:

$$\frac{L_p(n)}{L_p(n-1)}=\frac{L_p(n-1)+L_p(n-p-1)}{L_p(n-p-1)}$$
$$=1+\frac{1}{\frac{L_p(n-1)}{L_p(n-p-1)}}=1+\frac{1}{\frac{L_p(n-1)\cdot L_p(n-2)\cdots L_p(n-p)}{L_p(n-2)\cdot L_p(n-3)\ \cdots L_p(n-p-1)}}. \tag{4.204}$$

Taking into consideration the definition (4.203), for the case $n\to\infty$ we can exchange the formula (4.204) by the algebraic equation (4.42) with the positive root τ_p - the golden p-proportion. It follows from this reasoning that

$$\lim_{n\to\infty}\frac{L_p(n)}{L_p(n-1)}=\tau_p. \tag{4.205}$$

4.11. The "Metallic Means Family" by Vera W. de Spinadel

4.11.1. *The Generalized Fibonacci Sequences*

Recently the Fibonacci numbers and the golden mean were generalized by the Argentinean mathematician Vera W. de Spinadel who is the author of the original book in this regard [42]. Spinadel generalized the Fibonacci recursive relation $F(n+1)=F(n)+F(n-1)$ as follows:

$$G(n+1)=pG(n)+qG(n-1), \tag{4.206}$$

where p and q are natural numbers.

Consider the examples of the generalized Fibonacci sequences of the kind (4.206). If we assume that $p=2$ and $q=1$ in (4.206) and begin from the seeds $G(1)=G(2)=1$, then the recursive relation (4.206) generates the following generalized Fibonacci numbers:

$$1, 1, 3, 7, 17, 41, 99, 140, \ldots . \tag{4.207}$$

For the case $p=3$ and $q=1$, and $G(1)=G(2)=1$ the generalized Fibonacci numbers are:

$$1, 1, 1 , 4, 13, 43, 142, 469, \ldots . \tag{4.208}$$

We can represent the recursive relation (4.206) in the following form:

$$\frac{G(n+1)}{G(n)}=p+q\frac{G(n-1)}{G(n)}=p+\frac{q}{\frac{G(n)}{G(n-1)}}. \tag{4.209}$$

If we denote by x the limit of the ratio of two adjacent generalized Fibonacci numbers, that is,

$$x=\lim_{n\to\infty}\frac{G(n+1)}{G(n)},$$

then we can represent the expression (4.209) as follows:

$$x = p + \frac{q}{x} \tag{4.210}$$

or

$$x^2\text{-}px\text{-}q=0. \tag{4.211}$$

This algebraic equation has the following positive solution:

$$x = \frac{p+\sqrt{p^2+4q}}{2}. \tag{4.212}$$

This means that

$$\lim_{n\to\infty}\frac{G(n+1)}{G(n)} = \frac{p+\sqrt{p^2+4q}}{2}. \tag{4.213}$$

4.11.2. *The Metallic Means Family*

Above we have introduced the quadratic algebraic equation (4.211). Vera W. de Spinadel proved that Eq. (4.211) gives an infinite set of positive quadratic irrationals for the different values of p and q. They are all given by the formula (4.212) and together make the *Metallic Means Family* (MMF).

Let us denote any member of the MMF by σ_p^q, where p and q take their values from the set of natural numbers, that is, p=1, 2, 3, ... ; q=1, 2, 3, ...

Consider special cases of Eq. (4.211). We start from the case q=1, that is, from the following algebraic equation:

$$x^2\text{-}px\text{-}1=0. \tag{4.214}$$

It is convenient to represent the members of the MMF in the form of a continued fraction. It is clear that for the case p=1 Eq. (4.214) is reduced to the simplest algebraic equation

$$x^2\text{-}x\text{-}1=0 \tag{4.215}$$

with the positive root equal to the golden mean.

Equation (4.215) can be written in the form:

$$x = 1+\frac{1}{x}. \tag{4.216}$$

By substituting the golden mean τ for x in Eq. (4.216), we obtain the following representation of τ:

$$\tau = 1+\frac{1}{\tau}. \tag{4.217}$$

It comes from (4.217) that the following representation of the golden mean in the form of a continued fraction:

$$\tau = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}. \tag{4.218}$$

Note that the representation (4.218) was considered in Chapter 1.

In mathematics, for the compact representation of the continued fractions the following representation is used:

$$\varphi = [1,1,1,\ldots] = [\overline{1}]. \tag{4.219}$$

Now, assume that p=2 in Eq. (4.214), that is, consider the following equation:

$$x^2-2x-1=0. \tag{4.220}$$

Then, according to the above definition we can denote the positive root of Eq. (4.220) by σ_2^1. We can represent Eq. (4.220) in the form:

$$x = 2 + \frac{1}{x}. \tag{4.221}$$

If we substitute 2+(1/x) for x on the right-hand part of (4.221), we obtain the following representation of x:

$$x = 2 + \cfrac{1}{2 + \cfrac{1}{x}}. \tag{4.222}$$

Continuing this process ad infinitum, we obtain the representation of σ_2^1 in the form of the following continued fraction:

$$\sigma_2^1 = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}. \tag{4.223}$$

The positive quadratic irrational number σ_2^1 given by (4.223) was named by Spinadel the *Silver Mean*. By analogy with the golden mean (4.219), the *Silver Mean* (4.223) can be represented in the following compact form:

$$\sigma_2^1 = [2,2,2,\ldots] = [\overline{2}]. \tag{4.224}$$

Using the formula (4.212), we can write the analytical representation of the silver mean σ_2^1 as follows:

$$\sigma_2^1 = 1 + \sqrt{2} = [\overline{2}]. \tag{4.225}$$

If we assume p=3, then Eq. (4.214) takes the following form:

$$x^2-3x-1=0. \tag{4.226}$$

The positive root of this equation is called the *Bronze Mean* σ_3^1; it can be represented in the form of the following continued fraction:

$$\sigma_3^1 = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \ldots}}} \tag{4.227}$$

or

$$\sigma_2^1 = [3, 3, 3, \ldots] = [\overline{3}]. \tag{4.228}$$

Using the formula (4.212), we can write the analytical representation of the bronze mean σ_3^1 as follows:

$$\sigma_3^1 = \frac{3 + \sqrt{13}}{2} = [\overline{3}]. \tag{4.229}$$

For the cases where p=4, 5, 6, 7, 8, 9, 10, respectively, we can find the following analytical representations of the corresponding *Metallic Means*:

$$\sigma_4^1 = 2 + \sqrt{5}; \quad \sigma_5^1 = \frac{5 + \sqrt{29}}{2}; \quad \sigma_6^1 = 3 + \sqrt{10}; \quad \sigma_7^1 = \frac{7 + \sqrt{53}}{2};$$

$$\sigma_8^1 = 4 + \sqrt{17}; \quad \sigma_9^1 = \frac{9 + \sqrt{85}}{2}; \quad \sigma_{10}^1 = 5 + \sqrt{26}.$$

It is clear that all *Metallic Means* of the kind σ_p^1 have the following general representation in the form of the periodic continued fraction:

$$\sigma_p^1 = p + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \ldots}}} = [\overline{p}]. \tag{4.230}$$

4.11.3. *Other Types of the Metallic Means*

If we assume that p=1 in Eq. (4.211), then we obtain the following equation:

$$x^2 - x - q = 0, \tag{4.231}$$

where q is a natural number. The positive roots of this equation generate a new class of the *Metallic Means* denoted by σ_1^q.

Note that for the case q=1, Eq. (4.231) is reduced to the golden equation (4.215). For the case q=2, Eq. (4.231) takes the following form:

$$x^2 - x - 2 = 0. \tag{4.232}$$

The number $\sigma_1^q = 2$ is a positive root of this equation. Spinadel calls this number the *Copper Mean*. Using a traditional representation of the continued fraction, we can represent the *Copper Mean* as follows:

$\sigma_1^q = 2 = \left[2, \bar{0}\right]$.

If we assume that q=3 in Eq. (4.231), then we obtain the following equation:

$$x^2 - x - 3 = 0. \tag{4.233}$$

This equation results in the *Nickel Mean*

$$\sigma_1^3 = \frac{1+\sqrt{13}}{2} = \left[2, \bar{3}\right]. \tag{4.234}$$

By analogy, we can find the next representations of the other *Metallic Means* of the type σ_1^q.

4.11.4. *Pisot-Vijayaraghavan Numbers and Metallic Means*

Vera W. de Spinadel paid attention to the fact that her *Metallic Means* have a direct relation to *Pisot-Vijayaraghavan numbers* or PV-numbers [152]. It is well known that the PV-numbers are positive roots of the following algebraic equation:

$$x^m = a_{m-1}x^{m-1} + \ldots + a_i x^i + \ldots + a_1 x + a_0, \tag{4.235}$$

where a_i are integers.

The golden mean $\tau = \left(1+\sqrt{5}\right)/2$ is the example of the PV-numbers because the golden mean is the positive root of Eq. (4.215), which is a partial case of (4.235). Also, all the golden p-proportions τ_p are PV-numbers because Eq. (4.42) is a partial case of (4.235).

The number Q_1=1.324...- a positive root of the equation x^3-x-1=0 - is also a PV-number. This number is called a *Plastic Constant*.

It is proved that Eq. (4.215) with the root $\tau = \left(1+\sqrt{5}\right)/2$ appears in quasi-crystal structures. In addition, the following algebraic equations - which are partial cases of (4.235) - appear in quasi-crystal structures:

$$x^2 - 2x - 1 = 0 \quad \rightarrow \quad \gamma = 1+\sqrt{2} \tag{4.236}$$

$$x^2 - 4x + 1 = 0 \quad \rightarrow \quad \delta = 1+\sqrt{3}. \tag{4.237}$$

Note that the particular PV-number $\tau = \left(1+\sqrt{5}\right)/2$ (the golden mean) corresponds to *pentagonal* and *decagonal* quasi-lattices, while another PV-number $\gamma = 1+\sqrt{2}$ (the *Silver Mean*) corresponds to the *octagonal* quasi-lattice, and the PV-number $\delta = 1+\sqrt{3}$ corresponds to the case of the *dodecagonal* quasi-lattice.

The above examples demonstrate that Spinadel's *Metallic Means* are of great theoretical importance for algebra and crystallography.

4.11.5. *"Silver Means" by Jay Kappraff*

The American researcher Jay Kappraff – the author of the interesting books [47, 50] – developed an approach to the generalization of Fibonacci numbers and the golden mean similar to Spinadel's *Metallic Means*. He considered the recursive relation

$$G(n+1)=N\,G(n)+G(n-1) \tag{4.238}$$

that is a partial case of the recursive relation (4.206) for the case q=1 and p=N. This recursive relation gives different generalized Fibonacci numbers at different seeds. From the recursive relation (4.238) Kappraff derived the following algebraic equation

$$x-\frac{1}{x}=N, \tag{4.239}$$

which is the other form of the equation (4.214).

It is clear that for the case N=1 and the seeds $G(0)=0$ and $G(1)=1$ the recursive relation (4.238) generates the classical Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, … . The ratios of the adjacent numbers in this series converge to the golden mean τ - a positive root of the golden equation x^2-x-1=0. Kappraff names the classical golden mean τ the *1st Silver Mean of Type 1* and denotes it by $SM_1(1)$.

For the case N=2, the recursive relation (4.238) at the seeds $G(1)=1$ and $G(2)=2$ generates *Pell's Sequence* 1,2,5,12,29,… . The ratios of the adjacent numbers in Pell's sequence converge to the positive root of the equation x^2-2x-1=0 Kappraff names the positive root of this equation $\theta=1+\sqrt{2}=2.414...$ the *2nd Silver Mean of Type 1* and denotes it by $SM_1(2)$.

4.12. Gazale Formulas

4.12.1. *The Generalized Fibonacci m-Numbers*

Independently of Spinadel and Kappraff, the idea of generalized Fibonacci numbers was developed in the book [45] written by Egyptian mathematician and engineer Midchat Gazale. Gazale considers the recursive relation

$F_m(n+2)=m\ F_m(n+1)+F_m(n),$ (4.240)

where m is a positive real number and $n=0,\pm1,\pm2,\pm3,\ldots$. Note that the recursive formula (4.240) is similar to the recursive formula (4.206) used by Spinadel, when we take $p=m$ and $q=1$ in (4.206). However, in contrast to (4.206), where the coefficients p and q are integers, the coefficient m in the recursive formula (4.240) used by Gazale is a positive real number $m>0$.

We will name a positive real number m, used in the recursive relation (4.240), an *order* of the recursive relation (4.240). If we take the seeds

$F_m(0)=0, F_m(1)=1$ (4.241)

and then use the recursive relation (4.240) for a given $m>0$, we obtain an infinite number of the recursive numerical sequences called *Generalized Fibonacci Numbers of Order m* or simply *Fibonacci m-numbers*.

Note that for the case $m=1$ the recursive relation (4.240) and the seeds (2.41) can be represented, respectively, as follows:

$F_1(n+2)=F_1(n+1)+F_1(n)$ (4.242)

$F_1(0)=0, F_1(1)=1.$ (4.243)

It is clear that the recursive relation (4.242) with the seeds (4.243) generates the classical Fibonacci numbers.

For the case of $m=2$ the recursive formula (4.240) and the seeds (4.241) are reduced to the following:

$F_2(n+2)=2F_2(n+1)+F_2(n)$ (4.244)

$F_2(0)=0, F_2(1)=1.$ (4.245)

This case generates the so-called *Pell numbers*: 0, 1, 2, 5, 12, 29, ...

If we take $m=\sqrt{2}$, then the recursive relation (4.240) at the seeds (4.241) generates the following recursive numerical sequence:

$$0,1,\sqrt{2},3,4\sqrt{2},11,15\sqrt{2},41,56\sqrt{2},\ldots.$$

4.12.2. *The Generalized Golden Mean of Order m*

Let us represent the recursive relation (4.240) as follows:

$$\frac{F_m(n+2)}{F_m(n+1)}=m+\frac{1}{\frac{F_m(n+1)}{F_m(n)}}. \tag{4.246}$$

For the case $n\to\infty$, the expression (4.246) is reduced to the following quadratic equation:

$x^2-mx-1=0.$ (4.247)

Equation (4.247) has two roots - a positive root

$$x_1 = \frac{\sqrt{4+m^2}+m}{2} \tag{4.248}$$

and a negative root

$$x_2 = \frac{-\sqrt{4+m^2}+m}{2}. \tag{4.249}$$

If we sum up (4.248) and (4.249), we obtain:

$$x_1 + x_2 = m. \tag{4.250}$$

If we substitute the root (4.248) for x in Eq. (4.247), we obtain the following identity:

$$x_1^2 = mx_1 + 1. \tag{4.251}$$

If we multiply or divide repeatedly all terms of the identity (4.251) by x_1, we obtain the following general identity:

$$x_1^n = mx_1^{n-1} + x_1^{n-2}, \tag{4.252}$$

where $n=0,\pm1,\pm2,\pm3,\ldots$.

Using similar reasoning for the root x_2, we obtain the following identity for the root x_2:

$$x_2^n = mx_2^{n-1} + x_2^{n-2}. \tag{4.253}$$

Denote a positive root x_1 by Φ_m and name it the *Golden Mean of Order m* or the *Golden m-Proportion*. The golden m-proportion Φ_m has the following analytical expression:

$$\Phi_m = \frac{\sqrt{4+m^2}+m}{2}. \tag{4.254}$$

Note that for the case $m=1$, the golden m-proportion coincides with the classical golden mean $\Phi_1 = \left(1+\sqrt{5}\right)/2$.

Let us express the root x_2 by the golden m-proportion Φ_m. After simple transformation of (4.249) we can write the root x_2 as follows:

$$x_2 = \frac{-\sqrt{4+m^2}+m}{2} = \frac{-4}{2(\sqrt{4+m^2}+m)} = -\frac{1}{\Phi_m}. \tag{4.255}$$

Substituting Φ_m for x_1 and $(-1/\Phi_m)$ for x_2 in (4.250), we obtain:

$$m = \Phi_m - 1/\Phi_m, \tag{4.256}$$

where Φ_m is given by (4.254) and $1/\Phi_m$ is given by the formula:

$$\frac{1}{\Phi_m} = \frac{\sqrt{4+m^2}-m}{2}. \tag{4.257}$$

Using the formulas (4.254) and (4.257), we can write the following identity:

$$\Phi_m + \frac{1}{\Phi_m} = \sqrt{4+m^2}. \tag{4.258}$$

It is also easy to prove the following identity:

$$\Phi_m^n = m\Phi_m^{n-1} + \Phi_m^{n-2}, \tag{4.259}$$

where n=0, ±1, ±2, ±3,... .

4.12.3. *Two Surprising Representations of the Golden m-Proportion*

For the case n=2, the identity (4.259) can be represented in the form:

$$\Phi_m^2 = 1 + m\Phi_m. \tag{4.260}$$

The following representation of the golden m-proportion Φ_m comes from (4.260):

$$\Phi_m = \sqrt{1+m\Phi_m}. \tag{4.261}$$

Substituting $\sqrt{1+m\Phi_m}$ for Φ_m on the right-hand part of (4.261), we obtain:

$$\Phi_m = \sqrt{1+m\sqrt{1+\Phi_m}}. \tag{4.262}$$

Continuing this process ad infinitum, that is, substituting repeatedly $\sqrt{1+m\Phi_m}$ for Φ_m on the right-hand part of (4.262), we obtain the following surprising representation of Φ_m:

$$\Phi_m = \sqrt{1+m\sqrt{1+m\sqrt{1+m\sqrt{\ldots}}}}. \tag{4.263}$$

Now, represent the identity (4.260) in the form:

$$\Phi_m = m + \frac{1}{\Phi_m}. \tag{4.264}$$

Substituting m+(1/Φ_m) for Φ_m on the right-hand part of (4.264), we obtain:

$$\Phi_m = m + \frac{1}{m+\frac{1}{\Phi_m}}. \tag{4.265}$$

Continuing this process ad infinitum we obtain the following surprising representation of the golden m-proportion:

$$\Phi_m = m + \frac{1}{m+\frac{1}{m+\frac{1}{m+\ldots}}}. \tag{4.266}$$

Note that for the case of $m=1$, the representations (4.263) and (4.28) coincide with the well-known representations of the classical golden mean in the forms (1.20) and (1.24), respectively.

4.12.4. *A Derivation of the Gazale Formula*

The formula (4.240) at the seeds (4.241) defines the Fibonacci m-numbers $F_m(n)$ by recursion. We can represent the numbers $F_m(n)$ in analytical form using the golden m-proportion Φ_m.

Let us represent the Fibonacci number $F_m(n)$ by the roots x_1 and x_2 in the form:

$$F_m(n) = k_1 x_1^n + k_2 x_2^n, \tag{4.267}$$

where k_1 and k_2 are constant coefficients that are the solutions to the following system of algebraic equations:

$$\begin{cases} F_m(0) = k_1 x_1^0 + k_2 x_2^0 = k_1 + k_2 \\ F_m(1) = k_1 x_1^1 + k_2 x_2^1 = k_1 \Phi_m - k_2 (1/\Phi_m) \end{cases}. \tag{4.268}$$

Taking into consideration that $F_m(0)=0$ and $F_m(1)=1$, we can rewrite the system (4.268) as follows:

$$k_1 = -k_2 \tag{4.269}$$

$$k_1 \Phi_m + k_1 (1/\Phi_m) = k_1 (\Phi_m + 1/\Phi_m) = 1. \tag{4.270}$$

Taking into consideration (4.269) and (4.270) and also the identity (4.258), we can find the following formulas for the coefficients k_1 and k_2:

$$k_1 = \frac{1}{\sqrt{4+m^2}}; \quad k_2 = -\frac{1}{\sqrt{4+m^2}}. \tag{4.271}$$

Taking into consideration (4.271), we can write the formula (4.267) as follows:

$$F_m(n) = \frac{1}{\sqrt{4+m^2}} x_1^n - \frac{1}{\sqrt{4+m^2}} x_2^n = \frac{1}{\sqrt{4+m^2}} \left(x_1^n - x_2^n \right). \tag{4.272}$$

Taking into consideration that $x_1=\Phi_m$ and $x_2=-1/\Phi_m$, we can rewrite the formula (4.272) as follows:

$$F_m(n) = \frac{\Phi_m^n - (-1/\Phi_m)^n}{\sqrt{4+m^2}} \tag{4.273}$$

or

$$F_m(n) = \frac{1}{\sqrt{4+m^2}} \left[\left(\frac{m+\sqrt{4+m^2}}{2} \right)^n - \left(\frac{m-\sqrt{4+m^2}}{2} \right)^n \right]. \tag{4.274}$$

For the partial case m=1, formula (4.274) is reduced to the formula:

$$F_1(n)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{4.275}$$

called the *Binet formula*. This formula was obtained by Binet in 1843, although the result was known to Euler, Daniel Bernoulli, and de Moivre more than one century earlier. In particular, de Moivre obtained this formula in 1718.

For the case m=2, the formula (4.274) takes the following form:

$$F_2(n)=\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n\right]. \tag{4.276}$$

Note that this formula was obtained for the first time by the English mathematician John Pell (1610-1685).

For the cases m=3 and $m=\sqrt{2}$, the formula (4.274) takes the following forms, respectively:

$$F_3(n)=\frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^n-\left(\frac{3-\sqrt{13}}{2}\right)^n\right] \tag{4.277}$$

$$F_{\sqrt{2}}(n)=\frac{1}{\sqrt{6}}\left[\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)^n-\left(\frac{\sqrt{2}-\sqrt{6}}{2}\right)^n\right]. \tag{4.278}$$

Thus, the Egyptian mathematician Midhat J. Gazale obtained [45] the unique mathematical formula (4.274), which includes as partial cases the Binet formula for Fibonacci numbers (4.275) for the case m=1 and Pell's formula (4.276) for the case m=2. This formula generates an infinite number of the Fibonacci m-numbers because every positive real number m generates its own *Generalized Binet Formula* (4.274). Taking into consideration the uniqueness of the formulas (4.272)-(4.274) we will name this formula the *Gazale Formula for the Fibonacci m-numbers.*

4.13. Fibonacci and Lucas *m*-Numbers

4.13.1. *Fibonacci m-Numbers*

Let us prove that Gazale formulas (4.272)-(4.274) really express all Fibonacci m-numbers given by the recursive formula (4.240) at the seeds (4.241). In fact, for the case n=0 the formula (4.274) gives the Fibonacci m-number

$F_m(0)=0$ that corresponds to the seeds (4.241). For the case $n=1$, we can rewrite the formula (2.274) as follows:

$$F_m(1)=\frac{1}{\sqrt{4+m^2}}\left(\frac{m+\sqrt{4+m^2}}{2}-\frac{m-\sqrt{4+m^2}}{2}\right)=1$$

that corresponds to the seeds (4.241).

This means that the formula (4.276) does express the seeds (4.241).

Suppose that the formula (4.274) is valid for a given n (the inductive hypothesis) and prove that this formula is valid for the case $n+1$, that is,

$$F_m(n+1)=\frac{1}{\sqrt{4+m^2}}\left(x_1^{n+1}-x_2^{n+1}\right). \tag{4.279}$$

Using the identities (4.252) and (4.253), we can represent the formula (4.279) as follows:

$$\begin{aligned}F_m(n+1)&=\frac{m}{\sqrt{4+m^2}}\left(x_1^n-x_2^n\right)+\frac{1}{\sqrt{4+m^2}}\left(x_1^{n-1}-x_2^{n-1}\right)\\&=mF_m(n)+F_m(n-1).\end{aligned} \tag{4.280}$$

Thus, the formula (4.280), in fact, defines the Fibonacci m-numbers given by the recursive relation (4.240) at the seeds (4.241).

Note that the formula (4.274) defines all Fibonacci m-numbers in the range $n=0, \pm1, \pm2, \pm3, \ldots$. Let us find some surprising properties of the Fibonacci m-numbers. First of all, compare $F_m(n)$ and $F_m(-n)$. We can write the formula (4.273) as follows:

$$F_m(n)=\frac{\Phi_m^n-(-1)^n\Phi_m^{-n}}{\sqrt{4+m^2}}. \tag{4.281}$$

Let us consider the formula (4.281) for the negative values of n, that is,

$$F_m(-n)=\frac{\Phi_m^{-n}-(-1)^{-n}\Phi_m^n}{\sqrt{4+m^2}}. \tag{4.282}$$

Comparing the expression (4.281) and (4.282) for the even ($n=2k$) and odd ($n=2k+1$) values of n, we find:

$$F_m(2k)=-F_m(-2k) \text{ and } F_m(2k+1)=F_m(-2k-1). \tag{4.283}$$

This means that the sequences of the Fibonacci m-numbers in the range $n=0, \pm1, \pm2, \pm3, \ldots$ are symmetrical sequences with respect to the Fibonacci m-number $F_m(0)=0$ except that the Fibonacci m-numbers $F_m(2k)$ and $F_m(-2k)$ are opposite in sign.

In Table 4.5 we can see the Fibonacci m-numbers for the cases $m=1, 2, 3, 4$.

Table 4.5. Fibonacci m-numbers (m=1, 2, 3, 4)

m	-4	-3	-2	-1	0	1	2	3	4
1	-3	2	-1	1	0	1	1	2	3
2	-12	5	-2	1	0	1	2	5	12
3	-33	10	-3	1	0	1	3	10	33
4	-72	17	-4	1	0	1	4	17	72

4.13.2. *The Generalized Cassini Formula*

Next let's find a fundamental formula that connects three adjacent Fibonacci m-numbers. For the case m=1, this formula is known as the *Cassini Formula*. Remember that this formula for the classical Fibonacci numbers $F_1(n)$ has the following form:

$$F_1^2(n)-F_1(n-1)\times F_1(n+1)=(-1)^{n+1}. \tag{4.284}$$

It is easy to prove the following general identity for the Fibonacci m-numbers:

$$F_m^2(n)-F_m(n-1)\times F_m(n+1)=(-1)^{n+1}. \tag{4.285}$$

For example, for the case m=2 the Fibonacci m-numbers $F_2(-5)=29$, $F_2(-4)=-12$ and $F_2(-3)=5$ are connected by the following correlation: $(-12)^2-(29\times5)=-1$ and for the case m=3 the Fibonacci m-numbers $F_3(4)=33$, $F_2(3)=10$ and $F_3(3)=3$ are connected by the following correlation: $(10)^2-(33\times3)=1$. Note that both examples correspond to the general formula (4.285).

4.13.3. *Lucas m-Numbers*

Once again, consider the formula (4.267) that defines the Fibonacci m-numbers. By analogy with the classical Lucas numbers we can consider the formula

$$L_m(n)=x_1^n+x_2^n. \tag{4.286}$$

It is clear that for the case m=1, this formula defines the classical Lucas numbers: 2, 1, 3, 4, 7, 11, 18, ... Let us assume that this formula defines *the Generalized Lucas Numbers of Order m* or simply *Lucas m-Numbers*. For a given m we can find some peculiarities of the Lucas m-numbers. First of all, calculate the seeds of the Lucas m-numbers given by (4.286). In fact, for the cases n=0 and n=1 we have from (4.286), respectively:

$$L_m(0)=x_1^0+x_2^0=1+1=2 \tag{4.287}$$

$$L_m(1)=x_1^1+x_2^1=\Phi_m+(-1/\Phi_m)=m. \tag{4.288}$$

Note that for the case m=1, the seeds (4.287) and (4.288) come for the seeds of the classical Lucas numbers: $L_1(0)=2$, $L_1(1)=1$.

Using (4.252) and (4.253), we can represent the formula (4.286) as follows:

$$L_m(n)=x_1^n+x_2^n=\left(mx_1^{n-1}+x_1^{n-1}\right)+\left(mx_2^{n-1}+x_2^{n-1}\right)$$
$$=m\left(x_1^{n-1}+x_2^{n-1}\right)+\left(x_1^{n-1}+x_2^{n-1}\right). \tag{4.289}$$

Taking into consideration the definition (4.286), we can rewrite (4.289) in the following form of a recursive relation:

$$L_m(n)=mL_m(n-1)+L_m(n-2). \tag{4.290}$$

It is clear that the recursive relation (4.290) at the seeds (4.287) and (4.288) gives the Lucas m-numbers in recursive form.

If we substitute $x_1=\Phi_m$ and $x_2=-1/\Phi_m$ in the formula (4.286), we can represent the Lucas m-numbers in the following analytical form:

$$L_m(n)=\Phi_m^n+\left(-1/\Phi_m\right)^n. \tag{4.291}$$

Although this formula is absent in Gazale's book [45], we will name this important formula the *Gazale Formula for Lucas m-Numbers.*

We can rewrite the formula (4.291) as follows:

$$L_m(n)=\Phi_m^n+(-1)^n\Phi_m^{-n}. \tag{4.292}$$

Let us write the formula (4.292) for the negative values of n, that is,

$$L_m(-n)=\Phi_m^{-n}+(-1)^n\Phi_m^n. \tag{4.293}$$

Comparing the expressions (4.292) and (4.293) for even ($n=2k$) and odd ($n=2k+1$) values of n, we obtain:

$$L_m(2k)=L_m(-2k) \text{ and } L_m(2k+1)=-L_m(-2k-1). \tag{4.294}$$

This means that the sequences of Lucas m-numbers in the range $n=0, \pm1, \pm2, \pm3,\ldots$ are symmetrical sequences with respect to the Lucas m-number $L_m(0)=2$ except that the Lucas m-numbers $L_m(2k+1)$ and $L_m(-2k-1)$ are opposite by sign.

In Table 4.6 we can see the Lucas m-numbers for the cases m=1, 2, 3, 4.

Table 4.6. The Lucas m-numbers (m=1, 2, 3, 4)

m	-4	-3	-2	-1	0	1	2	3	4
1	7	-4	3	-1	2	1	3	4	7
2	34	-14	6	-2	2	2	6	14	34
3	119	-36	11	-3	2	3	11	36	119
4	322	-76	18	-4	2	4	18	76	322

Note that for the case m=1 the Lucas m-numbers coincide with the classical Lucas numbers and for the case m=2 coincides with the Pell-Lucas numbers. Note similar results obtained by Kappraff and Adamson in [153].

4.14. On the *m*-Extension of the Fibonacci and Lucas *p*-Numbers

4.14.1. *A Recursive Relation for the Fibonacci (p,m)-Numbers*

Now, we define the recursive relation for the m-extension of the Fibonacci p-numbers [154]. For a given integer p>0 and positive real number m>0 the recursive relation is given as follows

$$F_{p,m}(n)=mF_{p,m}(n-1)+F_{p,m}(n-p-1) \tag{4.295}$$

with initial conditions

$$F_{p,m}(0)=a_0,\ F_{p,m}(1)=a_1,\ F_{p,m}(2)=a_2,\ \ldots,\ F_{p,m}(p)=a_p,$$

where $a_0, a_1, a_2, \ldots, a_p$ are integer, real or complex numbers.

In particular, we can take these initial conditions as follows

$$F_{p,m}(0)=0,\ F_{p,m}(k)=m^{k-1}, \tag{4.296}$$

where k=1,2,3,...,p.

We name a new class of recursive numerical sequences given by (4.295) at the seeds (4.296) an *m-extension of the Fibonacci p-numbers* or simply *Fibonacci (p, m)-numbers.*

It is clear that the recursive formula (4.295) at the seeds (4.296) defines a more general class of recursive numerical sequences than the Fibonacci p-numbers or the Fibonacci m-numbers. Note that for the case m=1 the Fibonacci (p,m)-numbers coincide with the Fibonacci p-numbers, that is, $F_{p,1}(n)= F_p(n)$, and for the case p=1, the Fibonacci (p,m)-numbers coincide with the Fibonacci m-numbers, that is, $F_{1,m}(n)=F_m(n)$. For the cases p=1 and m=1, the Fibonacci (p, m)-numbers coincide with the classical Fibonacci numbers.

4.14.2. *Some Properties of the Fibonacci (p,m)-Numbers*

Let us consider the m-extension of the Fibonacci p-numbers given by (4.295) with the initial conditions of (4.296). Let us calculate the set of Fibonacci (p,m)-numbers for the negative values of the argument m

$$F_{p,m}(-1),\ F_{p,m}(-2),\ \ldots,\ F_{p,m}(-p),\ \ldots,\ F_{p,m}(-2p+1).$$

According to (4.296), we have: $F_{p,m}(p+1)=m^p$ and $F_{p,m}(p)=m^{p-1}$, thus $F_{p,m}(0)=0$. Continuing in this way, we obtain

$$F_{p,m}(-1)=F_{p,m}(-2)=...=F_{p,m}(-p+1)=0. \tag{4.297}$$

Let us write the m-extension of the Fibonacci p-number $F_{p,m}(1)$ in the form

$F_{p,m}(1)=mF_{p,m}(0)+F_{p,m}(-p)$.

Then, we obtain

$$F_{p,m}(-p)=1. \tag{4.298}$$

Using (4.295), (4.296) and (4.298), we have

$$F_{p,m}(-p-1)= F_{p,m}(-p-2)=...= F_{p,m}(-2p+1)= 0. \tag{4.299}$$

Also, we have

$$F_{p,m}(-2p)=m,\ F_{p,m}(-2p-1)=1,\ F_{p,m}(-2p-2)= 0,.... \tag{4.300}$$

For the case $m=2$ we obtain the 2-extension of Fibonacci p-numbers called Pell p-numbers. Table 4.7 gives the values of Pell p-numbers for the cases of $p=1, 2, 3, 4$.

Table 4.7. Pell p-numbers ($m=2$; $p=1,2,3,4$)

n	5	4	3	2	1	0	-1	-2	-3	-4	-5
$F_{1,2}(n)$	29	12	5	2	1	0	1	-2	5	-12	29
$F_{2,2}(n)$	22	9	4	2	1	0	0	1	0	-2	1
$F_{3,2}(n)$	17	8	4	2	1	0	0	0	1	0	0
$F_{4,2}(n)$	16	8	4	2	1	0	0	0	0	1	0

4.14.3. *Characteristic Algebraic Equation for the Fibonacci (p, m)-Numbers*

Let us represent the recursive relation (4.295) in the form:

$$\frac{F_{p,m}(n)}{F_{p,m}(n-1)}=\frac{mF_{p,m}(n-1)+F_{p,m}(n-p-1)}{F_{p,m}(n-1)}$$
$$=m+\frac{1}{\frac{F_{p,m}(n-1)}{F_{p,m}(n-p-1)}}=m+\frac{1}{\frac{F_{p,m}(n-1)\cdot F_{p,m}(n-2)\cdots F_{p,m}(n-p)}{F_{p,m}(n-2)\cdot F_{p,m}(n-3)\ \cdots F_{p,m}(n-p-1)}}. \tag{4.301}$$

Suppose that

$$\lim_{n\to\infty}\frac{F_{p,m}(n)}{F_{p,m}(n-1)}=x. \tag{4.302}$$

Taking into consideration the definition (4.302) and directing $n \to \infty$, we can replace the expression (4.301) with the following algebraic equation

$$x = m + \frac{1}{x^p},$$

whence it appears

$$x^{p+1} - mx^p - 1 = 0. \tag{4.303}$$

We will name the equation (4.303) a *Characteristic Equation* for the Fibonacci (p, m)-numbers.

Note that for the case $m=1$, Eq. (4.303) is reduced to Eq. (4.42) – the characteristic equation for the Fibonacci p-numbers – and for the case $p=1$ to Eq. (4.247) - the characteristic equation for the Fibonacci m-numbers.

Equation (4.303) has $p+1$ roots $x_1, x_2, ..., x_p, x_{p+1}$. If we substitute the root x_k ($k=1, 2, 3, ..., p+1$) for x into Eq. (4.303), we obtain the following identity for the root x_k:

$$x_k^{p+1} = mx_k^p + 1. \tag{4.304}$$

If we multiply and divide all terms of the identity (4.304) by x_k repeatedly, we come to the following general identity

$$x_k^n = mx_k^{n-1} + x_k^{n-p-1} = x_k \times x_k^{n-1}, \tag{4.305}$$

where $n=0,\pm1,\pm2,\pm3,...$ and x_k ($k=1,2,3,..., p+1$) is a root of Eq. (4.303).

4.14.4. *The Golden (p,m)-Proportions*

According to Descartes' "rule of signs," Eq. (4.303) has the only positive root. Suppose without loss of generality that the root x_1 is a positive root of Eq. (4.303). Let us denote the root x_1 by $\Phi_{p,m}$ and name it a *Golden (p, m)-Proportion*. Substituting $\Phi_{p,m}$ for x_k ($k=1$) into the identities (4.304) and (4.305), we obtain two important identities for $\Phi_{p,m}$:

$$\Phi_{p,m}^{p+1} = m\Phi_{p,m}^p + 1 \tag{4.306}$$

$$\Phi_{p,m}^n = \Phi_{p,m}^{n-1} + m\Phi_{p,m}^{n-p-1} = \Phi_{p,m} \times \Phi_{p,m}^{n-1}, \tag{4.307}$$

where $n=0,\pm1,\pm2,\pm3,....$

If we divide all terms of the identity (4.306) by $\Phi_{p,m}^p$, we obtain the following remarkable property of the golden (p,m)-proportions

$$\Phi_{p,m} = m + 1/\Phi_{p,m}^p \tag{4.308}$$

or

$$\Phi_{p,m} - m = 1/\Phi_{p,m}^p. \tag{4.309}$$

Note that the golden (p,m)-proportions $\Phi_{p,m}$ is a wide generalization of the golden p-proportions (m=1), the generalized golden m-proportions (p=1) and the classical golden mean (p=1, m=1). Also the identities (4.308) and (4.309) are a wide generalization of the corresponding identities for the golden p-proportions, the golden m-proportions, and the classical golden mean.

4.14.5. *Properties of the Roots of the Characteristic Equation*

Theorem 4.4. For a given integer $p>0$ and a positive real number m, the following correlations for the roots of the golden algebraic equation $x^{p+1}-mx^p-1=0$ are valid:

$$x_1+x_2+x_3+...+x_p+x_{p+1}=m \tag{4.310}$$

$$\begin{gathered}\left(x_1x_2+...+x_1x_{p+1}\right)+\left(x_2x_3+...+x_2x_{p+1}\right)+...+\left(x_{p-1}x_p+x_{p-1}x_{p+1}\right)+x_px_{p+1}=0\\ \left(x_1x_2x_3+...+x_1x_px_{p+1}\right)+\left(x_2x_3x_4+...+x_2x_px_{p+1}\right)+...+x_{p-1}x_px_{p+1}=0\\ ...\\ x_1x_2\cdots x_{p-2}x_{p-1}x_p+x_1x_3x_4\cdots x_{p-1}x_px_{p+1}+...+x_2x_3x_4\cdots x_{p-1}x_px_{p+1}=0\end{gathered} \tag{4.311}$$

$$x_1x_2x_3\cdots x_{p-1}x_px_{p+1}=(-1)^p. \tag{4.312}$$

Theorem 4.4 is proved by analogy with Theorem 4.1. We use the "Basic Theorem of Algebra" that is given by (4.74) to represent the characteristic equation (4.303) in the form:

$$\begin{aligned}&x^{p+1}-mx^p-1=(x-x_1)(x-x_2)\cdots(x-x_{p+1})=x^{p+1}-\left(x_1+x_2+...+x_{p+1}\right)x^p\\ &+\left(x_1x_2+...+x_1x_{p+1}+x_2x_3+...+x_2x_{p+1}+...+x_{p-1}x_{p+1}+x_px_{p+1}\right)x^{p-1}\\ &-\left(x_1x_2x_3+...+x_1x_px_{p+1}+x_2x_3x_4+...+x_2x_px_{p+1}+...+x_{p-1}x_px_{p+1}\right)x^{p-2}\\ &+\left(x_1x_2x_3x_4+x_1x_2x_3x_5+...+x_{p-2}x_{p-1}x_px_{p+1}\right)x^{p-3}\\ &...\\ &+\left(x_1x_2x_3\cdots x_{p-2}x_{p-1}x_p+...+x_2x_3x_4\cdots x_{p-1}x_px_{p+1}\right)x-x_1x_2x_3\cdots x_{p-1}x_px_{p+1}=0.\end{aligned} \tag{4.313}$$

We can give some explanations regarding the identities (4.310), (4.311), and (4.312) that connect the roots of Eq. (4.303). It is evident from (4.310) that the sum of the roots of Eq. (4.303) is identically equal to m. The expression (4.311) gives the values for every possible sum of the roots of Eq. (4.303) taken by two, three, ..., or p roots from the $(p+1)$ roots of Eq. (4.303). According to (4.311), each of these sums is equal to 0 ! At last, the expression (4.312) gives the value of the product of all roots of Eq. (4.303). According to (4.312) this product is equal to 1 (for the even p) or -1 (for the odd p).

Theorem 4.5. For a given integer p=0,1,2,3,... and for the condition, when k takes its values from the set {1,2,3,..., p}, the following identity is valid for the roots of the algebraic equation $x^{p+1}-mx^p-1=0$:

$$\left(x_1+x_2+x_3+...+x_p+x_{p+1}\right)^k=x_1^k+x_2^k+x_3^k+...+x_p^k+x_{p+1}^k=m^k. \tag{4.314}$$

Theorem 4.5 is proven by analogy with Theorem 4.2. For the proof we may consider the following expression:

$$\left(x_1+x_2+x_3+...+x_p+x_{p+1}\right)^k, \tag{4.315}$$

where k takes its values from the set $\{1,2,3,..., p\}$.

Taking into consideration the identity (4.310), we can write:

$$\left(x_1+x_2+x_3+...+x_p+x_{p+1}\right)^k=m^k. \tag{4.316}$$

On the other hand, if we factorize the expression (4.315) and take into consideration the identities (4.311), we obtain:

$$\left(x_1+x_2+x_3+...+x_p+x_{p+1}\right)^k=x_1^k+x_2^k+x_3^k+...+x_{p+1}^k, \tag{4.317}$$

whence the identity (4.314) appears.

4.14.6. *Generalized Binet Formulas for the Fibonacci (p,m)-Numbers*

For a given $p>0$, the generalized Binet formula for the Fibonacci (*p,m*)-numbers is

$$F_{p,m}(n)=k_1(x_1)^n+k_2(x_2)^n+...+k_{p+1}\left(x_{p+1}\right)^n, \tag{4.318}$$

where $x_1, x_2, ..., x_{p+1}$ are the roots of the characteristic equation (4.303) and k_1, $k_2, ..., k_{p+1}$ are constant coefficients that depend on the initial terms (4.296) of the Fibonacci (*p,m*)-sequence.

In order to calculate the values of the coefficients $k_1, k_2, ..., k_{p+1}$ in (4.318), we consider solutions of the following system of equations

$$\begin{cases} F_{p,m}(0)=k_1+k_2+\cdots+k_{p+1}=0 \\ F_{p,m}(1)=k_1x_1+k_2x_2+\cdots+k_{p+1}x_{p+1}=1 \\ F_{p,m}(2)=k_1x_1^2+k_2x_2^2+\cdots+k_{p+1}x_{p+1}^2=m \\ \vdots \\ F_{p,m}(p)=k_1x_1^p+k_2x_2^p+\cdots+k_{p+1}x_{p+1}^p=m^{p-1} \end{cases}. \tag{4.319}$$

Solving the system of the equations (4.319), we can obtain the numerical values of the coefficients $k_1, k_2, ..., k_{p+1}$ for different values of p.

For the case p=1, the Fibonacci (*p, m*)-numbers are reduced to the Fibonacci m-numbers, and we obtain the formula similar to the Gazale formula (4.281), that is,

$$F_{1,m}(n)=\frac{\Phi_{1,m}^n-(-1)^n\Phi_{1,m}^{-n}}{\sqrt{4+m^2}}, \tag{4.320}$$

where $\Phi_{1,m}$ is the golden $(1, m)$-proportion.

For the case p=2, the recursive relation (4.295), the seeds (4.296), and the characteristic equation (4.303) take the following forms, respectively,

$$F_{2,m}(n)=mF_{2,m}(n-1)+F_{2,m}(n-3) \tag{4.321}$$

$$F_{2,m}(0)=0,\ F_{2,m}(1)=1,\ F_{2,m}(2)=m \tag{4.322}$$

$$x^3-mx^2-1=0. \tag{4.323}$$

Equation (4.323) has one real root and two complex roots. The roots of Eq. (4.323) are:

$$x_1=\frac{h^2+2mh+4m^2}{6h}, \tag{4.324}$$

$$x_2=-\frac{h^2-2mh+4m^2}{6h}+i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right), \tag{4.325}$$

$$x_3=-\frac{h^2-2mh+4m^2}{6h}-i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right), \tag{4.326}$$

where $h=\sqrt[3]{108+8m^3+12\sqrt{81+12m^3}}$.

The Binet formula for the Fibonacci $(2,m)$-numbers is

$$F_{2,m}(n)=k_1(x_1)^n+k_2(x_2)^n+k_3(x_3)^n. \tag{4.327}$$

The values of k_1, k_2, k_3 are solutions of the system

$$\begin{aligned} F_{2,m}(0)&=k_1+k_2+k_3=0,\\ F_{2,m}(1)&=k_1x_1+k_2x_2+k_3x_3=1,\\ F_{2,m}(2)&=k_1(x_1)^2+k_2(x_2)^2+k_3(x_3)^2=m. \end{aligned} \tag{4.328}$$

Solving the system, we obtain

$$k_1=\frac{2h(h+2m)}{h^3+8m^3},\quad k_2=\frac{h[-(h+2m)-i\sqrt{3}(h-2m)]}{h^3+8m^3}$$

$$k_3=\frac{h[-(h+2m)+i\sqrt{3}(h-2m)]}{h^3+8m^3}.$$

Taking p=2,3,4,5,... we can derive the generalized Binet formulas for the Fibonacci (p,m)-numbers. For example, the Binet formula for the Fibonacci $(2, m)$-numbers is the following:

$$F_{2,m}(n)=\frac{2h(h+2m)}{h^3+8m^3}\left(\frac{h^2+2mh+4m^2}{6h}\right)^n$$
$$+\frac{h[-(h+2m)-i\sqrt{3}(h-2m)]}{h^3+8m^3}\left(-\frac{h^2+4m^2-4hm}{12h}+i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right)\right)^n \tag{4.329}$$
$$+\frac{h[-(h+2m)+i\sqrt{3}(h-2m)]}{h^3+8m^3}\left(-\frac{h^2+4m^2-4hm}{12h}-i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right)\right)^n$$

4.14.7. *Generalized Binet Formulas for the Lucas (p,m)-Numbers*

By analogy to the formula (4.161), we introduce the following formula that defines the *Lucas* (p,m) *-numbers*:

$$L_{p,m}(n)=(x_1)^n+(x_2)^n+...+(x_{p+1})^n, \tag{4.330}$$

where $x_1, x_2, ..., x_{p+1}$ are the roots of the characteristic equation (4.303).

Calculate the value of the initial $(p+1)$ terms of the number sequence defined by (4.330). For the case $n=0$, we have:

$$L_{p,m}(0)=(x_1)^0+(x_2)^0+...+(x_{p+1})^0=p+1. \tag{4.331}$$

Let us take k from the set $\{1,2,3,...,p\}$ and represent the formula (4.330) in the form:

$$L_{p,m}(k)=(x_1)^k+(x_2)^k+...+(x_{p+1})^k. \tag{4.332}$$

Using the identity (4.314), we can write:

$$L_{p,m}(k)=m^k, \tag{4.333}$$

where k takes its values from the set $\{1,2,3,..., p\}$.

The expressions (4.331) and (4.333) can be considered as the seeds of the Lucas (p, m)-numbers that are given by the following recursive formula:

$$L_{p,m}(n)=mL_{p,m}(n-1)+L_{p,m}(n-p-1). \tag{4.334}$$

Let us prove that the formula (4.330) gives the same numerical sequence as well as the recursive relation (4.334) at the seeds (4.331) and (4.333). Suppose that the formula (4.330) defines the same number $L_{p,m}(n)$ that is calculated according to the recursive relation (4.334). Consider the formula (4.330) for the case of $n+1$, that is,

$$L_{p,m}(n+1)=(x_1)^{n+1}+(x_2)^{n+1}+...+(x_{p+1})^{n+1}. \tag{4.335}$$

Using the identity (4.305), we can write the expression (4.335) as follows

$$L_{p,m}(n+1) = m\left[(x_1)^n + (x_2)^n + \ldots + (x_{p+1})^n\right] \\ +(x_1)^{n-p-1} + (x_2)^{n-p-1} + \ldots + (x_{p+1})^{n-p-1}, \tag{4.336}$$

whence it appears

$$L_{p,m}(n+1)=mL_{p,m}(n)+L_{p,m}(n-p) \tag{4.337}$$

that corresponds to the recursive relation (4.334).

This means that our reasoning results in the discovery of a new class of numerical sequences – the *m-extension of the Lucas p-numbers* or simply the *Lucas (p, m)-numbers.*

Note that for the case m=1 the recursive relation (4.334) and the seeds (4.331) and (4.333) are reduced to the following:

$$L_{p,1}(n)=L_{p,1}(n-1)+L_{p,1}(n-p-1) \tag{4.338}$$

$$L_{p,1}(0)=p+1 \text{ and } L_{p,1}(k)=1\ (k=1,2,3,\ldots,p). \tag{4.339}$$

It is clear that the recursive relation (4.338) at the seeds (4.339) gives the above Lucas p-numbers.

For the case m=2, the recursive relation (4.334) and the seeds (4.331) and (4.333) are reduced to the following:

$$L_{p,2}(n)=2L_{p,2}(n-1)+L_{p,2}(n-p-1) \tag{4.340}$$

$$L_{p,2}(0)=p+1;\ L_{p,2}(k)=2^k\ (k=1,2,3,\ldots,p). \tag{4.341}$$

For the case p=1, the Lucas (p,m)-numbers are reduced to the Lucas m-numbers and we obtain the Binet formula similar to (4.292), that is,

$$L_{1,m}(n) = \Phi_{1,m}^n + (-1)^n \Phi_{1,m}^{-n}, \tag{4.342}$$

where $\Phi_{1,m}$ is the golden (1, m)-proportion.

For the case p=2, the recursive relation (4.334), the seeds (4.331) and (4.333), the characteristic equation (4.303), and Binet formula (4.330) take the following forms, respectively:

$$L_{2,m}(n)=mL_{2,m}(n-1)+L_{2,m}(n-3) \tag{4.343}$$

$$L_{2,m}(0)=3,\ L_{2,m}(1)=2,\ L_{2,m}(2)=4 \tag{4.344}$$

$$x^3-mx^2-1=0 \tag{4.345}$$

$$L_{2.m}(n) = (x_1)^n + (x_2)^n + (x_3)^n. \tag{4.346}$$

If we substitute the expressions (4.324)-(4.326) for the roots x_1, x_2, x_3 into (4.346), we obtain the following analytical expression of the Binet formula for the Lucas (2,m)-numbers that are called the Pell-Lucas p-numbers:

$$L_{2,m}(n)=\left(\frac{h}{6}+\frac{2m^2}{3h}+\frac{m}{3}\right)^n+\left[-\frac{h}{12}-\frac{m^2}{3h}+\frac{m}{3}+\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2m^2}{3h}\right)\right]^n$$
$$+\left[-\frac{h}{12}-\frac{m^2}{3h}+\frac{m}{3}-\frac{i\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2m^2}{3h}\right)\right]^n. \qquad (4.347)$$

Table 4.8 gives the 2-extension of the Lucas p-numbers (Pell-Lucas p-numbers) for the cases p=1,2,3,4.

Table 4.8. Pell-Lucas p-numbers (m=2, p=1,2,3,4)

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$L_{1,2}(n)$	-82	34	-14	6	-2	2	2	6	14	34	82
$L_{2,2}(n)$	-10	8	3	-4	0	3	2	4	11	24	52
$L_{3,2}(n)$	0	4	-6	0	0	4	2	4	8	20	42
$L_{4,2}(n)$	5	-8	0	0	0	5	2	4	8	16	37

It is clear that our reasoning resulted in a wide generalization of the Fibonacci and Lucas p-numbers and the Fibonacci and Lucas m-numbers. The Fibonacci and Lucas (p,m)-numbers are of theoretical interest for discrete mathematics and open new perspectives for the development of theoretical physics because new recursive numerical sequences and new mathematical constants that follow from this approach may be discovered in some physical processes.

4.15. Structural Harmony of Systems

4.15.1. *Soroko's Law of the Structural Harmony of Systems*

Belorussian philosopher Eduardo Soroko was one of the first researchers who used the Fibonacci p-numbers and golden p-proportions for simulation of the processes in self-organizing systems [25, 56]. Soroko's main idea is to consider real systems from the dialectical point of view. As is well known, any natural object can be represented as the dialectical unity of the two opposite sides A and B. This dialectical connection may be expressed in the following form:

$A+B=U$ (universum). (4.348)

The equality (4.348) is the most general expression of the so-called *Conservation Law*. Here A and B are distinctions inside of the Unity, logically disjoint classes of the Whole. There is one requirement that A and B need to be measured by the same measure and members of the relation that underlies the unity. Probability and improbability of events, mass and energy, the nucleus of an atom and its envelope, substance and field, anode and cathode, animals and plants, spiritual and material beginnings in a value system, and profit and cost are various examples of (4.348).

The identity (4.348) may be reduced to the following normalized form:

$$\overline{A}+\overline{B}=1, \tag{4.349}$$

where $\overline{A}$ and $\overline{B}$ are the relative "weights" of the parts A and B that make up some Unity.

The *Law of Information Conservation* is a partial case of (4.348):

$$I+H=\log N, \tag{4.350}$$

where I is an information quantity and H is the entropy of the system having N different states.

The normalized form of the law (4.350) is the following:

$$R+\overline{H}=1, \tag{4.351}$$

where $R=1/\log N$ is a relative redundancy, $\overline{H}=H/\log N$ is a relative entropy.

Let us consider the process of system self-organization. This one is reduced to the passage of the system into some "harmonious" state called the state of *thermodynamic equilibrium*. There is some correlation or proportion between the sides A and B of the dialectical contradiction (4.348) for the state of thermodynamic equilibrium. This correlation has a strictly regular character and is the cause of system stability. Soroko uses the *Principle of Multiple Relations* to find a connecting law between A and B in the state of thermodynamic equilibrium. This principle is well known in chemistry as *Dalton's Law* and in crystallography as the *Law of Rational Parameters*.

Soroko proposes the hypothesis that the *Principle of Multiple Relations* is a general principle of the Universe. That is why, in accordance with this principle there is the following connection between the components R and $\overline{H}$ in the identity (4.351):

$$\log R=(s+1)\overline{H} \tag{4.352}$$

or

$$\log \overline{H}=(s+1)R. \tag{4.353}$$

The equalities of (4.352) and (4.353) may be represented in exponential form:

$$R=\left(\overline{H}\right)^{s+1} \tag{4.354}$$

$$\overline{H}=R^{s+1}, \tag{4.355}$$

where the number s is called the *Range of Multiplicity* and takes the following values: 0,1,2,3,... .

Substituting the expressions (4.354) and (4.355) into the equality (4.351), we obtain the following algebraic equations, respectively:

$$\left(\overline{H}\right)^{s+1}+\overline{H}-1=0 \tag{4.356}$$

$$R^{s+1}+R-1=0. \tag{4.357}$$

If we denote the variables $\overline{H}$ and R in the equations (4.356) and (4.357) by y, we obtain the following algebraic equation:

$$y^{s+1}+y-1=0. \tag{4.358}$$

Let us introduce the new variable $x=1/y$ and apply it to the equation (4.358). Inserting the new variable into (4.358), we obtain the following algebraic equation:

$$x^{s+1}+x^{s}-1=0. \tag{4.359}$$

We can see that the equation (4.359) coincides with the algebraic equation of the golden p-proportion that is given by (4.42). The positive root of the equation (4.358) is the reciprocal of the golden p-proportion:

$$\beta_s=1/\tau_s, \tag{4.360}$$

where τ_s is a positive root of the equation (4.359).

In accordance with Soroko's concept, the roots of the equation (4.358) that are equivalent to equation (4.359), express the *Law of the Structural Harmony of Systems*:

"The generalized golden proportions are invariants that allow natural systems in the process of their self-organization to find harmonious structure, a stationary regime for their existence, and structural and functional stability."

What is peculiar about Soroko's Law? Since Pythagoras, scientists combined the concept of Harmony solely with the classical golden mean $\tau=\left(1+\sqrt{5}\right)/2$. Soroko's Law asserts that the harmonious state that corresponds to the classical golden mean is not unique to a system. Soroko's Law asserts that there are an infinite number of harmonious states of the system that correspond to the numbers τ_s or to the inverse numbers $\beta_s=1/\tau_s$ (s=1,2,3,...) that are positive roots of the general algebraic equations (4.358) and (4.359).

Table 4.9 gives the values of Soroko's "structural invariants" for the initial terms of s.

Table 4.9. Soroko's numerical invariants

s	1	2	3	4	5	6	7
β_s	0.618	0.682	0.724	0.755	0.778	0.796	0.812

4.15.2. *Application of Soroko's Law to Thermodynamic and Information Systems*

The thermodynamic or informational state of a system is expressed by *entropy*, which is the principal concept of thermodynamics and information theory. The expression for the entropy of an information source with the alphabet $A=\{a_1, a_2, ..., a_N\}$ has the following form:

$$H=-\sum_{k=1}^{N} p_k \log p_k\ , \tag{4.361}$$

where $p_1, p_2, ..., p_N$ are the probabilities of the letters $a_1, a_2, ..., a_N$, N is a number of the letters.

It is well known that entropy (4.361) reaches its maximum value

$$H_{\max}=\log N \tag{4.362}$$

for the case, when the probabilities of the letters are equal among themselves, that is,

$p_1=p_2= ...= p_N=1/N.$

Using the concept of relative entropy

$$\overline{H}=H/\log N\,, \tag{4.363}$$

we can write the following evident equality:

$$\overline{H}\log N=H=-\sum_{k=1}^{N} p_k \log p_k. \tag{4.364}$$

In accordance with the *Law of the Structural Harmony of Systems,* any system reaches its harmonious state when its relative entropy (4.363) satisfies the equality (4.353). The following expression for the entropy of the harmonious system follows from this consideration:

$$H=-\sum_{k=1}^{N} p_k \log p_k=\beta_s \log N. \tag{4.365}$$

It is clear that for any given s, the problem of obtaining the optimal set of values p_i (i=1, 2, 3, ..., N) corresponding to the optimal ("harmonious") state of the system, has many solutions. However, the expression (4.365) plays a role in the function's "aim" towards the solution of various scientific and technical problems.

In his book, *Structural Harmony of Systems* [25], Soroko gave a number of interesting examples from different fields of science to confirm the *Law of the Structural Harmony of Systems*. For example, let us consider such a natural object as "dry air" that is the basis for life on Earth. We can ask the question: does the "dry air" have an optimal or harmonious structure? Soroko's theory gives a positive answer to this question. In fact, the chemical compound of the "dry air" is the following: nitrogen – 78.084%; oxygen – 20.948%; argon – 0.934%; carbon dioxide – 0.031%; neon – 0.002%; and helium – 0.001%. If we calculate the entropy of the "dry air" according to the formula of (4.361) and then its relative entropy according to (4.363) taking into consideration that $\log N = \log 6$, then the value of the relative entropy of the "dry air" will be equal to 0.683. With a high degree of precision this value corresponds to the invariant $\beta_2 = 0.682$. This means that in the process of self-organization the "dry air" reaches its optimal, harmonious structure. This example is very typical and demonstrates that "Soroko's Law" can be used today for checking the biosphere state, in particular, the states of air and water.

It is clear thus the practical use of the *Law of the Structural Harmony of Systems* provides a clear advantage to the solution of many technological, economical, ecological and other problems. In particular, it can help improve the technology of structurally-complicated products, including the monitoring of the biosphere and so forth.

4.16. Conclusion

In recent years the generalization of Fibonacci and Lucas numbers and the golden section has been an important trend in the development of Fibonacci number theory. One may cite various ways in which this generalization takes place. One way, for example, is a generalization based on Pascal's triangle. In the latter half of the 20th century several eminent mathematicians (including Martin Gardner [12], George Polya [17], Alfred Renyi [23] and others) each independently discovered a connection of Fibonacci numbers with Pascal's triangle and binomial coefficients. This finding confirms a fundamental connection of Fibonacci and golden mean based Harmony Mathematics with combinatorial analysis suggesting a future line of development.

In the early 1970s, in his doctoral dissertation [19], Alexey Stakhov introduced the so-called Fibonacci p-numbers (p=0,1,2,3,...) that follow from the diagonal sums in Pascal's triangle. By studying the Fibonacci p-numbers, Stakhov generalized the golden section problem and introduced the golden p-proportions that are positive roots of the characteristic equation $x^{p+1}=x^p+1$. Later by continuing this research, Stakhov formulated the *Generalized Principle of the Golden Section* [107] and generalized the Euclidean problem of the "division in extreme and mean ratio" (the golden section) [20]. Later Stakhov and Rozin generalized the Binet formulas and introduced the generalized Lucas p-numbers [111].

Fibonacci numbers were further generalization through a consideration of the generalized recursive relation $G(n+1)=pG(n)+qG(n-1)$ (p and q are integers or real numbers) that coincides with the classical recursive Fibonacci relation for the case $p=q=1$. This became the source of many original discoveries made by Vera W. de Spinadel [42], Midhat Gazale [45] and Jay Kappraff [47]. By using this approach, Vera W. de Spinadel introduced the *Metallic Means* [42]. By using the recursive relation $F_m(n+1)=mF_m(n)+F_m(n-1)$ and the characteristic equation $x^2-mx-1=0$, where m is a positive real number, Midhat Gazale further generalized the Binet formula for Fibonacci m-numbers $F_m(n)$. Gazale formulas allow one to represent all Fibonacci and Lucas m-numbers by the golden m-proportion F_m that is a generalization of the classical golden mean.

Recently, E. Gokcen, Naim Tuglu and Alexey Stakhov [154] introduced the m-extension of the Fibonacci and Lucas p-numbers, generating the new class of Fibonacci and Lucas (p,m)-numbers. This new class of characteristic equations $x^{p+1}-mx^p-1=0$, for generalized Fibonacci and Lucas (p,m)-numbers and the golden mean, infinitely extend algebraic equations and mathematical constants, which can be used in contemporary mathematics, theoretical physics and computer science.

Chapter 5

Hyperbolic Fibonacci and Lucas Functions

5.1. The Simplest Elementary Functions

5.1.1. *Trigonometric Functions*

In mathematics, an *Elementary Function* is built up from a finite number of exponentials, logarithms, constants, variables, and roots of equations by using the four elementary operations (addition, subtraction, multiplication and division). *Logarithms, Exponential Functions* (including *Hyperbolic Functions*), *Power Functions,* and *Trigonometric Functions* are the best known amongst them.

We begin to study the elementary functions starting with trigonometric functions. Trigonometry and trigonometric functions *Sine, Cosine, Tangent, Cotangent, Secant,* and *Cosecant* are well known to many of us from secondary school. Trigonometry was developed in antiquity initially as a branch of astronomy, as a computing tool for practical purposes. Application of trigonometry to astronomy explains why spherical trigonometry arose first, before planar trigonometry. Ancient Greek astronomers successfully solved some problems of trigonometry. However, these scientists (Ptolemy, Menelaus, etc.) had not studied trigonometric functions (sine, cosine, etc.) but line segments and spans. The span that connects together a double arch 2α played a role of a sine of the angle α. Ptolemy had found a formula for the definition of the span as the sum and difference of two arches, the span of the half arch, and so on.

Sine and cosine were first introduced by Indian scientists. In India, the doctrine of trigonometric values was named *Goniometry* as it started to develop. The further development of the doctrine of trigonometric values continued in the 9th-15th centuries in the countries of the Middle and the Near East. The Arabian mathematicians introduced all basic trigonometric functions derivative from sine and cosine: tangent, cotangent, secant and cosecant. They had proved the basic relations amongst trigonometric functions.

For the first time Arabian mathematicians began to develop trigonometry as a part of mathematics independent from astronomy.

In the 13th-15th centuries, the further development of trigonometry in Europe continued following the translation of mathematical and astronomical works of Arabian and Greek science into Latin. The German scientist Regiomontanus (1436-1476) was the most famous European mathematician of this epoch in the field of trigonometry. His trigonometric work *Five Books about Triangles of All Kinds* was of great importance for the further development of trigonometry during the 16th-17th centuries.

On the threshold of the 17th century, an analytical direction in trigonometry started to develop. Before the 17th century, the direction regarding triangles and the calculation of the elements of geometric figures was the main objective of trigonometry and the doctrine of trigonometric functions had developed along geometrical lines. Whereas in the 17th-19th centuries, trigonometry gradually became one of the branches of mathematical analysis. Trigonometry finds wide application in mechanics, physics and engineering, especially in the study of oscillatory movements and other periodic processes. The development of the doctrine of oscillatory movements, of sound, light and electromagnetic waves became central to the basic contents of trigonometry. It is well known from physics that the equation of harmonious fluctuation (for example, of a variable electric current) appears as:

$$y = A\sin(\omega t + \alpha).$$

Sinusoids are graphs of harmonious fluctuations; therefore, in physics and engineering the harmonious fluctuations are often called *Sinusoidal Fluctuations*.

Newton and Euler developed an analytical approach to trigonometry, and in the first half of the 19th century the French scientist Fourier proved that any periodic movement can be presented (with any degree of accuracy) in the form of the sum of simple harmonious fluctuations. Presently trigonometry is no longer considered an independent part of mathematics. Its major section – the *Doctrine about Trigonometric Function* – is a part of the theory of functions; and its other section –the *Decision of Triangles* – is considered to be part of geometry (planar and spherical).

5.1.2. *The Power Function*

The *Power Function* plays an important role in mathematics. The *Power* concept originally meant the product of a finite number of equal coefficients (a power with a natural parameter), that is,

$$a^n = a\times a\times\ldots\times a \text{ (} n \text{ times).} \tag{5.1}$$

This definition possesses the following mathematical properties:

$$\begin{aligned}&(ab)^n = a^n \times b^n;\ (a/b)^n = a^n / b^n;\\ &(a^m)^n = a^{mn};\ a^m / a^n = a^{m-n}.\end{aligned} \tag{5.2}$$

Over the centuries, this concept was repeatedly generalized and enriched by its contents. The concept of the 2nd and 3rd powers a^2 and a^3 appeared in connection with the definition of the area of a square and the volume of a cube. Already the Babylonians had made use of tables of the squares and cubes of numbers. The titles *Square* and *Cube* for the 2nd and 3rd powers are of ancient Greek origin. The practice of solving more and more complex algebraic problems, led to the necessity to generalize the power concept and its extension by means of the introduction of the powers of zero, and negative and fractional numbers. The powers of zero and also negative and fractional parameters were defined so that the actions of the power with a natural basis given by (5.2) were applied to them. The principle here observed in the generalization of mathematical concepts is referred to as "a permanence principle."

The power concept (5.1) depends on two parameters, the numbers of a and n. Depending on what parameter is chosen as a variable, it is possible to give two generalizations of the power concept (5.1). The first generalization is the power function:

$$y=x^\sigma, \tag{5.3}$$

where x is a variable and σ is a given real number.

Note that for the cases $\sigma=0$ and $\sigma=1$ according to (5.3) we have, respectively:

$$y=1 \text{ and } y=x. \tag{5.4}$$

The graph of the function $y=1$ is a straight line parallel to the axis OX, and the graph of the function $y=x$ is a bisector of the 1st and 3d coordinate angles.

For the case $\sigma=2$ the graph of the function (5.3) is a *Parabola* $y=x^2$. For the case $\sigma=3$ the graph of the function (5.3) is a *Cubic Parabola* $y=x^3$. This curve had been used by French mathematician Gaspard Monge (1746-1818), the father of descriptive geometry, for finding the real roots of the cubic equations.

A derivative of the power function (5.3) is equal to the product of the power parameter α *multiplied* by the power function with the parameter (α-1), that is,

$$\frac{d}{dy}x^\alpha = \alpha x^{\alpha-1}.$$

The power function (5.3) is a basic mathematical tool that is used widely in many other fields as well, including economics, biology, chemistry, physics, and computer science, with such applications as compound interest, population growth, chemical reactions, wave behavior, and public key cryptography.

5.1.3. *An Exponential Function*

However, there is another generalization of the power concept (5.1). Let us examine the so-called exponential function given by the following expression:

$$y=a^x, \tag{5.5}$$

where x is a variable and $a>0$ is any real number.

The exponential function (5.5) has many applications in the study of natural and social phenomena. It is known, for example, that the decay of radioactive substance is described by the function (5.5). If we denote by t_0 a period of the half-decay, that is, a time interval necessary to get half of the initial mass m_0, then the mass of the substance via t years can be expressed as follows:

$$\frac{d}{dy}x^{\alpha}=\alpha x^{\alpha-1}.$$

The exponential functions (5.5) posses the following *Exponential Laws*:

$$a^0=1; a^1=a;\ a^{x+y}=a^x a^y;\ a^{xy}=\left(a^x\right)^y;$$

$$1/a^x=a^{-x};\ a^x b^x=\left(ab\right)^x.$$

These correlations are valid for all positive real numbers a and b and all values of the variables x and y. The expressions that involve fractions and roots can often be simplified by using exponential functions because:

$$1/a=a^{-1};\quad \sqrt[n]{a^b}=\left(\sqrt[n]{a}\right)^b=a^{\frac{b}{n}},$$

where n is a natural number.

In mathematics, the following partial case of the exponential function is widely used:

$$y=e^x, \tag{5.6}$$

where e=2.71828183 is the base of natural logarithms. The graph of the exponential function (5.6) is represented in Fig. 5.1.

The exponential function is climbing slowly with the negative values of x, it is climbing quickly with the positive values of x, and is equal to 1 when x is equal to 0. As a function (5.6) is the function of real variable x, the graph of (5.6) is positive for all values of x and is increasing (by viewing left-to-right). This graph never touches the x-axis, although it approaches the x-axis; thus, the x-axis is a horizontal asymptote to the graph.

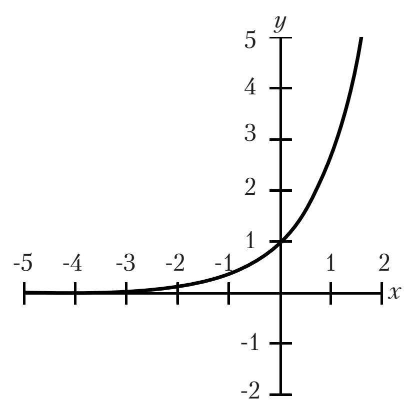


Figure 5.1. Exponential function

A derivative of the exponential function (5.6) coincides with the function (5.6), that is,

$$\frac{d}{dx}e^x = e^x.$$

Note that only the function (5.6) has this unique property.

For the exponential functions with other bases we have:

$$\frac{d}{dx}a^x = (\ln a)a^x.$$

5.1.4. *Logarithms*

The theory of logarithms is based on the following simple reasoning. Let us examine the exponential function (5.5). First we represent the variable x in (5.5) in the form of a special function of y as follows:

$$x = \log_a y = \log_a\left(a^x\right).$$

We name this function a *Logarithm* with base a. Depending upon the choice of the base a there are different kinds of logarithms. For the case a=2 we have *Binary Logarithms* $\log_2 x = \lg x$, for the case a=10 we have *Decimal Logarithms* $\log_{10} x$ and finally, for the case $a=e$ we have *Natural Logarithms* $\log_e x = \ln x$.

Logarithms are used widely in many areas of science and engineering where the magnitudes vary over a large range. Logarithmic scales are used, for example, in the decibel scale for the loudness of sound, the Richter scale of earthquake magnitudes, and the astronomical scale of stellar brightness.

A derivative and an indefinite integral of $\log_a z$ are given respectively by

$$\frac{d}{dz}\log_a z = \frac{1}{z\ln a}; \quad \int \log_a dz = \frac{z(\ln z - 1)}{\ln a} + c.$$

5.2. Hyperbolic Functions

5.2.1. *Definition of the Hyperbolic Functions*

The function

$$sh(x) = \frac{e^x - e^{-x}}{2} \qquad (5.7)$$

is called the *Hyperbolic Sine* and the function

$$ch(x)=\frac{e^x+e^{-x}}{2} \tag{5.8}$$

is called the *Hyperbolic Cosine*.

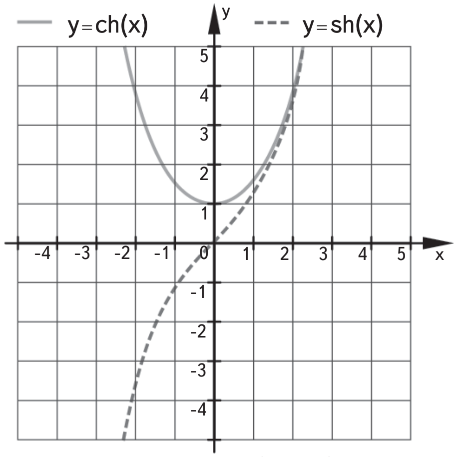


Figure 5.2. Graphs of the hyperbolic sine and cosine

There is a similarity between trigonometric and hyperbolic functions. Like the trigonometric sine and cosine that are the coordinates of the points on a circle, the hyperbolic sine and cosine are the coordinates of the points on a hyperbola. The hyperbolic functions are defined on all numerical axes. The hyperbolic sine is an odd function that is increasing on all numerical axes. The hyperbolic cosine is an even function that is decreasing on the interval (-∞; 0) and increasing on the interval (0; +∞). The point (0; 1) is the minimum of this function (see Fig. 5.2).

By analogy with the trigonometric functions we can define hyperbolic tangent and cotangent:

$$th(x)=\frac{sh(x)}{ch(x)},\quad cth(x)=\frac{ch(x)}{sh(x)}.$$

Analytical definitions (5.7) and (5.8) of the hyperbolic functions can be used for obtaining some important identities of the hyperbolic trigonometry. It is well known that there are *Trigonometric Identities*, for example, the *Pythagorean Theorem for the Trigonometric Functions*:

$$\cos^2 a+\sin^2 a=1. \tag{5.9}$$

We can prove similar identities for the hyperbolic functions:

$$\begin{aligned}ch^2x-sh^2x&=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2\\&=\frac{e^{2x}+2+e^{-2x}}{4}-\frac{e^{2x}-2+e^{-2x}}{4}=1\end{aligned} \tag{5.10}$$

$$\begin{aligned}ch^2x+sh^2x&=\left(\frac{e^x+e^{-x}}{2}\right)^2+\left(\frac{e^x-e^{-x}}{2}\right)^2\\&=\frac{e^{2x}+2+e^{-2x}}{4}+\frac{e^{2x}-2+e^{-2x}}{4}=\frac{e^{2x}+e^{-2x}}{2}=sh2x.\end{aligned} \tag{5.11}$$

We can prove the following properties for derivatives and integrals:

$$(shx)' = chx;\ (chx)' = shx;\ (thx)' = 1/ch^2 x$$

$$\int sh(x)dx = ch(x) + C$$

$$\int ch(x)dx = sh(x) + C$$

$$\int th(x)dx = \ln[ch(x)] + C$$

$$\int \frac{1}{ch^2(x)} dx = th(x) + C$$

$$\int \frac{1}{sh^2(x)} dx = -cth(x) + C$$

5.2.2. *History and Applications of the Hyperbolic Functions*

5.2.2.1. *Lambert and Riccati*

Although *Johann Heinrich Lambert* (1728-1777), a French mathematician, is often credited with introducing hyperbolic functions, it was actually *Vincenzo Riccati* (1707-1775), an Italian mathematician, who did this in the middle of the 18th century. He studied these functions and used them to obtain solutions for *cubic equations*. Riccati found the standard *Addition* formulas, similar to trigonometric identities, for hyperbolic functions as well as their derivatives. He revealed the relationship between the hyperbolic functions and the exponential function. For the first time Riccati used the symbols *sh* and *ch* for the hyperbolic sine and cosine.

5.2.2.2. *Non-Euclidean Geometry*

Among the mathematical works of ancient science, *The Elements* by Euclid is of special importance. This famous work contains the fundamentals of ancient mathematics: elementary geometry, number theory, algebra, the general theory of relations and calculation of areas and volumes. Euclid summarized Greek mathematics and created a stable foundation for further mathematical progress. *The Elements* by Euclid is constructed as a deductive mathematical system: at first the definitions, the postulates and the axioms are given, then the theorems are formulated and their proofs are given. Among the Euclidean axioms, the 5th Euclidean axiom, the *Axiom about Parallels* is the most famous.

During almost 2 millennia many mathematicians tried to deduce this axiom as a theorem from other Euclidean axioms. The problem of the 5-th Euclidean axiom was brilliantly solved for the first time by a professor of Kazan University, the Great Russian mathematician *Nikolay Lobachevsky* (1792-1856) who developed a non-Euclidean geometry in 1826. Lobachevsky's geometry is also named *Hyperbolic Geometry* because it is based on the hyperbolic functions. Independently Lobachevsky, the young Hungarian mathematician *Janosh Bolyai* had also developed a similar non-Euclidean geometry. The first published work on non-Euclidean geometry, Lobachevsky's article *About the Geometry Beginnings*, was published in 1829 in *The Kazan Bulletin*. Three years later Janosh Bolyai's work on non-Euclidean geometry, called the *Appendix*, was published in Latin. After Gauss' death it was clear that he also had developed a geometry similar to those of Lobachevsky and Bolyai.

Lobachevsky had formulated a new axiom about parallel lines that, in contrast to the Euclidean 5th axiom, was formulated as follows:

"Through any point outside of a given straight line, it is possible to draw at least two different 'parallel' straight lines that are mutually disjoint with the given straight line."

Having replaced the 5th Euclidean axiom with this new axiom, Lobachevsky developed a non-Euclidean geometry that is as logically correct as Euclidean geometry.

We will not stop to detail all features of Lobachevsky's geometry, as its study goes far outside the limits of "Elementary mathematics," studied in high school. It is important to emphasize that, by studying trigonometric relations of his geometry, Lobachevsky used the above hyperbolic functions, that is, Lobachevsky's geometry is an important confirmation of the fundamental character of hyperbolic functions in the development of new geometric models of the Universe.

5.2.2.3. *Minkowski's Four-dimensional World*

The 20th century became a new stage in the evolution of spatial ideas in all scientific spheres. Physics stimulated the global process of change regarding geometric spatial ideas. Until the creation of non-Euclidean geometry, it was not necessary to prove the mutual relation of Newton's mechanics and Euclidean geometry. This fact was considered to be obvious. However, at the end of the 19th century and the beginning of the 20th century many new physical facts and observations, which did not fit the classical spatial representations, were gathered together. Maxwell's research on electro-dynamics, Michelson's experiments on the measurement of the speed of light and other scientific facts resulted in the problem of the correspondence of Euclidean geometry to the real physical world. Einstein's theory of relativity resulted in the explanation of new physical facts, which concern relations between physics and classical geometry. The conclusion about the non-Euclidean character of real spatial geometry became the main outcome of Einstein's special theory of relativity.

The need for attracting new geometric ideas arose within 20th century physics. In particular, Einstein's special relativity theory (1905) was a cause for this research. In this theory for the first time in the history of physics, a mutual relation between space and time became the objects of physical research. Relativity theory proves that the metrics of real space and time are not absolute: they depend on dynamical conditions, in which the spatial and temporal measurements are carried out. Relativity theory for the first time showed that space and time are continuum and that their properties are integrally interconnected.

In 1908, that is, in three years after the promulgation of Einstein's special theory of relativity, the German mathematician Herman Minkowski presented the geometric substantiation of the special relativity theory [37]. Minkowski's idea is characterized by two essential peculiarities. In the first place, the geometric spatial-temporal model that was offered by him was four-dimensional: in this model the spatial and temporal coordinates are connected in the common coordinate system. The position of the material point in Minkowski's model was determined by the point $M(x, y, z, t)$ called the *World Point*. Secondly, the geometric connection between the spatial and temporal coordinates in Minkowski's system had a non-Euclidean character, that is, the given model reflected certain special properties of real space-time that could not be simulated in the framework of the "traditional" Euclidean geometry.

Geometrically, a connection between the spatial (x) and temporal (t) coordinates in Minkowski's model is given by the *Hyperbolic Turn*, the movement that is similar to the traditional turn of the Cartesian system in Euclidean geometry. Here, the coordinates x and t of any point can be transformed according to the formulas:

$$x' = xch\psi + tsh\psi,\ \ t' = xsh\psi + tch\psi,$$

where ψ is the angle of the hyperbolic turn and ch and sh are the hyperbolic sine and cosine, respectively.

The original geometric interpretation of the well-known *Lorentz transformations* follows from the given model:

$$x' = (x + vt)\Big/\sqrt{1-(v/c)^2}\,;$$

$$t' = \left[t + \left(vx/c^2\right)\right]\Big/\sqrt{1-(v/c)^2}\,,$$

where v is the speed of the system, c is the speed of light.

Note that in Minkowski's geometry *Lorentz transformations* are relations of the hyperbolic trigonometry expressed in physics terms. Minkowski's geometry uncovers the hyperbolic nature of all mathematical formulas of relativity theory.

5.2.2.4. *Vernadsky's Ideas*

Up until the time of relativity theory the mathematical apparatus for biological research practically did not exist. Nevertheless the response of biologists to discoveries in geometric physics was practically instantaneous.

Russian scientist Vladimir Vernadsky was one of the first scientists to give serious attention to the geometric problems of biology [37]. Vernadsky's peculiar approach was a holistic vision of the "space-time" problem. Vernadsky expanded on the concepts of the "space-time" problem. He took into consideration the biological specificity of its study and discussed the realization in Nature of general geometric (spatial-temporal) laws. Vernadsky gave attention to the problem of biological symmetry and considered this to be the key biological problem. He came to the conclusion that the explanation of biological symmetry is connected to the non-Euclidean character of the spatial geometry of living substance. He also assumed that the geometry of biological "space-time" differs from the geometry of "space-time" relativity theory, known as the "Four-dimensional Minkowski world" [37].

According to his opinion, the issue of how non-Euclidean geometry is embodied in living nature is the key problem of biological research. Vernadsky's ideas became a leitmotiv for scientific research in subsequent periods of development in biology resulting in the creation of a new direction, mathematical biology.

5.3. Hyperbolic Fibonacci and Lucas Functions (Stakhov-Tkachenko's Definition)

5.3.1. *A Definition of Hyperbolic Fibonacci and Lucas Functions*

Fibonacci and Lucas functions were introduced, for the first time, in 1993 by Alexey Stakhov and Ivan Tkachenko [98]. If we compare the hyperbolic functions (5.7) and (5.8) with Binet formulas (2.66) and (2.67), we can see that Binet formulas are similar to the hyperbolic functions in their mathematical structure.

For a strong definition of the hyperbolic Fibonacci and Lucas functions we can rewrite the formulas (2.68) and (2.67) respectively, as follows:

$$F_{2k}=\left(\tau^{2k}-\tau^{-2k}\right)/\sqrt{5} \tag{5.12}$$

$$F_{2k+1}=\left(\tau^{2k+1}+\tau^{-(2k+1)}\right)/\sqrt{5} \tag{5.13}$$

$$L_{2k+1}=\tau^{2k+1}-\tau^{-(2k+1)} \tag{5.14}$$

$$L_{2k}=\tau^{2k}+\tau^{-2k}, \tag{5.15}$$

where the discrete variable k takes its values from the set $\{0,\pm1,\pm2,\pm3,...\}$.

Comparing the formulas (5.12)-(5.15) to the hyperbolic functions (5.7) and (5.8), we can see that formulas (5.12) and (5.14) correspond in their structure to the hyperbolic sine (5.7) and the formulas (5.13) and (5.15) correspond to the hyperbolic cosine (5.8). This simple analogy underlies the *Hyperbolic Fibonacci and Lucas Functions* [98].

By substituting the discrete variable k in the formulas (5.12)-(5.15) for the continuous variable x that takes its values from the set of real numbers, we obtain the following four new elementary functions:

Hyperbolic Fibonacci sine

$$sF(x)=\left(\tau^{2x}-\tau^{-2x}\right)/\sqrt{5}\ . \tag{5.16}$$

Hyperbolic Fibonacci cosine

$$cF(x)=\left(\tau^{2x+1}+\tau^{-(2x+1)}\right)/\sqrt{5}\,. \tag{5.17}$$

Hyperbolic Lucas sine

$$sL(x)=\tau^{2k+1}-\tau^{-(2k+1)}\,. \tag{5.18}$$

Hyperbolic Lucas cosine

$$cL(x)=\tau^{2k}+\tau^{-2k}. \tag{5.19}$$

Note that for the discrete values $x=k$ the hyperbolic Fibonacci and Lucas functions (5.16)-(5.19) coincide with the Fibonacci and Lucas numbers, that is,

$$sF\left(k\right)=F_{2k};\,cF\left(k\right)=F_{2k+1};\,sL\left(k\right)=L_{2k+1};\,cL\left(k\right)=L_{2k}. \tag{5.20}$$

This means that the extended Fibonacci and Lucas numbers coincide with the hyperbolic Fibonacci and Lucas functions for the discrete points of the continuous variable $x=k$ ($k=0,\pm1,\pm2,\pm3,...$), that is, the extended Fibonacci and Lucas numbers are "discrete analogs" of the hyperbolic Fibonacci and Lucas sine and cosine. This property (5.20) is a rather characteristic peculiarity of the above hyperbolic Fibonacci and Lucas functions (5.16)-(5.19) in comparison with the classical hyperbolic functions (5.7), (5.8) that, however, do not have "discrete analogs" similar to (5.20).

5.3.2. *The Hyperbolic Fibonacci and Lucas Tangent and Cotangent*

Let us introduce the definitions of the hyperbolic Fibonacci and Lucas tangents and cotangents by analogy to the classical hyperbolic tangent and cotangent.

Hyperbolic Fibonacci tangent:

$$tFx = \frac{sFx}{cFx}. \tag{5.21}$$

Taking into consideration (5.16) and (5.17), the hyperbolic Fibonacci tangent can be represented as follows:

$$tFx = \frac{\tau^{2x} - \tau^{-2x}}{\tau^{2x+1} + \tau^{-(2x+1)}} = \frac{(\tau^{4x} - 1)\tau^{2x+1}}{\tau^{2x}(\tau^{4x+2} + 1)} = \frac{\tau(\tau^{4x} - 1)}{\tau^{4x+1} + 1}. \tag{5.21}$$

Hyperbolic Fibonacci cotangent:

$$ctFx = \frac{cFx}{sFx}.$$

Taking into consideration (5.16) and (5.17), the expression for the hyperbolic Fibonacci cotangent can be represented as follows:

$$ctFx = \frac{\tau^{4x+2} + 1}{\tau(\tau^{4x} - 1)}. \tag{5.22}$$

Hyperbolic Lucas tangent:

$$tLx = \frac{sLx}{cLx} = \frac{\tau^{4x+2} - 1}{\tau(\tau^{4x} + 1)}. \tag{5.23}$$

Hyperbolic Lucas cotangent:

$$ctLx = \frac{cLx}{sLx} = \frac{\tau(\tau^{4x} + 1)}{\tau^{4x+2} - 1}. \tag{5.24}$$

Thus, the main result that follows from our study is the introduction of two new kinds of elementary functions, the *Hyperbolic Fibonacci Functions* and the *Hyperbolic Lucas Functions*. By their form, these functions are similar to the classical hyperbolic functions, but differ from them by one important feature. In contrast to the classical hyperbolic functions, the hyperbolic Fibonacci and Lucas functions have numerical analogs - the classical Fibonacci and Lucas numbers. In particular, the Fibonacci numbers with the even indices are discrete analogs for the hyperbolic Fibonacci sine (5.16), the Fibonacci numbers with the odd indices are discrete analogs for the hyperbolic Fibonacci cosine (5.17), the Lucas numbers with the odd indices are discrete analogs for the hyperbolic Lucas sine (5.18), and the Lucas numbers with the even indices are discrete analogs for the hyperbolic Fibonacci cosine (5.19).

5.3.3. *Some Properties of the Hyperbolic Fibonacci and Lucas Functions*

5.3.3.1. *Hyperbolic Fibonacci Sine*

The function $y=sF(x)$ is odd function because

$$sF(-x)=\left(\tau^{-2x}-\tau^{-(-2x)}\right)/\sqrt{5}=-\left(\tau^{2x}-\tau^{-2x}\right)/\sqrt{5}=-sF(x).$$

Its value at the point $x=0$ is equal to

$$sF(0)=\left(\tau^{0}-\tau^{0}\right)/\sqrt{5}=0.$$

5.3.3.2. *Hyperbolic Fibonacci Cosine*

The function $y=sF(x)$ is given by (5.17). Let us find the intersection points of the function graph with the coordinate axes. As $cF(x)=\left(\tau^{2x+1}+\tau^{-(2x+1)}\right)/\sqrt{5}>0$ for any $x\in\{-\infty,+\infty\}$ this means that the graph of the function does not intersect the x-axes at any point.

For $x=0$ we have

$$cF(0)=\left(\tau^{1}+\tau^{-1}\right)/\sqrt{5}=1,$$

that is, the graph of the function $y=sF(x)$ intersects the y-axis at the point $y=1$.

Now, let us examine the function $y=cF(x)$ on the extremum; for this purpose we find its first derivative:

$$y'=(cFx)'=\frac{2\ln\tau}{\sqrt{5}}\left(\tau^{2x+1}-\tau^{-(2x+1)}\right)=\frac{2}{\sqrt{5}}\ln\tau\ sLx.$$

Note that $y'=0$ for the condition:

$$\tau^{2x+1}-\tau^{-(2x+1)}=0;\ \tau^{4x+2}=1;\ 4x+2=0;\ x=-1/2.$$

Thus, we have the extremum at the point with abscissa $x=-1/2$.

It is easy to prove that the function $y=cF(x)$ is symmetric about the line $x=-1/2$ because

$$cF\left[x-(1/2)\right]=\left(\tau^{2x}+\tau^{-2x}\right)/\sqrt{5}=cF\left[-x-(1/2)\right].$$

It is important to note that the function $y=cF(x)$ is not symmetric about the y-axis.

5.3.3.3. *Hyperbolic Lucas Sine*

The hyperbolic Lucas sine $y=sL(x)$ is given by (5.18). Let us calculate the value of the function at the point $x=0$:

$$sL(0)=\tau-\tau^{-1}=\left(1+\sqrt{5}\right)/2+\left(1-\sqrt{5}\right)/2=1.$$

Now, let us calculate the value of x where the function $y=sL(x)$ is equal to 0:
$\tau^{2x+1}-\tau^{-(2x+1)}=0;\ \tau^{4x+2}=1;\ 4x+2=0;\ x=-1/2.$

As $sL(-1/2)=0$, this means that the graph of the function $y=sL(x)$ intersects the x-axis at the point $x=-1/2$.

5.3.3.4. *Hyperbolic Lucas Cosine*

The function $y=cL(x)$ is given by (5.19). By using the calculus methods [154], we can find the following properties of the function $y = cL(x)$. The function is an even function because

$cL(-x)=\tau^{-2x}+\tau^{-(-2x)}=\tau^{-2x}+\tau^{2x}=cLx.$

Let us find the intersection points of the function graph with the coordinate axes. As $cL(x)=\tau^{2x}+\tau^{-2x}>0$ for any $x\in\{-\infty,+\infty\}$, this means that the graph of the function does not intersect the x-axes at any point.

For the case $x=0$ we have: $cL(0)=\tau^{0}+\tau^{-0}=2$; therefore, the graph of the function intersects the y-axis at the point $y=2$.

Let us examine the function $y=cL(x)$ on the extremum; with this purpose we find the first derivative of the function:

$y'=\left[cL(x)\right]'=2\ln\tau\left(\tau^{2x}-\tau^{-2x}\right).$

The first derivative y' turns out to be 0 only under the condition that $\tau^{2x}-\tau^{-2x}=0$, this is possible only for the case $x=0$. This means that at the point $x=0$ the function has an extremum. The extreme value of the function $y=cL(x)$ at the point $x=0$ is equal to $cL(0)=2$.

By analogy, we can examine other hyperbolic Fibonacci and Lucas functions, in particular, tangent and cotangent, secant and cosecant.

5.4. Integration and Differentiation of the Hyperbolic Fibonacci and Lucas Functions and their Main Identities

5.4.1. *Integration of the Function $y=sF(x)$*

Represent the function $y=sF(x)$ given by (5.16) in the form:

$$sF(x)=\left(\tau^{2x}-\tau^{-2x}\right)/\sqrt{5}=\left(e^{2x\ln\tau}-e^{-2x\ln\tau}\right)/\sqrt{5}. \qquad (5.25)$$

Then, by using the representation (5.25), we obtain the following expression for the integral of the function $y=sF(x)$:

$$\int sF(x)\,dx = \int\left[\left(e^{2x\ln\tau} - e^{-2x\ln\tau}\right)/\sqrt{5}\right]dx$$
$$= \left(1/\sqrt{5}\right)\left(\int e^{2x\ln\tau}dx - \int e^{-2x\ln\tau}dx\right)$$
$$= \left(1/\sqrt{5}\right)\left[(1/2\ln\tau)e^{2x\ln\tau} + (1/2\ln\tau)e^{-2x\ln\tau}\right]$$
$$= \left(1/2\sqrt{5}\ln\tau\right)\left(e^{2x\ln\tau} + e^{-2x\ln\tau}\right)$$
$$= \left(1/2\sqrt{5}\ln\tau\right)\left(\tau^{2x} + \tau^{-2x}\right)$$
$$= \left(1/2\sqrt{5}\ln\tau\right)cLx = (1/2\ln\tau)\,cF\left[x-(1/2)\right].$$

5.4.2. *Integration of the Function* $y=cF(x)$

By using the above approach, we obtain the following expression for the integral of the function $y=cF(x)$ given by (5.17):

$$\int cF(x)\,dx = \int\left[\left(e^{(2x+1)\ln\tau} + e^{-(2x+1)\ln\tau}\right)/\sqrt{5}\right]dx$$
$$= \left(1/\sqrt{5}\right)\left(\int e^{(2x+1)\ln\tau}dx + \int e^{-(2x+1)\ln\tau}dx\right)$$
$$= \left(1/\sqrt{5}\right)\left[(1/2\ln\tau)e^{(2x+1)\ln\tau} - (1/2\ln\tau)e^{-(2x+1)\ln\tau}\right]$$
$$= \left(1/2\sqrt{5}\ln\tau\right)\left(e^{(2x+1)\ln\tau} - e^{-(2x+1)\ln\tau}\right)$$
$$= \left(1/2\sqrt{5}\ln\tau\right)\left(\tau^{2x+1} - \tau^{-(2x+1)}\right)$$
$$= \left(1/2\sqrt{5}\ln\tau\right)sLx = (1/2\ln\tau)\,sF\left[x+(1/2)\right].$$

5.4.3. *Integration of the Function* $y=sL(x)$

$$\int sL(x)\,dx = \int\left(e^{(2x+1)\ln\tau} - e^{-(2x+1)\ln\tau}\right)dx$$
$$= (1/2\ln\tau)e^{(2x+1)\ln\tau} + (1/2\ln\tau)e^{-(2x+1)\ln\tau}$$
$$= (1/2\ln\tau)\left[e^{(2x+1)\ln\tau} + e^{-(2x+1)\ln\tau}\right]$$
$$= (1/2\ln\tau)\left[\tau^{2x+1} + \tau^{-(2x+1)}\right]$$
$$= \left(\sqrt{5}/2\ln\tau\right)cFx = (1/2\ln\tau)\,cL\left[x+(1/2)\right].$$

5.4.4. *Integration of the Function* $y=cL(x)$

$$\int cL(x)\,dx = \int\left(e^{2x\ln\tau} + e^{-2x\ln\tau}\right)dx$$
$$= (1/2\ln\tau)e^{2x\ln\tau} - (1/2\ln\tau)e^{-2x\ln\tau}$$
$$= (1/2\ln\tau)\left[e^{2x\ln\tau} - e^{-2x\ln\tau}\right]$$
$$= (1/2\ln\tau)\left[\tau^{2x} - \tau^{-2x}\right]$$
$$= \left(\sqrt{5}/2\ln\tau\right)sFx = (1/2\ln\tau)\,sL\left[x-(1/2)\right].$$

The results of integration are presented in Table 5.1.

Table 5.1. Formulas for the integrals of the hyperbolic Fibonacci and Lucas functions

y	$sF(x)$	$cF(x)$
$\int ydx$	$\frac{1}{2\sqrt{5}\ln\tau}cL(x)=\frac{1}{2\ln\tau}cF\left(x-\frac{1}{2}\right)$	$\frac{1}{2\sqrt{5}\ln\tau}sL(x)=\frac{1}{2\ln\tau}sF\left(x+\frac{1}{2}\right)$
y	$sL(x)$	$cL(x)$
$\int ydx$	$\frac{\sqrt{5}}{2\ln\tau}cF(x)=\frac{1}{2\ln\tau}cL\left(x+\frac{1}{2}\right)$	$\frac{\sqrt{5}}{2\ln\tau}sF(x)=\frac{1}{2\ln\tau}sL\left(x-\frac{1}{2}\right)$

5.4.5. *Differentiation of the Function* $y=sF(x)$

By using the expression (5.25), we obtain the following expressions for the derivatives of the function $y=sF(x)$:

$$y'=\left(2\ln\tau\ e^{2x\ln\tau}+2\ln\tau\ e^{-2x\ln\tau}\right)/\sqrt{5}$$
$$=\left(2\ln\tau/\sqrt{5}\right)\left(\tau^{2x}+\tau^{-2x}\right)=\left(2\ln\tau/\sqrt{5}\right)cL(x);$$

$$y''=\left(2\ln\tau/\sqrt{5}\right)\left(2\ln\tau\, e^{2x\ln\tau}-2\ln\tau\, e^{-2x\ln\tau}\right)$$
$$=\left[(2\ln\tau)^2/\sqrt{5}\right]\left(\tau^{2x}-\tau^{-2x}\right)=(2\ln\tau)^2\, sF(x);$$

$$y'''=\left[(2\ln\tau)^2/\sqrt{5}\right]\left(2\ln\tau\, e^{2x\ln\tau}+2\ln\tau\, e^{-2x\ln\tau}\right)$$
$$=\left[(2\ln\tau)^3/\sqrt{5}\right]\left(\tau^{2x}+\tau^{-2x}\right)$$
$$=\left[(2\ln\tau)^3/\sqrt{5}\right]cL(x)=(2\ln\tau)^2\, y';$$
$$\ldots$$
$$y^{(2k+1)}=\frac{(2\ln\tau)^{2k+1}}{\sqrt{5}}cL(x)=(2\ln\tau)^{2k}\, y';$$
$$y^{(2k)}=(2\ln\tau)^{2k}\, sF(x).$$

5.4.6. *Differentiation of the Function* $y=cF(x)$

By representing the function $y=cF(x)$ in the form

$$y=cF(x)=\left[\tau^{(2x+1)}+\tau^{-(2x+1)}\right]/\sqrt{5}$$
$$=\left[e^{(2x+1)\ln\tau}+e^{-(2x+1)\ln\tau}\right]/\sqrt{5} \tag{5.26}$$

and repeatedly carrying out the differentiation of (5.26), we obtain the following expressions for the derivatives of the function $y=cF(x)$:

$$y' = \left(2\ln\tau/\sqrt{5}\right)sL(x);$$
$$y'' = (2\ln\tau)^2\, cF(x);$$
$$y''' = \left[(2\ln\tau)^3/\sqrt{5}\right]sL(x) = (2\ln\tau)^2\, y';$$
$$y^{(IV)} = (2\ln\tau)^4\, cF(x);$$
$$\ldots$$
$$y^{(2k+1)} = \left[(2\ln\tau)^{2k+1}/\sqrt{5}\right]sL(x) = (2\ln\tau)^{2k}\, y';$$
$$y^{(2k)} = (2\ln\tau)^{2k}\, cF(x).$$

5.4.7. *Differentiation of the Function* $y=sLx$

By representing the function $y=sLx$ given by (5.18) in the form

$$y = sL(x) = \tau^{2x+1} - \tau^{-(2x+1)} = e^{(2x+1)\ln\tau} - e^{-(2x+1)\ln\tau}, \tag{5.27}$$

and repeatedly carrying out the differentiation of (5.27), we obtain the following expressions for the derivatives of the function $y=sLx$:

$$y' = 2\sqrt{5}\ln\tau\, cF(x);$$
$$y'' = (2\ln\tau)^2\, sL(x);$$
$$y''' = (2\ln\tau)^3\sqrt{5}cF(x) = (2\ln\tau)^2\, y';$$
$$y^{(IV)} = (2\ln\tau)^4\, sL(x);$$
$$y^{(V)} = (2\ln\tau)^5\sqrt{5}cF(x) = (2\ln\tau)^4\, y';$$
$$\ldots$$
$$y^{(2k+1)} = \sqrt{5}(2\ln\tau)^{2k+1}\, cF(x) = (2\ln\tau)^{2k}\, y';$$
$$y^{(2k)} = (2\ln\tau)^{2k}\, sL(x).$$

5.4.8. *Differentiation of the Function* $y=cLx$

By representing the function $y=cLx$ in the form

$$y = cL(x) = \tau^{2x} + \tau^{-2x} = e^{2x\ln\tau} + e^{-2x\ln\tau} \tag{5.28}$$

and repeatedly carrying out the differentiation of (5.28), we obtain the following expressions for the derivatives of the function $y=cLx$:

$$y' = 2\sqrt{5}\ln\tau\; sF(x);$$
$$y'' = (2\ln\tau)^2\, cL(x);$$

$$y = \sqrt{5}\left(2\ln\tau\right)^3 sF\left(x\right) = \left(2\ln\tau\right)^2 y';$$
$$y^{(IV)} = \left(2\ln\tau\right)^4 cL\left(x\right);$$
$$y^{(V)} = \sqrt{5}\left(2\ln\tau\right)^5 sF\left(x\right) = \left(2\ln\tau\right)^4 y';$$
...
$$y^{(2k+1)} = \sqrt{5}\left(2\ln\tau\right)^{2k+1} sF\left(x\right) = \left(2\ln\tau\right)^{2k} y';$$
$$y^{(2k)} = \left(2\ln\tau\right)^{2k} cL\left(x\right).$$

The results of differentiation are given in Table 5.2.

Table 5.2. Formulas for the derivatives of hyperbolic Fibonacci and Lucas functions

y	$sF(x)$	$cF(x)$	$sL(x)$	$cL(x)$
y'	$\frac{2\ln\tau}{\sqrt{5}} cL(x)$	$\frac{2\ln\tau}{\sqrt{5}} sL(x)$	$2\sqrt{5}\ln\tau cF(x)$	$2\sqrt{5}\ln\tau sF(x)$
y''	$(2\ln\tau)^2 sF(x)$	$(2\ln\tau)^2 cF(x)$	$(2\ln\tau)^2 sL(x)$	$(2\ln\tau)^2 cL(x)$
y'''	$(2\ln\tau)^2 y'$	$(2\ln\tau)^2 y'$	$(2\ln\tau)^2 y'$	$(2\ln\tau)^2 y'$
$y^{(IV)}$	$(2\ln\tau)^4 sF(x)$	$(2\ln\tau)^4 cF(x)$	$(2\ln\tau)^4 sL(x)$	$(2\ln\tau)^4 cL(x)$
$y^{(V)}$	$(2\ln\tau)^4 y'$	$(2\ln\tau)^4 y'$	$(2\ln\tau)^4 y'$	$(2\ln\tau)^4 y'$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$y^{(2k)}$	$(2\ln\tau)^{2k} sF(x)$	$(2\ln\tau)^{2k} cF(x)$	$(2\ln\tau)^{2k} sL(x)$	$(2\ln\tau)^{2k} cL(x)$
$y^{(2k+1)}$	$(2\ln\tau)^{2k} y'$	$(2\ln\tau)^{2k} y'$	$(2\ln\tau)^{2k} y'$	$(2\ln\tau)^{2k} y'$

5.4.9. *The Main Identities for the Hyperbolic Fibonacci and Lucas Functions*

The hyperbolic Fibonacci and Lucas functions (5.16)-(5.19) are a generalization of the classical Fibonacci and Lucas numbers given by Binet formulas (5.12)-(5.15). These hyperbolic Fibonacci and Lucas functions (5.16)-(5.19) are connected with the classical Fibonacci and Lucas numbers through simple correlations (5.20). Using a geometric representation of the hyperbolic Fibonacci and Lucas functions, we result in a very simple geometric interpretation of the correlations (5.20). The classical Fibonacci and Lucas numbers are as if inscribed into the graphs of the hyperbolic Fibonacci and Lucas functions at the discrete points of the variable $x=0,\pm1,\pm2,\pm3,...$. Thus, the hyperbolic Fibonacci and Lucas functions are an extension of the classical Fibonacci and Lucas numbers for the continuous domain. From this it appears that the hyperbolic Fibonacci and Lucas functions possess *recursive* properties similar to those of the classical Fibonacci and Lucas numbers.

On the other hand, functions (5.16)-(5.19) are similar to the classical hyperbolic functions (5.7) and (5.8). We can assume by analogy that the hyperbolic Fibonacci and Lucas functions possess *hyperbolic* properties similar to the properties of classical hyperbolic functions.

Let us examine these analogies in greater detail, starting with the simplest recursive relation that gives classical Fibonacci numbers:

$$F_{n+1}= F_n + F_{n-1}. \tag{5.29}$$

If we accept n=2k or n=2k+1, where k=0,±1,±2,±3,..., we can rewrite the recursive relation (5.29) in two ways:

$$F_{2k+1}= F_{2k} + F_{2k-1} \tag{5.30}$$

$$F_{2k+2}=F_{2k+1}+F_{2k}. \tag{5.31}$$

Using the correlations (5.20), we can rewrite the recursive relations (5.30) and (5.31) in terms of hyperbolic Fibonacci and Lucas functions as follows:

$$cF(k)=sF(k)+cF(k-1), \tag{5.32}$$

$$sF(k+1)=cF(k)+sF(k), \tag{5.33}$$

where k is a discrete variable that takes its values from the set {0,±1,±2,±3,...}.

Substituting the discrete variable k with the continuous variable x in the recursive relations (5.32) and (5.33), we obtain two important identities for the hyperbolic Fibonacci and Lucas functions:

$$cF(x)=sF(x)+cF(x-1) \tag{5.34}$$

$$sF(x+1)=cF(x)+sF(x). \tag{5.35}$$

Let us represent the recursive relation $L_{n+1}= L_n + L_{n-1}$ for the classical Lucas numbers in the form of two recursive relations

$$L_{2k+1}=L_{2k}+L_{2k-1} \tag{5.36}$$

$$L_{2k+2}=L_{2k+1}+L_{2k}, \tag{5.37}$$

where k is a discrete variable that takes its values from the set {0,±1,±2,±3,...}. Then we represent (5.36) and (5.37) in terms of the hyperbolic Fibonacci and Lucas functions, rewriting the recursive relations (5.36) and (5.37) as follows:

$$sL(k)=cL(k)+sL(k-1) \tag{5.38}$$

$$cL(k+1)=sL(k)+cL(k). \tag{5.39}$$

Substituting the continuous variable x for the discrete variable k in the recursive relations (5.38) and (5.39), we obtain two important identities:

$$sL(x)=cL(x)+sL(x-1) \tag{5.40}$$

$$cL(x+1)=sL(x)+cL(x). \tag{5.41}$$

Note that there is another way to obtain the identities (5.34), (5.35), (5.40), and (5.41). For example, let us prove the identity (5.35) by using the definitions (5.16) and (5.17):

$$sF(x)+cF(x)=\left(\tau^{2x}-\tau^{-2x}\right)\Big/\sqrt{5}+\left[\tau^{2x+1}+\tau^{-(2x+1)}\right]\Big/\sqrt{5}$$

$$=\left[\tau^{2x}\left(\tau+1\right)+\tau^{-2x}\left(\tau^{-1}-1\right)\right]\Big/\sqrt{5}=\left[\tau^{2(x+1)}-\tau^{-2(x+1)}\right]\Big/\sqrt{5}=sF\left(x+1\right).$$

By analogy we can prove the other identities (5.34), (5.40) and (5.41).

The above examination has very unusual consequences for the Fibonacci number theory [13, 16, 28]. We can assume the following hypothesis: all well-known identities for the Fibonacci and Lucas numbers have the "hyperbolic interpretations" in the form of corresponding identities for the hyperbolic Fibonacci and Lucas functions. For example, let us give the "hyperbolic interpretation" of the famous Cassini formula that connects three adjacent Fibonacci numbers:

$$F_n^2-F_{n-1}F_{n+1}=\left(-1\right)^{n+1}, \tag{5.42}$$

where $n=0,\pm1,\pm2,\pm3,...$ is a discrete variable.

We can represent the Cassini formula (5.42) in the form of two formulas for the even ($n=2k$) and odd ($n=2k+1$) values of the discrete variable n as follows:

$$F_{2k}^2-F_{2k-1}F_{2k+1}=-1 \tag{5.43}$$

$$F_{2k+1}^2-F_{2k}F_{2k+2}=1, \tag{5.44}$$

where k is a discrete variable that takes its values from the set $\{0,\pm1,\pm2,\pm3,...\}$.

By using (5.20), we can represent the identities (5.43) and (5.44) as follows:

$$sF^2\left(k\right)-cF\left(k-1\right)cF\left(k\right)=-1 \tag{5.45}$$

$$cF^2\left(k\right)-sF\left(k\right)sF\left(k+1\right)=1. \tag{5.46}$$

Substituting the continuous variable x for the discrete variable k in the formulas (5.45) and (5.46), we obtain two important identities for the hyperbolic Fibonacci and Lucas functions:

$$sF^2\left(x\right)-cF\left(x-1\right)cF\left(x\right)=-1 \tag{5.47}$$

$$cF^2\left(x\right)-sF\left(x\right)sF\left(x+1\right)=1. \tag{5.48}$$

We can prove the identities (5.47) and (5.48) by using the definitions (5.16) and (5.17). For example, let us prove the identity (5.48):

$$\left(\left(\tau^{2x+1}+\tau^{-(2x+1)}\right)/\sqrt{5}\right)^2-\left(\left(\tau^{2x}-\tau^{-2x}\right)/\sqrt{5}\right)\times\left(\left(\tau^{2x+2}-\tau^{-(2x+2)}\right)/\sqrt{5}\right)$$

$$=\left(\left(\tau^{4x+2}+2\tau^{2x+1}\tau^{-(2x+1)}+\tau^{-(4x+2)}\right)/5\right)-\left(\left(\tau^{4x+2}-\tau^2-\tau^{-2}+\tau^{-(4x+2)}\right)/5\right)$$

$$=\left(2+\tau^2+\tau^{-2}\right)/5. \tag{5.49}$$

In Chapter 2 we derived the Binet formulas for Lucas numbers given by (2.67). For the even $n=2k$ the formula (2.67) can be written as follows:

$$L_{2k}=\tau^{2k}+\tau^{-2k}. \tag{5.50}$$

For $k=1$ we have:

$$L_2=\tau^2+\tau^{-2}=3. \tag{5.51}$$

Substituting (5.51) into (5.49), we obtain the identity (5.48).

It is important to note that the hyperbolic Fibonacci and Lucas functions are a generalization and extension of the Fibonacci and Lucas numbers for the continuous domain. This means that the classical Fibonacci number theory [13, 16, 28] is reducible to a more general theory - the theory of the hyperbolic Fibonacci and Lucas functions. All identities for the classical Fibonacci and Lucas numbers have their continuous analogs in the form of the corresponding identities for the hyperbolic Fibonacci and Lucas functions, and conversely.

As practical examples for students we can consider the following "recursive" identities for the hyperbolic Fibonacci and Lucas functions:

$$sF(x)cF(x)=\frac{1}{5}\left[sL(2x)-1\right]$$

$$sF(x)sL(x)=\sqrt{5}sF(x)sF\left(x+\frac{1}{2}\right)$$

$$cF(x)cL(x)=\sqrt{5}\,cF(x)cF\left(x-\frac{1}{2}\right)$$

$$sF(x)=\frac{1}{\sqrt{5}}sL\left(x-\frac{1}{2}\right)$$

$$cF\left(x-\frac{1}{2}\right)=\frac{1}{\sqrt{5}}cL(x)$$

$$cF(x)+cF(y)=\sqrt{5}\left(cL^2\left(\frac{2x+1}{4}\right)+sL^2\left(\frac{2y-1}{4}\right)\right)$$

$$sF(3x)=sF\left(\frac{x}{2}\right)\left[cL(x)+1\right]$$

$$cL(3x)=sF\left(\frac{x}{2}\right)\left[cL(x)+1\right]$$

Let us next consider some "hyperbolic" properties of the hyperbolic Fibonacci and Lucas functions. Consider these important properties for the classical hyperbolic functions:

Evenness property

$$sh(-x)=-sh(x);\, ch(-x)=ch(x);\, th(-x)=th(x).$$

Formulas for addition

$$sh(x+y)=sh(x)ch(y)+sh(y)ch(x);$$

$$ch(x+y)=ch(x)ch(y)+sh(y)sh(x).$$

Formulas for the double angle

$$sh(2x)=2ch(x)sh(x);\, ch(2x)=2ch^2(x)-1$$

It is easy to prove that the hyperbolic Fibonacci and Lucas functions possess similar "hyperbolic" properties, for example,

$$sF(2x)=sF(x)cL(x)$$

$$cF(2x+1)=cF(x)sL(x)$$

$$sF(x)sL(x)=sF(2x)-1$$

$$cF(x)cL(x)=cF(2x)+1$$

$$sL(2x)=sL(x)cL(x)+1$$

$$sF(y)sF(x)=\frac{1}{5}\left[sL(x+y)-sL(x-y)\right]$$

$$sL(x)cL(x)=sL(x+y)+sL(x-y)$$

$$sF(x)+sF(y)=sF\left(\frac{x+y}{2}\right)cL\left(\frac{x-y}{2}\right)$$

$$sF(x)-sF(y)=sF\left(\frac{x-y}{2}\right)cL\left(\frac{x+y}{2}\right)$$

$$cF(x)+cF(y)=cF\left(\frac{x+y}{2}\right)cL\left(\frac{x-y}{2}\right)$$

$$cF(x)-cF(y)=sF\left(\frac{x-y}{2}\right)sL\left(\frac{x+y}{2}\right)$$

$$ctF(2x)=\frac{sF^2(x)+cL^2(x)}{2\,sF(x)cL(x)}$$

The above formulas produce only a small portion of the huge number of mathematical identities for the hyperbolic Fibonacci and Lucas functions. Their proof can become a fascinating pastime for students interested in Fibonacci mathematics and its applications.

5.5. Symmetric Hyperbolic Fibonacci and Lucas Functions (Stakhov - Rozin Definition)

The above hyperbolic Fibonacci and Lucas functions given by (5.16)-(5.19) have an essential shortcoming relative to the classical hyperbolic functions. In particular, they don't possess the evenness property of classical hyperbolic functions:

$$sh(-x)=-sh(x);\ ch(-x)=ch(x);\ th(-x)=th(x). \tag{5.52}$$

To overcome this shortcoming, Stakhov and Rozin introduced [106] the so-called *Symmetric Hyperbolic Fibonacci and Lucas Functions*:

<u>Symmetric hyperbolic Fibonacci sine</u>

$$sFs(x)=\frac{\tau^x-\tau^{-x}}{\sqrt{5}} \tag{5.53}$$

<u>Symmetric hyperbolic Fibonacci cosine</u>

$$cFs(x)=\frac{\tau^x+\tau^{-x}}{\sqrt{5}} \tag{5.54}$$

<u>Symmetric hyperbolic Lucas sine</u>

$$sLs(x)=\tau^x-\tau^{-x} \tag{5.55}$$

<u>Symmetric hyperbolic Lucas cosine</u>

$$cLs(x)=\tau^x+\tau^{-x} \tag{5.56}$$

The Fibonacci and Lucas numbers are determined identically by the symmetric hyperbolic Fibonacci and Lucas functions as follows:

$$F_n=\begin{cases} sFs(n), & for \quad n=2k \\ cFs(n), & for \quad n=2k+1 \end{cases}$$
$$L_n=\begin{cases} cLs(n), & for \quad n=2k \\ sLs(n), & for \quad n=2k+1 \end{cases}. \tag{5.57}$$

It is easy to prove that the function (5.53) is an odd function because

$$sFs(-x)=\frac{\tau^{-x}-\tau^x}{\sqrt{5}}=-\frac{\tau^x-\tau^{-x}}{\sqrt{5}}=-sFs(x). \tag{5.58}$$

On the other hand,

$$cFs(-x)=\frac{\tau^{-x}+\tau^x}{\sqrt{5}}=cFs(x), \tag{5.59}$$

that is, the symmetric hyperbolic Fibonacci cosine (5.54) is an even function. By analogy we can prove the following properties of the hyperbolic Lucas functions:

$$sL(-x)=-sL(x) \tag{5.60}$$

$$cL(-x)=cL(x). \tag{5.61}$$

The properties (5.58)-(5.61) show that the symmetric hyperbolic Fibonacci and Lucas functions (5.52)-(5.55) possess the evenness property (5.52).

It is easy to construct the graphs of symmetric hyperbolic Fibonacci and Lucas sines and cosines (Fig. 5.3 and Fig. 5.4). Their graphs have symmetric form and in this respect are similar to the classical hyperbolic functions.

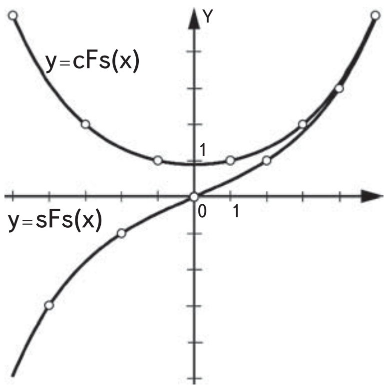


Figure 5.3. A graph of the symmetric hyperbolic Fibonacci sine and cosine

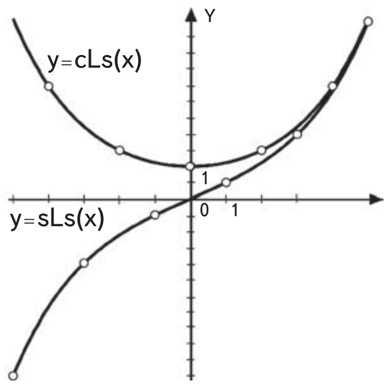


Figure 5.4. A graph of the symmetric hyperbolic Lucas sine and cosine

Here it is necessary to point out that at x=0 the symmetric hyperbolic Fibonacci cosine $cFs(x)$ takes the value $cFs(0)=2/\sqrt{5}$, and the symmetric hyperbolic Lucas cosine $cLs(x)$ takes the value $cLs(0)$=2. It is also important to emphasize that the Fibonacci numbers F_n with even indices (n=0,±2,±4,±6,...) are "inscribed" into the graph of symmetric hyperbolic Fibonacci sine $sFs(x)$ at the discrete points (x=0,±2,±4,±6,...) and the Fibonacci numbers with odd indices (n=±1,±3,±5,...) are "inscribed" into the symmetric hyperbolic Fibonacci cosine $cFs(x)$ at the discrete points (x=±1,±3,±5,...). On the other hand, the Lucas numbers L_n with the even indices are "inscribed" into the graph of symmetric hyperbolic Lucas cosine $cLs(x)$ at the discrete points (n=0,±2,±4,±6,...) and the Lucas numbers with odd indices are "inscribed" into the graph of the symmetric hyperbolic Lucas cosine $sLs(x)$ at the discrete points (x=±1,±3,±5,...).

The symmetric hyperbolic Fibonacci and Lucas functions are connected amongst themselves by the following simple correlations:

$$sFs(x)=\frac{sLs(x)}{\sqrt{5}};\ cFs(x)=\frac{cLs(x)}{\sqrt{5}}.$$

Also we can introduce the notions of symmetric hyperbolic Fibonacci and Lucas tangents and cotangents.

Symmetric hyperbolic Fibonacci tangent

$$tFs(x)=\frac{sFs(x)}{cFs(x)}=\frac{\tau^x-\tau^{-x}}{\tau^x+\tau^{-x}} \tag{5.62}$$

Symmetric hyperbolic Fibonacci cotangent

$$ctFs(x)=\frac{cFs(x)}{sFs(x)}=\frac{\tau^x+\tau^{-x}}{\tau^x-\tau^{-x}} \tag{5.63}$$

Symmetric hyperbolic Lucas tangent

$$tLs(x)=\frac{sLs(x)}{cLs(x)}=\frac{\tau^x-\tau^{-x}}{\tau^x+\tau^{-x}} \tag{5.64}$$

Symmetric hyperbolic Lucas cotangent

$$ctLs(x)=\frac{cLs(x)}{sLs(x)}=\frac{\tau^x+\tau^{-x}}{\tau^x-\tau^{-x}} \tag{5.65}$$

We conclude from a comparison of the functions (5.62) with (5.63), and (5.64) with (5.65) that these functions coincide, that is, we have:

$$tFs(x)=tLs(x)\text{ and }ctFs(x)=ctLs(x). \tag{5.66}$$

It is easy to prove that the functions (5.62) and (5.63) are odd functions because

$$tFs(-x)=\frac{\tau^{-x}-\tau^{x}}{\tau^{-x}+\tau^{x}}=-\frac{\tau^x-\tau^{-x}}{\tau^x+\tau^{-x}}=-tFs(x)$$

$$ctFs(-x)=\frac{\tau^{-x}+\tau^{x}}{\tau^{-x}-\tau^{x}}=-\frac{\tau^x+\tau^{-x}}{\tau^x-\tau^{-x}}=-ctFs(x).$$

Taking into consideration (5.66), we can write:

$$tLs(-x)=-tLs(x);ctLs(-x)=-ctLs(x),$$

that is, the functions (5.64) and (5.65) are also odd functions.

Next let us compare the classical hyperbolic functions (5.7), (5.8) with the symmetric hyperbolic Fibonacci functions (5.53)-(5.56). It follows from this comparison that the symmetric hyperbolic Fibonacci and Lucas functions possess all the important properties of classical hyperbolic functions. Thus, the symmetric hyperbolic Fibonacci and Lucas functions have *Hyperbolic Properties*.

On the other hand, a comparison of Fibonacci and Lucas numbers with the symmetric hyperbolic Fibonacci and Lucas functions show that according to (5.61) these functions are a generalization of Fibonacci and Lucas numbers for the continuous domain. This means that the symmetric hyperbolic Fibonacci and Lucas functions possess *Recursive Properties* similar to the properties of Fibonacci and Lucas numbers.

5.6. Recursive Properties of the Symmetric Hyperbolic Fibonacci and Lucas Functions

Consider the recursive properties of the symmetric hyperbolic Fibonacci and Lucas functions in comparison with the analogous properties of Fibonacci and Lucas numbers.

Theorem 5.1. The following correlations, that are analogous to the recursive correlation for the Fibonacci numbers $F_{n+2}=F_{n+1}+F_n$, are valid for the symmetric hyperbolic Fibonacci functions:

$$\begin{aligned} sFs(x+2) &= cFs(x+1)+sFs(x) \\ cFs(x+2) &= sFs(x+1)+cFs(x). \end{aligned} \tag{5.67}$$

Proof:

$$cFs(x+1)+sFs(x)=\frac{\tau^{x+1}+\tau^{-(x+1)}}{\sqrt{5}}+\frac{\tau^x-\tau^{-x}}{\sqrt{5}}$$

$$=\frac{\tau^x(\tau+1)-\tau^{-x}(1-\tau)}{\sqrt{5}}=\frac{\tau^x\times\tau^2-\tau^{-x}\times\tau^{-2}}{\sqrt{5}}$$

$$=\frac{\tau^{x+2}-\tau^{-(x+2)}}{\sqrt{5}}=sFs(x+2);$$

$$sFs(x+1)+cFs(x)=\frac{\tau^{x+1}-\tau^{-(x+1)}}{\sqrt{5}}+\frac{\tau^x+\tau^{-x}}{\sqrt{5}}$$

$$=\frac{\tau^x(\tau+1)+\tau^{-x}(1-\tau)}{\sqrt{5}}=\frac{\tau^x\times\tau^2+\tau^{-x}\times\tau^{-2}}{\sqrt{5}}$$

$$=\frac{\tau^{x+2}+\tau^{-(x+2)}}{\sqrt{5}}=cFs(x+2).$$

Theorem 5.2. The following correlations, that are analogous to the recursive equation for the Lucas numbers $L_{n+2}=L_{n+1}+L_n$, are valid for the symmetric hyperbolic Lucas functions:

$$sLs(x+2)=cLs(x+1)+sLs(x)$$
$$cLs(x+2)=sLs(x+1)+cLs(x). \tag{5.68}$$

The proof is analogous to Theorem 5.28.

Theorem 5.3 (a generalization of Cassini's formula). The following correlations, that are similar to Cassini's formula $F_n^2-F_{n+1}F_{n-1}=(-1)^{n+1}$, are valid for the symmetric hyperbolic Fibonacci functions:

$$[sFs(x)]^2-cFs(x+1)cFs(x-1)=-1$$
$$[cFs(x)]^2-sFs(x+1)sFs(x-1)=1. \tag{5.69}$$

Proof:

$$[sFs(x)]^2-cFs(x+1)cFs(x-1)$$
$$=\left(\frac{\tau^x-\tau^{-x}}{\sqrt{5}}\right)^2-\frac{\tau^{x+1}+\tau^{-(x+1)}}{\sqrt{5}}\times\frac{\tau^{x-1}+\tau^{-(x-1)}}{\sqrt{5}}$$
$$=\frac{\tau^{2x}-2+\tau^{-2x}-\left(\tau^{2x}+\tau^2+\tau^{-2}+\tau^{-2x}\right)}{5}=-1;$$

$$[cFs(x)]^2-sFs(x+1)sFs(x-1)$$
$$=\left(\frac{\tau^x+\tau^{-x}}{\sqrt{5}}\right)^2-\frac{\tau^{x+1}-\tau^{-(x+1)}}{\sqrt{5}}\times\frac{\tau^{x-1}-\tau^{-(x-1)}}{\sqrt{5}}$$
$$=\frac{\tau^{2x}+2+\tau^{-2x}-\left(\tau^{2x}-\tau^2-\tau^{-2}+\tau^{-2x}\right)}{5}=1.$$

Note that for the proof we used the Binet formula for the Lucas numbers: $\tau^2+\tau^{-2}=L(3)=3.$

Theorem 5.4. The following correlations, that are similar to the identity $L_n^2-2(-1)^n=L_{2n}$, are valid for the symmetric hyperbolic Lucas functions:

$$[sLs(x)]^2+2=cLs(2x);\quad [cLs(x)]^2-2=sLs(2x). \tag{5.70}$$

The proof is analogous to Theorem 5.3.

Theorem 5.5. The following correlations, that are similar to the identity $F_{n+1}+F_{n-1}=L_n$, are valid for the symmetric hyperbolic Fibonacci and Lucas functions:

$$cFs(x+1)+cFs(x-1)=cLs(x)$$
$$sFs(x+1)+sFs(x-1)=sLs(x). \tag{5.71}$$

The proof is analogous to Theorem 5.1.

Theorem 5.6. The following correlations, that are similar to the identity $F_n+L_n=2F_{n+1}$, are valid for the symmetric hyperbolic Fibonacci and Lucas functions:

$$\begin{aligned} &cFs(x)+sLs(x)=2sFs(x+1)\\ &sFs(x)+cLs(x)=2cFs(x). \end{aligned} \tag{5.72}$$

The proof is analogous to Theorem 5.1.

Based on the definitions (5.53) - (5.56), we can prove different identities for the symmetric hyperbolic Fibonacci and Lucas functions. We can see some of these identities in Table 5.3.

Table 5.3. The identities for Fibonacci and Lucas numbers and for symmetric hyperbolic Fibonacci and Lucas functions

$F_{n+2}=F_{n+1}+F_n$	$sFs(x+2)=cFs(x+1)+sFs(x)$	$cFs(x+2)=sFs(x+1)+cFs(x)$
$L_{n+2}=L_{n+1}+L_n$	$sLs(x+2)=cLs(x+1)+sLs(x)$	$cLs(x+2)=sLs(x+1)+cLs(x)$
$F_n=(-1)^{n+1}F_{-n}$	$sFs(x)=-sFs(-x)$	$cFs(x)=cFs(-x)$
$L_n=(-1)^n L_{-n}$	$sLs(x)=sLs(-x)$	$cLs(x)=-cLs(-x)$
$F_{n+3}+F_n=2F_{n+2}$	$sFs(x+3)+cFs(x)=2cFs(x+2)$	$cFs(x+3)+sFs(x)=2sFs(x+2)$
$F_{n+3}-F_n=2F_{n+1}$	$sFs(x+3)-cFs(x)=2sFs(x+1)$	$cFs(x+3)-sFs(x)=2cFs(x+1)$
$F_{n+6}-F_n=4F_{n+3}$	$sFs(x+6)-cFs(x)=4sFs(x+3)$	$cFs(x+6)-sFs(x)=4cFs(x+3)$
$F_n^2-F_{n+1}F_{n-1}=(-1)^{n+1}$	$[sFs(x)]^2-cFs(x+1)cFs(x-1)=-1$	$[cFs(x)]^2-sFs(x+1)sFs(x-1)=1$
$F_{2n+1}=F_{n+1}^2+F_n^2$	$cFs(2x+1)=[cFs(x+1)]^2+[sFs(x)]^2$	$sFs(2x+1)=[sFs(x+1)]^2+[cFs(x)]^2$
$L_n^2-2(-1)^n=L_{2n}$	$[sLs(x)]^2+2=cLs(2x)$	$[cLs(x)]^2-2=cLs(2x)$
$L_n+L_{n+3}=2L_{n+2}$	$sLs(x)+cLs(x+3)=2sLs(x+2)$	$cLs(x)+sLs(x+3)=2cLs(x+2)$
$L_{n+1}L_{n-1}-L_n^2=-5(-1)^n$	$sLs(x+1)sLs(x-1)-[cLs(x)]^2=-5$	$cLs(x+1)cLs(x-1)-[sLs(x)]^2=5$
$F_{n+3}-2F_n=L_n$	$sFs(x+3)-2cFs(x)=sLs(x)$	$cFs(x+3)-2sFs(x)=cLs(x)$
$L_{n-1}+L_{n+1}=5F_n$	$sLs(x-1)+sLs(x+1)=5sFs(x)$	$cLs(x-1)+cLs(x+1)=5cFs(x)$
$L_n+5F_n=2L_{n+1}$	$sLs(x)+5cFs(x)=2cLs(x+1)$	$cLs(x)+5sFs(x)=2sLs(x+1)$
$L_{n+1}^2+L_n^2=5F_{2n+1}$	$[sLs(x+1)]^2+[cLs(x)]^2=5cFs(2x+1)$	$[cLs(x+1)]^2+[sLs(x)]^2=5sFs(2x+1)$

We can see from Table 5.3 that two identities for hyperbolic Fibonacci and Lucas functions correspond to one identity for Fibonacci and Lucas numbers. This fact can be explained very easily. This all depends on the evenness of the index n of the Fibonacci and Lucas numbers. For example, consider the simplest identity $F_{n+2}=F_{n+1}+F_n$. If $n=2k$ (n is even), we should use the first identity $sFs(x+2)=cFs(x+1)+sFs(x)$; in the opposite case ($n=2k+1$) we should use the other identity $cFs(x+2)=sFs(x+1)+cFs(x)$.

5.7. Hyperbolic Properties of the Symmetric Hyperbolic Fibonacci and Lucas Functions and Formulas for Their Differentiation and Integration

5.7.1. *Hyperbolic Properties*

The symmetric hyperbolic Fibonacci and Lucas functions possess "hyperbolic" properties similar to classical hyperbolic functions. Consider some of them in contrast to certain properties of classical hyperbolic functions.

Theorem 5.7. The following identity, that is similar to the identity $[ch(x)]^2$-$[ch(x)]^2=1$, is valid for the symmetric hyperbolic Fibonacci function:

$$\left[cFs(x)\right]^2-\left[sFs(x)\right]^2=\frac{4}{5}. \tag{5.73}$$

Proof:

$$\left[cFs(x)\right]^2-\left[sFs(x)\right]^2=\left(\frac{\tau^x+\tau^{-x}}{\sqrt{5}}\right)^2-\left(\frac{\tau^x-\tau^{-x}}{\sqrt{5}}\right)^2$$

$$=\frac{\tau^{2x}+2+\tau^{-2x}-\tau^{2x}+2-\tau^{-2x}}{5}=\frac{4}{5}.$$

By analogy, we can prove the following theorem for symmetric hyperbolic Lucas functions.

Theorem 5.8.

$$\left[cLs(x)\right]^2-\left[sLs(x)\right]^2=4. \tag{5.74}$$

Theorem 5.9. The following identity, that is similar to the identity $ch(x+y)=ch(x)ch(y)+sh(x)sh(y)$, is valid for the symmetric hyperbolic Fibonacci function:

$$\frac{2}{\sqrt{5}}cFs(x+y)=cFs(x)cFs(y)+sFs(x)sFs(y). \tag{5.75}$$

Proof:

$$cFs(x)cFs(y)+sFs(x)sFs(y)=\frac{\tau^x+\tau^{-x}}{\sqrt{5}}\times\frac{\tau^y+\tau^{-y}}{\sqrt{5}}+\frac{\tau^x-\tau^{-x}}{\sqrt{5}}\times\frac{\tau^y-\tau^{-y}}{\sqrt{5}}$$

$$=\frac{\tau^{x+y}+\tau^{x-y}+\tau^{-x+y}+\tau^{-x-y}+\tau^{x+y}-\tau^{x-y}-\tau^{-x+y}+\tau^{-x-y}}{\sqrt{5}}$$

$$=\frac{2\left(\tau^{x+y}+\tau^{-x-y}\right)}{\sqrt{5}}=\frac{2}{\sqrt{5}}cFs(x+y).$$

Theorem 5.10. The identity, that is similar to the identity $ch(x\text{-}y)=ch(x)ch(y)\text{-}sh(x)sh(y)$, is valid for the symmetric hyperbolic Fibonacci function:

$$\frac{2}{\sqrt{5}}cFs(x-y)=cFs(x)cFs(y)-sFs(x)sFs(y). \tag{5.76}$$

By analogy we can prove the following theorems for the symmetric hyperbolic Lucas functions.

Theorem 5.11.

$$2cLs(x\pm y)=cLs(x)cLs(y)\pm sLs(x)sLs(y). \tag{5.77}$$

Theorem 5.12. The following identities, that are similar to the identity $ch(2x)=[ch(x)]^2+[sh(x)]^2$, are valid for the symmetric hyperbolic Fibonacci and Lucas functions:

$$\frac{2}{\sqrt{5}}cFs(2x)=\left[cFs(x)\right]^2+\left[sFs(x)\right]^2 \tag{5.78}$$

$$2cLs(2x)=\left[cLs(x)\right]^2+\left[sLs(x)\right]^2. \tag{5.79}$$

Theorem 5.13. The following identities, that are similar to the identity $sh(2x)=2sh(x)ch(x)$, are valid for the symmetric hyperbolic Fibonacci and Lucas functions:

$$\frac{1}{\sqrt{5}}sFs(2x)=sFs(x)cFs(x) \tag{5.80}$$

$$sLs(2x)=sLs(x)cLs(x). \tag{5.81}$$

Theorem 5.14. The following formulas, that are similar to Moivre's formulas $[ch(x)\pm sh(x)]^n=ch(nx)\pm sh(nx)$, are valid for the symmetric hyperbolic Fibonacci and Lucas functions:

$$\left[cFs(x)\pm sFs(x)\right]^n=\left(\frac{2}{\sqrt{5}}\right)^{n-1}\left[cFs(nx)\pm sFs(nx)\right] \tag{5.81}$$

$$\left[cLs(x)\pm sLs(x)\right]^n=2^{n-1}\left[cFs(nx)\pm sFs(nx)\right]. \tag{5.82}$$

5.7.2. *Formulas for Differentiation and Integration*

It is easy to prove the following formulas for differentiation and integration of symmetric hyperbolic Fibonacci and Lucas functions.

Theorem 5.15 (formulas for differentiation). The following correlations similar to the n^{th} derivatives of the classical hyperbolic functions

$$\left[ch(x)\right]^{(n)}=\begin{cases} sh(x), & for \quad n=2k+1 \\ ch(x), & for \quad n=2k \end{cases};$$

$$\left[sh(x)\right]^{(n)}=\begin{cases}ch(x), & for \quad n=2k+1\\ sh(x), & for \quad n=2k\end{cases}$$

are valid for the derivatives of the symmetric hyperbolic Fibonacci functions:

$$\left[cFs(x)\right]^{(n)}=\begin{cases}(\ln\tau)^n\, sFs(x), & for \quad n=2k+1\\ (\ln\tau)^n\, cFs(x), & for \quad n=2k\end{cases} \tag{5.83}$$

$$\left[sFs(x)\right]^{(n)}=\begin{cases}(\ln\tau)^n\, cFs(x), & for \quad n=2k+1\\ (\ln\tau)^n\, sFs(x), & for \quad n=2k\end{cases}. \tag{5.84}$$

Theorem 5.16 (formulas for integration). The following correlations similar to the integrals of the classical hyperbolic functions

$$\underbrace{\iiint}_{n} ch(x)dx \quad =\begin{cases}sh(x), & for \quad n=2k+1\\ ch(x), & for \quad n=2k\end{cases};$$

$$\underbrace{\iiint}_{n} sh(x)dx \quad =\begin{cases}ch(x), & for \quad n=2k+1\\ sh(x), & for \quad n=2k\end{cases}$$

are valid for the symmetric hyperbolic Fibonacci and Lucas functions:

$$\underbrace{\iiint}_{n} cFs(x)dx \quad =\begin{cases}(\ln\tau)^{-n}\, sFs(x), & for \quad n=2k+1\\ (\ln\tau)^{-n}\, cFs(x), & for \quad n=2k\end{cases} \tag{5.85}$$

$$\underbrace{\iiint}_{n} cLs(x)dx \quad =\begin{cases}(\ln\tau)^{-n}\, sLs(x), & for \quad n=2k+1\\ (\ln\tau)^{-n}\, cLs(x), & for \quad n=2k\end{cases} \tag{5.86}$$

$$\underbrace{\iiint}_{n} sFs(x)dx \quad =\begin{cases}(\ln\tau)^{-n}\, cFs(x), & for \quad n=2k+1\\ (\ln\tau)^{-n}\, sFs(x), & for \quad n=2k\end{cases} \tag{5.87}$$

$$\underbrace{\iiint}_{n} sLs(x)dx \quad =\begin{cases}(\ln\tau)^{-n}\, cLs(x), & for \quad n=2k+1\\ (\ln\tau)^{-n}\, sLs(x), & for \quad n=2k\end{cases}. \tag{5.88}$$

5.8. The Golden Shofar

5.8.1. *The Quasi-sine Fibonacci and Lucas Functions*

Consider Binet formulas for Fibonacci and Lucas numbers represented in the following form:

$$F_n = \frac{\tau^n - (-1)^n \tau^{-n}}{\sqrt{5}} \tag{5.89}$$

$$L_n = \tau^n + (-1)^n \tau^{-n}, \tag{5.90}$$

where $\tau = \left(1+\sqrt{5}\right)/2$ is the golden mean and $n = 0, \pm1, \pm2, \pm3, \ldots$.

By comparing Binet formula (5.89) and (5.90) with the symmetric hyperbolic Fibonacci and Lucas functions (5.53)-(5.56), we can see that the continuous functions τ^x and τ^{-x} in the formulas (5.53)-(5.56) correspond to the discrete sequences τ^n and τ^{-n} in formulas (5.89) and (5.90). Then we set up in correspondence to the alternating sequence $(-1)^n$ in the Binet formulas (5.89) and (5.90) some continuous function, taking the values -1 and 1 at the discrete points $x = 0, \pm1, \pm2, \pm3, \ldots$. The trigonometric function $\cos(\pi x)$ is the simplest of them. This reasoning is the basis for introducing the new continuous function that is connected with Fibonacci and Lucas numbers.

Definition 5.1. The following continuous function is called the *Quasi-sine Fibonacci Function (QSFF)*:

$$Q_F(x) = \frac{\tau^x - \cos(\pi x)\tau^{-x}}{\sqrt{5}}. \tag{5.91}$$

There is the following correlation between Fibonacci numbers F_n given by (5.89) and the quasi-sine Fibonacci function given by (5.91):

$$F_n = Q_F(n) = \frac{\tau^n - \cos(\pi n)\tau^{-n}}{\sqrt{5}}, \tag{5.92}$$

where $n = 0, \pm1, \pm2, \pm3, \ldots$.

Definition 5.2. The following continuous function is called the *Quasi-sine Lucas Function (QSLF)*:

$$Q_L(x) = \tau^x + \cos(\pi x)\tau^{-x}. \tag{5.93}$$

The graph of the QSFF is a quasi-sine curve that passes through all points corresponding to the Fibonacci numbers that are given by (5.89) on the coordinate plane (Fig. 5.5). The symmetric hyperbolic Fibonacci functions (5.53) and (5.54) (Fig. 5.3) are the envelopes of the quasi-sine Fibonacci function $Q(x)$.

The graph of the QSLF is a quasi-sine curve that passes through all points corresponding to the Lucas numbers that are given by (5.90) on the coordinate plane (Fig. 5.6). The symmetric hyperbolic Lucas functions (5.55) and (5.56) (Fig. 5.4) are the envelopes of the quasi-sine Lucas function $Q_L(x)$.

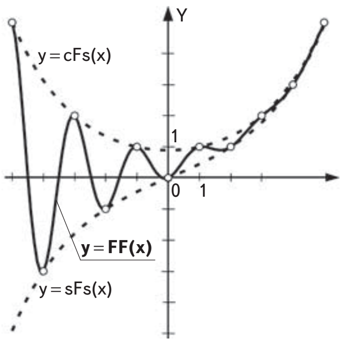


Figure 5.5. A graph of the quasi-sine Fibonacci function

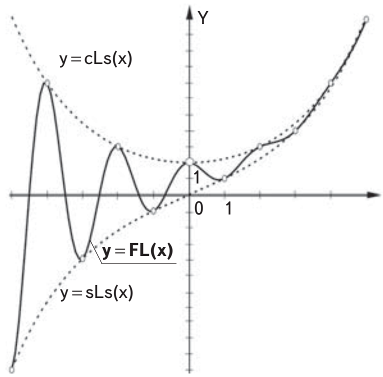


Figure 5.6. A graph of the quasi-sine Lucas function

5.8.2. *Recursive Properties of the Quasi-sine Fibonacci and Lucas Functions*

It is easy to prove the following theorems for the quasi-sine Fibonacci and Lucas functions.

Theorem 5.17. For the quasi-sine Fibonacci function there is the following correlation similar to the recursive relation for the Fibonacci numbers $F_{n+2}=F_{n+1}+F_n$:

$$Q_F(x+2)=Q_F(x+1)+Q_F(x). \tag{5.94}$$

Theorem 5.18. For the quasi-sine Lucas function there is the following correlation similar to the recursive relation for the Lucas numbers $L_{n+2}=L_{n+1}+L_n$:

$$Q_L(x+2)=Q_L(x+1)+Q_L(x). \tag{5.95}$$

Theorem 5.19. For the quasi-sine Fibonacci function there is the following correlation similar to Cassini's formula $F_n^2-F_{n+1}F_{n-1}=(-1)^{n+1}$:

$$\left[Q_F(x)\right]^2-Q_F(x+1)Q_F(x-1)=-\cos(\pi x). \tag{5.96}$$

By analogy with Theorems 5.17 - 5.19 we can prove other identities for the quasi-sine Fibonacci and Lucas functions (Table 5.4).

Table 5.4. The identities for the quasi-sine Fibonacci and Lucas functions

$F_{n+2}=F_{n+1}+F_n$	$Q_F(x+2)=Q_F(x+1)+Q_F(x)$
$F_{n+3}+F_n=2F_{n+2}$	$Q_F(x+3)+Q_F(x)=2Q_F(x+2)$
$F_{n+3}-F_n=2F_{n+1}$	$Q_F(x+3)-Q_F(x)=2Q_F(x+1)$
$F_{n+6}-F_n=4F_{n+3}$	$Q_F(x+6)-Q_F(x)=4Q_F(x+3)$
$F_n^2-F_{n+1}F_{n-1}=(-1)^{n+1}$	$[Q_F(x)]^2-Q_F(x+1)Q_F(x-1)=-\cos(\pi x)$
$F_{2n+1}=F_{n+1}^2+F_n^2$	$Q_L(2x+1)=[Q_L(x+1)]^2+[Q_L(x)]^2$
$L_{n+2}=L_{n+1}+L_n$	$Q_F(x+2)=Q_F(x+1)+Q_F(x)$
$L_{n+3}+L_n=2L_{n+2}$	$Q_L(x+3)+Q_L(x)=2Q_L(x+2)$
$L_{n-1}+L_{n+1}=5F_n$	$Q_L(x-1)+Q_L(x+1)=5Q_F(x)$
$L_n+5F_n=2L_{n+1}$	$Q_L(x)+5Q_F(x)=2Q_L(x+1)$
$F_{n+3}-2F_n=L_n$	$Q_F(x+3)-2Q_F(x)=Q_L(x)$
$L_{n+1}L_{n-1}-L_n^2=-5(-1)^n$	$Q_L(x+1)Q_L(x-1)-[Q_L(x)]^2=-5\cos(\pi x)$

5.8.3. *Three-dimensional Fibonacci Spiral*

It is well known that the trigonometric sine and cosine can be defined as a horizontal projection of the translational movement of a point on the surface of an infinite rotating cylinder with radius 1 and the symmetry center coinciding with the axis *OX*. Such three-dimensional spiral is described by the complex function $f(x)=\cos(x)+i\sin(x)$, where $i=\sqrt{-1}$. The sine function is its projection on a plane.

If we assume that the quasi-sine Fibonacci function (5.91) is a projection of the three-dimensional spiral on some funnel-shaped surface, we can then define the so-called *Three-dimensional Fibonacci Spiral.*

Definition 5.3. The following function is called the *Three-dimensional Fibonacci Spiral*:

$$S_F(x)=\frac{\tau^x-[\cos(\pi x)]\tau^{-x}}{\sqrt{5}}+i\frac{[\sin(\pi x)]\tau^{-x}}{\sqrt{5}}. \tag{5.97}$$

This function, by its shape, reminds one of a spiral that is drawn on a well with the end bent (Fig. 5.7).

It is easy to prove the following theorem for the three-dimensional Fibonacci spiral.

Theorem 5.20. For the three-dimensional Fibonacci spiral the following correlation, similar to the recursive relation for Fibonacci numbers $F_{n+2}=F_{n+1}+F_n$, is valid:

$$S_F(x+2)=S_F(x+1)+S_F(x).$$

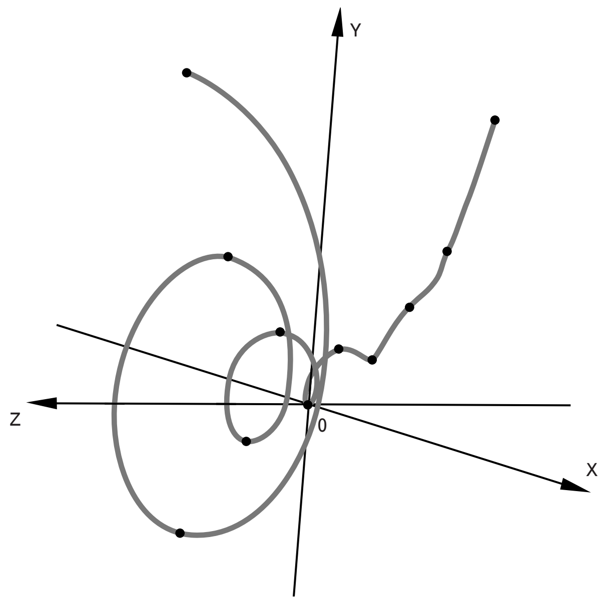


Figure 5.7. The three-dimensional Fibonacci spiral

5.8.4. *The Golden Shofar*

We can separate the real and imaginary parts of the three-dimensional Fibonacci spiral (5.97):

$$\mathrm{Re}\left[S_F(x)\right]=\frac{\tau^x-\left[\cos(\pi x)\right]\tau^{-x}}{\sqrt{5}} \tag{5.98}$$

$$\mathrm{Im}\left[S_F(x)\right]=\frac{\left[\sin(\pi x)\right]\tau^{-x}}{\sqrt{5}}. \tag{5.99}$$

The following system of equations can be obtained from (5.97), (5.98), and (5.99) if we consider the axis OY as a real axis and the axis OZ as an imaginary axis:

$$\begin{cases} y(x)-\dfrac{\tau^x}{\sqrt{5}}=-\dfrac{\left[\cos(\pi x)\right]\tau^{-x}}{\sqrt{5}} \\ z(x)=\dfrac{\left[\sin(\pi x)\right]\tau^{-x}}{\sqrt{5}} \end{cases} \tag{5.100}$$

Let us square both expressions of the equation system (5.100) and add them together. Taking y and z as independent variables, we obtain a curvilinear surface of the second degree called the *Golden Shofar*.

Definition 5.4. The following curvilinear function of the second degree is the *Golden Shofar*:

$$\left(y-\frac{\tau^x}{\sqrt{5}}\right)^2+z^2=\left(\frac{\tau^{-x}}{\sqrt{5}}\right)^2. \tag{5.101}$$

We can see in Fig. 5.8 a three-dimensional surface corresponding to (5.101). It is similar to the horn or well with a narrow end. In the Hebrew language the word "Shofar" means horn which is a symbol of power.

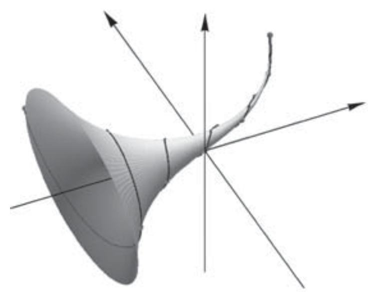


Figure 5.8. The Golden Shofar

The formula for the Golden Shofar can be represented in the following form:

$$z^2=\left[cFs(x)-y\right]\times\left[sFs(x)+y\right], \tag{5.102}$$

where $sFs(x)$ and $cFs(x)$ are the symmetric hyperbolic Fibonacci sine and cosine, respectively.

A projection of the Golden Shofar on the plane *XOY* is shown in Fig. 5.9. The Golden Shofar is a projection into the space between the graphs of the symmetric hyperbolic Fibonacci sine and cosine (Fig. 5.3).

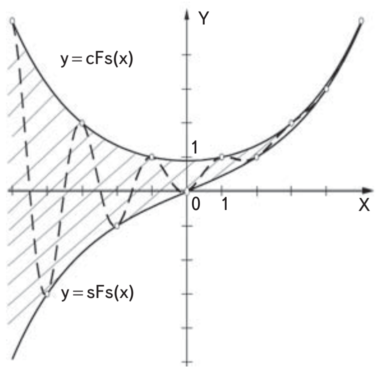


Figure 5.9. A projection of the Golden Shofar on the plane *XOY*

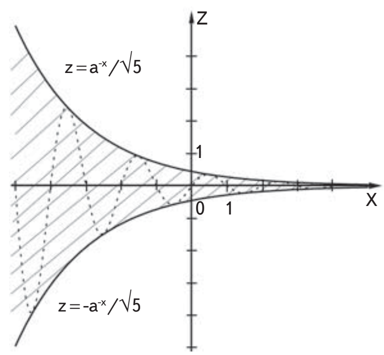


Figure 5.10. A projection of the Golden Shofar on the plane *XOZ*

The function (5.97) lies on the Golden Shofar and "pierces" the plane *XOY* at the points that correspond to the Fibonacci sequence (Fig. 5.9).

A projection of the Golden Shofar on the plane *XOZ* is shown in Fig. 5.10. The Golden Shofar is a projection into the space between the graphs of the two exponent functions $\left(-\tau^x/\sqrt{5}\right)$ and $\left(\tau^{-x}/\sqrt{5}\right)$.

By cutting the Golden Shofar by the planes that are parallel to the plane *YOZ*, the circles with the center $\left(0;\tau^x/\sqrt{5}\right)$ and the radius $\tau^{-x}/\sqrt{5}$ are obtained.

It is possible to say that the function $y(x)=\tau^x/\sqrt{5}$ is the pseudo-axis of symmetry (or the axis of pseudo-symmetry) of the Golden Shofar (Fig. 5.8).

It is possible to assume that the Golden Shofar is a new model of the field with curvilinear structure and similar to the model of the gravitational well used in the general theory of relativity.

5.8.5. *A General Model of the Hyperbolic Space with a "Shofarable" Topology*

Based on experimental data obtained in 2003 by the NASA Wilkinson Microwave Anisotropy Probe (WMAP), a new hypothesis about the structure of the Universe was developed. According to [155], the geometry of the Universe is similar in shape to a horn or a pipe with an extended bell. As a result of this discovery we can make the following claim:

The Universe has a "shofar-like" topology as shown in Fig. 5.11.

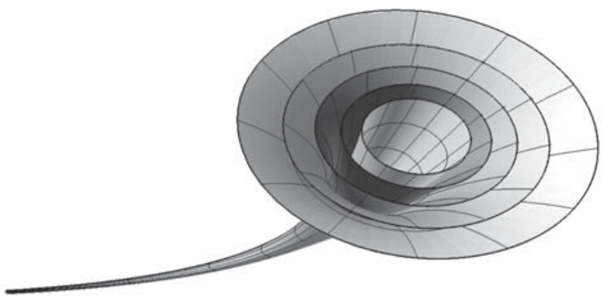


Figure 5.11. The "shofar-like" topology

5.9. A General Theory of the Hyperbolic Functions

5.9.1. *A Definition of the Hyperbolic Fibonacci and Lucas m-Functions*

Alexey Stakhov and Boris Rozin introduced [106] a new class of hyperbolic functions, the *Symmetric Hyperbolic Fibonacci and Lucas Functions*, based on an analogy between Binet formulas and the classical hyperbolic functions. By using this approach, Alexey Stakhov introduced [118] the *Hyperbolic Fibonacci and Lucas Functions of the Order m* or simply *Hyperbolic Fibonacci and Lucas m-Functions*. These functions are based on an analogy between Gazale formulas that are given by (4.281) and (4.292) with the classical hyperbolic functions (5.7) and (5.8). Consider the new class of hyperbolic Fibonacci and Lucas functions introduced in [118].

Hyperbolic Fibonacci m-sine

$$sF_m(x)=\frac{\Phi_m^x-\Phi_m^{-x}}{\sqrt{4+m^2}}=\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x-\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right] \quad (5.103)$$

Hyperbolic Fibonacci m-cosine

$$cF_m(x)=\frac{\Phi_m^x+\Phi_m^{-x}}{\sqrt{4+m^2}}=\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x+\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right] \quad (5.104)$$

Hyperbolic Lucas m-sine

$$sL_m(x)=\Phi_m^x-\Phi_m^{-x}=\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x-\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x} \quad (5.105)$$

Hyperbolic Lucas m-cosine

$$cL_m(x)=\Phi_m^x+\Phi_m^{-x}=\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x+\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x} \quad (5.106)$$

It is easy to prove that the Fibonacci and Lucas m-numbers are determined identically by the hyperbolic Fibonacci and Lucas m-functions as follows:

$$F_m(n)=\begin{cases} sF_m(n) & for\ n=2k \\ cF_m(n) & for\ n=2k+1 \end{cases}$$

$$L_m(n)=\begin{cases} cL_m(n) & for\ n=2k \\ sL_m(n) & for\ n=2k+1 \end{cases} \quad (5.107)$$

Graphs of the hyperbolic Fibonacci and Lucas m-functions are similar to graphs of the classical hyperbolic functions. Here it is important to note that at the point x=0, the hyperbolic Fibonacci m-cosine $cF_m(x)$ (5.104) takes on the value $cF_m(x)=2/\sqrt{4+m^2}$, and the hyperbolic Lucas m-cosine $cL_m(x)$ takes on the value $cL_m(0)=2$. It is also important to emphasize that the Fibonacci m-numbers $F_m(n)$ with the even indices $n=0,\pm2,\pm4,\pm6,...$ are "inscribed" into the graph of the hyperbolic Fibonacci m-sine $sF_m(x)$ at the discrete points $x=0,\pm2,\pm4,\pm6,...$ and the Fibonacci m-numbers $F_m(n)$ with odd indices $n=\pm1,\pm3,\pm5,...$ are "inscribed" into the hyperbolic Fibonacci m-cosine $cF_m(x)$ at the discrete points $x=\pm1,\pm3,\pm5,...$. On the other hand, the Lucas m-numbers $L_m(n)$ with even indices are "inscribed" into the graph of the hyperbolic Lucas m-cosine $cL_m(x)$ at the discrete points $x=0,\pm2,\pm4,\pm6,...$ and the Lucas m-numbers $L_m(n)$ with odd indices are "inscribed" into the graph of the hyperbolic Lucas m-sine $cL_m(x)$ at the discrete points $x=\pm1,\pm3,\pm5,...$.

We can also introduce the notions of hyperbolic Fibonacci and Lucas m-tangents and m-cotangents.

Hyperbolic Fibonacci m-tangent

$$tF_m(x) = \frac{sF_m(x)}{cF_m(x)} = \frac{\Phi_m^x - \Phi_m^{-x}}{\Phi_m^x + \Phi_m^{-x}} \tag{5.108}$$

Hyperbolic Fibonacci m-cotangent

$$ctF_m(x) = \frac{cF_m(x)}{sF_m(x)} = \frac{\Phi_m^x + \Phi_m^{-x}}{\Phi_m^x - \Phi_m^{-x}} \tag{5.109}$$

Hyperbolic Lucas m-tangent

$$tL_m(x) = \frac{sL_m(x)}{cL_m(x)} = \frac{\Phi_m^x - \Phi_m^{-x}}{\Phi_m^x + \Phi_m^{-x}} \tag{5.110}$$

Hyperbolic Lucas m-cotangent

$$ctL_m(x) = \frac{cL_m(x)}{sL_m(x)} = \frac{\Phi_m^x + \Phi_m^{-x}}{\Phi_m^x - \Phi_m^{-x}} \tag{5.111}$$

By analogy we can introduce other hyperbolic Fibonacci and Lucas m-functions, in particular, secant, cosecant, and so on.

5.9.2. *General Properties of the Hyperbolic Fibonacci and Lucas m-Functions*

It is easy to prove that the function (5.103) is an odd function because

$$sF_m(-x) = \frac{\Phi_m^{-x} - \Phi_m^x}{\sqrt{4+m^2}} = -\frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}} = -sF_m(x). \tag{5.112}$$

On the other hand,

$$cF_m(-x) = \frac{\Phi_m^{-x} + \Phi_m^x}{\sqrt{4+m^2}} = \frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}} = cF_m(x), \tag{5.113}$$

that is, the hyperbolic Fibonacci m-cosine (5.104) is an even function.

By analogy, we can prove that the hyperbolic Lucas m-sine (5.105) is an odd function and the hyperbolic Lucas m-cosine (5.106) is an even function, that is,

$$sL_m(x) = -sL_m(-x) \tag{5.114}$$

$$cL_m(x) = cL_m(-x). \tag{5.115}$$

The properties (5.112)-(5.15) show that the symmetric hyperbolic Fibonacci and Lucas m-functions (5.103)-(5.106) possess the property of evenness (5.52).

By making the pair-wise comparison of the functions (5.108) with (5.110), and (5.109) with (5.111), we can conclude that the hyperbolic Fibonacci and Lucas m-tangents and m-cotangents are coincident, respectively, that is, we have:

$$tF_m(x) = tL_m(x) \text{ and } ctF_m(x) = ctL_m(x). \tag{5.116}$$

It is easy to prove that the functions (5.108) and (5.109) are odd functions because

$$tF_m(-x) = \frac{\Phi_m^{-x} - \Phi_m^{x}}{\Phi_m^{-x} + \Phi_m^{x}} = -tF_m(x)$$

$$ctF_m(-x) = \frac{\Phi_m^{-x} + \Phi_m^{x}}{\Phi_m^{-x} - \Phi_m^{x}} = -ctF_m(x).$$

Taking into consideration (5.116) we can write:

$$tL_m(-x) = -tL_m(x)$$

$$ctL_m(-x) = -ctL_m(x),$$

that is, the functions (5.110) and (5.111) are also odd functions.

5.9.3. *Partial Cases of the Hyperbolic Fibonacci and Lucas m-Functions*

Consider the partial cases of the hyperbolic Fibonacci and Lucas m-functions (5.103)-(5.106) for different values of the order m.

Hyperbolic Fibonacci and Lucas 1-functions

$$sF_1(x) = \frac{\Phi_1^x - \Phi_1^{-x}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1+\sqrt{5}}{2}\right)^{-x}\right] \tag{5.117}$$

$$cF_1(x) = \frac{\Phi_1^x + \Phi_1^{-x}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^x + \left(\frac{1+\sqrt{5}}{2}\right)^{-x}\right] \tag{5.118}$$

$$sL_1(x) = \Phi_1^x - \Phi_1^{-x} = \left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1+\sqrt{5}}{2}\right)^{-x} \tag{5.119}$$

$$cL_1(x) = \Phi_1^x + \Phi_1^{-x} = \left(\frac{1+\sqrt{5}}{2}\right)^x + \left(\frac{1+\sqrt{5}}{2}\right)^{-x} \tag{5.120}$$

Hyperbolic Fibonacci and Lucas 2-functions

$$sF_2(x) = \frac{\Phi_2^x - \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x}\right] \tag{5.121}$$

$$cF_2(x) = \frac{\Phi_2^x + \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}\right] \tag{5.122}$$

$$sL_2(x)=\Phi_2^x-\Phi_2^{-x}=\left(1+\sqrt{2}\right)^x-\left(1+\sqrt{2}\right)^{-x} \tag{5.123}$$

$$cL_2(x)=\Phi_2^x+\Phi_2^{-x}=\left(1+\sqrt{2}\right)^x+\left(1+\sqrt{2}\right)^{-x} \tag{5.124}$$

Hyperbolic Fibonacci and Lucas 3-functions

$$sF_3(x)=\frac{\Phi_3^x-\Phi_3^{-x}}{\sqrt{13}}=\frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^x-\left(\frac{3+\sqrt{13}}{2}\right)^{-x}\right] \tag{5.125}$$

$$cF_3(x)=\frac{\Phi_3^x+\Phi_3^{-x}}{\sqrt{13}}=\frac{1}{\sqrt{13}}\left[\left(\frac{3+\sqrt{13}}{2}\right)^x+\left(\frac{3+\sqrt{13}}{2}\right)^{-x}\right] \tag{5.126}$$

$$sL_3(x)=\Phi_3^x-\Phi_3^{-x}=\left(\frac{3+\sqrt{13}}{2}\right)^x-\left(\frac{3+\sqrt{13}}{2}\right)^{-x} \tag{5.127}$$

$$cL_3(x)=\Phi_3^x+\Phi_3^{-x}=\left(\frac{3+\sqrt{13}}{2}\right)^x+\left(\frac{3+\sqrt{13}}{2}\right)^{-x} \tag{5.128}$$

Note that a list of these functions can be continued ad infinitum.

It is easy to see that the functions (5.103)-(5.106) are connected by very simple correlations:

$$sF_m(x)=\frac{sL_m(x)}{\sqrt{4+m^2}};\quad cF_m(x)=\frac{cL_m(x)}{\sqrt{4+m^2}}. \tag{5.129}$$

This means that the hyperbolic Lucas m-functions (5.103) and (5.104) coincide with the hyperbolic Fibonacci m-functions (5.105) and (5.106) to within the constant coefficient $1/\sqrt{1+m^2}$.

5.9.4. *Comparison of the Classical Hyperbolic Functions with the Hyperbolic Lucas m-Functions*

Let us compare the hyperbolic Lucas m-functions (5.105) and (5.106) with the classical hyperbolic functions (5.7) and (5.8). For the case

$$\Phi_m=\frac{\sqrt{4+m^2}+m}{2}=e \tag{5.130}$$

the hyperbolic Lucas m-functions (5.105) and (5.106) coincide with the classical hyperbolic functions (5.7) and (5.8) to within the constant coefficient 1/2, that is,

$$sh(x)=\frac{sL_m(x)}{2}\ \text{and}\ ch(x)=\frac{cL_m(x)}{2}. \tag{5.131}$$

By using (5.130), after simple transformations we can calculate the value

m_e, for which the equality (5.130) is valid:

$$m_e = e - \frac{1}{e} \approx 2.35040238... \quad (5.132)$$

Thus, according to (5.131) the classical hyperbolic functions (5.7) and (5.8) are a partial case of the hyperbolic Lucas m-functions, if m is equal to (5.132).

As the classical hyperbolic functions (5.7) and (5.8) are a partial case of (5.105) and (5.106), we have the right to assert that the formulas (5.103) through (5.106) represent a general class of hyperbolic functions.

5.9.5. *Recursive Properties of the Hyperbolic Fibonacci and Lucas m-Functions*

The hyperbolic Fibonacci and Lucas m-functions possess recursive properties similar to Fibonacci and Lucas m-numbers that are given by the recursive relations (4.250) and (4.302). On the other hand, they possess all of the hyperbolic properties similar to the properties of the classical hyperbolic functions. Let us first prove the recursive properties for the hyperbolic Fibonacci and Lucas m-functions.

Theorem 5.21. The following correlations that are similar to the recursive relation for the Fibonacci m-numbers $F_m(n+2)=mF_m(n+1)+F_m(n)$ are valid for the hyperbolic Fibonacci m-functions:

$$sF_m(x+2) = mcF_m(x+1) + sF_m(x) \quad (5.133)$$

$$cF_m(x+2) = msF_m(x+1) + cF_m(x). \quad (5.134)$$

Proof:

$$mcF_m(x+1) + sF_m(x) = m\frac{\Phi_m^{x+1} + \Phi_m^{-x-1}}{\sqrt{4+m^2}} + \frac{\Phi_m^{x} - \Phi_m^{-x}}{\sqrt{4+m^2}}$$

$$= \frac{\Phi_m^{x}(m\Phi_m + 1) - \Phi_m^{-x}(1 - m\Phi_m^{-1})}{\sqrt{4+m^2}}. \quad (5.135)$$

As $m\Phi_m + 1 = \Phi_m^2$ and $1 - \Phi_m^{-1} = \Phi_m^{-2}$, we can represent (5.135) as follows:

$$mcF_m(x+1) + sF_m(x) = \frac{\Phi_m^{x+2} - \Phi_m^{-x-2}}{\sqrt{4+m^2}} = sF_m(x+2)$$

that proves the identity (5.133).

By analogy, we can prove the identity (5.134).

Theorem 5.22 (a generalization of Cassini formula).

The following correlations, that are similar to the Cassini formula $F_m^2(n) - F_m(n-1)F_m(n+1) = (-1)^{n+1}$, are valid for the hyperbolic Fibonacci m-functions:

$$\left[sF_m(x)\right]^2 - cF_m(x+1)cF_m(x-1) = -1 \tag{5.136}$$

$$\left[cF_m(x)\right]^2 - sF_m(x+1)sF_m(x-1) = 1. \tag{5.137}$$

Proof:

$$\left[sF_m(x)\right]^2 - cF_m(x+1)cF_m(x-1)$$

$$=\left(\frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}}\right)^2 - \frac{\Phi_m^{x+1}+\Phi_m^{-x-1}}{\sqrt{4+m^2}} \times \frac{\Phi_m^{x-1}+\Phi_m^{-x+1}}{\sqrt{4+m^2}}$$

$$=\frac{\Phi_m^{2x} - 2 + \Phi_m^{-2x} - \left(\Phi_m^{2x}+\Phi_m^2+\Phi_m^{-2}+\Phi_m^{-2x}\right)}{4+m^2} \tag{5.138}$$

$$=\frac{-2-\left(\Phi_m^2+\Phi_m^{-2}\right)}{4+m^2}.$$

By using the Binet formula (4.292), for the case n=2 we can write:

$$L_m(2) = \Phi_m^2 + \Phi_m^{-2}. \tag{5.139}$$

By using the recursive formula (4.290) and the seeds (4.287) and (4.288), we can represent the Lucas m-number $L_m(2)$ as follows:

$$L_m(2) = mL_m(1) + L_m(0) = m\times m + 2 = m^2 + 2. \tag{5.140}$$

Taking into consideration (5.140), we can conclude from (5.138) that the identity (5.136) is valid.

By analogy, we can prove the identity (5.137).

5.9.6. *Hyperbolic Properties of the Hyperbolic Fibonacci and Lucas m-Functions*

Theorem 5.23. The following identity, that is similar to the identity $[ch(x)]^2$-$[sh(x)]^2$=1 for the classical hyperbolic functions, is valid for the hyperbolic Fibonacci m-functions:

$$\left[cF_m(x)\right]^2 - \left[sF_m(x)\right]^2 = \frac{4}{4+m^2}. \tag{5.141}$$

Proof:

$$\left[cF_m(x)\right]^2 - \left[sF_m(x)\right]^2 = \left(\frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}}\right)^2 - \left(\frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}}\right)^2$$

$$=\frac{\Phi_m^{2x}+2+\Phi_m^{-2x}-\Phi_m^{2x}+2-\Phi_m^{-2x}}{4+m^2} = \frac{4}{4+m^2}.$$

Theorem 5.24. The following identity, that is similar to the identity $[ch(x)]^2$- $[sh(x)]^2$=1 for the classical hyperbolic functions, is valid for the hyperbolic Lucas m-functions:

$$\left[cL_m(x)\right]^2-\left[sL_m(x)\right]^2=4. \tag{5.142}$$

The proof is analogous to Theorem 5.23.

Theorem 5.25. The following identity, that is similar to the identity $ch(x+y)=ch(x)ch(y)+sh(x)sh(y)$ for the classical hyperbolic functions, is valid for the hyperbolic Fibonacci m-functions:

$$\frac{2}{\sqrt{4+m^2}}cF_m(x+y)=cF_m(x)cF_m(y)+sF_m(x)sF_m(y). \tag{5.143}$$

Proof:

$$cF_m(x)cF_m(y)+sF_m(x)sF_m(y)$$
$$=\frac{\Phi_m^x+\Phi_m^{-x}}{\sqrt{4+m^2}}\times\frac{\Phi_m^y+\Phi_m^{-y}}{\sqrt{4+m^2}}+\frac{\Phi_m^x-\Phi_m^{-x}}{\sqrt{4+m^2}}\times\frac{\Phi_m^y-\Phi_m^{-y}}{\sqrt{4+m^2}}$$
$$=\frac{\Phi_m^{x+y}+\Phi_m^{x-y}+\Phi_m^{-x+y}+\Phi_m^{-x-y}+\Phi_m^{x+y}-\Phi_m^{x-y}-\Phi_m^{-x+y}+\Phi_m^{-x-y}}{4+m^2}$$
$$=\frac{2\left(\Phi_m^{x+y}+\Phi_m^{-x-y}\right)}{\sqrt{4+m^2}\times\sqrt{4+m^2}}=\frac{2}{\sqrt{4+m^2}}cF_m(x+y).$$

Theorem 5.26. The following identity, that is similar to the identity $ch(x-y)=ch(x)ch(y)-sh(x)sh(y)$ for the classical hyperbolic functions, is valid for the hyperbolic Fibonacci m-functions:

$$\frac{2}{\sqrt{4+m^2}}cF_m(x-y)=cF_m(x)cF_m(y)-sF_m(x)sF_m(y). \tag{5.144}$$

The proof is analogous to Theorem 5.25.

By analogy, we can prove the following theorems for the hyperbolic Fibonacci and Lucas m-functions.

Theorem 5.27. The following identities, that are similar to the identity $ch(2x)=[ch(x)]^2+[sh(x)]^2$ for the classical hyperbolic functions, are valid for the hyperbolic Fibonacci and Lucas m-functions:

$$\frac{2}{\sqrt{5}}cF_m(2x)=\left[cF_m(x)\right]^2+\left[sF_m(x)\right]^2 \tag{5.145}$$

$$2cL_m(2x)=\left[cL_m(x)\right]^2+\left[sL_m(x)\right]^2. \tag{5.146}$$

Theorem 5.28. The following identities, that are similar to the identity $sh(2x)=2sh(x)ch(x)$ for the classical hyperbolic functions, are valid for the hyperbolic Fibonacci and Lucas m-functions:

$$\frac{1}{\sqrt{4+m^2}}sF_m(2x)=sF_m(x)cF_m(x) \tag{5.147}$$

$$sL_m(2x) = sL_m(x)cL_m(x). \tag{5.148}$$

Theorem 5.29. The following formulas, that are similar to Moivre's formulas $[ch(x)\pm sh(x)]^n = ch(nx)\pm sh(nx)$ for the classical hyperbolic functions, are valid for the hyperbolic Fibonacci and Lucas m-functions:

$$\left[cF_m(x)\pm sF_m(x)\right]^n = \left(\frac{2}{\sqrt{4+m^2}}\right)^{n-1}\left[cF_m(nx)\pm sF_m(nx)\right] \tag{5.149}$$

$$\left[cL_m(x)\pm sL_m(x)\right]^n = 2^{n-1}\left[cF_m(nx)\pm sF_m(nx)\right]. \tag{5.150}$$

Thus, our research results in a general theory of hyperbolic functions based on the Gazale formulas (4.281) and (4.291). Over several centuries, science, in particular, mathematics and theoretical physics, made wide use of classical hyperbolic functions with base e. These functions were used by Lobachevsky in his non-Euclidean geometry and Minkowski in his geometric interpretation of Einstein's relativity theory. More recently Ukrainian mathematicians Stakhov, Tkachenko and Rozin [51, 98, 106, 116, 118, 119] broke monopoly on classical hyperbolic functions in contemporary mathematics and theoretical physics. It is now clear that the above hyperbolic Fibonacci and Lucas m-functions based on the Gazale formulas infinitely extendable to new hyperbolic models of Nature. It is difficult to imagine that the set of new hyperbolic functions has the same cardinality as the set of real numbers because every positive real number m generates its own kind of hyperbolic functions! And all of them possess unique recursive and hyperbolic properties similar to the properties of classical hyperbolic functions and the symmetric hyperbolic Fibonacci and Lucas functions introduced in [106, 116, 119].

5.10. A Puzzle of Phyllotaxis

5.10.1. *The Phenomenon of Phyllotaxis*

In Chapter 2 we described the botanical phenomenon known as *Phyllotaxis*. This phenomenon is inherent in many biological objects. On the surface of the bio-organs, these objects (sprouts of plants and trees, seeds on the disks of sunflower heads and pine cones, etc.) are arranged in clockwise and counterclockwise spirals. There are two types of phyllotaxis. The first type concerns the disposition of branches of plants and trees, and the second type concerns the dense-packed phyllotaxis objects such as that of a pinecone, pineapple, or the disk of a sunflower head. In the first case, the regularities of phyllotaxis are described by the ratios of adjacent Fibonacci numbers taken through one, that is,

$$\left[cL_m(x) \pm sL_m(x)\right]^n = 2^{n-1}\left[cF_m(nx) \pm sF_m(nx)\right]. \tag{5.151}$$

Each plant and tree possesses its own characteristic ratio taken from (5.151). This ratio is named the *Phyllotaxis Order* of a given plant or tree. The phyllotaxis orders are different for different plants and trees, for example, for linden, elm, beech and cereals the phyllotaxis order is equal to (2/1); for alder, hazel and grape (3/1); for oak and cherry (5/2); for raspberry, pear, poplar and barberries (8/3); and for almonds (13/5).

Note that trees and plants are subject primarily to the laws of Fibonacci phyllotaxis (5.151); in rare cases the plants are subject to Lucas phyllotaxis (Lucas numbers 2, 1, 3, 4, 7, 11, 18, 29, ...) or in a few cases phyllotaxis based on the numerical sequence: 5, 2, 7, 9, 16, 25, ... that also satisfies the Fibonacci recursive formula.

The other type of phyllotaxis is represented in the form of densely-packed botanical objects such as the head of a sunflower, pinecone, pineapple or cactus, etc. For such phyllotaxis objects, it is used usually the number ratios of the left-hand and right-hand spirals observed on the surface of the phyllotaxis objects. These ratios are equal to the ratios of adjacent Fibonacci numbers, that is,

$$\frac{F_{n+1}}{F_n}: \quad \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots \to \tau = \frac{1+\sqrt{5}}{2}. \tag{5.152}$$

For example, the head of a sunflower can have the phyllotaxis orders given by Fibonacci ratios: 89/55, 144/89 and even 233/144.

5.10.2. *Dynamic Symmetry*

When observing phyllotaxis the question arises: how do Fibonacci spirals forming on the surface during growth? This problem is one of the most intriguing *Puzzles of Phyllotaxis*. Its essence consists in the fact that the majority of bio-forms change their phyllotaxis orders during their growth. It is known, for example, that sunflower disks that are located on different levels of the same stalk have different phyllotaxis orders; moreover, the greater the age of the disk, the higher its phyllotaxis order tends to be. This means that during the growth of the phyllotaxis object, a natural modification (or increase) of the symmetry relation occurs, and this modification of symmetry obeys the law:

$$\frac{2}{1} \to \frac{3}{2} \to \frac{5}{3} \to \frac{8}{5} \to \frac{13}{8} \to \frac{21}{13} \to \ldots. \tag{5.153}$$

The modification of the phyllotaxis orders according to (5.153) is called *Dynamic Symmetry* [37]. Many scientists who study this puzzle of phyllotaxis believe that the phenomenon of dynamic symmetry (5.153) is of fundamental interdisciplinary importance. Recall that, in Vernadsky's opinion, the problem of biological symmetry is the key problem of biology.

Thus, the phenomenon of dynamic symmetry (5.153) plays a special role in the geometric problem of phyllotaxis. One may assume that this numerical regularity (5.153) reflects some general geometric laws that hide the secret of the dynamic mechanism of phyllotaxis, and uncovering it would be of great importance for understanding the phyllotaxis phenomenon in general.

A new geometric theory of phyllotaxis was developed recently by the Ukrainian architect Oleg Bodnar. This original theory is stated in Bodnar's books [37, 52]. However, in order to understand Bodnar's research more fully, we have to get a better understanding of the geometric theory of hyperbolic functions [156].

5.11. A Geometric Theory of the Hyperbolic Functions

5.11.1. *Compression and Expansion*

We start the study of geometric theory of hyperbolic functions from the important geometric transformation of the hyperbolic geometry. This is a geometric transformation named the *Compression to a Straight Line o* with the *Compression Coefficient k* (Fig. 5.12). This transformation consists of the following: Every point A of the plane passes into the point A' that lies on the ray PA perpendicular to o, here the ratio $PA':PA=k$ or $PA'=kPA$ (Fig. 5.12-*a*). If the compression coefficient $k>1$, then $PA'>PA$ (Fig. 5.12-*b*); in this case the transformation could be named an *Expansion from a Straight Line o*. It is clear that the expansion using the coefficient k is equivalent to the compression with coefficient $(1/k)$.

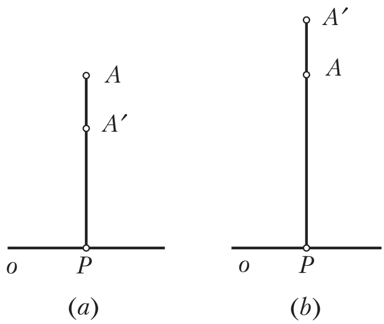


Figure 5.12. The compression of a point A to a straight line (*a*) and the expansion from a straight line (*b*)

A compression and an expansion possess a number of important properties [156]:

1. At the compression (expansion) every straight line passes on into a straight line.
2. At the compression (expansion) all parallels pass on into parallels.
3. At the compression (expansion) the ratio of the segments lying on one straight line remain constant.
4. At the compression (expansion), the areas of all figures change in a constant ratio equal to the compression coefficient k.

5.11.2. *Hyperbola*

Consider the important geometric curve called a *Hyperbola*. It is described by the following equality:

$y=a/x$ or $xy=a$. (5.154)

A graph of the hyperbola is represented in Fig. 5.13.

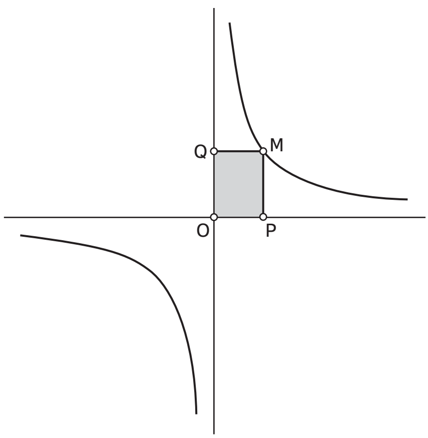


Figure 5.13. Hyperbola

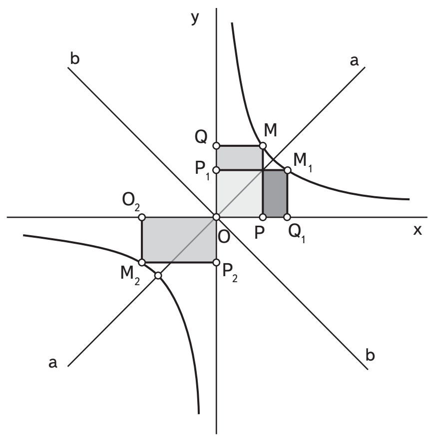


Figure 5.14. The axes of a hyperbola

It follows from (5.154) and Fig. 5.13 that a graph of the hyperbola consists of two branches that are located for the case $a>0$ in the first quadrant (x and y are positive) and in the third quadrant (x and y are negative) of the coordinate system. Geometrically, the branches of the hyperbola aim towards the coordinate axes, but they never intersect themselves. This means that the coordinate axes are *asymptotes* of the hyperbola. Note that the equation $xy=a$ has a simple geometric interpretation: the area of the rectangles $MQOP$ or $M'Q'O'P'$ that are bounded by the coordinate axes and the straight lines that are drawn through any points M and M' of the hyperbola in parallel to the coordinate axes (Fig. 5.13) is equal to a, that is, this area does not depend on the choice of the points M and M'. We will name these rectangles $MQOP$ and $M'Q'O'P'$ with area equal to a the *Coordinate Rectangles* of the points M and M'. Then we can give the following geometric definition of the hyperbola [156]:

"A hyperbola is the geometric location of the points that lie in the first and third quadrants of the coordinate system with coordinate rectangles that have constant area."

It is easy to prove that the origin of coordinate O is the *Centre of Symmetry* of the hyperbola, that is, the branches of the hyperbola are symmetric one to other with respect to the origin of coordinates *O*. The hyperbola also has the *Axes of Symmetry*, the bisectors of the coordinate angles *aa* and *bb* (Fig. 5.14). The centre of symmetry *O* and the axes of symmetry *aa* and *bb* are frequently called simply the *Center* and the *Axes* of the hyperbola; the points *A* and *B*, in which the hyperbola is intersected with the axis *aa*, are called *Tops* of the hyperbola.

Hereinafter, we will use analogies between hyperbola and circle. With this purpose in mind we will introduce, first of all, the concept of the *Diameter* of a hyperbola; every line segment passing through the centre of the hyperbola and connecting the points of the opposite branches of the hyperbola is called a *Diameter of the Hyperbola* (it is similar to the diameter of a circle passing through its centre). Let us also introduce the concept of a *radius* of the hyperbola; a line segment, going from the centre of the hyperbola up to the crossing point with the hyperbola, is called a *radius* of the hyperbola (that is, the radii of the hyperbola are determined similarly to the radii of the circle).

5.11.3. *Geometric Definition of the Hyperbolic Functions*

The hyperbola in Fig. 5.13 and Fig. 5.14 is the basis for a geometric definition of hyperbolic functions. The geometric theory of *Hyperbolic Functions* or *Hyperbolic Trigonometric Functions* is similar to the theory of traditional *Circular Trigonometric Functions*. In order to emphasize an analogy between hyperbolic and circular trigonometric functions, we will state the theory of the hyperbolic functions in parallel with the theory of circular trigonometric functions. We can choose the axis of symmetry of the hyperbola by the coordinate axes as is shown in Fig. 5.15 and then we use this geometric representation of the hyperbola for the geometric definition of the hyperbolic functions.

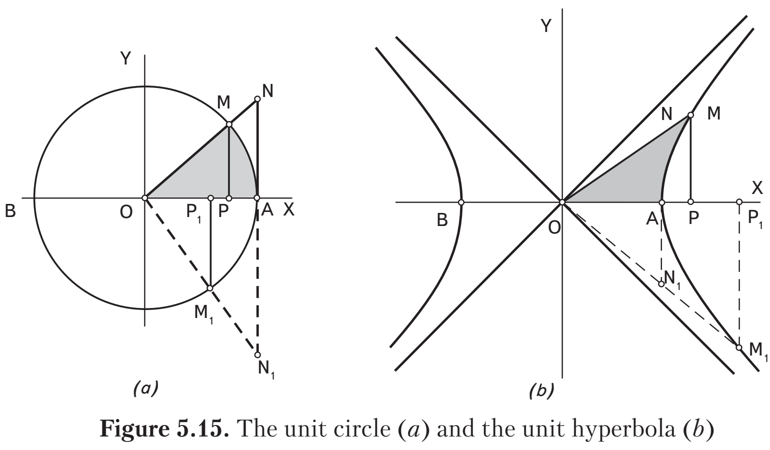


Figure 5.15. The unit circle (*a*) and the unit hyperbola (*b*)

Let us examine the unit circle (Fig. 5.15-*a*). We can see in Fig. 5.15-a a *Circular Sector OMA* bounded by the radii *OM, OA* and the arc *MA*. The number that is equal to the length of the arc *AM* or equal to the double area of the sector, bounded by the radii *OM* and *OA* and the arc *MA,* is called a *Radian Angle* α between the radii *OA* and *OM* of the circle. We can now drop the perpendicular *MP* to the diameter *OA* from the point *M* of the circle; at the point *A* we draw a tangent to the circle up to its intersection with the diameter *OM* at the point *N*. The line segment *PM* of the perpendicular is called a *Line of Sine*, the line segment *OP* of the diameter is called a *Line of Cosine* and the line segment *AN* is called a *Line of Tangent*. The lengths of the line segments *PM, OP* and *AN* are equal respectively to the *Sine, Cosine* and *Tangent* of the angle α, that is,

$PM=\sin\alpha$, $OP=\cos\alpha$, $AN=\mathrm{tg}\alpha$.

Now, let us examine the unit hyperbola (Fig. 5.15-*b*) $X^2 - Y^2 = 1$. We can see in Fig.20-b the *Hyperbolic Sector OMA* bounded by the hyperbolic radii *OM, OA* and the hyperbolic arc *MA*. Then, the number that is equal to the double area of the hyperbolic sector, bounded by these radii *OM, OA* and the arc *MA* of the hyperbola, is called the *Hyperbolic Angle t* between the hyperbolic radii *OA* and *OM*. We can now drop the perpendicular *MP* from the point *M* of the hyperbola to the diameter *OA*, which is a symmetry axis, intersecting the hyperbola at the top *A*. Next we draw a tangent to the hyperbola to its intersection with the radius *OM* at point *N*. The line segment *PM* of the perpendicular is called a *Line of Hyperbolic Sine*, the line segment *OP* of the axis *X* is called a *Line of Hyperbolic Cosine* and the line segment *AN* is called a *Line of Hyperbolic Tangent*. The lengths of the line segments *PM, OP* and *AN* are equal respectively to the *Hyperbolic Sine, Hyperbolic Cosine* and *Hyperbolic Tangent* of the hyperbolic angle *t,* that is,

$PM=\mathrm{sh}t$, $OP=\mathrm{ch}t$, $AN=\mathrm{th}t$.

As is well-known, the circular trigonometric functions are changing periodically with the period 2π. In contrast with this the hyperbolic functions are not periodic. It follows from Fig. 5.15-*b* that the hyperbolic angle *t* changes from 0 up to ∞. It follows from the definition of hyperbolic functions that at the change of the hyperbolic angle from 0 up to ∞, the hyperbolic sine sh*t* is changing from 0 up to ∞, the hyperbolic cosine ch*t* is changing from 1 up

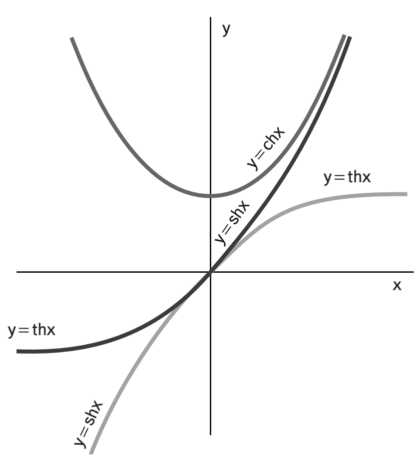


Figure 5.16. Graphs of the hyperbolic functions

to ∞ and the hyperbolic tangent th$t$ is changing from 0 up to 1. The graphs of these functions are represented in Fig. 5.16.

Based upon this geometric approach, it is easy to obtain the basic relations for the circular and hyperbolic trigonometric functions.

It follows from the similarity of the triangles OMP and ONA (Fig. 5.15-a) that

$$\frac{AN}{OA}=\frac{PM}{OP}.$$

However, $AN/OA=\mathrm{tg}\alpha$ (because $OA=1$), and $PM/OP=\sin\alpha/\cos\alpha$. Thus, we have: $\mathrm{tg}\alpha=\sin\alpha/\cos\alpha$.

Further, the coordinates of any point M of the circle are equal $OP=X$, $PM=Y$. However, then the following important identity follows from the unit circle equation $X^2+Y^2=1$:

$$OP^2+PM^2 = 1$$

or

$$\cos^2\alpha+\sin^2\alpha=1.$$

By dividing both parts of the obtained identity at first by $\cos^2\alpha$ and then by $\sin^2\alpha$ we get the following remarkable formulas for the trigonometric functions:

$$1+\mathrm{tg}^2\alpha=1/\cos^2\alpha$$

$$\mathrm{ctg}^2\alpha+1=1/\sin^2\alpha.$$

It follows from a similarity of the triangles OMP and ONA (Fig. 5.15-b) that

$$\frac{AN}{OA}=\frac{PM}{OP}.$$

However, $AN/OA=\mathrm{tg}t$ (because $OA=1$), and $PM/OP=\mathrm{sh}t/\mathrm{ch}t$. Thus, we have: $\mathrm{tg}t=\mathrm{sh}t/\mathrm{ch}t$.

Further, the coordinates of any point M of the hyperbola are equal $OP=X, PM=Y$. However, then the following important identity follows from the unit hyperbola equation

$$X^2-Y^2=1:$$

$$OP^2-PM^2 = 1$$

or

$$\mathrm{ch}^2t - \mathrm{sh}^2t=1.$$

By dividing both parts of the obtained identity at first by ch^2t and then by sh^2t we get the following remarkable formulas for the hyperbolic functions:

$$1-\mathrm{th}^2t=1/\mathrm{ch}^2t$$

$$\mathrm{cth}^2t-1=1/\mathrm{sh}^2t.$$

By using the geometric approach, we can prove many other identities for the trigonometric and hyperbolic functions.

5.11.4. *Hyperbolic Rotation*

Next we will study the hyperbola $xy=a$. First we make the compression of a plane to the axis x with the compression coefficient k. In this case the hyperbola $xy=a$ passes on into the hyperbola $xy=ka$ because the abscissa x remains without change and the ordinate y is replaced by yk. Then, we make one more compression of a plane to the axis y with coefficient $1/k$. Note that the compression with coefficient $1/k$ is equivalent to the expansion with coefficient k. After the fulfilment of the compression to axis y with coefficient $1/k$ it is equivalent to the extension from axis y with the same coefficient k, the hyperbola $xy=ka$ passes on into the hyperbola $xy=(ka/k)=a$, because the ordinate y of each point for the case of new compression to the axis y does not vary, and the abscissa x passes on into x/k. Thus, we can see that the sequential compression of the plane to the axis x with the compression coefficient k and then to the axis y with the compression coefficient $1/k$ transforms the hyperbola $xy=a$ into itself. A sequence of these two compressions of the plane to a straight line represents the important geometric transformation called *Hyperbolic Rotation*. The title of the hyperbolic rotation reflects the fact that in such transformation all points of the hyperbola act as though they "glide on a curve," that is, the hyperbola acts as though it "rotates."

Once again, note that the hyperbolic rotation is the sequential fulfilment of the two geometric transformations, at first the compression of a plane with the coefficient k to the axis x and then the expansion of a plane from the axis y with the same coefficient k.

From the above properties of the "compression" and "extension," the following properties of the hyperbolic rotation result:

1. At the hyperbolic rotation every straight line passes on into a straight line.
2. At the hyperbolic rotation the coordinate axes (the asymptotes of the hyperbola) pass on into themselves.
3. At the hyperbolic rotation parallels pass on into parallels.
4. At the hyperbolic rotation the ratios of all segments, lying on one and the same straight line, remain constant.
5. At the hyperbolic rotation, the areas of all transferred figures remain constant.

It is very important to emphasize that by means of the choice of the appropriate value of the coefficient k, and by means of the hyperbolic rotation we can transfer each point of the hyperbola into any other point of the same hyperbola. In fact, the compression to the axis x with a given coefficient k transfers the point (x,y) of the hyperbola $xy=a$ into the point (x,ky) of the other hyperbola $xy=ka$; after that the extension of the point (x,ky) from the axis y with the same coefficient k transfers the point (x,ky) of the hyperbola $xy=ka$ into the point $(x/y,ky)$ of the initial hyperbola. Thus, a result of the hyperbolic rotation the point (x,y) passes on into the point $(x/y,ky)$ of the initial hyperbola. It follows from here that by means of a suitable hyperbolic rotation we can transfer the point (x,y) of the hyperbola into the point (x_1,y_1) of the same hyperbola provided we take the compression coefficient $k=x/x_1$.

5.12. Bodnar's Geometry

5.12.1. *Structural - numerical Analysis of Phyllotaxis Lattices*

As previously noted, the Ukrainian architect Oleg Bodnar recently made an attempt to uncover the *Puzzle of Phyllotaxis* with a new phyllotaxis geometry ("Bodnar's geometry") presented in his books [37, 52].

To understand Bodnar's geometry let us study a cedar cone as a characteristic example of a phyllotaxis object (Fig. 5.17-a).

On the surface of the cedar cone every seed is blocked by the adjacent seeds in three directions. As a result, we can see the picture consisting of three types of spirals equal to the Fibonacci numbers: 3, 5 and 8. In an effort to simplify the geometric model of the phyllotaxis object in a Fig. 5.17-*a, b*, we can represent the phyllotaxis object in a cylindrical form (Fig.

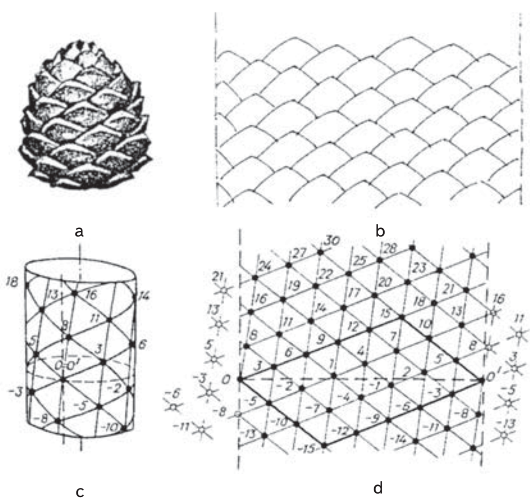


Figure 5.17. Analysis of structure-numerical properties of the phyllotaxis lattice

5.17-*c*). If we cut the surface of the cylinder in Fig. 5.17-*c* by the vertical straight line and then unroll the cylinder on a plane (Fig. 5.17-*d*), we obtain a fragment of the phyllotaxis lattice that is bounded by two parallel straight lines that are traces of the cutting line. We can see that the three groups of parallel straight lines in Fig. 5.17-*d*, namely, the three straight lines 0-21, 1-16, 2-8 with the right-hand small declination; the five straight lines 3-8, 1-16, 4-19, 7-27, 0-30 with the left-hand declination; and the eight straight lines 0-24, 3-27, 6-30, 1-25, 4-25, 7-28, 2-18, 5-21 with the right-hand abrupt declination, correspond to three types of spirals on the surface of the cylinder in Fig. 5.17-*c*.

We use the following method of numbering the lattice nodes in Fig. 5.17-*d*, introducing the following system of coordinates. We use the direct line OO' as the abscissa axis and the vertical trace that passes through the point O as the ordinate axis. Taking the ordinate of the point 1 as the length unit, the number that is ascribed to some point of the lattice is then equal to its ordinate. The lattice that is numbered by the indicated method has some characteristic properties. Any pair of the points gives a certain direction in the lattice system and, finally, the set of the three parallel directions of the phyllotaxis lattice. We can see that the lattice in Fig. 5.17-*d* consists of triangles. The vertices of the triangles are indicated by the letters *a, b and c*. It is clear that the lattice in Fig. 5.17-*d* consists of the set triangles of the kind {*c,b,a*}, for example, {0,3,8}, {3,6,11}, {3,8,11}, {6,11,14} and so on. It is important to note that the sides of the triangle {*c,b,a*} are equal to the differences between the values *a,b,c* of the triangle {*a,b,c*} and are the adjacent Fibonacci numbers: 3,5,8. For example, for the triangle {0,3,8} we have the following differences: 3-0=3; 8-3=5; 8-0=8. This means that the sides of the triangle {0,3,8} are equal to, respectively, 3, 5, 8. For the triangle {3,6,11} we have: 6-3=3; 11-6=5; 11-3=8. This means that its sides are equal to 3, 5, 8, respectively. Here each side of the triangle defines one of three declinations of the straight lines that build up the lattice in Fig. 5.17-*d*. In particular, the side of length 3 defines the right-hand small declination, the side of length 5 defines the left-hand declination, and the side of length 8 defines the right-hand abrupt declination. Thus, Fibonacci numbers 3, 5, 8 determine the structure of the phyllotaxis lattice in Fig. 5.17-*d*.

The second property of the lattice in Fig. 5.17-*d* is the following. The line segment OO' can be considered as a diagonal of the parallelogram constructed on the basis of the straight lines corresponding to the left-hand declination and the right-hand small declination. Thus, the given parallelogram allows one to evaluate the symmetry of the lattice without having to use digital numbering. This parallelogram is called a *Coordinate Parallelogram*. Note that coordinate parallelograms of different sizes correspond to lattices with different symmetries.

5.12.2. *The Dynamic Symmetry of Phyllotaxis Objects*

Here we begin an analysis of the phenomenon of dynamic symmetry. The idea of analysis involves the comparison of a series of phyllotaxis lattices (the unrolling of the cylindrical lattice) with different symmetries (Fig. 5.18).

In Fig. 5.18 a variance of Fibonacci phyllotaxis is illustrated, where we observe the following modification of the dynamic symmetry of the phyllotaxis object during its growth:

1:2:1→ 2:3:1→ 2:5:3 → 3:8:3 → 5:13:8

Note that the lattices, represented in Fig. 5.18, are considered as the sequential stages (5 stages) of the transformation of one and the same phyllotaxis object. There is a question: how are the transformations of the lattices being carried out, that is, which geometric movement can be used to provide the sequential passing of all illustrated stages of the phyllotaxis lattice?

5.12.3. *The Key Idea of Bodnar's Geometry*

Here we will not go deeply into Bodnar's original reasoning that resulted in a new geometrical theory of phyllotaxis, but direct readers to the remarkable Bodnar books [37, 52] for a more detailed acquaintance with his original geometry. Rather we focus our attention solely on two key ideas that underlie his geometry.

We first begin with an analysis of the phenomenon of dynamic symmetry. The idea of analysis involves a comparison of the series of the phyllotaxis lattices of different symmetries (Fig. 5.18). We start with the comparison of stages I and II. At these stages the lattice can be transformed by the compression of the plane along the 0-3 direction up to the position where the line segment 0-3 arrives at the edge of the lattice.

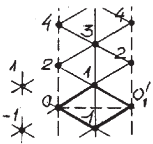
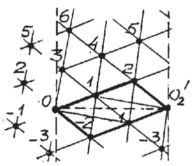
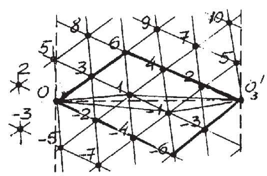
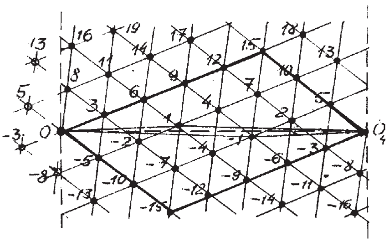
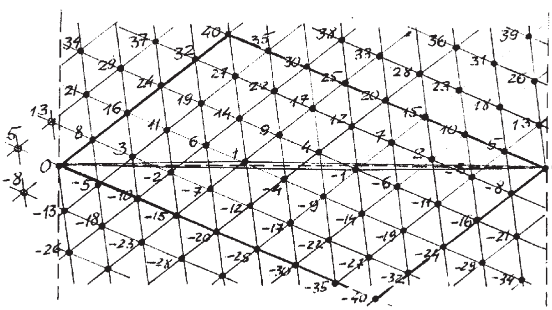






Figure 5.18. An analysis of dynamic symmetry of a phyllotaxis object

Simultaneously, the expansion of the plane should occur in the 1-2 direction perpendicular to the compression direction. At the passing on from stage II to stage III, the compression should be made along the *O*-5 direction and the expansion along the perpendicular 2-3 direction. The next passage is accompanied by similar deformations of the plane in the *O*-8 direction (compression) and in the perpendicular 3-5 direction (expansion).

But we know from the prior consideration that the compression of a plane to any straight line with the coefficient k and the simultaneous expansion of a plane in the perpendicular direction with the same coefficient k are nothing but *Hyperbolic Rotation*. A scheme of hyperbolic transformation of the lattice fragment is presented in Fig. 5.19. The scheme corresponds to stage II of Fig. 5.18. Note that the hyperbola of the first quadrant has the equation xy=1 and the hyperbola of the fourth quadrant has the equation xy=-1.

The transformation of the phyllotaxis lattice in the process of its growth is carried out by means of the hyperbolic rotation, the main geometric transformation of hyperbolic geometry, following from a consideration of the first key idea of Bodnar's geometry.

This transformation is accompanied by a modification of dynamic symmetry, which can be simulated by the sequential passage from the object with smaller symmetry order to the object with larger symmetry order.

However, this idea does not give the answer to the question: why are the phyllotaxis lattices in Fig. 5.18 based on Fibonacci numbers?

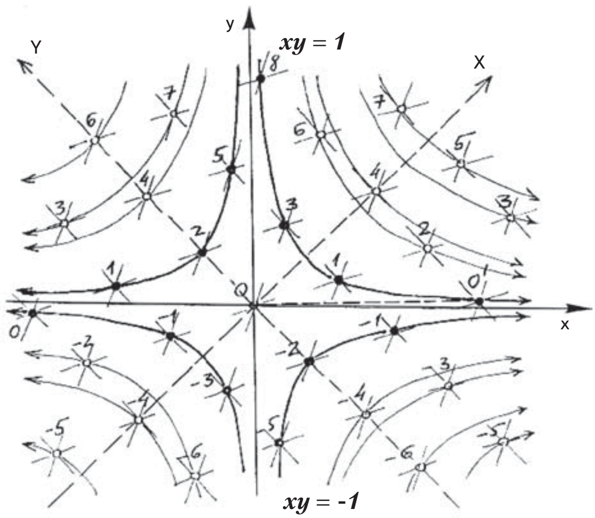


Figure 5.19. The general scheme of phyllotaxis lattice transformation in a system of equatorial hyperboles

5.12.4. *The "Golden" Hyperbolic Functions*

For a more detailed study of the metric properties of the lattice in Fig. 5.19 we can examine its fragment represented in Fig. 5.20. Here the disposition of the points is similar to Fig. 5.19.

Let us pay attention to the basic peculiarities of the disposition of points in Fig. 5.20: (1) the points M_1 and M_2 are symmetrical relative to the bisector of the right angle YOX; (2) the geometric figures $OM_1M_2N_1$, $OM_2N_2N_1$, and $OM_2M_3N_2$ are parallelograms; and (3) the point A is the vertex of the hyperbola $xy=1$, that is, $x_A=1, y_A=1$, therefore $OA=\sqrt{2}$.

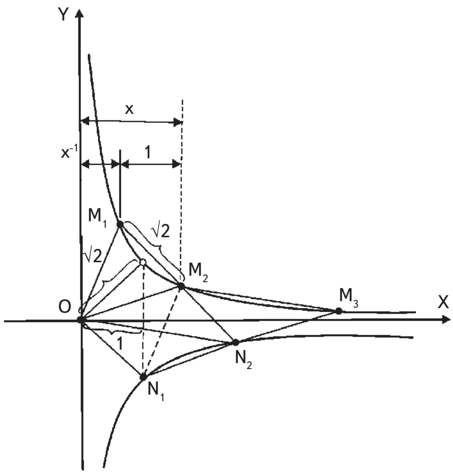


Figure 5.20. Analysis of the metric properties of the phyllotaxis lattice

Let us evaluate the abscissa of point M_2 denoted by $x_{M_2}=x$. Taking into consideration the symmetry of points M_1 and M_2, we can write: $x_{M_1}=x^{-1}$. It follows from the symmetry condition of these points that the line segment M_1M_2 is tilted to the coordinate axis under the angle of 45°. The line segment M_1M_2 is parallel to the line segment ON_1; this means that the line segment ON_1 is tilted to the coordinate axis under the angle of 45°. Therefore, the point N_1 is the top of the lower branch of the hyperbola; here $x_{N_1}=1$, $y_{N_1}=1$, $ON_1=OA=\sqrt{2}$. It is clear that $ON_1=M_1M_2=\sqrt{2}$. It is obvious that the difference between the abscissas of the points M_1 and M_2 is equal to 1.

These considerations result in the following equation for the calculation of the abscissa of the point M_2, that is, $x_{M_2}=x$:

$$x-x^{-1}=1 \text{ or } x^2-x-1=0. \tag{5.155}$$

This means that the abscissa $x_{M_2}=x$ is the positive root of the famous "golden" algebraic equation (5.155):

$$x_{M_2}=\tau=\frac{1+\sqrt{5}}{2}. \tag{5.156}$$

Thus, study of the metric properties of the phyllotaxis lattice in Fig. 5.20 unexpectedly leads us to the golden mean. And this fact is the *Second Key Outcome of Bodnar's Geometry*. This result was used by Bodnar for a detailed study of the phenomenon phyllotaxis. By developing this idea, Bodnar concluded that for a mathematical simulation of the phyllotaxis phenomenon we need to use a special class of the hyperbolic functions, *the "Golden" Hyperbolic Functions*:

The "golden" hyperbolic sine

$$Gshn=\frac{\tau^n-\tau^{-n}}{2}. \tag{5.157}$$

The "golden" hyperbolic cosine

$$Gchn = \frac{\tau^n + \tau^{-n}}{2}. \tag{5.158}$$

Furthermore, Bodnar found a fundamental connection of the "golden" hyperbolic functions with Fibonacci numbers:

$$F_{2k-1} = \frac{2}{\sqrt{5}} Gch(2k-1) \tag{5.159}$$

$$F_{2k} = \frac{2}{\sqrt{5}} Gsh(2k). \tag{5.160}$$

Using the correlations (5.159) and (5.160), Bodnar gave a very simple explanation for the "puzzle of phyllotaxis": why do Fibonacci numbers occur with such persistent constancy on the surface of phyllotaxis objects? The main reason is the fact that the geometry of "Living Nature," in particular, the geometry of phyllotaxis, is a non-Euclidean geometry; however, this geometry differs substantially from Lobachevsky's geometry and Minkowski's four-dimensional world based on the classical hyperbolic functions. This difference consists in the fact that the main correlations of this geometry are described with the help of the "golden" hyperbolic functions (5.157) and (5.158) that are connected with Fibonacci numbers by the simple correlations (5.159) and (5.160).

It is important to emphasize that Bodnar's model of the dynamic symmetry of the phyllotaxis object illustrated in Fig. 5.20 is brilliantly confirmed by real-life phyllotaxis pictures of botanic objects.

5.12.15. *The Connection of Bodnar's "Golden" Hyperbolic Functions with the Hyperbolic Fibonacci Functions*

Comparing the expressions for symmetric hyperbolic Fibonacci and Lucas sines and cosines given by formulas (5.57) - (5.60) with expressions for Bodnar's "golden" hyperbolic functions given by formulas (5.155) and (5.156), we discover the following simple correlations between the indicated groups of the formulas:

$$Gsh(x) = \frac{\sqrt{5}}{2} sFs(x) \tag{5.161}$$

$$Gch(x) = \frac{\sqrt{5}}{2} cFs(x) \tag{5.162}$$

$$Gsh(x) = 2sLs(x) \tag{5.163}$$

$$Gch(x)=2cLs(x). \tag{5.164}$$

The analysis of these correlations allows us to conclude that the "golden" hyperbolic sine and cosine introduced by Oleg Bodnar [37], and the symmetric hyperbolic Fibonacci and Lucas sines and cosines introduced by Stakhov and Rozin [106], coincide within constant coefficients. The question of the use of the "golden" hyperbolic functions or the hyperbolic Fibonacci and Lucas functions for the simulation of phyllotaxis objects has no particular significance because the final result is the same: it always results in an unexpected appearance of Fibonacci or Lucas numbers on the surfaces of phyllotaxis objects.

5.13. Conclusion

1. The discovery of Lobachevsky's geometry became an epoch-making event in the development not only mathematics, but also of science in general. Academician Kolmogorov appreciated the role of this discovery in the development of mathematics in the following words [1]: "... It is difficult to overrate the importance of the reorganization of the entire warehouse of mathematical thinking, which happened in the 19th century. In this connection, Lobachevsky's geometry was the most significant mathematical discovery at the start of the 19th century. Based upon this geometric insight, the belief in the absolute stability of mathematical axioms was overthrown. This allowed for the creation of essentially new and original abstract mathematical theories and, at last, to demonstrate that similar abstract theories can result hereafter in wider and more concrete applications." After Lobachevsky's discovery, the "hyperbolic ideas" started to penetrate widely into various spheres of science. After the promulgation of the special theory of relativity by Einstein in 1905 and its "hyperbolic interpretation," given by Minkowski in 1908, the "hyperbolic ideas" became universally recognized. Thus, a comprehension of the "hyperbolic character" of the processes in the physical world surrounding us became the major result in the development of science during the 19th and 20th centuries.

The mathematical correlations of Lobachevsky's geometry are or course based upon the classical hyperbolic functions. Why did Lobachevsky use these functions, introduced by Vincenzo Riccati in the late 18th century, in his geometry? Apparently, Lobachevsky understood that these functions provide the best way to model the "hyperbolic character" of his geometry; however, he used them because other hyperbolic functions at that moment simply did not exist. It is necessary to note that Lobachevsky's geometry, based on classical hyperbolic functions, is historically the first "hyperbolic model" of physical space. Lobachevsky's geometry and Minkowski's geometry had put forward the hyperbolic functions as the basic plan for modern science.

2. At the end of the 20th century, the Ukrainian architect Oleg Bodnar [37] and the Ukrainian mathematicians Alexey Stakhov and Ivan Tkachenko [98] broke the monopoly of classical hyperbolic functions in modern science. They introduced a new class of hyperbolic functions based on the golden mean. Later Alexey Stakhov and Boris Rozin developed the symmetrical hyperbolic Fibonacci and Lucas functions [108]. However, Bodnar, Stakhov, Tkachenko and Rozin each used their own unique methods to arrive at a new class of hyperbolic functions. Oleg Bodnar found the "golden" hyperbolic functions thanks to his scientific intuition, which resulted in the "golden" hyperbolic functions as the "key" idea in the study of the phyllotaxis phenomenon. These functions were used by him for the creation of the original geometric theory of phyllotaxis (*Bodnar's Geometry*). The approach of Alexey Stakhov, Ivan Tkachenko and Boris Rozin was based on an analogy between the Binet formulas and hyperbolic functions. This approach resulted in the discovery of a new class of hyperbolic functions, *Hyperbolic Fibonacci and Lucas Functions*.

3. The hyperbolic Fibonacci and Lucas functions are a generalization of the Fibonacci and Lucas numbers "extended" to the continuous domain. There is a direct analogy between the Fibonacci and Lucas number theory and the theory of hyperbolic Fibonacci and Lucas functions because the "extended" Fibonacci and Lucas numbers coincide with the hyperbolic Fibonacci and Lucas functions at discrete values of the variable x (x=0, ±1, ±2, ±3, ...). Besides, every identity for the Fibonacci and Lucas numbers has its continuous analog in the form of a corresponding identity for the hyperbolic Fibonacci and Lucas functions, and conversely. This outcome is of special significance for the Fibonacci number theory [13, 16, 28, 38] because this theory is as if it were transformed into the theory of hyperbolic Fibonacci and Lucas functions [98, 108, 116, 119]. Thanks to this approach, the Fibonacci and Lucas numbers became one of the most important numerical sequences of hyperbolic geometry.

4. Bodnar's geometry [37, 52] demonstrates that the "golden" hyperbolic world exists objectively and independently of our consciousness and persistently appears in Living Nature, in particular, in pine cones, pineapples, cacti, heads of sunflowers and baskets of various flowers in the form of Fibonacci and Lucas spirals on the surface of these biological objects (the phyllotaxis law). Furthermore Bodnar's geometry is a demonstration of the "physical practica-

bility" of the hyperbolic Fibonacci and Lucas functions, which underlie all of Living Nature. However, the promulgation of the new geometrical theory of phyllotaxis, made by the Ukrainian architect Oleg Bodnar [37], had demonstrated that in addition to "Lobachevsky's geometry." Nature also uses other variants of the so-called "hyperbolic models of Nature" [98, 108]. The use of hyperbolic Fibonacci and Lucas functions allowed for the solution to the "riddle of phyllotaxis," that is, an explanation of, how Fibonacci and Lucas spirals appear on the surface of phyllotaxis objects.

5. However, perhaps the final step in the development of the "hyperbolic models" of Nature was made by Alexey Stakhov in his article [118]. The hyperbolic Fibonacci and Lucas m-functions are a wide generalization of the symmetric hyperbolic Fibonacci and Lucas functions introduced by Stakhov and Rozin in an earlier article [106]. They are based on the Gazale formulas and extend ad infinitum a number of new hyperbolic models of Nature. It is difficult to imagine that the set of new hyperbolic functions is infinite! The hyperbolic Fibonacci and Lucas m-functions do complete a general theory of hyperbolic functions, started by Johann Heinrich Lambert and Vincenzo Riccati, and opens new perspectives for the development of new "hyperbolic ideas" in modern science.

The development of a general theory of hyperbolic functions [118], based on the Gazale formulas [45], gives us the opportunity to put forward the following unusual hypothesis. Apparently, we can assume that theoretically there are an infinite number of "hyperbolic models of Nature." One of the possible types of the hyperbolic functions given by the general formulas (5.105) - (5.108) underlies these models. Each type of hyperbolic function meets some positive real number m, called the order of the function. This number generates new mathematical constants, *the generalized golden m-proportions*, given by the formula (4.266). Each type of hyperbolic function, in turn, generates new classes of the recursive numerical sequences, the *generalized Fibonacci and Lucas m-numbers* given by the recursive relations (4.251) and (4.302).

Nature intelligently uses one or another type of hyperbolic function for introducing objects into this or that "hyperbolic world." It is necessary to note that "Lobachevsky's geometry" is one of the possible variants of the realization of the "hyperbolic world," which probably is preferable for the objects of "mineral Nature." Apparently in the case of "Living Nature," "Bodnar's geometry" based on the hyperbolic Fibonacci and Lucas functions, is preferable. This fact is confirmed by the "law of phyllotaxis" met in many botanical objects. But "Lobachevsky's geometry" and "Bodnar's geometry" are not unique variants of the realization of the "hyperbolic worlds." It is possible to expect that new types of hyperbolic functions, described in [118], will result in the discovery of new "hyperbolic worlds" which can be embodied in natural structures at different levels of organization in the Universe. Importantly, the occurrence of characteristic recursive numerical sequences given by the general recursive formulas (4.251) and (4.302) are "external attributes", embodied in Nature's "hyperbolic worlds," found in the search for corresponding types of hyperbolic functions (5.105) - (5.108).

Chapter 6

Fibonacci and Golden Matrices

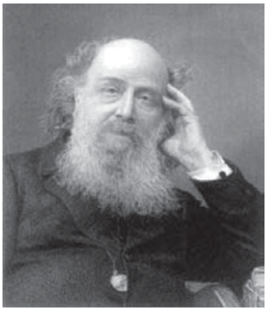
6.1. Introduction to Matrix Theory

6.1.2. *A History of Matrices*

The history of matrices goes back to ancient times. But the term "matrix" was not applied to the subject until 1850 by James Joseph Sylvester. "Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean, more generally, any place in which something is formed or produced.

The origin of mathematical matrices arose with the study of systems of simultaneous linear equations. An important Chinese text from between 300 BC and 200 AD, *Nine Chapters of the Mathematical Art* (Chiu Chang Suan Shu), is the first known example of the use of matrix methods to solve simultaneous equations. In the treatise's seventh chapter, *Too Much and Not Enough,* the concept of a determinant first appears, nearly two millennia before its supposed invention by the Japanese mathematician Seki Kowa in 1683, or alternatively, his German contemporary Gottfried Leibnitz (who is also credited with the invention of differential calculus, independently though simultaneously with Isaac Newton). More uses of matrix-like arrangements of numbers appear in chapter eight, *Methods of Rectangular Arrays,* in which a method is given for solving simultaneous equations. This method is mathematically identical to the modern matrix method of solution outlined by Carl Friedrich Gauss (1777-1855), also known as Gaussian elimination.

James Joseph Sylvester (1814 – 1897) was an English mathematician. He made fundamental contributions to matrix theory, invariant theory, number theory, partition theory and combinatorics. He played a leading role in American mathematics in the latter half of the 19th century as a professor at Johns Hopkins University and as founder of the American Journal of Mathematics. At his death, he was a professor at Oxford.



James Joseph Sylvester
(1814 - 1897)

Sylvester began his study of mathematics at St. John's College, Cambridge in 1831. In 1838 he became professor of natural philosophy at University College London. In 1841 he moved to the United States to become a professor at the University of Virginia, but soon returned to England. On his return to England he studied law, alongside British lawyer/mathematician Arthur Cayley, with whom he made significant contributions to matrix theory. Poetry was one of Sylvester's lifelong passions; he read and translated works out of the original French, German, Italian, Latin and Greek, and many of his mathematical papers contain illustrative quotes from classical poetry. In 1870, following his early retirement, Sylvester published a book titled *The Laws of Verse* in which he attempted to codify a set of poetry laws. In 1877 he again crossed the Atlantic to become the inaugural professor of mathematics at the new Johns Hopkins University in Baltimore, Maryland. In 1878 he founded the *American Journal of Mathematics*. And in 1883, Sylvester returned to England to become Savilian Professor of Geometry at Oxford University.

Since their first appearance in ancient China, matrices have remained important mathematical tools. Today, they are used not only for solving systems of simultaneous linear equations, but also for describing quantum atomic structure, designing computer game graphics, analyzing relationships, and in numerous other capacities.

The elevation of the matrix from mere tool to important mathematical theory owes a lot to the work of female mathematician Olga Taussky Todd (1906-1995), who began by using matrices to analyze vibrations on airplanes and became the torch-bearer for matrix theory.

6.1.2. *Definition of a Matrix*

In mathematics, a *Matrix* (plural *Matrices*) is a rectangular table of numbers or, more generally, a table consisting of abstract quantities that can be added and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformations, and to record data that depend on two parameters. Matrices can be added, multiplied, and decomposed in various ways, making them a key concept in linear algebra and matrix theory [157].

Let us consider, for example, a matrix A

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The horizontal lines in a matrix are called *Rows* and the vertical lines are called *Columns*. A matrix with m rows and n columns is called an m-by-n matrix (written $m \times n$) and m and n are called its *Dimensions*. The dimensions of a matrix are always given with the number of rows first, then the number of columns. The entry of a matrix A that lies in the i-th row and the j-th column is called the (i,j)-th entry of A. This is written as $A_{i,j}$ or $A[i,j]$. As is indicated, the row is always noted first, then the column.

A matrix, where one of the dimensions is equal to 1, is often called a *Vector*. The $(1\times n)$-matrix (one row and n columns) is called a *Row Vector*, and the $(m\times 1)$-matrix (m rows and one column) is called a *Column Vector*.

6.1.3. *Matrix Addition and Scalar Multiplication*

Let A and B be two matrices with the same size, i.e. the same number of rows and of columns. The sum of A and B, written $A+B$ is the matrix obtained by adding corresponding elements from A and B:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \ldots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \ldots & a_{2n}+b_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \ldots & a_{mn}+b_{mn} \end{pmatrix}.$$

The product of a scalar k and a matrix A, written kA or Ak is the matrix obtained by multiplying each element of A by k:

$$k\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}=\begin{pmatrix} ka_{11} & ka_{12} & \ldots & ka_{1n} \\ ka_{21} & ka_{22} & \ldots & ka_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ ka_{m1} & ka_{m2} & \ldots & ka_{mn} \end{pmatrix}.$$

We also define:

$-A=(-1)A$ and $A-B=A+(-B)$

Note that the matrix $(-A)$ is negative to the matrix A.

Let us consider the following theorem for matrix addition and scalar multiplication.

Theorem 6.1. Let A, B and C be matrices with the same size and let k and k' be scalars. Then we have:

$(A+B)+C = A+(B+C)$, i.e., matrix addition is *associative*;

$(A+B) = (B+A)$, i.e., matrix addition is *commutative*;

$A+0=0+A$; $A+(-A)=0$; $k(A+B)=kA+kB$;

$(k+k')A = kA+k'A$; $(k\times k')A = k(k'A)$; $1\times A = A$.

6.1.4. *Matrix Multiplication*

Suppose A and B are two matrices such that the number of columns of A is equal to the number of rows of B, say A is an $(m\times p)$-matrix and B is a $(p\times n)$-matrix. Then the product of A and B, written AB is the $(m\times n)$-matrix whose (i, j)-entry is obtained by multiplying the elements of the i-th row of A by the corresponding elements of the j-th column of B and then adding:

$$\begin{pmatrix} a_{11} & \dots & a_{1p} \\ \dots & \dots & \dots \\ a_{i1} & \dots & a_{ip} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ \dots & c_{ij} & \dots \\ \dots & \dots & \dots \\ c_{m1} & \dots & c_{mn} \end{pmatrix},$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

Matrix multiplication satisfies the following properties:

Theorem 6.2.

$(AB)C = A(BC)$; $A(B+C) = AB+AC$;

$k(AB) = (kA)B = A(kB)$,

where k is a scalar.

6.1.5. *Square Matrices*

6.1.5.1. *Definition of a Square Matrix*

A matrix with the same number of rows and columns is called a *Square Matrix*. A square matrix with n rows and n columns is said to be of order n, and is called an n-square matrix. The *Main Diagonal*, or simply *Diagonal*, of a square matrix $A=(a_{ij})$ consists of the numbers:

$(a_{11}, a_{22}, \dots, a_{nn})$.

The n-square matrix with 1's along the main diagonal and 0's elsewhere, e.g.

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is called the *Unit* or *Identity Matrix* and will be denoted by I. The unit matrix I plays the same role in matrix multiplication as the number 1 does in the usual multiplication of numbers.

Specifically,

$AI=IA=A$

for any square matrix A.

Square matrices can be raised to various powers. We can define powers of the square matrix A as follows:

$A^2=AA$, $A^3= A^2A$, ..., and $A^0=I$.

6.1.5.2. *Invertible Matrices*

A square matrix A is said to be *Invertible* if there exists a matrix B with the property that

$AB=BA=I$.

Such a matrix B is unique; it is called the *Inverse* of A and is denoted by A^{-1}.

6.1.5.3. *Determinants*

To each n-square matrix $A=(a_{ij})$ we assign a specific number called the *Determinant* of A, denoted by $Det(A)$ or $|A|$.

The determinants of order one, two and three are defined as follows:

$$Det(a_{11}) = |a_{11}| = a_{11}$$

$$Det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

An important determinant property is given by the following theorems.

Theorem 6.3. For any two n-square matrices A and B we have

$Det(A\times B)=DetA\times DetB$

Theorem 6.4

$$Det(A^n)=(DetA)^n. \tag{6.1}$$

6.2. Fibonacci *Q*-Matrix

6.2.1. *A Definition of the Q-Matrix*

In recent decades the theory of Fibonacci numbers [13, 16, 28] has been supplemented by the theory of the so-called Fibonacci Q-matrix [9]. The latter is the (2×2)-matrix of the following form:

$$Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}. \tag{6.2}$$

Note that the determinant of the Q-matrix is equal to -1:

$$DetQ=-1. \tag{6.3}$$

The article [158] devoted to the memory of Verner E. Hoggatt, founder of the Fibonacci Association, contained the history and an extensive bibliography of the Q-matrix and emphasized Hoggatt's contribution in its development. Although the name of the Q-matrix was introduced before Verner E. Hoggatt, it was from Hoggatt's articles that the idea of the Q-matrix "caught on like wildfire among Fibonacci enthusiasts. Numerous papers have appeared in 'The Fibonacci Quarterly' authored by Hoggatt and/or his students and other collaborators where the Q-matrix method became a central tool in the analysis of Fibonacci properties" [158].

We will consider here a theory of the Q-matrix developed in Hoggatt's book [16].

6.2.2. *Properties of the Q-Matrix*

The following theorem connects the Q-matrix to Fibonacci numbers.

Theorem 6.5. For a given integer n the nth power of the Q-matrix is given by

$$Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n-1}\end{pmatrix}, \tag{6.4}$$

where F_{n-1}, F_n, F_{n+1}, are Fibonacci numbers.

Proof. We use mathematical induction. Clearly, for n=1,

$$Q^1=\begin{pmatrix}F_2 & F_1\\ F_1 & F_0\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}.$$

Suppose that $Q^k=\begin{pmatrix}F_{k+1} & F_k\\ F_k & F_{k-1}\end{pmatrix}$, then

$$Q^{k+1}=Q^k\times Q=\begin{pmatrix}F_{k+1} & F_k\\ F_k & F_{k-1}\end{pmatrix}\times\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}F_{k+1}+F_k & F_{k+1}\\ F_k+F_{k-1} & F_k\end{pmatrix}=\begin{pmatrix}F_{k+2} & F_{k+1}\\ F_{k+1} & F_k\end{pmatrix}.$$

The theorem is proved.

The next theorem gives a formula for the determinant of the matrix (6.5).

Theorem 6.6. For a given integer n we have:

$$Det(Q^n)=(-1)^n. \tag{6.5}$$

Proof. Using (6.1) and (6.5), we can write:

$Det(Q^n)=(DetQ)^n=(-1)^n$.

The theorem is proved.

The following remarkable property for Fibonacci numbers follows from Theorem 6.5:

$$DetQ^n=F_{n-1}F_{n+1}-F_n^2=(-1)^n. \tag{6.6}$$

Recall that the identity (6.6) is one of the most important identities for Fibonacci numbers. This one is called the *Cassini formula* in honor of the famous French astronomer Giovanni Domenico Cassini (1625-1712), who discovered this formula for the first time.

Now, represent the matrix (6.4) in the following recursive form:

$$Q^n=\begin{pmatrix}F_n+F_{n-1} & F_{n-1}+F_{n-2}\\ F_{n-1}+F_{n-2} & F_{n-2}+F_{n-3}\end{pmatrix}=\begin{pmatrix}F_n & F_{n-1}\\ F_{n-1} & F_{n-2}\end{pmatrix}+\begin{pmatrix}F_{n-1} & F_{n-2}\\ F_{n-2} & F_{n-3}\end{pmatrix} \tag{6.7}$$

or

$$Q^n=Q^{n-1}+Q^{n-2}. \tag{6.8}$$

We can represent the recursive relation (6.8) in the following form:

$$Q^{n-2}=Q^n-Q^{n-1}. \tag{6.9}$$

The explicit forms of the matrices Q^n and Q^{-n} $(n=0,\pm1,\pm2,\pm3,\ldots)$ obtained by means of the use of the recursive relations (6.8) and (6.9) are given in Table 6.1.

Table 6.1. Q-matrices

n	0	1	2	3	4	5
Q^n	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$	$\begin{pmatrix}3 & 2\\ 2 & 1\end{pmatrix}$	$\begin{pmatrix}5 & 3\\ 3 & 2\end{pmatrix}$	$\begin{pmatrix}8 & 5\\ 5 & 3\end{pmatrix}$
Q^{-n}	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}$	$\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$	$\begin{pmatrix}-1 & 2\\ 2 & -3\end{pmatrix}$	$\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}$	$\begin{pmatrix}-3 & 5\\ 5 & -8\end{pmatrix}$

6.2.3. *Binet Formulas for the Q-Matrix*

We can find the analytical expressions for the following sum and difference:

$$Q^n+Q^{-n} \text{ and } Q^n-Q^{-n}. \tag{6.10}$$

We can start from the following examples for the even (n=2k) and odd (n=2k+1) powers. For the case n=5 we have the following expression for the matrix sum (6.10):

$$Q^5+Q^{-5}=\begin{pmatrix}8&5\\5&3\end{pmatrix}+\begin{pmatrix}-3&5\\5&-8\end{pmatrix}=\begin{pmatrix}5&10\\10&-5\end{pmatrix}=5\begin{pmatrix}1&2\\2&-1\end{pmatrix}=F_5T, \tag{6.11}$$

where F_5 is a Fibonacci number and

$$T=\begin{pmatrix}1&2\\2&-1\end{pmatrix}. \tag{6.12}$$

We have the following expression for the matrix difference (6.10):

$$Q^5-Q^{-5}=\begin{pmatrix}8&5\\5&3\end{pmatrix}-\begin{pmatrix}-3&5\\5&-8\end{pmatrix}=\begin{pmatrix}11&0\\0&11\end{pmatrix}=11\begin{pmatrix}1&0\\0&1\end{pmatrix}=L_5I, \tag{6.13}$$

where L_5 is a Lucas number, I is the identity matrix.

For the case n=6 we have:

$$Q^6+Q^{-6}=\begin{pmatrix}13&8\\8&5\end{pmatrix}+\begin{pmatrix}5&-8\\-8&13\end{pmatrix}=\begin{pmatrix}18&0\\0&18\end{pmatrix}=18\begin{pmatrix}1&0\\0&1\end{pmatrix}=L_6I \tag{6.14}$$

$$Q^6-Q^{-6}=\begin{pmatrix}13&8\\8&5\end{pmatrix}-\begin{pmatrix}5&-8\\-8&13\end{pmatrix}=\begin{pmatrix}8&16\\16&-8\end{pmatrix}=8\begin{pmatrix}1&2\\2&-1\end{pmatrix}=F_6T, \tag{6.15}$$

where the matrix T is given by (6.12).

The expressions (6.11), (6.13), (6.14), (6.15) are the partial cases of the following general formulas valid for an arbitrary n (n=0, ±1, ±2, ±3,...):

$$Q^{2k+1}-Q^{-(2k+1)}=L_{2k+1}I \tag{6.16}$$

$$Q^{2k}-Q^{-2k}=L_{2k}I \tag{6.17}$$

$$Q^{2k+1}+Q^{-(2k+1)}=F_{2k+1}T \tag{6.18}$$

$$Q^{2k}+Q^{-2k}=F_{2k+1}T \tag{6.19}$$

Note that the formulas (6.16) - (6.19) are the matrix equivalents of the Binet formulas (2.67) and (2.68). The formulas (6.16) and (6.17) are the matrix equivalent of the Binet formula (2.67) for Lucas numbers and the formulas (6.18) and (6.19) are the matrix equivalents of the Binet formula (2.68) for Fibonacci numbers. It is clear that the Q-matrix (6.2) in these formulas plays the role of the golden ratio τ in the formulas (2.67) and (2.68). For all that, the special matrix (6.12) plays the role of the irrational number $\sqrt{5}$ in the Binet formula (2.68) for Fibonacci numbers.

This analogy between the golden ratio τ and Q-matrix (6.2) shows that there is an isomorphism between the golden mean theory and the Q-matrix theory. We can prove the following theorems confirming this isomorphism. First of all, we can write the following trivial property for the golden ratio powers:

$$\tau^n\tau^m = \tau^m\tau^n = \tau^{n+m}.$$

Note that we do not need to prove for the number τ the following obvious equality: $\tau^n\tau^m = \tau^m\tau^n$, however, for the matrices we need to prove the similar equality.

We can prove the following theorem for the Q-matrices.

Theorem 6.7.

$$Q^nQ^m = Q^mQ^n = Q^{n+m}. \tag{6.20}$$

Proof. At first, we prove the identity

$$Q^nQ^m = Q^{n+m}.$$

In order to prove this identity we can write the product of the matrices Q^nQ^m as follows:

$$Q^nQ^m = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \times \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix}$$
$$= \begin{pmatrix} F_{n+1}F_{m+1} + F_nF_m & F_{n+1}F_m + F_nF_{m-1} \\ F_nF_{m+1} + F_{n-1}F_m & F_nF_{m-1} + F_{n-1}F_{m-1} \end{pmatrix}. \tag{6.21}$$

For the proof we use the formula for the generalized Fibonacci numbers G_n described in Section 2.5:

$$G_{n+m} = G_{n+1}F_m + G_nF_{m-1}. \tag{6.22}$$

If $G_i = F_m$, the formula (6.22) takes the following form:

$$F_{n+m} = F_{n+1}F_m + F_nF_{m-1}. \tag{6.23}$$

Using (6.23), we can represent the elements of the matrix (6.21) as follows:

$$F_{n+1}F_{m+1} + F_nF_m = F_{n+m+1} \tag{6.24}$$

$$F_{n+1}F_m + F_nF_{m-1} = F_{n+m} \tag{6.25}$$

$$F_nF_{m+1} + F_{n-1}F_m = F_{n+m} \tag{6.26}$$

$$F_nF_{m-1} + F_{n-1}F_{m-1} = F_{n+m-1}. \tag{6.27}$$

Taking into consideration the formulas (6.24) - (6.27), we can represent the matrix (6.21) as follows:

$$Q^nQ^m = \begin{pmatrix} F_{n+m+1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{pmatrix} = Q^{n+m}.$$

By analogy we can prove the validity of the following identity:

$$Q^mQ^n = Q^{n+m}.$$

It follows from the identity (6.20) that the matrices Q^n and Q^m possess the property of *multiplication commutativity*.

Thus, the above property of the Q-matrix, given by (6.20), gives us the right to affirm that the Fibonacci matrix $Q=\begin{pmatrix}1&1\\1&0\end{pmatrix}$ is a special matrix, which plays in the theory of two-by-two matrices a particular role similar to the role of the golden mean in the theory of real numbers.

6.3. Generalized Fibonacci Q_p-Matrices

6.3.1. *A Definition of the Q_p-Matrices*

Note that the Q-matrix is a generating matrix for the classical Fibonacci numbers given by the recursive relation (2.3). In the article [103] Alexey Stakhov generalized the concept of the Q-matrix [16, 158] and introduced the generalized Fibonacci Q_p*-matrices*. For a given p (p=1, 2, 3, ...) the generalized Fibonacci Q_p-matrix has the following form:

$$Q_p=\begin{pmatrix}1&1&0&0&\cdots&0&0\\0&0&1&0&\cdots&0&0\\0&0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&\cdots&1&0\\0&0&0&0&\cdots&0&1\\1&0&0&0&\cdots&0&0\end{pmatrix} \tag{6.28}$$

The Q_p-matrix (6.28) is a square $(p+1)\times(p+1)$-matrix. It consists of the identity $(p\times p)$-matrix bordered by the last row consisting of 0's and the leading 1, and the first column consisting of 0's embraced by a pair of 1's. We can list the Q_p -matrices for the case p=1, 2, 3, ... as follows:

$$Q_1=\begin{pmatrix}1&1\\1&0\end{pmatrix}=Q;\quad Q_2=\begin{pmatrix}1&1&0\\0&0&1\\1&0&0\end{pmatrix};\quad Q_3=\begin{pmatrix}1&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix};\quad Q_4=\begin{pmatrix}1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\end{pmatrix}. \tag{6.29}$$

6.3.2. *The Main Theorems for the Q_p-Matrices*

Next let us raise the Q_p-matrix (6.28) to the n-th power and find the analytical expression for the matrix Q_p^n. Let us prove the following theorem.

Theorem 6.8. For a given integer p= 1, 2, 3, ... we have:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}, \tag{6.30}$$

where $F_p(n)$ is the Fibonacci p-number, n=0, ±1, ±2, ±3, ...

Before the proof of Theorem 6.8 we analyze the matrix (6.30). Note that all entries of the matrix are Fibonacci p-numbers given by the recursive relation (4.18) at the seeds (4.19). The matrix (6.30) consists of (p+1) rows and (p+1) columns. The first row is the following sequence of the Fibonacci p-numbers:

$F_p(n+1), F_p(n),..., F_p(n\text{-}p+2), F_p(n\text{-}p+1),$ (6.31)

the second row consists of the Fibonacci p-numbers:

$F_p(n\text{-}p+1), F_p(n\text{-}p),..., F_p(n\text{-}2p+2), F_p(n\text{-}2p+1),$ (6.32)

the p-th row consists of the Fibonacci p-numbers:

$F_p(n\text{-}1), F_p(n\text{-}2),..., F_p(n\text{-}p), F_p(n\text{-}p\text{-}1),$ (6.33)

the (p+1)-th row consists of the following Fibonacci p-numbers:

$F_p(n), F_p(n\text{-}1),..., F_p(n\text{-}p+1), F_p(n\text{-}p).$ (6.34)

Note that every one of the rows given by (6.31) – (6.34) is the sequence of the Fibonacci p-numbers consisting of (p+1) sequential Fibonacci p-numbers. The first row (6.31) begins with the Fibonacci p-number $F_p(n+1)$ and finishes with the Fibonacci p-number $F_p(n\text{-}p+1)$, the second row (6.32) begins with the Fibonacci p-number $F_p(n\text{-}p+1)$ and finishes with the Fibonacci p-number $F_p(n\text{-}2p+1)$ and finally, the (p+1)-th row begins from the Fibonacci p-number $F_p(n)$ and finishes with the Fibonacci p-number $F_p(n\text{-}p)$. It is important to emphasize that the second Fibonacci p-number of the first row (6.31) and the first Fibonacci p-number of the last row (6.34) are equal to $F_p(n)$, the third Fibonacci p-number of the second row (6.32), the fourth Fibonacci p-number of the third row, ..., and the (p+1)-th Fibonacci p-number of the p-th row (6.33) are equal to $F_p(n\text{-}p\text{-}1)$. Now we can start a proof of Theorem 6.8.

Proof. For a given p we can use the induction method.

(a) The basis of the induction. At first we prove the basis of the induction, that is, we prove that the statement (6.30) is valid for the case n=1. It is clear that for the case n=1 the matrix (6.30) takes the following form:

$$\begin{pmatrix} F_p(2) & F_p(1) & \cdots & F_p(-p+3) & F_p(-p+2) \\ F_p(-p+2) & F_p(-p+1) & \cdots & F_p(-2p+3) & F_p(-2p+2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(0) & F_p(-1) & \cdots & F_p(-p+1) & F_p(-p) \\ F_p(1) & F_p(0) & \cdots & F_p(-p+2) & F_p(-p+1) \end{pmatrix}. \tag{6.35}$$

Taking into consideration the values of the initial terms of the Fibonacci p-numbers given by (4.19) and the mathematical properties of the extended Fibonacci p-numbers given by (4.29), (4.31) and (4.32) we can write:

$F_p(2)=F_p(1)=F_p(-p)=1.$

Note that all the remaining entries of the matrix (6.34) are equal to 0. This means that the matrix (6.35) coincides with the Q_p-matrix given by (6.28). The basis of the induction is proved.

(b) Inductive hypothesis. Suppose that the statement (6.30) is valid for arbitrary n, and then prove it for n=1. For this purpose we shall consider the product of the matrices

$$Q_p^n \times Q_p = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (6.36)$$

Let us now consider the matrix (6.36). If we multiply the first row of the matrix Q_p^n by the matrix Q_p in the example (6.36) we obtain the entry a_{11}:

$a_{11}= F_p(n+1)+ F_p(n-p+1)= F_p(n+2).$

Continuing the process of the matrix multiplication for the example (6.36), we obtain the following entries of the first row of the matrix Q_p^{n+1}:

$$F_p(n+2), F_p(n+1),..., F_p(n-p+3), F_p(n-p+2). \quad (6.37)$$

Next let us compare the first row of the matrix Q_p^n given by (6.31) with the first row of the matrix Q_p^{n+1} given by (6.37). We can see that all arguments of the Fibonacci p-numbers in the sequence (6.37) differ by 1 from the arguments of the corresponding Fibonacci p-numbers, which form the first row (6.31) of the matrix (6.30). By analogy we can show that this rule is valid for all entries of the matrix Q_p^{n+1}, which is formed from the matrix Q_p^n by means of its multiplication by the matrix Q_p in accordance with (6.36). This consideration proves the validity of Theorem 6.8.

It is easy to prove the following identity for the product of the following Q_p -matrices:

$$Q_p^{n+1} = Q_p^n \times Q_p = Q_p \times Q_p^n. \quad (6.38)$$

Using (6.38), we can prove the following theorem.

Theorem 6.9.

$Q_p^n \times Q_p^m = Q_p^m \times Q_p^n = Q_p^{m+n}.$

Decomposing each entry of the matrix (6.30) - the Fibonacci p-number - in accordance with the basic recursive relation (4.18), we obtain the following result given by Theorem 6.10.

Theorem 6.10.

$$Q_p^n = Q_p^{n-1} + Q_p^{n-p-1}. \tag{6.39}$$

This recursive relation (6.39) can also be represented in the following form:

$$Q_p^{n-p-1} = Q_p^n - Q_p^{n-1}. \tag{6.40}$$

For example, consider the matrices Q_p^n corresponding to the cases p=2 and p=3

$$Q_2^n = \begin{pmatrix} F_2(n+1) & F_2(n) & F_2(n-1) \\ F_2(n-1) & F_2(n-2) & F_2(n-3) \\ F_2(n) & F_2(n-1) & F_2(n-2) \end{pmatrix} \tag{6.41}$$

$$Q_3^n = \begin{pmatrix} F_3(n+1) & F_3(n) & F_3(n-1) & F_3(n-2) \\ F_3(n-2) & F_3(n-3) & F_3(n-4) & F_3(n-5) \\ F_3(n-1) & F_3(n-2) & F_3(n-3) & F_3(n-4) \\ F_3(n) & F_3(n-1) & F_3(n-2) & F_3(n-3) \end{pmatrix}. \tag{6.42}$$

Using the matrices (6.41) and (6.42) and the general matrix (6.30), we obtain some special matrices, in particular, the matrices of the kind Q_p^p and the inverse matrices of the kind Q_p^{-1}. For example, for the cases p=2, 3 , 4, 5 the matrices of the kind Q_p^p and Q_p^{-1} have the following form:

***p*=2**

$$Q_2^2 = \begin{pmatrix} F_2(3) & F_2(2) & F_2(1) \\ F_2(1) & F_2(0) & F_2(-1) \\ F_2(2) & F_2(1) & F_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{6.43}$$

$$Q_2^{-1} = \begin{pmatrix} F_2(0) & F_2(-1) & F_2(-2) \\ F_2(-2) & F_2(-3) & F_2(-4) \\ F_2(-1) & F_2(-2) & F_2(-3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{6.44}$$

***p*=3**

$$Q_3^3 = \begin{pmatrix} F_3(4) & F_3(3) & F_3(2) & F_3(1) \\ F_3(1) & F_3(0) & F_3(-1) & F_3(-2) \\ F_3(2) & F_3(1) & F_3(0) & F_3(-1) \\ F_3(3) & F_3(2) & F_3(1) & F_3(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{6.45}$$

$$Q_3^{-1} = \begin{pmatrix} F_3(0) & F_3(-1) & F_3(-2) & F_3(-3) \\ F_3(-3) & F_3(-4) & F_3(-5) & F_3(-6) \\ F_3(-2) & F_3(-3) & F_3(-4) & F_3(-5) \\ F_3(-1) & F_3(-2) & F_3(-3) & F_3(-4) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.46}$$

***p*=4**

$$Q_4^4=\begin{pmatrix}1&1&1&1&1\\1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\end{pmatrix}\quad Q_4^{-1}=\begin{pmatrix}0&0&0&0&1\\1&0&0&0&-1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}\tag{6.47}$$

***p*=5**

$$Q_5^5=\begin{pmatrix}1&1&1&1&1&1\\1&0&0&0&0&0\\1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&0&0\\1&1&1&1&1&0\end{pmatrix}\quad Q_5^{-1}=\begin{pmatrix}0&0&0&0&0&1\\1&0&0&0&0&-1\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{pmatrix}.\tag{6.48}$$

Looking at the matrices (6.43) - (6.48), we can see that they have a strong regular form. This allows us to construct similar matrices for arbitrary p. Comparing the matrices of the kind $Q_p^p\ (p=1,2,3,...)$ we can see that each matrix Q_p^p includes in itself all the preceding matrices of the kind $Q_{p-1}^{p-1}, Q_{p-2}^{p-2}, ..., Q_1^1$. For example, the matrix Q_{p-1}^{p-1} is built up from the matrix Q_p^p by means of striking out the last row and last column of the matrix Q_p^p.

The last rows and columns of all matrices of the kind Q_p^p have a strong regular structure. In particular, each last row has the following form (111...10) (p+1 entries) and each last column begins with the top 1, but all the remaining entries are equal to zero.

Comparing the inverse matrices of the kind $Q_p^{-1}\ (p=1,2,3,...)$, we can find the following regularity. The main diagonal of the matrix Q_p^{-1} consists of zeros. All entries, forming the diagonal under the main diagonal, are equal to 1. The last column starts with the entries 1 and −1. All of the remaining entries of the matrix Q_p^{-1} are equal to 0.

There is the following connection between the next matrices Q_p^{-1} and Q_{p-1}^{-1}. The matrix Q_{p-1}^{-1} can be obtained from the matrix Q_p^{-1}, if we strike out the last row and the next to the last column of the matrix Q_p^{-1}.

6.4. Determinants of the Q_p-Matrices and their Powers

Now let us calculate the determinants of the Q_p-matrices (6.28). It is clear that $DetQ_1$=-1. Calculate the determinant of the Q_2-matrix

$$Q_2=\begin{pmatrix}1&1&0\\0&0&1\\1&0&0\end{pmatrix}.\tag{6.49}$$

Comparing the Q_2-matrix (6.49) with the Q_1-matrix

$$Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q, \tag{6.50}$$

we can see that the Q_2-matrix (6.49) is reduced to the Q_1-matrix (6.50) if we strike out from the Q_2-matrix (6.49) the 2nd row and the 3rd column. Note that the sum of these numbers 2+3=5. According to the matrix theory [157] we have:

$$DetQ_2=1\times(-1)^5\times DetQ_1=1\times(-1)\times(-1)=1. \tag{6.51}$$

Now let us calculate the determinant of the Q_3-matrix

$$Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.52}$$

Comparing the matrix (6.52) with the matrix (6.49) we can see that the Q_3-matrix (6.52) is reduced to the Q_2-matrix (6.49) if we strike out the 3rd row and the 4th column (3+4=7) in the matrix (6.52). This means [157] that

$$DetQ_3=1\times(-1)^7\times DetQ_2=1\times(-1)\times1=-1. \tag{6.53}$$

By analogy we can show that

$$DetQ_4=1\times(-1)^9\times DetQ_3=1\times(-1)\times(-1)=1. \tag{6.54}$$

Ccontinuing this process, that is, generating the formulas similar to (6.51), (6.53), and (6.54) we can write the following recursive relation, which connects the determinants of the adjacent matrices Q_p and Q_{p-1}:

$$DetQ_p=1\times(-1)^{2p+1}\times DetQ_{p-1}. \tag{6.55}$$

It is easy to show that $DetQ_p$ is equal to 1 for the even values of $p=2k$ and is equal to (-1) for the odd values of $p=2k+1$. We can formulate this result in the form of the following theorem.

Theorem 6.11. For a given p=1,2,3,... we have:

$$DetQ_p=(-1)^p . \tag{6.56}$$

Let us calculate the determinants of the inverse matrices of the kind Q_p^{-1}. For this purpose we consider the well-known correlation, which connects the "direct" and "inverse" matrices:

$$Q_p^{-1}\times Q_p^1 = I, \tag{6.57}$$

where I is the identity matrix with

$$DetI=1. \tag{6.58}$$

Applying to (6.57) the well-known rule given by Theorem 6.3 for the determinant of the product of two matrices and taking into consideration (6.58), we can write:

$$Det\, Q_p^{-1} \times Det\, Q_p^1 = Det\, I = 1. \tag{6.59}$$

In accordance with (6.56) the determinant of the matrix Q_p^1 takes one of the two values, 1 or -1; it follows from (6.59) that

$$Det\, Q_p^{-1} = Det\, Q_p^1 = 1. \tag{6.60}$$

Let us calculate the determinant of the matrix Q_p^n using (6.1):

$$Det\, Q_p^n = \left(Det\, Q_p\right)^n, \tag{6.61}$$

where n=0,±1,±2,±3,... .

Taking into consideration the result of Theorem 6.11, we can formulate the following theorem.

Theorem 6.12. For given p=1,2,3,... and n=0,±1,±2,±3,..., the determinant of the matrix Q_p^n is given by the following expression:

$$Det\, Q_p^n = (-1)^{pn}. \tag{6.62}$$

It is clear that Theorem 6.12 is a generalization of the well-known Theorem 6.6 for the Q-matrix that corresponds to the case p=1.

Note that Theorems 6.8 and 6.12 are a source for new results in the field of Fibonacci number theory [16].

For example, consider the matrix Q_2^n for the case p=2

$$Q_2^n = \begin{pmatrix} F_2(n+1) & F_2(n) & F_2(n-1) \\ F_2(n-1) & F_2(n-2) & F_2(n-3) \\ F_2(n) & F_2(n-1) & F_2(n-2) \end{pmatrix} \tag{6.63}$$

where n=0,±1,±2,±3,... .

By calculating the determinant of the matrix (6.63) and by using the identity (6.62), we can write the case for p=2:

$$Det\, Q_2^n = 1. \tag{6.64}$$

It is easy to prove the following theorem.

Theorem 6.13. For p=2 and n=0,±1,±2,±3,..., we have the following identity connecting the adjacent Fibonacci 2-numbers:

$$\begin{aligned} Det\, Q_2^n = {} & F_2(n+1)\left[F_2(n-2)F_2(n-2) - F_2(n-1)F_2(n-3)\right] \\ & + F_2(n)\left[F_2(n)F_2(n-3) - F_2(n-1)F_2(n-2)\right] \\ & + F_2(n-1)\left[F_2(n-1)F_2(n-1) - F_2(n)F_2(n-2)\right] = 1. \end{aligned} \tag{6.65}$$

It is clear that Theorem 6.8 and 6.12 give a theoretically infinite number of correlations similar to (6.6) and (6.65). If we remember that Fibonacci p-numbers were obtained in the study of the diagonal sums of Pascal's triangle, we can assert that the identities similar to (6.6) and (6.65) express some unusual mathematical properties of Pascal's triangle!

6.5.The “Direct” and “Inverse” Fibonacci Matrices

Again, let us consider Table 6.1, which gives the “direct” and “inverse” Q-matrices. By comparing the “direct” (Q^n) and “inverse (Q^{-n}) Fibonacci Q-matrices, it is easy to find a very simple method that allows us to obtain the “inverse” matrix Q^{-n} from its “direct” matrix Q^n.

In fact, if the power n of the “direct” matrix Q^n given by (6.4) is even (n=2k), then for obtaining its inverse matrix Q^{-n} it is simply necessary to interchange the places of the diagonal elements F_{n+1} and F_{n-1} in (6.4) and to change the sign of the diagonal elements F_n. This means that for the case n=2k the “inverse” matrix Q^{-n} has the following form:

$$Q^{-2k}=\begin{pmatrix} F_{2k-1} & -F_{2k} \\ -F_{2k} & F_{2k+1} \end{pmatrix}. \tag{6.66}$$

In order to obtain the “inverse” matrix Q^{-n} from the “direct” matrix Q^n given by (6.4) for the case n=2k+1, it is necessary to interchange the places of the diagonal elements F_{n+1} and F_{n-1} in (6.4) and give them the opposite sign, that is:

$$Q^{-2k-1}=\begin{pmatrix} -F_{2k} & F_{2k+1} \\ F_{2k+1} & -F_{2k+2} \end{pmatrix}. \tag{6.67}$$

Another way to obtain the matrices Q^n follows directly from the expression (6.4). Here we can represent two Fibonacci series F_{n+1} and F_{n-1} shifted by one number with respect to the other (Table 6.2).

Table 6.2. Fibonacci series F_{n+1} and F_{n-1}

n	**5**	4	3	2	**1**	0	–1	–2	–3	–4	**-5**	–6
F_{n+1}	**8**	**5**	3	2	**1**	**1**	0	1	–1	2	**-3**	**5**
F_n	**5**	**3**	2	1	**1**	**0**	1	–1	2	–3	**5**	**-8**

If we select the number n=1 in the first row of Table 6.2 and then select the four Fibonacci numbers in the lower two rows under the number 1 and to the right with respect to it, then the totality of these Fibonacci numbers form the Q-matrix. The Q-matrix is singled out in bold type in Table 6.2. If we move in Table 6.2 to the left with respect to the Q-matrix, then we obtain the matrices Q^2, Q^3,..., Q^n, respectively. If we move in Table 6.2 to the right with respect to the Q-matrix, then we obtain the matrices Q^0, Q^{-1},..., Q^{-n}, respectively. Also the Fibonacci matrices Q^5 and the “inverse” to it, Fibonacci matrix Q^{-5} are singled out in bold type in Table 6.2. Note that the matrix Q^0 is an identity matrix.

This principle of the Q-matrices design can be used for the general case of Q_p-matrices. The analysis of the matrix (6.30) for the case p=2 shows that all matrices of the kind Q_2^n can be obtained from (6.30), provided we represent three series of the adjacent Fibonacci 2-numbers $F_2(n+1), F_2(n-1), F_2(n)$, shifted with respect to each other as is shown in Table 6.3.

Table 6.3. Fibonacci 2-numbers $F_2(n+1), F_2(n-1), F_2(n)$

n	**5**	4	3	2	**1**	0	–1	–2	–3	–4	**-5**	–6	–7
$F_2(n+1)$	**4**	**3**	**2**	1	**1**	**1**	**0**	0	1	0	**-1**	**1**	**1**
$F_2(n-1)$	**2**	**1**	**1**	1	**0**	**0**	**1**	0	–1	1	**1**	**-2**	**0**
$F_2(n)$	**3**	**2**	1	1	**1**	**0**	**0**	1	0	–1	**1**	**1**	**-2**

Select the number n=1 in the first row of Table 6.3 and then select the known Fibonacci 2-numbers in the lower rows with respect to the number 1 as is shown in Table 6.3 (they are singled out by bold type). It is clear that the totality of singled out Fibonacci 2-numbers form the Q_2-matrix.

If we move to the left in Table 6.3 with respect to the Q_2-matrix, then we obtain the matrices $Q_2^2, Q_2^3, ..., Q_2^n$, respectively. If we move to the right in Table 6.3 with respect to the Q_2-matrix, then we obtain the matrices $Q_2^0 = I, Q_2^{-1}, ..., Q_2^{-n}$, respectively.

By analogy, using (6.30), we can construct the table of the Fibonacci 3-numbers that give the matrices of the kind Q_3^n (see Table 6.4).

Table 6.4. Fibonacci 3-numbers $F_3(n+1), F_3(n-2), F_3(n-1), F_3(n)$

n	5	4	3	2	**1**	0	–1	–2	–3	–4	–5	–6	–7
$F_3(n+1)$	3	2	1	1	**1**	**1**	**0**	**0**	0	1	0	0	–1
$F_3(n-2)$	1	1	1	0	**0**	**0**	**1**	**0**	0	–1	1	0	1
$F_3(n-1)$	1	1	1	1	**0**	**0**	**0**	**1**	0	0	–1	1	0
$F_3(n)$	2	1	1	1	**1**	**0**	**0**	**0**	1	0	0	–1	1

6.6. Fibonacci G_m-Matrices

6.6.1. *A Definition of the Fibonacci G_m-Matrix*

In Chapter 4 we introduced the generalized Fibonacci m-numbers given by the recursive relation

$$F_m(n+2)=mF_m(n+1)+F_m(n) \tag{6.68}$$

at the seeds

$$F_m(0)=0, F_m(1)=1, \tag{6.69}$$

where $m>0$ is a given real number and $n=0,\pm1,\pm2,\pm3,\ldots$.

Similar to the Fibonacci Q-matrix (6.2), which is a generating matrix for the classical Fibonacci numbers, we can introduce the G_m-matrix [118] that is a generating matrix for the Fibonacci m-numbers given by the recursive relation (6.68) at the seeds (6.69).

G_m-matrix

$$G_m = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.70}$$

Note that the determinant of the G_m-matrix (6.70) is equal to -1:

$$Det\, G_m = m\times 0 - 1\times 1 = -1. \tag{6.71}$$

The following theorem gives a connection of the G_m-matrix (6.70) to the Fibonacci m-numbers given by (6.68) and (6.69) .

Theorem 6.14. For a given integer $n=0,\pm1,\pm2,\pm3,\ldots$, the n-th power of the G_m-matrix is given by

$$G_m^n = \begin{pmatrix} F_m(n+1) & F_m(n) \\ F_m(n) & F_m(n-1) \end{pmatrix}, \tag{6.72}$$

where $F_m(n-1), F_m(n), F_m(n+1)$ are the Fibonacci m-numbers.

Proof. We use mathematical induction. Clearly, for $n=1$

$$G_m^1 = \begin{pmatrix} F_m(2) & F_m(1) \\ F_m(1) & F_m(0) \end{pmatrix}. \tag{6.73}$$

By using the seeds (6.69) and the recursive relation (6.68), we can write:

$$F_m(0)=0, F_m(1)=1, F_m(2)=mF_m(1)+F_m(0)=m. \tag{6.74}$$

It follows from (6.73) and (6.74) that

$$G_m^1 = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.75}$$

The basis of the induction is therefore proved.

Suppose that for a given integer k our inductive hypothesis is the following:

$$G_m^k = \begin{pmatrix} F_m^{k+1} & F_m^k \\ F_m^k & F_m^{k-1} \end{pmatrix}.$$

We can then write:

$$G_m^{k+1} = G_m^k \times G_m = \begin{pmatrix} F_m(k+1) & F_m(k) \\ F_m(k) & F_m(k-1) \end{pmatrix} \times \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} mF_m(k+1)+F_m(k) & F_m(k+1) \\ mF_m(k)+F_m(k-1) & F_m(k) \end{pmatrix}=\begin{pmatrix} F_m(k+2) & F_m(k+1) \\ F_m(k+1) & F_m(k) \end{pmatrix}.$$

Thus, the theorem is proved.

6.6.2. *Determinants of the G_m-Matrices*

The next theorem gives a formula for the determinant of the matrix (6.72).

Theorem 6.15. For a given integer n=0,±1,±2,±3,..., we have:

$$Det\ G_m^n=(-1)^n. \tag{6.76}$$

Proof. By using the property (6.1) and by taking into consideration (6.71), we can write:

$$Det\ G_m^n=(Det\ G_m)^n=(-1)^n.$$

Thus, the theorem is proved.

6.6.3. *G_m-Matrix and Cassini Formula for the Fibonacci m-Numbers*

If we calculate the determinant of the matrix (6.72) and take into consideration (6.76), we obtain the following identity:

$$Det\ G_m^n=F_m(n+1)\times F_m(n-1)-F_m^2(n)=(-1)^n. \tag{6.77}$$

Note that the identity (6.77) is one of the most important identities for the Fibonacci m-numbers. It is clear that the identity (6.77) is a generalization of the well-known Cassini formula (6.6).

6.6.4. *Some Properties of the G_m-Matrices*

Theorem 6.16.

$$G_m^n=mG_m^{n-1}+G_m^{n-2}. \tag{6.78}$$

Proof. By using the recursive relation (6.68), we can represent the matrix (6.72) as follows:

$$G_m^n=\begin{pmatrix} mF_m(n)+F_m(n-1) & mF_m(n-1)+F_m(n-2) \\ mF_m(n-1)+F_m(n-2) & mF_m(n-2)+F_m(n-3) \end{pmatrix}$$

$$=m\begin{pmatrix} F_m(n) & F_m(n-1) \\ F_m(n-1) & F_m(n-2) \end{pmatrix}+\begin{pmatrix} F_m(n-1) & F_m(n-2) \\ F_m(n-2) & F_m(n-3) \end{pmatrix}=mG_m^{n-1}+G_m^{n-2}.$$

Thus, the theorem is proved.

We can also represent the recursive relation (6.78) in the following form:

$$G_m^{n-2} = G_m^n - G_m^{n-1}. \tag{6.79}$$

Based on the recursive relations (6.78) and (6.79), we can construct the sequences of the G_m-matrices (6.72) similar to Table 6.1. Note that for the case m=1 the matrices G_1^n coincide with the matrices Q^n, that is, Table 6.1 gives a sequence of the matrices G_1^n.

Consider the case m=2. Remember that for this case a sequence of the Fibonacci m-numbers $F_{m=2}(n)$ looks similar to Table 6.5.

Table 6.5. A sequence of the Fibonacci m-numbers for the case m=2

n	6	5	4	3	2	1	0	–1	–2	–3	–4	–5	–6
$F_2(n)$	70	29	12	5	2	1	0	1	–2	5	–12	29	–70

Let us construct a sequence of the matrices G_2^n. For the case n=0, we define the matrix G_m^n as follows:

$$G_m^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.80}$$

Using the recursive relation (6.78) and taking into consideration the seeds (6.80) and (6.75), we can construct the matrices G_2^2, G_2^3, G_2^4 and so on as follows:

$$G_2^2 = 2\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \tag{6.81}$$

$$G_2^3 = 2\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} \tag{6.82}$$

$$G_2^4 = 2\begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix}. \tag{6.83}$$

Using the recursive relation (6.79) and taking into consideration the seeds (6.80) and (6.75), we can construct the matrices $G_2^{-1}, G_2^{-2}, G_2^{-3}$ and so on as follows:

$$G_2^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \tag{6.84}$$

$$G_2^{-2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \tag{6.85}$$

$$G_2^{-3} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ 5 & -12 \end{pmatrix}. \tag{6.86}$$

A sequence of the matrices G_2^n is represented in Table 6.6.

Table 6.6. A sequence of the matrices G_2^n

n	0	1	2	3	4	5
G_2^n	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$\begin{pmatrix}5&2\\2&1\end{pmatrix}$	$\begin{pmatrix}12&5\\5&2\end{pmatrix}$	$\begin{pmatrix}29&12\\12&5\end{pmatrix}$	$\begin{pmatrix}70&29\\29&12\end{pmatrix}$
G_2^{-n}	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&-2\end{pmatrix}$	$\begin{pmatrix}1&-2\\-2&5\end{pmatrix}$	$\begin{pmatrix}-2&5\\5&-12\end{pmatrix}$	$\begin{pmatrix}5&-12\\-12&29\end{pmatrix}$	$\begin{pmatrix}-12&29\\29&-70\end{pmatrix}$

Consider the case m=3. Remember that for the case m=3 a sequence of the Fibonacci m-numbers $F_m(n)$ looks similar to Table 6.7.

Table 6.7. A sequence of the Fibonacci m-numbers for the case m=3

n	6	5	4	3	2	1	0	–1	–2	–3	–4	–5	–6
$F_3(n)$	360	109	33	10	3	1	0	1	–3	10	–33	109	–360

Using the recursive formulas (6.78) and (6.79) at the seeds (6.75) and (6.80) for the case m=3 we can construct the matrices G_3^n (see Table 6.8).

Table 6.8. A sequence of the matrices G_3^n

n	0	1	2	3	4	5
G_3^n	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}3&1\\1&0\end{pmatrix}$	$\begin{pmatrix}10&3\\3&1\end{pmatrix}$	$\begin{pmatrix}33&10\\10&3\end{pmatrix}$	$\begin{pmatrix}109&33\\33&10\end{pmatrix}$	$\begin{pmatrix}360&109\\109&33\end{pmatrix}$
G_3^{-n}	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&-3\end{pmatrix}$	$\begin{pmatrix}1&-3\\-3&10\end{pmatrix}$	$\begin{pmatrix}-3&10\\10&-33\end{pmatrix}$	$\begin{pmatrix}10&-33\\-33&109\end{pmatrix}$	$\begin{pmatrix}-33&109\\109&-360\end{pmatrix}$

It is easy to verify that all square matrices of the kind G_m^n in Table 6.6 and Table 6.8 possess one surprising property: all their determinants are equal to +1 (for the even powers n) or (–1) (for the odd powers n). In fact, the determinant of the matrix $G_2^4=\begin{pmatrix}29&12\\12&5\end{pmatrix}$ is equal to 29×5–12×12=1 and the determinant of the matrix $G_3^{-5}=\begin{pmatrix}-33&109\\109&-360\end{pmatrix}$ is equal to (–33)×(–360)–109×109=–1.

Now, let us consider a general case of m. Remember that the number m takes its value from the range of positive real numbers, for example, m can take the following values: $m=\sqrt{2},\ \pi,\ e$ (the base of natural logarithms) and so on. Using the recursive relation (6.78) and taking into consideration the seeds (6.75) and (6.80), we can construct the matrices G_m^2, G_m^3, G_m^4 and so on for a general case of m:

$$G_m^2=m\begin{pmatrix}m&1\\1&0\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}m^2+1&m\\m&1\end{pmatrix} \tag{6.87}$$

$$G_m^3=m\begin{pmatrix}m^2+1&m\\m&1\end{pmatrix}+\begin{pmatrix}m&1\\1&0\end{pmatrix}=\begin{pmatrix}m^3+2m&m^2+1\\m^2+1&m\end{pmatrix} \tag{6.88}$$

$$G_m^4 = m\begin{pmatrix} m^3+2m & m^2+1 \\ m^2+1 & m \end{pmatrix} + \begin{pmatrix} m^2+1 & m \\ m & 1 \end{pmatrix} = \begin{pmatrix} m^4+3m^2 & m^3+2m \\ m^3+2m & m^2+1 \end{pmatrix}. \tag{6.89}$$

Using the recursive relation (6.79) and taking into consideration the seeds (6.75) and (6.80), we can construct the matrices $G_m^{-1}, G_m^{-2}, G_m^{-3}$ and so on for another general case of m:

$$G_m^{-1} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} - m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -m \end{pmatrix} \tag{6.90}$$

$$G_m^{-2} = \begin{pmatrix} 1 & -m \\ -m & m^2+1 \end{pmatrix} \tag{6.91}$$

$$G_m^{-3} = \begin{pmatrix} -m & m^2+1 \\ m^2+1 & -m^3-2m \end{pmatrix}. \tag{6.92}$$

6.6.5. *The Inverse Matrices* G_m^n

Again, let us consider Tables 6.6 and 6.8. They set the "direct" and "inverse" G_m-matrices. Compare the "direct" and the "inverse" G_m-matrices, G_m^n and G_m^{-n}. It is easy to find a very simple method to obtain the "inverse" matrix G_m^{-n} from its "direct" matrix G_m^n.

In fact, if the power n of the "direct" matrix G_m^n given by (6.72) is even (n=2k), then for obtaining its inverse matrix G_m^{-n} it is necessary to interchange the places of the diagonal elements $F_m(n+1)$ and $F_m(n-1)$ in (6.72) and to take the diagonal elements $F_m(n)$ in (6.72) with the opposite sign. This means that for the case n=2k the "inverse" matrix G_m^{-2k} has the following form:

$$G_m^{-2k} = \begin{pmatrix} F_m(2k-1) & -F_m(2k) \\ -F_m(2k) & F_m(2k+1) \end{pmatrix}. \tag{6.93}$$

To obtain the "inverse" matrix G_m^{-n} from the "direct" matrix G_m^n given by (6.72) for the case $n = 2k+1$, it is necessary to interchange the places of the diagonal elements $F_m(n+1)$ and $F_m(n-1)$ in (6.72) and to take them with the opposite sign, that is:

$$G_m^{-2k-1} = \begin{pmatrix} -F_m(2k-1) & F_m(2k) \\ F_m(2k) & -F_m(2k+1) \end{pmatrix}. \tag{6.94}$$

Another way of obtaining the matrices G_m^n follows directly from the expression (6.72). Let us consider the case m=2. In order to get the matrices G_2^n (n=0,±1,±2,±3,...) we have to represent two Fibonacci series $F_2(n+1)$ and $F_2(n)$ shifted by one number with respect to the other (Table 6.9).

Table 6.9. The shifted Fibonacci series $F_2(n+1)$ and $F_2(n)$

n	6	**5**	4	3	2	**1**	0	–1	–2	–3	–4	**-5**	–6
$F_2(n+1)$	169	**70**	**29**	12	5	**2**	**1**	0	1	–2	5	**-12**	**29**
$F_2(n)$	70	**29**	**12**	5	2	**1**	**0**	1	–2	5	–12	**29**	**-70**

If we select the number $n=1$ in the first row of Table 6.9 and then select four Fibonacci numbers in the lower two rows under the number 1 and to the right with respect to it, then a totality of these Fibonacci numbers build up the G_m-matrix (6.72). The G_m-matrix is singled out by bold type in Table 6.9. If we move in Table 6.9 to the left with respect to the G_m-matrix, then we obtain the matrices G_2^2, G_2^3, G_2^4 and so on. If we move in Table 6.9 to the right with respect to the G_m-matrix, we then obtain the matrices $G_2^0 = I, G_2^{-1}, G_2^{-2}$ and so on. Note that the matrix G_2^5 and the "inverse" to it matrix G_2^{-5} are singled out by bold type in Table 6.9. Note that the matrix $G_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix.

This method of obtaining the matrices G_m^n can be used for any arbitrary m.

6.7. Fibonacci $Q_{p,m}$-Matrices

6.7.1. *A Definition of the $Q_{p,m}$-Matrix*

In order to define the $Q_{p,m}$-matrix, let us return back to the recursive relation (4.295) at the seeds (4.296) that generate the Fibonacci (p,m)-numbers. By analogy with the Q_p-matrix (6.28), which is a generating matrix for the Fibonacci p-numbers $F_p(n)$, we introduce the $Q_{p,m}$-matrix (6.95), which is a generating matrix for the recursive relation (4.295) at the seeds (4.296). Given integer $p>0$ and real number $m>0$, the generating matrix for the Fibonacci (p, m)-numbers is

$$Q_{p,m} = \begin{pmatrix} m & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{6.95}$$

Note that the expression (6.95) defines an infinite number of square matrices because every positive real number m generates its own generating matrix of the kind (6.95). The $Q_{p,m}$-matrix (6.95) is a square $(p+1)\times(p+1)$-matrix. It consists of the identity $(p\times p)$-matrix bordered by the last row consisting of 0's and the leading 1, and the first column consisting of 0's embraced by the upper element of m and the lower element of 1. We can list the $Q_{p,m}$-matrices for the case $p=1, 2, 3, 4$ as follows:

$$Q_{1,m}=\begin{pmatrix} m & 1\\ 1 & 0\end{pmatrix}=G_m;\ Q_{2,m}=\begin{pmatrix} m & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix};\ Q_{3,m}=\begin{pmatrix} m & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{pmatrix};\ Q_{4,m}=\begin{pmatrix} m & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0\end{pmatrix}. \quad (6.96)$$

Note that the $Q_{p,m}$-matrix (6.95) is a wide generalization of the Q-matrix (6.2) ($p=1$, $m=1$), the Q_p-matrix (6.28) ($m=1$) and G_m-matrix (6.70) ($p=1$).

6.7.2. *The Main Theorems for the Powers of the $Q_{p,m}$-Matrices*

Now, let us raise the $Q_{p,m}$-matrix (6.95) to the n-th power and find the analytical expression for matrix $Q^n_{p,m}$. We have the following theorem for the $Q_{p,m}$-matrices.

Theorem 6.17. For a given integer $p=1,2,3,\ldots$, we have:

$$Q^n_{p,m}=\begin{pmatrix} F_{p,m}(n+1) & F_{p,m}(n) & \cdots & F_{p,m}(n-p+2) & F_{p,m}(n-p+1)\\ F_{p,m}(n-p+1) & F_{p,m}(n-p) & \cdots & F_{p,m}(n-2p+2) & F_{p,m}(n-2p+1)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ F_{p,m}(n-1) & F_{p,m}(n-2) & \cdots & F_{p,m}(n-p) & F_{p,m}(n-p-1)\\ F_{p,m}(n) & F_{p,m}(n-1) & \cdots & F_{p,m}(n-p+1) & F_{p,m}(n-p)\end{pmatrix}, \quad (6.97)$$

where $F_{p,m}(n)$ is the Fibonacci (p,m)-number, and $n=0,\pm1,\pm2,\pm3,\ldots$.

Proof. For a given p we can use an induction method.

(a) The basis of the induction. First, we will prove the basis of the induction, that is, we prove that the statement (6.97) is valid for the case $n=1$. It is clear that for the case $n=1$ the matrix (6.97) takes the following form:

$$\begin{pmatrix} F_{p,m}(2) & F_{p,m}(1) & \cdots & F_{p,m}(-p+3) & F_{p,m}(-p+2)\\ F_{p,m}(-p+2) & F_{p,m}(-p+1) & \cdots & F_{p,m}(-2p+3) & F_{p,m}(-2p+2)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ F_{p,m}(0) & F_{p,m}(-1) & \cdots & F_{p,m}(-p+1) & F_{p,m}(-p)\\ F_{p,m}(1) & F_{p,m}(0) & \cdots & F_{p,m}(-p+2) & F_{p,m}(-p+1)\end{pmatrix}. \quad (6.98)$$

Taking into consideration the values of the initial terms of the Fibonacci (p,m)-numbers given by (4.296) and their mathematical properties given by (4.297), (4.299) and (4.300), we can write:

$F_{p,m}(2)=m$, $F_{p,m}(2)=1$ and $F_{p,m}(-p)=1$.

Note that all the remaining entries of the matrix (6.98) are equal to 0. This means that the matrix (6.98) coincides with the $Q_{p,m}$-matrix given by (6.95). The basis of the induction is proved.

(b) Inductive hypothesis. Suppose that the statement (6.97) is valid for an arbitrary n and prove it for $n+1$. For this purpose we consider the product of the matrices

$$Q_{p,m}^{n+1} = Q_{p,m}^{n} \times Q_{p,m}$$

$$= \begin{pmatrix} F_{p,m}(n+1) & F_{p,m}(n) & \cdots & F_{p,m}(n-p+2) & F_{p,m}(n-p+1) \\ F_{p,m}(n-p+1) & F_{p,m}(n-p) & \cdots & F_{p,m}(n-2p+2) & F_{p,m}(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{p,m}(n-1) & F_{p,m}(n-2) & \cdots & F_{p,m}(n-p) & F_{p,m}(n-p-1) \\ F_{p,m}(n) & F_{p,m}(n-1) & \cdots & F_{p,m}(n-p+1) & F_{p,m}(n-p) \end{pmatrix} \times \begin{pmatrix} m & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (6.99)$$

Let us consider the matrix $Q_p^{n+1} = (a_{ij})$. In order to calculate the entry a_{11}, we multiply term by term the first row of the matrix $Q_{p,m}^n$ by the first column of the matrix $Q_{p,m}$ as follows:

$$a_{11}=mF_{p,m}(n+1)+F_{p,m}(n-p+1). \quad (6.100)$$

According to the recursive relation (4.295) we have: $a_{11}=F_{p,m}(n+2)$.

Continuing the process of the matrix multiplication for the example (6.99), we obtain the following entries of the first row of the matrix $Q_{p,m}^{n+1}$:

$$F_{p,m}(n+2),\ F_{p,m}(n+1),\ \ldots,\ F_{p,m}(n-p+3),\ F_{p,m}(n-p+2). \quad (6.101)$$

Now, let us compare the first row of the matrix $Q_{p,m}^n$ given by (6.97) with the first row of the matrix $Q_{p,m}^{n+1}$ given by (6.101). We can see that all arguments of the Fibonacci (p,m)-numbers in the sequence (6.101) differ by +1 from the arguments of the corresponding Fibonacci (p,m)-numbers, which build up the first row of the matrix (6.97). By analogy, we can show that this rule is valid for all entries of the matrix $Q_{p,m}^{n+1}$ that are formed from the matrix $Q_{p,m}^n$ by means of its multiplication by the matrix $Q_{p,m}$ in accordance with (6.99). This consideration proves the validity of Theorem 6.17.

Also, it is easy to prove the following identity for the product of these $Q_{p,m}^n$-matrices:

$$Q_{p,m}^{n+1} = Q_{p,m}^n \times Q_{p,m} = Q_{p,m} \times Q_{p,m}^n. \quad (6.102)$$

Using (6.102), we can prove the following theorem.

Theorem 6.18.

$$Q_{p,m}^n \times Q_{p,m}^k = Q_{p,m}^k \times Q_{p,m}^n = Q_{p,m}^{n+k}, \quad (6.103)$$

where n and k are integers.

Decomposing each entry of the matrix (6.97), that is, the corresponding Fibonacci (p,m)-number and taking into consideration the recursive relation (4.295), we obtain the following result given by Theorem 6.19.

Theorem 6.19.

$$Q_{p,m}^{n} = mQ_{p,m}^{n-1} + Q_{p,m}^{n-p-1}. \tag{6.104}$$

The recursive relation (6.104) can also be represented in the following form:

$$Q_{p,m}^{n-p-1} = Q_{p,m}^{n} - Q_{p,m}^{n-1}. \tag{6.105}$$

Using (6.104) and (6.105), we can construct different sequences of the $Q_{p,m}^{n}$-matrices. For example, for the case p=1 the sequences of the $Q_{1,m}^{n}$-matrices for the cases m=2 and m=3 are given by Tables 6.6 and 6.8, respectively.

6.8. Determinants of the $Q_{p,m}$-Matrices and their Powers

6.8.1. *Determinant of the $Q_{p,m}$-Matrix*

For the case p=1 the $Q_{p,m}$-matrix is reduced to the G_m-matrix, that is:

$$Q_{1,m} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} = G_m. \tag{6.106}$$

According to (6.71) we have:

$$Det Q_{1,m} = -1. \tag{6.107}$$

Now, let us calculate the determinant of the $Q_{2,m}$-matrix

$$Q_{2,m} = \begin{pmatrix} m & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.108}$$

By comparing the $Q_{2,m}$-matrix (6.108) with the $Q_{1,m}$-matrix (6.106), we can see that the $Q_{2,m}$-matrix (6.108) is reduced to the $Q_{1,m}$-matrix (6.106) if we strike out the 2nd row and the 3rd column from the $Q_{2,m}$-matrix (6.108). Note that the sum of these numbers 2+3=5. According to the matrix theory [157] we have:

$$Det\ Q_{2,m} = 1\times(-1)^5 Det\ Q_{1,m} = 1\times(-1)\times(-1) = 1. \tag{6.109}$$

Now, let us calculate the determinant of the $Q_{3,m}$-matrix

$$Q_{3,m} = \begin{pmatrix} m & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6.110}$$

Comparing the matrix (6.110) with the matrix (6.108), we can see that the $Q_{3,m}$-matrix (6.110) is reduced to the $Q_{2,m}$-matrix (6.108) if we strike out the 3rd row and the 4th column (3+4=7) in the matrix (6.110). This means [157] that

$$Det\ Q_{3,m}=1\times(-1)^7\times Det\ Q_{2,m}=1\times(-1)\times 1=-1. \qquad (6.111)$$

By analogy, we can show that

$$Det\ Q_{4,m}=1\times(-1)^9\times Det\ Q_{3,m}=1\times(-1)\times(-1)=1. \qquad (6.112)$$

Continuing this process, that is, generating the formulas similar to (6.109), (6.111), and (6.112), we can write the following recursive relation, which connects the determinants of the adjacent matrices $Q_{p,m}$ and $Q_{p-1,m}$:

$$Det\ Q_{p,m}=1\times(-1)^{2p+1}\times Det\ Q_{p-1,m}\ . \qquad (6.113)$$

It is clear that $Det\ Q_{p,m}$ is equal to 1 for the even values of $p=2k$ and to (-1) for the odd values of $p=2k+1$. We can formulate this result in the form of the following theorem.

Theorem 6.20. For a given p=1,2,3,..., we have:

$$Det\ Q_{p,m}=(-1)^p. \qquad (6.114)$$

Now, let us calculate the determinants of the inverse matrices of the kind $Q_{p,m}^{-1}$. For this purpose we consider this well-known correlation connecting the "direct" and "inverse" matrices:

$$Q_{p,m}^{-1}\times Q_{p,m}^{1}=I, \qquad (6.115)$$

where I is the identity matrix with $Det\ I=1$.

Applying to (6.115) the well-known rule that is given by Theorem 6.3, we can write:

$$Det\ Q_{p,m}^{-1}\times Det\ Q_{p,m}^{1}=Det\ I. \qquad (6.116)$$

In accordance with (6.114) the determinant of the matrix $Det\ Q_{p,m}$ takes one of the two values, 1 or -1. It follows from (6.116) that

$$Det\ Q_{p,m}^{-1}=Det\ Q_{p,m}. \qquad (6.117)$$

6.8.2. *Determinant of the Matrix* $Q_{p,m}^n$

Let us calculate the determinant of the matrix $Q_{p,m}^n$ given by (6.97). Using the property (6.1), we can write:

$$Det\ Q_{p,m}^n=\left(Q_{p,m}\right)^n, \qquad (6.118)$$

where $n=0,\pm1,\pm2,\pm3,\ldots$.

Taking into consideration the result of Theorem 6.20, we can formulate the following theorem.

Theorem 6.21. For given p=1,2,3,... and n=0,±1,±2,±3,... the determinant of the matrix $Q_{p,m}^{n}$ is given by the following formula:

$$Det\ Q_{p,m}^{n} = (-1)^{pm}. \tag{6.119}$$

6.9. The Golden *Q*-Matrices

6.9.1. *A Definition of the Golden Matrices*

We can represent the matrix (6.4) in the form of two matrices that are given for the even (n=2k) and odd (n=2k+1) values of n:

$$Q^{2k} = \begin{pmatrix} F_{2k+1} & F_{2k} \\ F_{2k} & F_{2k-1} \end{pmatrix} \tag{6.120}$$

$$Q^{2k+1} = \begin{pmatrix} F_{2k+2} & F_{2k+1} \\ F_{2k+1} & F_{2k} \end{pmatrix}. \tag{6.121}$$

In Chapter 5 we introduced the so-called *Symmetric Hyperbolic Fibonacci and Lucas Functions* (5.57)-(5.60) that are connected with the Fibonacci and Lucas numbers by the simple correlations (5.61). Consider once again the symmetric hyperbolic Fibonacci functions:

<u>Symmetric hyperbolic Fibonacci sine</u>

$$sFs(x) = \frac{\tau^x - \tau^{-x}}{\sqrt{5}} \tag{6.122}$$

<u>Symmetric hyperbolic Fibonacci cosine</u>

$$cFs(x) = \frac{\tau^x + \tau^{-x}}{\sqrt{5}} \tag{6.123}$$

Remember that the symmetric hyperbolic Fibonacci functions (6.122) and (6.123) are connected by the following surprising identities:

$$\begin{aligned} &[sFs(x)]^2 - cFs(x-1)cFs(x-1) = -1; \\ &[cFs(x)]^2 - sFs(x-1)sFs(x-1) = -1. \end{aligned} \tag{6.124}$$

Note that the identities (6.124) are a generalization of the Cassini formula for the continuous domain.

Remember that the Fibonacci numbers F_n are connected with the functions (6.122) and (6.123) by the simple correlation:

$$F_n = \begin{cases} sFs(n), & for\ n = 2k \\ cFs(n), & for\ n = 2k+1. \end{cases} \tag{6.125}$$

We can use the correlation (6.125) to represent the matrices (6.120) and (6.121) in terms of the symmetric hyperbolic Fibonacci functions (6.122) and (6.123):

$$Q^{2k} = \begin{pmatrix} cFs(2k+1) & sFs(2k) \\ sFs(2k) & cFs(2k-1) \end{pmatrix} \tag{6.126}$$

$$Q^{2k+1} = \begin{pmatrix} sFs(2k+2) & cFs(2k+1) \\ cFs(2k+1) & sFs(2k) \end{pmatrix}, \tag{6.127}$$

where k is a discrete variable, $k=0,\pm1,\pm2,\pm3,\ldots$.

If we exchange the discrete variable k in the matrices (6.126) and (6.127) for the continuous variable x, we obtain two unusual matrices that are functions of the continuous variable x:

$$Q^{2x} = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix} \tag{6.128}$$

$$Q^{2x+1} = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix}. \tag{6.129}$$

It is clear that the matrices (6.128) and (6.129) are a generalization of the Q-matrix (6.4) for the continuous domain. They have a number of unique mathematical properties. For example, for $x=1/4$ the matrix (6.128) takes the following form:

$$Q^{\frac{1}{2}} = \sqrt{Q} = \begin{pmatrix} cFs\left(\frac{3}{2}\right) & sFs\left(\frac{1}{2}\right) \\ sFs\left(\frac{1}{2}\right) & cFs\left(-\frac{1}{2}\right) \end{pmatrix}. \tag{6.130}$$

It is difficult to even imagine what a "square root of the Q-matrix" means. However, such an amazing "Fibonacci fantasy" follows directly from (6.130).

6.9.2. *The Inverse Golden Matrices*

In the above we introduced the "inverse" Fibonacci Q-matrices given by (6.66) and (6.67). We can represent the inverse of matrices (6.66) and (6.67) in terms of the symmetric hyperbolic Fibonacci functions (6.122) and (6.123) as follows:

$$Q^{-2k}=\begin{pmatrix} cFs(2k-1) & -sFs(2k) \\ -sFs(2k) & cFs(2k+1) \end{pmatrix} \quad (6.131)$$

$$Q^{-2k-1}=\begin{pmatrix} -sFs(2k) & cFs(2k+1) \\ cFs(2k+1) & -sFs(2k+2) \end{pmatrix}, \quad (6.132)$$

where k is a discrete variable, k=0,±1,±2,±3,... .

If we exchange the discrete variable k in the matrices (6.131) and (6.132) for the continuous variable x, we then obtain the following matrices that are functions of the continuous variable x:

$$Q^{-2x}=\begin{pmatrix} cFs(2x-1) & -sFs(2x) \\ -sFs(2x) & cFs(2x+1) \end{pmatrix} \quad (6.133)$$

$$Q^{-2x-1}=\begin{pmatrix} -sFs(2x) & cFs(2x+1) \\ cFs(2x+1) & -sFs(2x+2) \end{pmatrix}. \quad (6.134)$$

It is easy to prove that the matrices (6.133) and (6.134) are the inverse of the matrices (6.128) and (6.129), respectively, that is,

$Q^{2x}\times Q^{-2x}=I$ and $Q^{2x+1}\times Q^{-2x-1}=I$,

where I is an identity matrix.

6.9.3. *Determinants of the Golden Matrices*

Now, let us calculate the determinants of the matrices (6.128) and (6.129):

$$Det\,Q^{2x}=cFs(2x+1)cFs(2x-1)-\left[sFs(2x)\right]^2 \quad (6.135)$$

$$Det\,Q^{2x+1}=cFs(2x+2)cFs(2x)-\left[sFs(2x+1)\right]^2. \quad (6.136)$$

Compare formulas (6.135) and (6.136) with the identities (5.124) for the symmetric hyperbolic Fibonacci functions. As the identities (6.124) are valid for all values of the variable x, in particular, for the value of $2x$, the next identities follow from (6.135), (6.134) and (6.124):

$$Det Q^{2x}=1 \quad (6.137)$$

$$Det Q^{2x+1}=-1 \quad (6.138)$$

Note that the unusual identities (6.137) and (6.138) are a generalization of the "Cassini formula" for the continuous domain.

6.10. The Golden G_m-Matrices

6.10.1. *A Definition of the Golden G_m-Matrices*

The Fibonacci G_m-matrices (6.72) are a source for the wide generalization of "golden" matrices introduced above. We can represent the G_m-matrix (6.72) in the form of two matrices that are given for the even (n=2k) and odd (n=2k+1) values of n:

$$G_m^{2k}=\begin{pmatrix} F_m(2k+1) & F_m(2k) \\ F_m(2k) & F_m(2k-1) \end{pmatrix} \tag{6.139}$$

$$G_m^{2k+1}=\begin{pmatrix} F_m(2k+2) & F_m(2k+1) \\ F_m(2k+1) & F_m(2k) \end{pmatrix}. \tag{6.140}$$

Let us consider the hyperbolic Fibonacci and Lucas m-functions.

Hyperbolic Fibonacci m-functions

$$sF_m(x)=\frac{\Phi_m^x-\Phi_m^{-x}}{\sqrt{4+m^2}}=\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x-\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right] \tag{6.141}$$

$$cF_m(x)=\frac{\Phi_m^x+\Phi_m^{-x}}{\sqrt{4+m^2}}=\frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^x+\left(\frac{m+\sqrt{4+m^2}}{2}\right)^{-x}\right]. \tag{6.142}$$

The Fibonacci m-numbers are determined identically by the hyperbolic Fibonacci and Lucas m-functions as follows:

$$F_m(n)=\begin{cases} sF_m(n), & for\ n=2k \\ cF_m(n), & for\ n=2k+1. \end{cases} \tag{6.143}$$

As is shown in Chapter 5, the hyperbolic Fibonacci m-functions (6.141) and (6.142) possess the following unique properties:

$$\left[sF_m(x)\right]^2-cF_m(x+1)cF_m(x-1)=-1 \tag{6.144}$$

$$\left[cF_m(x)\right]^2-sF_m(x+1)sF_m(x-1)=1. \tag{6.145}$$

Using (6.143), we can represent the matrices (6.139) and (6.140) in terms of the hyperbolic Fibonacci m-functions (6.122) and (6.123) as follows:

$$G_m^{2k} = \begin{pmatrix} cF_m(2k+1) & sF_m(2k) \\ sF_m(2k) & cF_m(2k-1) \end{pmatrix} \tag{6.146}$$

$$G_m^{2k+1} = \begin{pmatrix} sF_m(2k+2) & cF_m(2k+1) \\ cF_m(2k+1) & sF_m(2k) \end{pmatrix}, \tag{6.147}$$

where k is a discrete variable, $k=0,\pm1,\pm2,\pm3,\ldots$.

If we exchange the discrete variable k in the matrices (6.146) and (6.147) for the continuous variable x, we then obtain two unusual matrices that are functions of the continuous variable x:

$$G_m^{2x} = \begin{pmatrix} cF_m(2x+1) & sF_m(2x) \\ sF_m(2x) & cF_m(2x-1) \end{pmatrix} \tag{6.148}$$

$$G_m^{2x+1} = \begin{pmatrix} sF_m(2x+2) & cF_m(2x+1) \\ cF_m(2x+1) & sF_m(2x) \end{pmatrix}. \tag{6.149}$$

Note that the "golden" G_m-matrices given by (6.148) and (6.149) provide a wide generalization of the "golden" Q-matrices given by (6.128) and (6.129). The "golden" Q-matrices (6.128) and (6.129) are partial cases of the matrices (6.148) and (6.149) for the case $m=1$, that is,

$$G_1^{2x} = Q^{2x} \text{ and } G_1^{2x+1} = Q^{2x+1}. \tag{6.150}$$

6.10.2. *Inverse Golden G_m-matrices*

We can represent the inverse G_m-matrices (6.93) and (6.94) in terms of the hyperbolic Fibonacci m-functions (5.105) and (5.106), that is,

$$G_m^{-2k} = \begin{pmatrix} cF_m(2k-1) & -sF_m(2k) \\ -sF_m(2k) & cF_m(2k+1) \end{pmatrix} \tag{6.151}$$

$$G_m^{-2k-1} = \begin{pmatrix} -sF_m(2k) & cF_m(2k+1) \\ cF_m(2k+1) & -sF_m(2k+2) \end{pmatrix}, \tag{6.152}$$

where k is a discrete variable, $k=0,\pm1,\pm2,\pm3,\ldots$.

If we exchange the discrete variable k in the matrices (6.151) and (6.152) for the continuous variable x, we then obtain the following matrices, which are functions of the continuous variable x:

$$G_m^{-2x} = \begin{pmatrix} cF_m(2x-1) & -sF_m(2x) \\ -sF_m(2x) & cF_m(2x+1) \end{pmatrix} \tag{6.153}$$

$$G_m^{-2x-1} = \begin{pmatrix} -sF_m(2x) & cF_m(2x+1) \\ cF_m(2x+1) & -sF_m(2x+2) \end{pmatrix}. \tag{6.154}$$

It is easy to prove that the matrices (6.153) and (6.154) are the inverse of the matrices (6.148) and (6.149), respectively, that is,

$G_m^{2x} \times G_m^{-2x} = I$ and $G_m^{2x+1} \times G_m^{-2x-1} = I$,

where I is an identity matrix.

6.10.3. *Determinants of the Golden G_m-Matrices*

Calculate the determinants of the "golden" m-matrices (6.148) and (6.149):

$$Det\ G_m^{2x} = cF_m(2x+1) \times cF_m(2x-1) - \left[sF_m(2x)\right]^2 \tag{6.155}$$

$$Det\ G_m^{2x+1} = sF_m(2x+2) \times sF_m(2x) - \left[cF_m(2x+1)\right]^2. \tag{6.156}$$

Let us compare the formulas (6.155) and (6.156) with the identities (5.136) and (5.137) for the hyperbolic Fibonacci m-functions. As the identities (5.136) and (5.137) are valid for all values of the variable x, in particular, for the value of $2x$, the following identities follow from this consideration:

$$Det\ G_m^{2x} = 1 \tag{6.157}$$

$$Det\ G_m^{2x+1} = -1. \tag{6.158}$$

Note that the unusual identities (6.157) and (6.158) are further generalizations of the Cassini formula for the continuous domain.

6.11. The Golden Genomatrices by Sergey Petoukhov

6.11.1. *The Genetic Code*

The discovery of the genetic code, which is general for all living organisms from a bacterium to a man, led us to the development of the informational point of view for living organisms. As it is emphasized in [59], "from this point of view all organisms are informational essences. They exist because they get hereditary information from ancestors and they live to transfer the informational genetic code to descendants. Given such an approach we may treat all other physical and chemical mechanisms presented in living organisms as auxiliary, helping promote the realization of this basic, informational problem."

The basis for the hereditary information language is amazingly simple. For recording genetic information in the ribonucleic acids (RNA) of any organism, the "alphabet" that consists of the four "letters" of nitrogenous bases, is used: *Adenine* (A), *Cytosine* (C), *Guanine* (G), *Uracil* (U) [in DNA instead of *Uracil(U)* the related *Thymine* (T) is used].

The genetic information transferred by molecules of heredity (DNA and RNA) defines the primary structure of the protein of the living organism. Each coded protein represents a chain that consists of 20 kinds of amino acids. The block formed from three adjacent nitrogen bases is known as a *Triplet*. It is possible to make $4^3=64$ triplets from the four-letter alphabet. The genetic code is called a *Degenerate* one because 64 triplets code only 20 amino acids. If any protein chain contains n amino acids, then the sequence of triplets that corresponds to it contains $3n$ nitrogen bases in the DNA molecule or, in other words, is given by a $3n$-triplet. Protein chains usually contain hundreds of amino acids and are accordingly represented by rather long poly-triplets.

Recently the Russian researcher Sergey Petoukhov made an original discovery in the genetic code [59]. This discovery shows the fundamental role of the golden ratio in the genetic code. It provides further evidence that the golden ratio underlies all Living Nature!

6.11.2. *Symbolic Genomatrices*

Petoukhov's basic idea [59] consists of the representation of the genetic code in matrix form. A square (2×2)-matrix P is an elementary matrix that is used for the representation of a system of four nitrogen bases ("letters") of the genetic alphabet:

$$P = \begin{pmatrix} C & A \\ U & G \end{pmatrix}. \tag{6.159}$$

Sergey Petoukhov further suggests that one may represent a family of genetic codes of identical length in the form of a corresponding family of matrices $P^{(n)}$ that are tensor (Kronecker) degrees of the initial matrix (6.159).

The matrices $P^{(n)}$ are named *Symbolic Genomatrices*. For a large enough n the given family of symbolic genomatrices $P^{(n)}$ represents all systems of genetic polyplets including *Monoplets* of the genetic alphabet (6.159) and *Triplets* (6.161) that are used for coding amino acids.

In each of the four quadrants of the genomatrix $P^{(n)}$ we can see all n-plets that begin from one of the letters C, A, U, G. If we ignore the first letter of the n-plets, we can see that every quadrant of the matrix $P^{(n)}$ reproduces the matrix $P^{(n-1)}$ of the preceding generation. On the other hand, every matrix

$P^{(n)}$ builds up a quadrant of the matrix $P^{(n-1)}$ of the next generation. Thus, a genomatrix of every new generation contains in itself in latent form all the information of the preceding generations. It is pertinent here to compare the genomatrices $P^{(n)}$ and the Q_p-matrices (6.28). Examples of the matrices $P^{(2)}$ and $P^{(3)}$ are represented below:

$$P^{(2)} = P \otimes P = \begin{pmatrix} CC & CA & AC & AA \\ CU & CG & AU & AG \\ UC & UA & GC & GA \\ UU & UG & GU & GG \end{pmatrix} \tag{6.160}$$

$$P^{(3)} = P \otimes P \otimes P = \begin{pmatrix} CCC & CCA & CAC & CAA & ACC & ACA & AAC & AAA \\ CCU & CCG & CAU & CAG & ACU & ACG & AAU & AAG \\ CUC & CUA & CGC & CGA & AUC & AUA & AGC & AGA \\ CUU & CUG & CGU & CGG & AUU & AUG & AGU & AGG \\ UCC & UCA & UAC & UAA & GCC & GCA & GAC & GAA \\ UCU & UCG & UAU & UAG & GCU & GCG & GAU & GAG \\ UUC & UUA & UGC & UGA & GUC & GUA & GGC & GGA \\ UUU & UUG & UGU & UGG & GUU & GUG & GGU & GGG \end{pmatrix}. \tag{6.161}$$

We can see that the matrices (6.159)-(6.161) possess the properties similar to the matrix (6.28) because any Q_p-matrix (6.28) comprises the information about all the previous matrices Q_{p-1}, Q_{p-2},..., Q_1, Q_0. However, on the other hand, the Q_p-matrix (6.28) is contained in all the next matrices Q_{p+1}, Q_{p+2}, Q_{p+3},... .

6.12.3. *Numerical Genomatrices*

If we substitute some numerical parameter for every symbol of nitrogen bases in the symbolic genomatrix, we obtain a *Numerical Genomatrix*. To form such numerical parameters, Petoukhov suggests that one uses the numerical parameters of the complementary hydrogen relation for the nitrogen bases of the genetic code. We are talking about the two or three hydrogen relations that connect complementary pairs of nitrogen bases in molecules of heredity. For the bases C and G the number of such nitrogen relations is equal to 3; however, for A and U it is equal to 2. Petoukhov suggested the following rule for obtaining the numerical genomatrix from the corresponding symbolic genomatrix.

Petoukhov's rule 1. To obtain the numerical genomatrix from the corresponding symbolic genomatrix, it is necessary to substitute every polyplet for the product of the numbers of hydrogen relations of its nitrogen bases, namely A=U=2 and C=G=3.

For example, triplet CGA in the octet matrix (6.164) is replaced by the product 3×3×2=18.

As a result of such substituting, the symbolic genomatrices (6.159)-(6.161) are converted, respectively, into the following numerical genomatrices P_{mult}:

$$P_{mult}^{(1)} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \tag{6.162}$$

$$P_{mult}^{(2)} = \begin{pmatrix} 9 & 6 & 6 & 4 \\ 6 & 9 & 4 & 6 \\ 6 & 4 & 9 & 6 \\ 4 & 6 & 6 & 9 \end{pmatrix} \tag{6.163}$$

$$P_{mult}^{(3)} = \begin{pmatrix} 27 & 18 & 18 & 12 & 18 & 12 & 12 & 8 & =125 \\ 18 & 27 & 12 & 18 & 12 & 18 & 8 & 12 & =125 \\ 18 & 12 & 27 & 18 & 12 & 8 & 18 & 12 & =125 \\ 12 & 18 & 18 & 27 & 8 & 12 & 12 & 18 & =125 \\ 18 & 12 & 12 & 8 & 27 & 18 & 18 & 12 & =125 \\ 12 & 18 & 8 & 12 & 18 & 27 & 12 & 18 & =125 \\ 12 & 8 & 18 & 12 & 18 & 12 & 27 & 18 & =125 \\ 8 & 12 & 12 & 18 & 12 & 18 & 18 & 27 & =125 \\ 125 & 125 & 125 & 125 & 125 & 125 & 125 & 125 & 1000 \end{pmatrix}. \tag{6.164}$$

It is easy to find a number of interesting properties of the numerical genomatrices (6.162) - (6.164). First of all, all numerical genomatrices (6.162) - (6.164) are symmetric with respect to both diagonals and therefore are called *bisymmetric* [141]. Further, the sum of numbers of each line and each column of the matrices (6.162) - (6.164) are equal to 5, 5^2=25, and 5^3=125, respectively, and the total sums of numbers in the matrices (6.162) - (6.164) are equal to 10, 10^2=100, and 10^3=1000, respectively. It is proven [141] that any numerical genomatrix $P_{mult}^{(n)}$ possesses similar properties because each such matrix is bisymmetric. Thus, the sum of numbers of each line and each column is equal to 5^n, and the total sum of numbers in the matrix is equal to 10^n. Already these surprising properties of Petoukhov's numerical genomatrices create the impression of "magic." But Petoukhov's discovery [59] of an improbable connection of these numerical genomatrices to the golden mean is absolutely astounding!

6.12.4. *The Golden Genomatrices*

Next let us consider the numerical genomatrix (6.162) that corresponds to the simplest symbolic genomatrix (6.159).

Consider a square matrix $\Phi^{(1)}$, where the numbers $\tau = \left(1+\sqrt{5}\right)/2$ (the golden mean) and $\tau^{-1} = \left(-1+\sqrt{5}\right)/2$ are its elements:

$$\Phi^{(1)} = \begin{pmatrix} \tau & \tau^{-1} \\ \tau^{-1} & \tau \end{pmatrix}. \tag{6.165}$$

If we square the matrix (6.165), we obtain the following matrix:

$$\left[\Phi^{(1)}\right]^2 = \begin{pmatrix} \tau & \tau^{-1} \\ \tau^{-1} & \tau \end{pmatrix} \times \begin{pmatrix} \tau & \tau^{-1} \\ \tau^{-1} & \tau \end{pmatrix} = \begin{pmatrix} \tau^2+\tau^{-2} & 2 \\ 2 & \tau^{-2}+\tau^2 \end{pmatrix}. \qquad (6.166)$$

Now, let us recall the Lucas numbers L_n (1, 3, 4, 7, 11, 18, ...) and the Binet formula for Lucas numbers. We can represent the Binet formula for Lucas numbers as follows:

$$L_n = \tau^n + (-1)^n \tau^{-n}, \qquad (6.167)$$

where $n=0,\pm1,\pm2,\pm3,...$.

For the case $n=2$ the identity (6.167) takes the following form:

$$\tau^2+\tau^{-2}=L_2=3. \qquad (6.168)$$

Taking into consideration the identity (6.168), we can represent the matrix (6.166) as follows:

$$\left(\Phi^{(1)}\right)^2 = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}. \qquad (6.169)$$

If we compare the matrices (6.162), (6.165), and (6.169), we can see a deep mathematical connection between the numerical genomatrix (6.162) and the “golden” genomatrix (6.165), because after squaring the “golden” genomatrix (6.165) we come to the numerical genomatrix (6.169).

Petoukhov proved [59] that every numerical genomatrix of the kind $P^{(n)}_{mult}$ has the “golden” genomatrix $\Phi^{(n)}$, which after its squaring coincides with the initial numerical genomatrix. Petoukhov suggested the following rule for obtaining the “golden” genomatrix from the corresponding symbolic genomatrix.

Petoukhov’s rule 2. To obtain the “golden” genomatrix from the corresponding symbolic genomatrix, it is necessary to substitute every polyplet for the product of the following values of its letters: $C=G=\tau$, $A=U=\tau^{-1}$.

If we apply this rule to the symbolic genomatrix (6.160), we obtain the following “golden” genomatrix:

$$\Phi^{(2)} = \begin{pmatrix} \tau^2 & \tau^0 & \tau^0 & \tau^{-2} \\ \tau^0 & \tau^2 & \tau^{-2} & \tau^0 \\ \tau^0 & \tau^{-2} & \tau^2 & \tau^0 \\ \tau^{-2} & \tau^0 & \tau^0 & \tau^2 \end{pmatrix}. \qquad (6.170)$$

If we square the matrix (6.170), we obtain:

$$\left[\Phi^{(2)}\right]^2 = \begin{pmatrix} \tau^4+\tau^0+\tau^0+\tau^{-4} & \tau^2+\tau^2+\tau^{-2}+\tau^{-2} & \tau^2+\tau^{-2}+\tau^2+\tau^{-2} & \tau^0+\tau^0+\tau^0+\tau^0 \\ \tau^2+\tau^2+\tau^{-2}+\tau^{-2} & \tau^0+\tau^4+\tau^{-4}+\tau^0 & \tau^0+\tau^0+\tau^0+\tau^0 & \tau^{-2}+\tau^2+\tau^{-2}+\tau^2 \\ \tau^2+\tau^{-2}+\tau^2+\tau^{-2} & \tau^0+\tau^0+\tau^0+\tau^0 & \tau^0+\tau^{-4}+\tau^4+\tau^0 & \tau^{-2}+\tau^{-2}+\tau^2+\tau^2 \\ \tau^0+\tau^0+\tau^0+\tau^0 & \tau^{-2}+\tau^{-2}+\tau^2+\tau^2 & \tau^{-2}+\tau^{-2}+\tau^2+\tau^2 & \tau^{-4}+\tau^0+\tau^0+\tau^{-4} \end{pmatrix}. \qquad (6.171)$$

If we use the Binet formula (6.167), we can write the following identities:

$\tau^2+\tau^{-2}=3$ and $\tau^4+\tau^{-4}=7$. (6.172)

Taking into consideration the identities (6.172), we can represent the matrix (6.171) as follows:

$$\left[\Phi^{(2)}\right]^2 = \begin{pmatrix} 9 & 6 & 6 & 4 \\ 6 & 9 & 4 & 6 \\ 6 & 4 & 9 & 6 \\ 4 & 6 & 6 & 9 \end{pmatrix}. \quad (6.173)$$

After comparison of (6.163) and (6.173) we can write:

$$\left[\Phi^{(2)}\right]^2 = P^{(2)}_{mult}, \quad (6.174)$$

that is, the "golden" genomatrix (6.170), which was obtained from the symbolic genomatrix (6.160) by using Petoukhov's rule 2, is converted after its squaring into the numerical genomatrix (6.173), which was obtained from the symbolic genomatrix (6.160) by using Petoukhov's rule 1.

Now, let us consider the "golden" genomatrix $\Phi^{(3)}$, which can be obtained from the symbolic genomatrix (6.163) by using Petoukhov's rule 2:

$$\Phi^{(3)} = \begin{pmatrix} \tau^3 & \tau^1 & \tau^1 & \tau^{-1} & \tau^1 & \tau^{-1} & \tau^{-1} & \tau^{-3} \\ \tau^1 & \tau^3 & \tau^{-1} & \tau^1 & \tau^{-1} & \tau^1 & \tau^{-3} & \tau^{-1} \\ \tau^1 & \tau^{-1} & \tau^3 & \tau^1 & \tau^{-1} & \tau^{-3} & \tau^1 & \tau^{-1} \\ \tau^{-1} & \tau^1 & \tau^1 & \tau^3 & \tau^{-3} & \tau^{-1} & \tau^{-1} & \tau^1 \\ \tau^1 & \tau^{-1} & \tau^{-1} & \tau^{-3} & \tau^3 & \tau^1 & \tau^1 & \tau^{-1} \\ \tau^{-1} & \tau^1 & \tau^{-3} & \tau^{-1} & \tau^1 & \tau^3 & \tau^{-1} & \tau^1 \\ \tau^{-1} & \tau^{-3} & \tau^1 & \tau^{-1} & \tau^1 & \tau^{-1} & \tau^3 & \tau^1 \\ \tau^{-3} & \tau^{-1} & \tau^{-1} & \tau^1 & \tau^{-1} & \tau^1 & \tau^1 & \tau^3 \end{pmatrix}. \quad (6.175)$$

If we square the matrix (6.175), then after simple transformation we obtain:

$$\left[\Phi^{(2)}\right]^2 = P^{(3)}_{mult}. \quad (6.176)$$

It appears that a similar regularity is valid for any numerical genomatrix $P^{(n)}_{mult}$. This means that Sergey Petoukhov proved the following important proposition named *Petoukhov's Discovery*.

Petoukhov's discovery.

Let A (adenine), C (cytosine), G (guanine), and U (uracil) be nitrogen bases ("letters") of the genetic alphabet that build up the initial symbolic matrix $P=\begin{pmatrix} C & A \\ U & G \end{pmatrix}$. Let $P^{(n)}$ be a symbolic genomatrix formed by means of a tensor (Kronecker) raising of the initial symbolic matrix P to the n-th power and let polyplets, which are built up from the "letters" *A, C, G* and *U,* be the elements of the symbolic matrix $P^{(n)}$. If we build a numerical genomatrix $P_{mult}^{(n)}$ from the symbolic genomatrix $P^{(n)}$ by substituting for each polyplet of the matrix $P^{(n)}$ the products of the numbers of hydrogen relations of its nitrogen bases according to the rule: $A=U=2$ and $C=G=3$, and if we build a "golden" genomatrix $\Phi^{(n)}$ from the symbolic genomatrix $P^{(n)}$ by substituting for each polyplet of the symbolic matrix $P^{(n)}$ the products of the following values of its "letters" according to the rule: $C=G=\tau$ and $A=U=\tau^{-1}$, where $\tau=\left(1+\sqrt{5}\right)/2$ is the golden mean, then there is the following fundamental relation between the numerical genomatrix $P_{mult}^{(n)}$ and the "golden" genomatrix $\Phi^{(n)}$:

$$\left[\Phi^{(n)}\right]^2=P_{mult}^{(n)}.$$

This discovery of the relation between the golden mean and the genetic code allowed Petoukhov to give a new "matrix-genetic" definition based upon the golden mean.

Petoukhov's definition.

The golden mean and its inverse number (τ and τ^{-1}) are the only matrix elements of the bisymmetric matrix Φ that is a root square of such bisymmetric numerical genomatrix P_{mult} of the second order whose elements are genetic numbers of hydrogen relations ($C=G=3$, $A=U=2$).

The "golden" genomatrices of Sergey Petoukhov are the most surprising application of the matrix approach to the golden section theory. Petoukhov's discovery [59] shows a fundamental role of the "golden mean" in the genetic code. This discovery gives further evidence that the golden mean underlies all Living Nature! It is difficult to estimate the full impact of Petoukhov's discovery for the development of modern science. It is clear that this scientific discovery is of equal importance to the discovery of the genetic code!

We would like to end this chapter by quoting from Petoukhov's paper [59] emphasizing the importance of the matrix approach in Fibonacci number and golden mean theory:

"The above formulated proposition about the matrix definition and essence of the golden section gives the possibility of considering all this material and its informative interpretation from the fundamentally new, matrix point of view. The author believes that many realizations of the golden section in both Organic and Inorganic Nature are connected precisely to the matrix essence and representation of the golden section. The mathematics of the golden matrices is a new branch of mathematics, which studies recurrence relations between the series of the golden matrices and also models natural systems and processes with their help."

6.12. Conclusion

1. Thus in this chapter we have developed a theory of the special class of square matrices that have exceptional mathematical properties. The beginning of this theory is connected with the name of the American mathematician Verner Hoggatt – founder of the Fibonacci Association. The theory of the Fibonacci Q-matrix was first stated in Hoggatt's book [16]. Although the name of the *Q-matrix,* which is a generating matrix for the classical Fibonacci numbers, was introduced before Verner E. Hoggatt, it was from Hoggatt's research that the idea of the Q-matrix "caught on like wildfire among Fibonacci enthusiasts. Numerous papers have appeared in 'The Fibonacci Quarterly' authored by Hoggatt and/or his students and other collaborators where the Q-matrix method became a central tool in the analysis of Fibonacci properties" [158].

2. Alexey Stakhov in his 1999 article [103] generalized the concept of the Q-matrix and introduced the *Generalized Fibonacci Q_p-matrix,* which is a generating matrix for the Fibonacci p-numbers. In 2006 Stakhov introduced [118] the concept of a *G_m-matrix,* which is a generating matrix for the Fibonacci m-numbers. E. Gokcen Kocer, Naim Tuglu and Alexey Stakhov introduced [154] the concept of the *$Q_{p,m}$-matrix,* which is a generating matrix for the Fibonacci (p,m)-numbers. The "golden" Q- and G_m-matrices are based on the symmetric hyperbolic Fibonacci functions and the hyperbolic Fibonacci m-functions. A general property of all the above matrices is the fact that their determinants are equal to +1 or -1. It is clear that a theory of similar matrices is of general mathematical interest.

3. However, the "golden" genomatrices of Sergey Petoukhov [59] are the most surprising application of the matrix approach to the golden mean theory. Petoukhov's discovery [59] demonstrates the fundamental role of the "golden mean" in the genetic code. This discovery gives further evidence that the golden mean underlies all Living Nature! Petoukhov's discovery may actually be scientifically equal in importance to the discovery of the genetic code itself!

Chapter 7

Algorithmic Measurement Theory

7.1. The Role of Measurement in the History of Science

7.1.1. *What is a Measurement?*

In the Great Soviet Encyclopedia we can find the following definition of the measurement notion:

"Measurement is an operation, by means of which the ratio of one magnitude to another homogeneous magnitude is determined; the number, which represents this ratio, refers to the numerical value of the subject magnitude."

A measurement is an important method of quantitative cognition of the objective world. The eminent Russian scientist Dmitry Mendeleev, the founder of the *Periodic System of Chemical Elements* and *Father of Russian Metrology*, maintained: "Science begins with a measurement. Exact science is inconceivable without a measure."

7.1.2. *The "Differentiation Principle" of "Measurement Science"*

The concept of a *Magnitude* and its *Measurement* belongs among the basic concepts of science. The theoretical study of this concept began developing in ancient Greece. During the 20th century the representatives of the different scientific disciplines, philosophy, physics, mathematics, information theory, psychology, economy, and so on, paid focused attention to this concept. That is why, the problem of the definition of the subject, contents and the position of measurement science in the system of contemporary sciences is of great importance. In this Section we make a methodological analysis of the ideas, principles, and scientific theories that can be united under the general name *Science of Measurement* or *Measurement Science*.

The existence of various directions of measurement study in modern science is a reflection of dialectic process of the Measurement Science *dif-*

ferentiation as a major principle of its development. Measurement as a method of quantitative reflection of the objective properties of the Universe is a dialectically many-sided concept. Every exact science studies measurement from its own specific point of view. It selects the measurement property that is most important for that given science, and a study of this measurement property results in a special measurement theory. For example, in quantum physics the most essential property of quantum measurement is the interaction between micro-object and the macro-measurement device. This problem underlies *Quantum-Mechanical Measurement Theory* [159]. In sociology, psychology, systems theory, and economics the measurement is reduced to the choice of the type of scale, to which the measurable magnitude can be referred, and therefore the *Scale Problem* underlies the *Psychological Measurement Theory* [160]. For technical and physical measurement, the main problem of measurement is the choice of the *System of Physical Units* and the decrease of *Measurement Errors*. Study of these aspects of measurement resulted in the creation of *Metrology* - the science of technical and physical measurements [161]. Measurement errors can be considered to be "random noises" in the "measurement channel;" this idea underlies the *Information Measurement Theory* [162]. The study of measurement, as some method or algorithm to get a numerical result, then results in the *Algorithmic Measurement Theory* [20, 21].

7.1.3. *Applied and Fundamental Theories of Measurement*

As we mentioned above, the general measurement theory concerns all branches of science and technology. On the one hand, *Metrology* is an applied science that concerns the ***Engineering Sciences*** and ***Applied Physics***. On the other hand, measurement concerns the fundamental problems of science, in particular, ***Mathematics*** and ***Theoretical Physics***. Therefore, at all stages of the development of science and technology, two levels or aspects of measurement study exist: *Applied* and *Fundamental*.

The *Fundamental Level* assumes a study of measurement as a fundamental problem that is a source of the development of the exact sciences (in particular, physics, mathematics, non-physical exact sciences); on this level the problem consists of revealing the most general properties and laws of measurement as the method of quantitative cognition of the objective world. Discovery of these measurement laws had a substantial influence on the development of the exact sciences. The proof of the existence of *Incommensurable Segments* made by the Pythagoreans is an example of just such a unique mathematical discovery. This discovery determined the development of mathematics over millennia because it resulted in *Irrational Numbers*, one of the most important mathematical concepts. *Heisenberg's "Uncertainty Relations"* that underlie quantum physics fundamentally limit the exactness of quantum-mechanical measurement and may be considered to be the outstanding physical idea in the field of the measurement.

The *Applied Level* assumes the study of measurement from the point of view of the practical, applied problems that appear in technology and applied physics. The problem of the creation of *Systems of Measurement Units* (metric system, Gauss' absolute system, etc.) that runs throughout the history of science and technology is the most typical example of the applied measurement problem.

7.1.4. *What is the Fundamental Distinction between Physics and Mathematics?*

Physics and mathematics education programs do not, as a rule, give much attention to this important question. Its answer lies, for example, in the works of the famous physicist Brilluen [163]. By pondering mathematical theorems and physical theories, Brilluen defines the essence of the distinction as follows. While mathematics begins with definitions of dimensionless points, infinitely thin curves, and continuous space-time, modern physics denies any real sense to such definitions. Brilluen noted that "for physicists the irrational numbers do not have any significance. It is assumed that the irrational numbers need to be defined with infinite precision because the exactness of say one in a hundred, one in a million or even one in a billion is insufficient for their definition. For the physical experiment, this does not make any sense; we can measure within the fifth or tenth digits of a decimal fraction, but there is no experiment, which would give twenty decimal digits. There is no sense in asking, whether the speed of light is a rational number, whether the ratio M/m (the ratio of the proton weight M to the electron weight m) is rational. All irrational numbers, like π, e, $\sqrt{2}$, etc., only follow from abstract mathematical definitions. However, as we have already said, mathematics is an aimless art. No physical objects can show real geometrical properties in those borders, which are assumed by mathematicians; we can even name such an approach, "wishful thinking" [163].

Thus, according to Brilluen, a feature of the "physical" approach to the study of the objective world consists in the postulation of some limit of "definiteness" of the "physical" magnitude that is completely ignored in the purely mathematical approach to the definition of the concept of magnitude.

7.1.5. *A Thesis about Inevitability of Measurement Errors*

Hence, according to Brilluen, the distinction between physical and mathematical approaches to measurement theory consists of different attitudes toward the *Thesis about the Inevitability of Measurement Errors*. The physical approach recognizes this thesis as the main postulate of the physical measurement theory. This axiom has an empirical, practical substantiation. The Russian mechanic and academician Krylov noted that "every measurement of any magnitude always has some error. Clearly, the less the error, the more exact the measurement, but the practice of measurement shows that it is impossible to avoid error entirely. This is confirmed by the fact that when iterating the same measurement many times by the same device we will get measurement results, which are different between themselves" [161].

This axiom finds its theoretical acknowledgement in the fact of the existence of some limit of definiteness (*negentropy*) of physical magnitude, which is known in quantum mechanics as *Heisenberg's Uncertainty Relation*, in wave mechanics as a *Relation of Time-and-Frequency Uncertainty*, and on the molecular level is given by the laws of thermodynamics (*Brilluen's Negentropy Principle of Information*). The existence of the limit of definiteness of physical magnitude and, as a consequence, the impossibility of the "absolutely exact" comparison (distinction) of the sizes of two physical magnitudes, which differ from each other in the size of the definiteness limit, is the basic thesis of "physical" measurements. Thus, the concept of "magnitude" in physics differs from a similar concept in mathematics, where we adopt the opposite thesis about the possibility of the "absolutely exact" comparison of two magnitudes independently of their sizes (comparison axioms of the mathematical theory of magnitudes).

7.1.6. *Purposes of the "Physical" Measurement Theory*

If we follow Brilluen's approach, we can divide the "fundamental" part of measurement theory into the *Physical* and *Mathematical* measurement theories. This approach allows us to formulate the basic purpose of the "physical" measurement theory which follows from the basic thesis about the inevitability of measurement errors. In quantum physics [159], the main purpose of measurement theory is the study of the interaction of the quantum-mechanical object with the macro-device used for the quantum-mechanical measurement. In metrology and the applied physics, the main purpose of the measurement theory is to study and model the errors of the physical measurements and in recent decades to study the sensitivity threshold of measurement devices. Of course, the doctrine about physical magnitudes and units [161] always was an important branch of the "physical" measurement theory.

Study of the character of physical measurement errors resulted in the formulation of two empirical axioms, which underlie the theory of random errors [161]:

1. *Randomness' axioms:* for the very big number of measurement results, the random errors, equal in terms of absolute value, but varying by sign, meet equally often; the number of the negative errors is equal to the number of the positive errors.
2. *Distribution's axiom*: smaller errors meet more often, than bigger; very big errors do not meet.

These empirical axioms resulted in the well-known *Normal Law of Probability Distribution* (Gauss' law):

$$\varphi(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}},$$

where σ^2 is a dispersion of probabilities; x is current value of error.

In the opinion of Russian mathematical authority Tutubalin [164], "the 'Normal Law' is some kind of "miracle" of probability theory, without which this theory almost would not have the original contents ...."

The correctness of the above axioms and the "normal law," which follows from them, is confirmed by the fact that all conclusions, based on them, are always consistent with experience. However, the non-strict character of the formulated axioms always causes a certain dissatisfaction with Gauss' law. This situation is expressed by one witty mathematician who sarcastically noted that "the experimenters believe in Gauss' law, by relying on the proofs of mathematicians, and the mathematicians believe in this law, by relying on the experimental verification."

7.2. Mathematical Measurement Theory

7.2.1. *Evolution of the Measurement Concept in Mathematics*

A measurement problem plays the same exceptional role in mathematics, as in other areas of science, particularly in technology, physics, and other exact sciences. The Russian mathematician and academician Kolmogorov in the foreword to the Russian edition of Lebesgue's *The Measurement of the Magnitudes* [3] expressed his outlook on this problem in the following words: "Where is the main interest of

Lebesgue's book? It seems to me, it is in the following: mathematicians have a propensity already owning the finished mathematical theory to be ashamed of its origin. In comparison with the crystal clearness of the development of a theory, beginning from its already prepared basic concepts and assumptions, it seems dirty and unpleasant to delve into the origin of these basic concepts and assumptions. The entire edifice of school algebra and all mathematical analysis can be constructed solely on the concept of real number without any reference to the measurement of concrete magnitudes (lengths, areas, time intervals, etc.). Therefore, on the different steps of education with a different degree of boldness, one and the same tendency emerges: as soon as it is possible to be done with the introduction of number concept and immediately to start discussing only numbers and relations between them. This is the tendency Lebesgue protests against this!"

Let us track now the evolution of the measurement concept in mathematics. It is well attested that, the measurement rules used by the Egyptian land surveyors were the first "measurement theory." The ancient Greeks testify that geometry was obliged by its origin (and the name) to those measurement rules, that is, to the "measurement problem." However, in Ancient Greece the separation of the measurement problems into two parts began – the applied problems related to "logistics," which was at that time referred to as a set of certain rules for the applied measurements and calculations; and fundamental problems related to geometry and number theory. These latter fundamental measurement problems became the main problems of Greek mathematics.

In the Greek period, a science of measurement was developing primarily as a mathematical theory. During this period, the Greek mathematicians made the following mathematical discoveries: *Incommensurable Segments*, *Eudoxus' Method of Exhaustion* and the *Measurement Axiom*. These outstanding mathematical discoveries later became the sources of number theory, integration and differentiation and other mathematical theories. These mathematical discoveries had a direct impact upon the "measurement problem." It gave the Bulgarian mathematician and academician Iliev the impetus to proclaim that "during the first epoch of mathematics development, from antiquity to the discovery of differential and integral calculus, investigating first of all the measurement problems, mathematics had created Euclidean geometry and number theory" [5].

7.2.2. *Incommensurable Line Segments*

From high school on we believe in the strictness and stability of mathematics. Therefore, for some of us it is a big surprise that in the process of its development, mathematics was experiencing different crises. Moreover, it may be a bigger surprise for others that, since the beginning of the 20th century, modern mathematics is in a deep crisis, and contemporary mathematicians cannot see any way out of this crisis yet.

The first crisis in the foundations of mathematics was connected directly to the "measurement problem." The early Pythagorean mathematics was based upon the so-called *Commensurability Principle*. Let us recall, that in mathematics, two nonzero real numbers a and b are said to be *Commensurable* if and only if a/b is a rational number. According to the commensurability principle, any two geometric magnitudes Q and V have a common measure, that is, both magnitudes are divisible by it. Thus, their ratio can be expressed as the ratio of two mutually prime natural numbers m and n:

$$\frac{Q}{V}=\frac{m}{n}. \tag{7.1}$$

According to the main Pythagorean philosophical doctrine that *Everything is a number*, all geometrical magnitudes can be expressed in the form (7.1), that is, their ratio is always rational. This means that geometry is reduced to a number theory because according to (7.1) all geometric relations can be expressed by rational numbers.

Let us examine the ratio of the diagonal b and the side a of the square (Fig. 7.1). According to Pythagorean Theorem

$$b^2=2a^2. \tag{7.2}$$

It follows from (7.2) that the ratio of the diagonal b to the side a of the square (Fig. 7.1) is equal

$$b/a=\sqrt{2}. \tag{7.3}$$

Suppose that the diagonal b and the side a of the square are commensurable. This means that we can represent their ratio in the form $\sqrt{2}=m/n$, where m and n are the mutually prime natural numbers. Then $m^2=2n^2$. Hence, it follows from here that the number m^2 is an even number. However, if a square of a number is even, this means that the number m is also an even number. As the numbers m and n are mutually prime natural numbers, then the number n is an odd number, according to the definition (7.1). However, if m is even number, the number m^2 can be divided by 4, and hence, n^2 is an even number. Thus, n is also an even number. However, n cannot be an even and odd number simultaneously! This contradiction shows that our premise about the commensurability of the diagonal and the side of the square is wrong and therefore the number $\sqrt{2}$ is irrational.

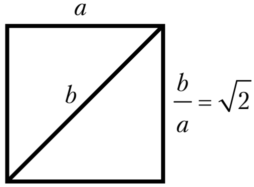


Figure 7.1. Incommensurable line segments.

The discovery of incommensurability staggered the Pythagoreans and caused the first crisis in the basis of mathematics, as this discovery refuted the initial Pythagorean doctrine about commensurability of all geometric magnitudes. A discovery of irrational numbers originated a complicated mathematical notion, which did not have direct connection with human experience. According to the legend, Pythagoras made a "hecatomb," that is, sacrificed one hundred oxen to the gods in honor of this discovery. However, according to the Brewer's Dictionary, "He sacrificed to the gods millet and honeycomb, but not animals. He forbade his disciples to sacrifice oxen." In any case, it was a worthwhile sacrifice because this discovery became the turning point in the development of mathematics. It ruined the former system created by the Pythagoreans (the commensurability principle) and became a source of new and remarkable theories. The importance of this discovery may be compared with the discovery of non-Euclidean geometry in the 19th century or to the theory of relativity at the beginning of 20th century. Similar to these theories, the problem of incommensurable line segments was well-known among educated people. Plato and Aristotle were also addressing the problem of incommensurability in their works.

7.2.3. *Eudoxus-Archimedes and Cantor's Axioms*

To overcome the first crisis in the mathematics foundations, the Great Greek mathematician Eudoxus developed the method of exhaustion and created a new theory of magnitudes.

Eudoxus of Cnidus (c.410/408 – 355/347 BC) was a Greek astronomer, mathematician, physicist, scholar and friend of Plato. Unfortunately, all of his own works were lost, therefore we obtained information about him from other sources, such as Aratus' poem on astronomy. Archytas from Athens was his mathematics teacher. He is famous due to the introduction of the astronomical globe, and his early contributions to the explanation of the planetary motion.

Eudoxus developed the method of exhaustion and used it for the creation of the theory of irrationals. Later this method was used masterfully by Archimedes. The essence of Eudoxus' method of exhaustion can be explained by the following practical example. If we have a barrel of beer and a beer mug, then this beer in the barrel will eventually be exhausted, even though the beer barrel is enormous and the beer mug very small.

Eudoxus' theory of incommensurability (see Book V of Euclid's *Elements*) could be considered one of the greatest achievements of mathematics and in general coincides with the modern theory of irrational numbers suggested by Dedekind in 1872.

The measurement theory of geometric magnitudes dates back to the incommensurable line segments. It is based on the group of so-called continuity axioms, which comprise both Eudoxus-Archimedes' axiom and Cantor's axiom (or Dedekind's axiom).

Eudoxus-Archimedes' axiom (the axiom of measurement). For any two segments A and B it is possible to specify such natural number n, which always results in the following non-equality:

$$nB>A. \tag{7.4}$$

In the 19th century, Dedekind and then Cantor made a final attempt to create a general theory of real numbers. For this purpose, they introduced the additional axioms into the group of the continuity axioms. For instance, let us consider *Cantor's Axiom*.

Cantor's continuity axiom (Cantor's principle of nested segments). If an infinite sequence of segments is given on a straight line A_0B_0, A_1B_1, A_2B_2,...,A_nB_n, ..., such that each next segment is nested within the preceding one, and the length of the segments tends to zero, then there exists a unique point which belongs to all the segments.

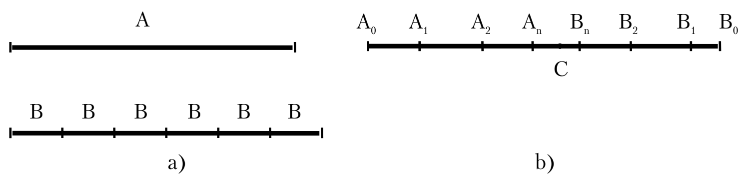


Figure 7.2. The continuity axioms: (a) Eudoxus-Archimedus' axiom (b) Cantor's axiom

The main result of the mathematical measurement theory that is based on the continuity axioms is a proof of the existence and uniqueness of the solution q of the *Basic Measurement Equality*:

$$Q=qV, \tag{7.5}$$

where V is a measurement unit, Q is a measurable segment, and q is any real number named a *Result of Measurement*.

The idea of the proof of the equality (7.5) consists of the following. Using Eudoxus-Archimedes' axiom and certain rules called a *Measurement Algorithm*, we can form from the measurement unit V some sequence of the nested segments, which can be compared with the measurable segment Q during the process of measurement. If we direct this process ad infinitum, then according to Cantor's axiom for the given line segments Q and V we always can find such nested segments, which coincides with the measurable segment Q. It follows from this proof that the idea about a measurement is one of a process running during an infinite time.

It is difficult to imagine that the continuity axioms and the mathematical measurement theory are the result of a more than 2000-year period in the development of mathematics. The continuity axioms and basic measurement equation (7.5) comprise in themselves a number of the important mathematical ideas that underlie different branches of mathematics.

It is necessary to note that the measurement axiom expressed by (7.4) is a reflection of Eudoxus' method of exhaustion in modern mathematics. The axiom generalizes mankind's millennia of experience regarding the measurement of distances, areas and time intervals. The simplest measurement algorithm of the line segment A by using the line segment B less than A underlies this axiom. This algorithm consists of the successive comparison of the line segment $B+B+...+B$ with the measurable line segment A. During the measurement process we count a number of the line segments B that are laid out on the measurable line segment A. This measurement algorithm is called a *Counting Algorithm*.

The counting algorithm is the source of the various fundamental notions of arithmetic and number theory, in particular, the notions of *Natural Number*, the *Prime* and *Composite* numbers, as well as the notion of *Multiplication*, *Division*, etc. In this connection the Euclidean definition of prime (the "first" number) and composite numbers ("the first number is measured only by 1," "the composite number is measured by some number") is of great interest. The measurement axiom originates a *Divisibility Theorem*, which plays a fundamental role in number theory. A *Theory of Divisibility* and a *Comparison Theory* are based on the *Divisibility Theorem*. Thus, the simplest measurement algorithm, a *Counting Algorithm,* originates natural numbers and all concepts and theories connected with them.

7.2.4. *A Contradiction in the Continuity Axioms*

The idea about measurement as a process running during infinite time is a brilliant example of the Cantorian style of mathematical thinking based on the concept of *Actual Infinity*. However, this concept was subjected to sharp criticism from the side of the representatives of constructive mathematics.

For the explanation of the distinctions between different interpretations of infinity used in Eudoxus-Archimedes' and Cantor's axioms, once again we return back to them. In Cantor's axiom all infinite sets of the nested segments are considered as the set given by all its objects simultaneously (Cantor's actual infinity). Eudoxus-Archimedes' axiom, which has an "empirical origin," is an example of a constructive axiom and in the implicit form relies on the abstraction of *Potential Practicability* because after each step it assumes the possibility of constructing the next line segment bigger than the previous one. The number of measurement steps in Eudoxus-Archimedes' axiom, necessary for realization of the condition $nB > A$, is always finite, but potentially with no limitation. Here we can see the example of the constructive understanding of infinity as a potential category. In this connection we can pay attention to the internal contradiction of the classical mathematical measurement theory (and as a consequence, of the theory of real numbers), which assumes in the initial suppositions (the continuity axioms) the coexistence of two opposite ideas about infinity, that is, the actual or completed infinity in Cantor's axiom and the potential or uncompleted infinity in the Eudoxus-Archimedes' axiom.

7.3. Evolution of the Infinity Concept

7.3.1. *An Infinity Concept*

The book *Philosophy of Mathematics and Natural Science* (1927, German) by the famous Germany mathematician Hermann Weyl (1885-1955) begins with the following words: "Mathematics is the science of infinity" [166]. Without a doubt, the infinity concept permeates throughout all mathematics, because mathematical objects are, as a rule, members of the classes or sets containing an uncountable set of elements of the same kind; examples include the set of natural numbers, the set of real numbers, the set of triangles, etc. That is why the infinity concept is necessary under a strict analysis.

Starting a discussion of the infinity concept, we should say a little about the concept of the "finite" that is opposite to the notion of "infinity." Our intellect determines that infinity is something that does not have an end, and the finite is something that does have an end. At first sight, the concept of the finite seems to us clear and self-obvious. However, in actuality it is not so simple. If we consider the finite as a self-obvious concept, then infinity can be seen as an unlimited accumulation of the finite. On the other hand, recognizing the infinite division of the finite object (for example, the geometric line segment that consists of points), we are compelled to state that the infinite is made from the finite and the finite is made from the infinite.

The secret of the infinity in space and of the "immortality" in time has always been the most high-powered catalyst that directs the human intellect to the search for truth. David Hilbert, one of the most brilliant mathematicians of the first half of the 20th century, said:

"No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite" [quoted from E. Manor, *To Infinity and Beyond*, Boston, 1987].

7.3.2. *The Origin of the Symbol* ∞

The symbol for "infinity" in the form of the "horizontal 8" was introduced for the first time in *Arithmetic of Infinite Values* by English mathematician John Vallis in 1665. Some historians of mathematics give the following explanation for the use of this infinity symbol: a similar symbol was used in Roman notation and was designated as a thousand which was identified as "very much." The Russian historian of mathematics Gleiser in his book [150] gives, however, another explanation of the origin of this symbol in Vallis' book. According to his opinion, the symbol ∞ can be considered as two zeros connected among themselves. This symbol was opposed by Vallis to the symbol "zero" (0) in the connection with the following expressions given in Vallis' book:

$$1/0=\infty, \quad 1/\infty=0. \tag{7.6}$$

However, at the present moment, the expressions (7.6) are considered to be an error. People should know and remember that the symbol ∞ does not mean any number and does not have any numerical sense. Therefore, it is necessary to use the symbols $+\infty$ and $-\infty$ with big caution; in particular, we cannot fulfill any arithmetical operations similar to (7.6) with them.

7.3.3. *Potential and Actual Infinity*

Although according to Hermann Weyl, "infinity" is a fundamental concept of mathematics, there is nevertheless no comprehensible definition of this important concept in mathematics. *Arithmetical* and *Geometric*, *Potential* and *Actual* infinity are used in mathematics. Let us consider these concepts in greater detail.

A sequence of natural numbers

$$1,2,3,... \tag{7.7}$$

is the first and most important example of *Arithmetical Infinity*. Since Hegel, the arithmetical infinity of natural numbers 1+1+1+ ... i.e. the endless "iteration of the same" is called the "bad infinity."

Geometric Infinity, for instance, consists of the unlimited bisection of a line segment. Pascal wrote about geometric infinity as follows: "There is no geometer who would not suppose, that any space can be divided ad infinitum. It is impossible to be devoid of this, similar to a person, who cannot be without a soul. And nevertheless there is no person who would understand that means an infinite divisibility" [166]. Except for the distinction between arithmetical and geometric infinity, there is a distinction between *Actual* and *Potential* infinity. For consideration of the difference between these concepts, once again, we address to the sequence of natural numbers (7.7). We can suppose this sequence as if "completed" sequence, that is, determined by all its members simultaneously. Such presentation about "infinity" is named *Actual Infinity*.

However, the sequence of natural numbers (7.7) can also be considered as the "developing" sequence that is generated according to the principle

$$n'=n+1. \tag{7.8}$$

This means that each natural number can be obtained from the previous one by means of summation of 1. Such presentation of the "infinity" is named *Potential Infinity*.

7.3.4. *An Origin and Application of the Infinity Idea in the Ancient Greek Mathematics*

Mathematics was turned into a deductive science in Ancient Greece. Many historians of mathematics believe that Greek mathematicians introduced into mathematics for the first time the concept of *Potential Infinity*.

"Zeno's paradoxes" played a major role in the development of the infinity concept. They demonstrated so well the logical difficulty of the hypotheses about the infinite division of geometric line segments and time intervals. Let us consider two of them, *Dichotomy* and *Achilles and the Tortoise*.

1. *Dichotomy* ("bisection" in Greek). In this antinomy Zeno asserts that movement is impossible. Really, if a solid moves from point A to point B, this solid must first traverse the line segments in 1/2, and before that 1/4, 1/8, 1/16 ... of the distance between points A and B. However, the sequence of such line segments is infinite. This means that one can never leave point A! The paradox, which results in an insuperable logical impasse, is based on the theorem that the sum of the infinite set of addends is finite.
2. *Achilles and the Tortoise*. Zeno asserts: "The fast-moving Achilles can never catch up to the tortoise." The proof comes to the following: let Achilles be n times faster than the tortoise and let the initial distance between them be equal to d. While Achilles overcomes this new distance, simultaneously with him the tortoise can move forward on the distance d/n; while Achilles overcomes new distance, the tortoise can move forward on the new distance d/n^2 and so on. Thus, the distance between Achilles and the tortoise will always be more than 0, that is, Achilles never can catch up to the tortoise.

Aristotle paid a great deal of attention to the concept of mathematical infinity. He strongly objected to the use of actual infinity in mathematics. The following well-known thesis is attributed to Aristotle: "Infinitum Actu Non Datur;" translated from Latin this means that the existence of actually-infinite objects in mathematics is impossible.

7.3.5. *Cantor's Theory of Infinite Sets*

The German mathematician George Cantor (1845-1918) is considered to be the prime disrupter of tranquility in 19th century mathematics. The history of set theory is quite different from the history of other areas of mathematics. For most areas, we can outline a long process, in which the ideas are evolving until the ultimate flash of inspiration produces a discovery of major importance, often in many mathematicians somewhat synchronously. That non-Euclidean geometry was discovered by Lobachevski and Bolyai independently at the same time is a brilliant example of such synchronous inspiration. As for set theory, it was the creation of one person, George Cantor.

Cantor's main idea is the study of infinite sets as actual-infinite sets. The idea of one-to-one correspondence between the elements of the comparable sets was used by Cantor in his research of infinite sets. If we can establish such correspondence between the elements of two sets, we can say that the sets have the same *Cardinality*, that is, they are equivalent. Cantor wrote - "In the case of the finite sets the cardinality coincides with the number of the elements." That is why, the cardinality is also named the cardinal (quantitative) number of the given set. This approach resulted in many paradoxical conclusions being in sharp contradiction to one's intuition. So, in contrast to the finite sets that comply with the Euclidean axiom, "The whole is more than its parts," the infinite sets do not comply with this axiom. For example, it is easy to prove that the set of natural numbers and some of its subsets are equivalent. In particular, we can establish the following one-to-one correspondence between the sets of natural numbers and the even numbers:

$$\begin{array}{cccccc} 1 & 2 & 3 & \ldots & n & \ldots \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & . \\ 2 & 4 & 6 & \ldots & 2n & \ldots \end{array}$$

This feature of the infinite sets can be used for their definition: a set is named infinite if the set is equivalent in extant to one of its subsets. A set is finite if the set is not equivalent to any of its subsets. Any set equivalent to the set of natural numbers is named *Denumerable* because all of its elements can be numbered.

The most amazing discovery was made by Cantor in 1873. He proved that all three of the most characteristic sets (natural numbers, rational numbers, and algebraic numbers) have one and the same cardinality, that is, the sets of rational and algebraic numbers are denumerable. Cantor also proved that the cardinality of real numbers is greater than the cardinality of natural numbers.

From the above consideration it is possible to conclude that in contrast to the majority of his predecessors, Cantor had undertaken for the first time a deep research into mathematical infinity that resulted in new and unexpected outcomes.

7.3.6. *Antinomies of Cantor's Theory of Infinite Sets*

It seemed for many mathematicians, that Cantor's theory of infinite sets produced a revolution in mathematics. The end of the 19th century culminated in the recognition of Cantor's theory of infinite sets. French mathematician Jacques Hadamard in his 1897 speech at the First International Congress of Mathematicians in Zurich officially proclaimed the set-theoretical ideas as the basis of mathematics. In his lecture, Hadamard emphasized that the most attractive reason of Cantor's set theory consists of supporting the fact that for the first time in mathematics history the classification of sets was made on the basis of a new concept of "cardinality" and the amazing mathematical outcomes inspired mathematicians to new and surprising discoveries.

However, very soon the "mathematical paradise" based on Cantor's set theory was destroyed. In the first few years after the First International Congress of Mathematicians, the paradoxes or antinomies appeared in set theory. Cantor discovered the first paradoxes at the end of the 19th century. Other paradoxes were discovered later by other scientists. These paradoxes became the basis for the next, or third (after the discovery of incommensurable line segments and substantiation of the limit theory) crisis in the foundation of mathematics. One of them, discovered by the English philosopher and mathematician Bertrand Russell in 1902, was connected with the foundations of set theory and the concept of the set of all sets. Each of the usual sets, which we met until now, does not contain itself as an element; so, for example, the set of all natural numbers is not a natural number. We name such sets *Ordinary Sets*. However, there are also such exotic sets that contain themselves as their own elements, for example, the set of all sets. We name such sets *Non-ordinary Sets*. Consider the set S of all ordinary sets and ask the following question: is S an element of S? Alternatively, is the set S ordinary or non-ordinary set? Suppose that S is an element of S; then it is necessary to recognize that S is a non-ordinary set. However, as S contains only ordinary sets, this means that S should also be an ordinary set. Thus, from this supposition it follows that S is an ordinary set, that is, we have a logical contradiction. Now, suppose that S is not an element of S, that is, S is the ordinary set; then S should be in S, as all ordinary sets. Thus, from the supposition that S is the ordinary set it follows that S is the non-ordinary set; again, we have contradiction!

Russell demonstrated this contradiction, known as the Barber Paradox, with the example of the village barber, who gave a promise to shave those and only those village inhabitants, who do not shave themselves. We can ask the question: Does the barber shave himself? If he shaves himself, this means that thereby he includes himself into the set of inhabitants, who shave themselves, and therefore he should not shave himself; if he does not shave himself, this means that he belongs already to those, who do not shave themselves, that is, he should shave himself. We again have a logical contradiction which of course is unacceptable in mathematics!

7.3.7. *Is Cantor's Actual Infinity the Greatest Mathematical Mystification?*

A discovery of the antinomies in Cantor's set theory caused a new crisis in the foundations of mathematics. Various attempts were made to overcome this crisis including *Constructive Analysis* in mathematics the most radical of them. The representatives of constructive analysis saw the main reason for paradoxes in Cantor's theory of sets to be the use of the concept of "actual infinity." Russian mathematician Markov, one of the brightest representatives of constructive analysis, wrote: "It is impossible to imagine an endless process as a completed process without rough violence to our intellect, in the rejection of such inconsistent fantasies." [167]

On the other hand, probably the works of Russian mathematician Alexander Zenkin took the final step in the dispute over Cantor's set theory. Zenkin found logical contradictions and errors in Cantor's set theory. The essence of Zenkin's approach, presented in a few main publications [168], consists of the fact that for the first time he formulated an almost obvious fact – the concept of potential infinity (PI) in the Aristotelian form – *"all infinite sets are potential sets,"* and the concept of actual infinity (AI) in Cantor's form – *"all infinite sets are actual sets"* - are AXIOMS, that is, they cannot be proved or refuted, but they can be either accepted, or rejected. For this reason, any discussions about "existence" or "non-existence" of the "actual infinity," which goes back to Aristotle, cannot convince anyone of anything. Here we can provide an analogy with the history of the 5th Euclidean axiom about parallel lines. However, there is also an essential difference – the correctness of the non-Euclidean geometry was proven – however, the correctness of Cantor's theory of sets based on "actual infinity" is not proven. Moreover, the antinomies of Cantor's theory of sets demonstrate that this theory is self contradictory.

After a thorough analysis of Cantor's continuum theorem, in which Zenkin gave the logical substantiation for legitimacy of the use of the "actual infinity" in mathematics, he derived the following unusual conclusion:

1. Cantor's proof of this theorem is neither a mathematical proof in Hilbert's sense nor in the sense of classical mathematics.
2. Cantor's conclusion about non-denumerability of the continuum is a jump through a potentially infinite stage, that is, Cantor's reasoning contains the fatal logical error of "unproved basis" (a leap to a "wishful conclusion").
3. Cantor's theorem does, in fact, strictly mathematically prove the potential, that is, the unfinished character of the infinity of the set of all "real numbers." Thus, Cantor proves mathematically the fundamental principle of classical logic and mathematics: *"Infinitum Actu Non Datur"* (Aristotle).

According to the opinion of Alexander Zenkin [168], Aristotle's famous thesis "Infinitum Actu Non Datur" – that is, the assertion about the impossibility of the existence of actually-infinite objects – had been supported during the last 2300 years by Aristotle's great followers Leibniz, Cauchy, Gauss, Kronecker, Poincare, Brouwer, Vail, Lusin and by many other famous founders of classical logic and modern classical mathematics! Every one of them professionally studied the problem of mathematical infinity, and there were not be any doubts, that they understood the true nature of the infinity in general no worse than Cantor did. Especially, if we take into consideration the important fact, that infinity does not depend on progress of technological equipment of science, infinity never was and never becomes an object of the technological research, because no contemporary computers, never, by definition, would ever be able to finish an enumeration of all elements of natural series 1, 2, 3, ...

For this reason – Alexander Zenkin asserted – all discussions about the possibility or impossibility of actual infinity during two millennia down to Cantor had only a speculative character. The infinity does not depend on the progress of science and technology!

Alexander Zenkin wrote [168]: "Why was such an analysis of Cantor's theorem not executed in time, i.e. at the end of the 19th century? It is a very crucial topic for fundamental research in the field of the psychology of scientific knowledge."

Thus, there is the impression that the long history (starting from Greek science) of the study of the infinity concept as one of the fundamental mathematical concepts is approaching its completion. The antinomies of Cantor's set theory, which caused the contemporary crisis in the foundations of mathematics, showed that the concept of actual infinity could not be a reliable basis for mathematical reasoning, because from the point of view of the representatives of constructive analysis the "actual infinity" concept is internally self-contradictory. On the other hand, the research of Russian mathematician Alexander Zenkin [168] testifies the existence of logical errors in Cantor's theorems, and this fact gives one the right to doubt the accuracy of Cantor set theory.

Thus, the concept of potential infinity introduced in the Greek mathematics can be the only real base for mathematical reasoning. In this connection Aristotle's thesis "*Infinitum Actu Non Datur*" should become the main axiom for the creation of new, constructive mathematics.

7.4. A Constructive Approach to Measurement Theory

7.4.1. *The Rejection of Cantor's Axiom from Mathematical Measurement Theory*

As the concept of "actual infinity" is an internally contradictory notion ("the completed infinity"), this concept cannot be a reasonable basis for the creation of constructive mathematical measurement theory. If we reject Cantor's axiom, we can try to construct mathematical measurement theory on the basis of the idea of potential infinity, which underlies the Eudoxus-Archimedes' axiom. By referring to the measurement theory, this means, that the number of steps for the given measurement is always finite, but potentially unlimited. The acceptance of the given approach at once results in the occurrence of the fundamentally irremovable measurement error called *Quantization Error*.

The constructive approach to measurement theory results in a change of the purpose of measurement theory. One of the essential moments at the proof of the equality (7.5) is a choice of the *Measurement Algorithm*, by means of which we carry out the formation of "nested segments" from the measurement unit V. With the actually infinite time, the measurement algorithm does not influence the measurement result q and its choice is arbitrary. With the given time of measurement – that is defined by the number of measurement steps n – and with the other equal conditions, the distinction between the measurement algorithms with respect to the "reached" measurement exactness appears.

With these conditions, the second constructive idea about the "efficiency" of the measurement algorithms comes into effect and the *Problem of the Synthesis of the Optimal Measurement Algorithms* is put forward as a central problem of the *Constructive (Algorithmic) Measurement Theory.*

7.4.2. *Bashet-Mendeleev's Problem*

The constructive approach, at once, results in the optimization problem in measurement theory which was not a topical problem for classical measurement theory. By studying the optimization problem in the measurement theory, we come unexpectedly to a combinatorial problem known as a *Problem about the Choice of the Best Weights System.* For the first time, this problem appeared in the 1202 in Fibonacci's *Liber Abaci.* From Fibonacci's work this problem moved to the 1494 book *Summa de Arithmetica, Geomeytria, Proprtioni et Proportionalita* by Luca Pacioli. After Pacioli's work, this problem again appeared in the 1612 book *Collection of the Pleasant and Entertaining Problems* by the French mathematician *Claude Gaspar Bachet de Meziriac.*


Claude Gaspar
Bachet de Meziriac
(1581-1638)

Claude Gaspar Bachet was born in Bourg-en-Bresse in Savoy (France). This was a region, which in different periods belonged to France, Spain or Italy. Therefore, the life of Claude Gaspar Bachet is connected with France, Spain and Italy of that period. He was a writer of books on mathematical puzzles and tricks that became a basis for all the books on mathematical puzzles. Bachet's 1612 book *Collection of the Pleasant and Entertaining Problems* contains different numerical problems. Fibonacci's "weighing" problem is one of them. This problem was formulated as follows: "What is the least number of weights used to weigh any whole number of pounds from 1 to 40 inclusively by using a balance, if the weights can be placed on both balance cups?"


Dmitri Mendeleev
(1834-1907)

In the Russian historical-mathematical literature [169] Fibonacci's "weighing" problem is known also under the name of the Bachet-Mendeleev problem in honor of Bachet de Meziriac and the famous Russian chemist Dmitri Mendeleev. However, this raises the question: why did the great Russian chemist become interested in the "weighing" problem? The answer to this question requires some little-known facts from the life of the great scientist. In 1892, Mendeleev was appointed the director of the Russian Depot of Standard Weights, which, according to Mendeleev's initiative, was transformed in 1893 to the Main Board of Weights and Measures of Russia. Mendeleev remained its director until the end of his life. Thus, the final stage of Mendeleev's life (since 1892 until his death in 1907) was connected with the development of metrology. During this period, Mendeleev was actively involved in various problems connected with measures, measurement and metrology; the "weighing" problem was one of them. Mendeleev's contribution to the development of metrology in Russia was so great, that he was named "the father of Russian metrology" and the "weighing" problem was named the Bachet-Mendeleev problem.

The essence of the problem consists of the following [169]. Suppose, we need to weigh on a balance any integer-valued weight Q in the range from 0 up to $Q_{\max}$ by using the n standard weights $\{q_1, q_2, ..., q_n\}$, where q_1=1 is a measurement unit; $q_i=k_i \times q_1$; k_i is any natural number. It is clear that the maximal weight $Q_{\max}$ is equal to the sum of all the standard weights, that is,

$$Q_{\max} = q_1 + q_2 + ... + q_n = (k_1 + k_2 + ... + k_n)\, q_1. \tag{7.9}$$

We have to find the *Optimal System of Standard Weights,* that is, such standard weight system, which gives the maximal value of the sum (7.9) with a given measurement unit q_1=1 among all possible variants.

There are two variants of the solution to the Bachet-Mendeleev problem. In the first case, the standard weights can be placed only on the free cup of the balance; in the second case, they can be placed on both cups of the balance. The *Binary Measurement Algorithm* with the binary system of standard weights {1, 2, 4, 8,..., 2^{n-1}} is the optimal solution for the first case. It is clear that the binary measurement algorithm generates the binary method of number representation:

$$Q = \sum_{i=0}^{n-1} a_i 2^i, \tag{7.10}$$

where $a_i \in \{0,1\}$ is a binary numeral.

Note that the binary numerals 0 and 1 have a precise measurement interpretation. The binary numerals 0 and 1 encodes one of the two possible positions of the balance; thus, a_i=1, if the balance remains in the initial position after putting a new standard weight on the free cup, otherwise a_i=0.

The ternary system of the standard weights {1, 3, 9, 27,..., 3^{n-1}} is the optimal solution to the second variant of the "weighing" problem when we can put the standard weights on both cups of the balance. It is clear that the ternary system of the standard weights results in the ternary method of number representation:

$$Q = \sum_{i=0}^{n-1} b_i 3^i, \tag{7.11}$$

where b_i is a ternary numeral that takes the values {-1, 0, 1}.

Hence, by solving the "weighing" problem, Fibonacci and his followers found a deep connection between measurement algorithms and positional methods of number representation. This is the key idea of algorithmic measurement theory [20, 21], which goes back to the "weighing" problem for its origin. Thus, from the measurement problem we come unexpectedly to the positional method of number representation - the greatest mathematical discovery of Babylonian mathematics.

7.4.3. *Asymmetry Principle of Measurement*

The principles of *Finiteness* and *Potential Feasibility* of measurement, underlying the constructive measurement theory, are "external" with respect to the measurement. They have such a general character that there is a danger of their reduction to some trivial result (for example, to the above binary measurement algorithm that leads the measurement to the consecutive comparison of the measurable weight with the binary standard weights $2^{n-1}, 2^{n-2}, ..., 2^0$).

For obtaining non-trivial results, the methodological basis of the constructive (algorithmic) theory of measurement should be added to by a certain general principle, which follows from the essence of measurement. Such principle follows directly from the analysis of the binary measurement algorithm, which is the optimal solution to Bachet-Mendeleev's problem.

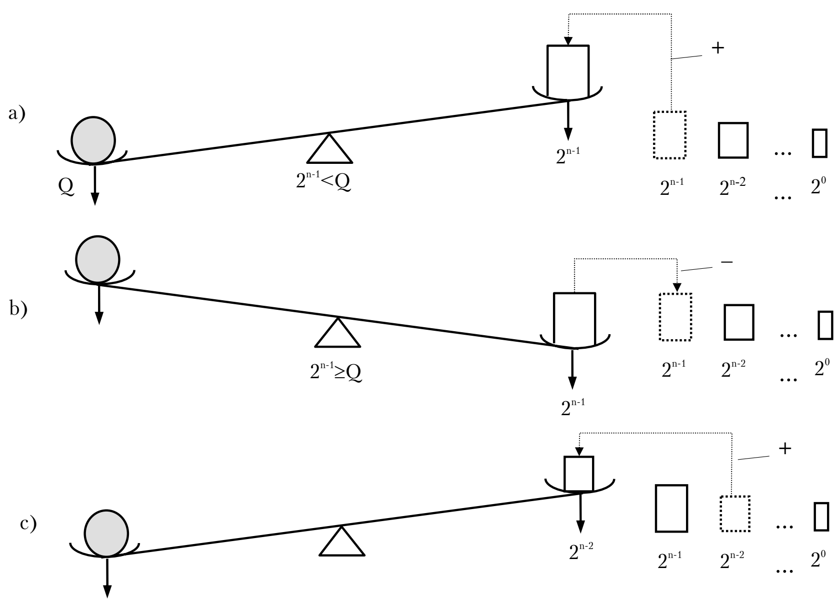


Figure 7.3. Asymmetry Principle of Measurement

We will analyze the above binary measurement algorithm by means of the use of the balance model (Fig. 7.3). This analysis allows for finding a measurement property of general character for any thinkable measurement, based on the comparison of the measurable weight Q with the standard weights.

We shall examine the weighing process of the weight Q on the balance, by using the binary standard weights. On the first step of the binary algorithm the largest standard weight 2^{n-1} is placed on the free cup of the balance (Fig. 6.3-a). The balance compares the unknown weight Q with the largest standard weight 2^{n-1}. After the comparison, we can get two situations: $2^{n-1}<Q$ (Fig. 7.3-a) and $2^{n-1}\geq Q$ (Fig. 6.3-b). In the first case (Fig. 7.3-a), the second step is to place the next standard weight 2^{n-2} on the free cup of the balance. In the second case (Fig.7.3-b), the weigher should perform two operations. First, one should remove the previous standard weight 2^{n-1} from the free cup of the balance (Fig. 7.3-b), after that, the balance should return to the initial position (Fig. 7.3-c). After returning the balance to the initial position, the next standard weight 2^{n-2} should be placed on the free cup of the balance (Fig. 7.3-c).

As we can see, both cases differ in their complexity. In the first case, the weigher fulfils only one operation, that is, he adds the next standard weight 2^{n-2} on the free cup of the balance. In the second case, the weigher's actions are determined by two factors. First, he has to remove the previous standard weight 2^{n-1} from the free cup of the balance, and after that, the balance is back to the initial position. After that, the weigher has to place the next standard weight on the free cup of the balance. Thus, for the first case the weigher has to perform only one operation. However, for the second case the weigher has to perform two sequential operations:

(1) remove the previous standard weight 2^{n-1} from the free cup;

(2) place a new standard weight 2^{n-2} on the free cup of the balance.

In the second case, the weigher's actions are more complicated in comparison with the first case because the weigher has to remove the previous standard weight taking into consideration the time necessary for the balance to return to its initial position. By synthesizing the optimal measurement algorithms, we have to consider these new data, which influence the process of measurement. We name this discovered property of measurement, the *Asymmetry Principle of Measurement* [20].

7.4.4. *A New Formulation of the Bachet-Mendeleev Problem*

Now, let us introduce the above measurement property into the Bachet-Mendeleev problem. For this purpose we will examine a measurement as a process, running during discrete periods of time; let the operation to "add the standard weight" be carried out within one unit of discrete time and the operation to "remove the standard weight" (which is followed by returning the balance into the initial position) be carried out within p units of discrete time, with $p \in \{0,1,2,3,...\}$.

It is clear that the parameter p simulates the *Inertness* of the balance. Note that the case $p=0$ corresponds to the "ideal situation" when we neglect the inertness of the balance. This case corresponds to the classical Bachet-Mendeleev problem. For the other cases $p>0$, we have some new variants of the Bachet-Mendeleev problem, which are studied in the algorithmic measurement theory [20, 21]. Below we study other generalizations of the Bachet-Mendeleev problem.

7.5. Mathematical Model of Measurement

7.5.1. *A Notion of the "Indicator Element"*

If we put forward a problem to create the *Algorithmic Measurement Theory*, we should define more exactly, what is a *Measurement*, what is the *Purpose of Measurement*, what is a *Measurement Algorithm,* and *How are the Measurement Algorithms carried out*.

First of all, we should note that, if we wish to measure something, we should know the range of the measurable magnitude. When we discuss "mathematical measurement," we are distracted from the physical nature of measurable magnitude. We will present a *Range of Measurable Magnitude* in the form of the geometric line segment AB. Clearly the measurable magnitude is one of the possible sizes of the magnitude that belong to the range of measurement, that is, prior to the measurement, a certain "indeterminacy" about the measurable size exists because otherwise the measurement would be senseless. We will represent this situation of "indeterminacy" by the "unknown" point X, which belongs to the line segment AB.

Now, let us formulate a purpose for the measurement. The purpose of the measurement is to find the length of the "unknown" segment AX. In practice this purpose is carried out by means of special devices, for example, "balances" or "comparators." The "comparators" perform a comparison of the measurable magnitude with some "measures," generated from the "measure unit." In dependence on the result of comparison, the device "indicates" the result of comparison in the form of a binary signal 0 or 1. Thus, the essence of the measurement consists of *Consecutive Comparisons* of the measurable magnitude with some "measures," which are formed at each step of the measurement. Moreover, each step of the measurement depends on the results of comparison on the preceding steps.

In order to model the process of comparison of the measurable segment *AX* with the "measures," we introduce the important concept of the *Indicator Element* (IE), which is an original model of the "comparator" or "balance," the basic tool of any measurement. Suppose that every IE can be put to any "known" point *C* of the segment *AB*. The indicator element provides the information about the mutual positioning the "unknown" point *X* and the "known" point *C*. If the IE is to the right of the point *X*, it "generates" the binary signal 0; otherwise, the binary signal 1.

7.5.2. *The (n, k, S)-algorithms*

Some *Conditions* or *Restrictions S* (that follow, for example, from the "Principle of Asymmetry of Measurement") can be imposed on the measurement process; thus, by means of the restrictions *S* the "inertness" property of the balance can be taken into consideration.

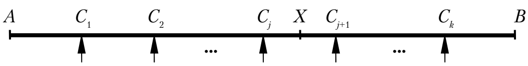


Figure 7.4. A geometric model of measurement

By using a concept of the indicator elements, we can describe the process of measurement in terms of geometry. Suppose we have a line segment *AB* with some "unknown" point *X* (Fig. 7.4).

The problem is to find the length of line segment *AX*. This is performed by means of the *k "Indicator Elements"* (IE). After we enclose the *j*-th IE (j=1,2,3,...,k) at some point C_j, the line segments *AX* and AC_j are compared, i.e. the relations "less than" ($AX< AC_j$) or "greater than or equal" ($AX\geq AC_j$) are defined. The relations "less than" and "greater than or equal" are encoded by the binary numerals 0 and 1, which are generated by the IE in the process of measurement. The problem of the measurement of a line segment *AX* by the *k* IE's is reduced to decreasing the *Indeterminacy Interval* about *X* according to the IE-"indications."

A measurement procedure consists of the *Measurement Steps*: in the first step the *k* IE's are enclosed to some points of the line segment *AB,* and the indeterminacy interval about *X* is diminished to the line segment A_1B_1 according to the IE-"indications"; in the next step the IE's are enclosed to the points of the line segment A_1B_1, and so on. The restrictions *S* are imposed on the measurement procedure. For the given restrictions *S* and the given *n* and *k,* the system of formal rules, which strictly define the points of the *k* IE's enclosures for each step in dependence on the IE-"indications" on the preceding steps, is called the (*n,k,S*)-*algorithm.*

7.5.3. *The Optimal (n, k, S)-algorithms*

A model of the indicator elements reduces the measurement problem to the problem of searching the point X (its coordinate is equal to the length of the line segment AX) within the line segment AB by means of the use of the k IE's in n steps. The above approach allows one to use the methods of one-dimensional search for the synthesis of the optimal measurement algorithms [20].

Let us consider the (n,k,S)-algorithm, which acts on the line segment AB. Some "indeterminacy interval" Δ_i, which contains the point X on the last step of the algorithm, is the result of the (n,k,S)-algorithm for the certain point X. Let us consider the action of the algorithm for all possible points $X \in AB$. As a result we obtain the set of line segments $\{\Delta_i\}$ called the *Partition P of the Line Segment AB,* i.e.

$$P=\{\Delta_1, \Delta_2, ..., \Delta_i, ..., \Delta_N\}. \tag{7.12}$$

For the general case, the partition (7.12) consists of the N line segments of the kind Δ_i. They are not equal to one another in the general case. Besides, there is the following correlation for the partition P:

$$AB=\Delta_1+\Delta_2+...+\Delta_N. \tag{7.13}$$

We choose from the partition (7.13) the largest line segment $\Delta_{\max}$ and define the effectiveness of the (n,k,S)-algorithm by means of the line segment $\Delta_{\max}$. Consider the ratio

$$T=\frac{AB}{\Delta_{\max}}, \tag{7.14}$$

which is called the (n,k)-*exactness of the* (n,k,S)-*algorithm.*

In accordance with (7.14), each (n,k,S)-algorithm is characterized by the number T, which is some numerical evaluation of the effectiveness of the (n,k,S)-algorithm. The availability of such a number T allows one to compare the different (n,k,S)-algorithms according to their effectiveness. By using the definition (7.14), we can introduce the notion of an *Optimal* (n,k,S)-*algorithm.*

Definition 7.1. For some given n, k, and S, the (n,k,S)-algorithm is called optimal, if it provides a maximal value of the (n,k)-exactness T given by (7.14) among all possible variants.

Note that this definition is based on the so-called *Mini-max Principle* [20]. This principle is used widely in the modern theory of optimal systems. According to this principle we refer some strategy to the "optimal" strategy, if it provides the maximal value of the efficiency criterion for the worst case.

It is easy to prove [20] that the optimal (n,k,S)-algorithm divides the line segment AB into N equal parts, i.e.

$$\Delta_1 = \Delta_2 = ... = \Delta_N = \Delta. \tag{7.15}$$

For the case (7.15) we have the following expression for the (n,k)-exactness:

$$T = \frac{AB}{\Delta} = N. \tag{7.16}$$

Thus, in accordance with (7.16) the (n,k)-exactness of the optimal (n,k,S)-algorithm coincides with the number N of the quantization levels that are provided by the optimal (n,k,S)-algorithm.

7.6. Classical Measurement Algorithms

Over the millennia, the practice of human measurement found a number of measurement algorithms that are widely used today in mathematics and measurement technology. The most common among them include the following: *Binary Algorithm, Counting Algorithm*, *Ruler Algorithm.* We will study these algorithms by using the above "indicator" model of measurement (Fig. 7.4).

7.6.1. *The "Binary" Algorithm*

The essence of the *"Binary" Algorithm* is demonstrated in Fig. 7.5. It consists of 3 steps (n steps in the general case) and uses only IE (k=1). By using 3 steps, the algorithm divides the initial line segment [0,8] into 8 equal parts.

The First Step is to enclose the IE to the middle of the initial line segment [0,8], i.e. to point 4. After the first step there appear two situations in dependence on the IE-"indication," [0,4] and [4,8].

The Second Step:

(a) If the IE-"indication" in point 4 is equal to 0 (the IE appears to the left), this means that the "unknown" point X is at the line segment [0,4]. For this situation the second step of the "binary" algorithm is to enclose the IE to the middle of the line segment [0,4], that is, to point 2.

(b) If the IE-"indication" in point 4 is equal to 1 (the IE appears to the right), this means that the "unknown" point X is at the line segment [4,8]. For this situation the second step of the "binary" algorithm is to enclose the IE to the middle of the line segment [4,8], i.e. to point 6.

It is clear that after the second step four situations may appear as shown in Fig. 7.5.

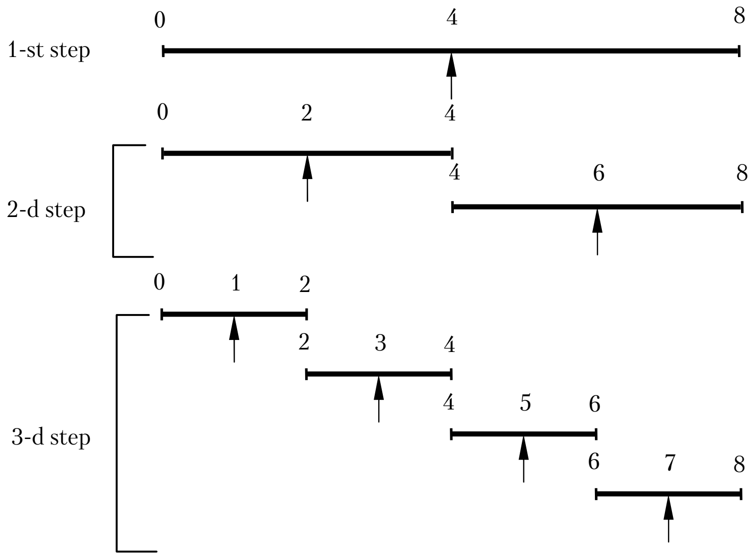


Figure 7.5. The "binary" algorithm

The Third Step. We can see from Fig. 7.5 that after the second step of the "binary" algorithm we have the four "indeterminacy intervals" [0,2], [2,4], [4,6], [6,8]. The third step of the "binary" algorithm is to enclose the IE at the middle of one of the "indeterminacy intervals" [0,2], [2,4], [4,6], [6,8]. It follows from Fig. 7.5 that the "binary" algorithm provides the following partition of the initial line segments [0,8]:

P={[0,1], [1,2], [2,3], [3,4], [4,5], [5,6], [6,7], [7,8]},

that is, it divides the line segment [0,8] into 8 equal parts.

It is clear that the above "binary" algorithm is the (3,1,S)-algorithm and its (3,1)-exactness is equal to T=2^3=8. It is easy to prove that the n-step "binary" algorithm provides the (n,1)-exactness given by the following formula:

$$T = 2^n. \tag{7.17}$$

7.6.2. *The "Counting" Algorithm*

Now, let us consider the so-called *Counting Algorithm* widely used in measurement practice. The "counting" algorithm (see Fig. 7.6) consists of 3 steps (n steps in the general case) and uses only the IE (k=1). It divides the initial line segment [0,4] into 4 equal parts in 3 steps.

The First Step is to enclose the IE at the point 1. After the first step there appear two situations, [0,1] and [1,4], depend on the IE-"indication" in point 1.

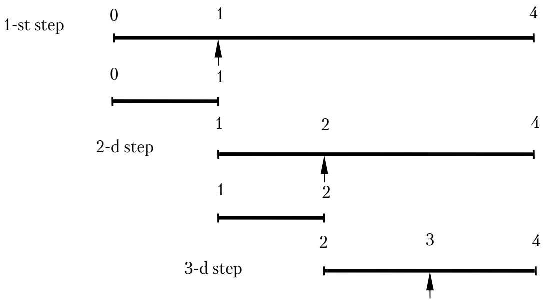


Figure 7.6. The "counting" algorithm

The Second Step:

(a) If the IE-"indication" in point 1 is equal to 0 (the IE appears to the left), then the "unknown" point X is at the line segment [0,1]. For this situation the measurement process is over because the X-coordinate ($X \in [0,1]$) is defined with "exactness" equal to measurement unit 1.

(b) If the IE-"indication" in point 1 is equal to 1 (the IE appears to the right), this means that the "unknown" point X is at the line segment [1,4]. For this situation the measurement process will continue and the IE is enclosed at point 2.

The Third Step. We can see from Fig. 7.6 that on the third step the IE is enclosed at point 3, which is from point 2 on the distance equal to the measurement unit 1.

From the example in Fig. 7.6 we can find the following general rule for the "counting" algorithm. If on the i-th step of the "counting" algorithm, which acts at the line segment AB, the IE was enclosed at point C, then we have two situations on the $(i+1)$-th step. If the IE appears at point C to the left, then the measurement procedure is over. If the IE appears at point C to the right, then we get the "indeterminacy interval" CB and the IE on the $(i+1)$-th step is enclosed at point $C'=C+1$.

It is clear that the above "counting" algorithm is the $(3,1,S)$-algorithm and its (3,1)-exactness is equal to $T=4$. It is easy to prove that the n-step "counting" algorithm has the $(n,1)$-exactness given by the following formula:

$$T = n + 1. \tag{7.18}$$

7.6.3. *The "Ruler" Algorithm*

Let's consider one more example of the measurement algorithm widely used in measurement practice. The latter underlies the traditional measurement ruler. We call this algorithm the *"Ruler" Algorithm.* The essence of the latter is demonstrated in Fig. 7.7.

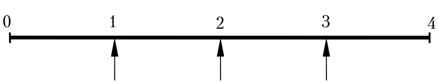


Figure 7.7. The "ruler" algorithm

The "ruler" algorithm is performed in 1 step and uses the k IE (the 3 IE in Fig. 7.7). The algorithm divides the initial line segment into

$$T = k + 1 \tag{7.19}$$

equal parts (into 4 parts in Fig. 7.7).

The formula (7.19) gives the $(1, k)$-exactness of the "ruler" algorithm. We can see that the latter refers to a class of the $(1,k,S)$-algorithms.

7.6.4. *The Restriction S*

The above definition of the (n,k,S)-algorithm includes a concept of the *Restriction S*. Clearly, any "restriction" limits the efficiency of the algorithm. However, each "restriction" (as show below) can result in rather unusual measurement algorithms, which are of theoretical and practical interest.

We start with the elementary "restrictions," which can be found in the comparative analysis of the classical measurement algorithms, in particular, the "binary" algorithm and the "counting" algorithm. Compare two 3-step (n,k,S)-algorithms, shown in Figs. 7.5 and 7.6. Both algorithms consist of 3 steps and use only the IE (k=1). What is the difference between them? The difference consists in the character of the movement of the IE along the line segment AB. Consider the restriction S for the "binary" algorithm in Fig. 6.6. Remember that the restriction S is imposed upon the movement of the IE along the line segment AB. We can see that the IE for the "binary" algorithms may be enclosed at each step to the left or to the right of the "unknown" point X. One may say that the movement of the IE for this case is performed without any "restrictions." Denote the restriction S for the "binary" algorithm by $S \equiv 0$. We will call the class of (n,k,S)-algorithms, which each individually satisfy the restriction $S \equiv 0$, the $(n,k,0)$*-algorithms.*

Consider the restriction S for the "counting" algorithm. We can see that for this case the IE is moving along the line segment [0,4] in only one direction, namely, from the point 0 to the point 4. We denote the restriction of this kind by $S \equiv 1$ and call the class of (n,k,S)-algorithms satisfying the restriction $S \equiv 1$, the $(n,k,1)$*-algorithms.*

Note that the restrictions $S \equiv 0$ and $S \equiv 1$ are not the only possible "restrictions." Below we will study the "restriction" S, which follows from the above formulated *Asymmetry Principle of Measurement*. It is important to emphasize, that each "restriction" results in the development of one or another class of the new measurement algorithms, which can be of theoretical or practical interest; therefore, a search of reasonable "restrictions" imposable on the measurement algorithm, is an important problem in algorithmic measurement theory.

7.6.5. *Connection between Measurement Algorithms and Positional Number Systems*

There is a deep connection between measurement algorithms and positional methods of number representation. In fact, the "binary" algorithm generates the binary representation of numbers:

$$A = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_i 2^i + \ldots + a_1 2^1 + a_0 2^0, \tag{7.20}$$

where a_i is a binary numeral {0,1}; 2^i is the weight of the ith digit (i=0,1,2, ..., n-1).

The formula (7.20) has the following "measurement" interpretation. The binary numerals a_{n-1}, a_{n-1}, ..., a_0 are the IE-"indications" on the first, second, ... , nth step of the "binary" algorithm, respectively.

One can readily see that the "counting" algorithm "generates" the following method of number representation that underlies the Euclidean definition of natural number:

$$N = \underbrace{1+1+\ldots+1}_{N}. \tag{7.21}$$

The latter is well known as a *Unitary Code* of the number N.

7.7. The Optimal Measurement Algorithms Originating Classical Positional Number Systems

7.7.1. *A General Method for the Synthesis of the Optimal (n,k,S)-algorithms*

For the synthesis of the optimal (n,k,S)-algorithm we use the *Method of Recurrence Relations* [170]. To obtain the recursive relation for (n,k)-exactness T of the optimal (n,k,S)-algorithm, we first formulate an *Inductive Assumption*: for any arbitrary n, k and S there is the optimal (n,k,S)-algorithm, which divides the line segment AB into T equal line segments of length D, where T is the (n,k)-exactness of the optimal (n,k,S)-algorithm. For a given restriction S, the (n,k)-exactness T depends on n and k. We can present this dependence in the form of some function of n and k, i.e.

$$T=F(n,k). \tag{7.22}$$

It is clear if the length D=1, then the line segment

$$AB=T=F(n,k). \tag{7.23}$$

Consider the first step of the optimal (n,k,S)-algorithm, which acts on the line segment AB (Fig. 7.8).

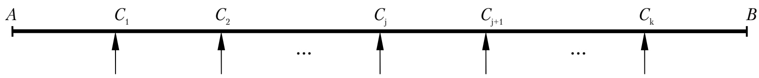


Figure 7.8. The first step of the optimal (n, k, S)-algorithm

Let the first step be to enclose the k IE's at some points $C_1, C_2, \ldots, C_j, C_{j+1}, \ldots, C_k$. Then, according to the IE-"indications" the following $(k+1)$ different situations may result:

$$(1)X \in AC_1; (2)X \in C_1C_2; \ldots; (j+1)X \in C_jC_{j+1}; \ldots; (k+1)X \in C_kB. \tag{7.24}$$

The second step of the optimal (n,k,S)-algorithm is to enclose the IE's at the points of the new "indeterminacy interval," which is one of the line segments (7.24). It is clear that for this situation we can use the optimal $(n-1)$-algorithm. However, the number of the IE's on the second step depends on the restriction S. Suppose, that in accordance with the restriction S we can use only the r IE's $(r \le k)$ on the second step of the algorithm. Note that the number r may vary for the different situations (7.24).

Suppose that the line segment C_jC_{j+1} is the "indeterminacy interval" on the second step. Since on the second step we have r IE's and $(n-1)$ steps of the algorithm, we can use the optimal $(n-1,r,S)$-algorithm on the line segment C_jC_{j+1}. In accordance with the inductive assumption, the $(n-1,r,S)$-algorithm will divide the line segment C_jC_{j+1} into $F(n-1,r)$ equal line segments of length $\Delta=1$, i.e.

$$C_jC_{j+1} = F(n-1,r). \tag{7.25}$$

Using the equality (7.13), we can write:

$$AB = AC_1 + C_1C_2 + \ldots + C_jC_{j+1} + \ldots + C_kB. \tag{7.26}$$

Using (7.23) and (7.25), we can get the recursive relation for the calculation of the function (7.23).

7.7.2. *The Optimal (n,k,0)-algorithms*

Now, we can use the above method for the synthesis of the optimal $(n,k,0)$-algorithm. Recall that the restriction $S\equiv 0$ means what each IE may be enclosed at any arbitrary point of the line segment of AB on each step.

According to the above method, one of the situations (7.24) appears after the first step of the optimal $(n,k,0)$-algorithm. Consider the situation: $X \in C_jC_{j+1}$, which can appear after the first step of the algorithm. Then in accordance with the restriction $S\equiv 0$, the second step of the optimal $(n,k,0)$-algorithm is to enclose all k IE's at the points of the new line segment C_jC_{j+1}. If we use the $(n-1,k,0)$-algorithm on the "indeterminacy interval" C_jC_{j+1}, then in accordance with the inductive assumption we can divide the line segment C_jC_{j+1} into $F(n\text{-}1,k)$ equal parts $\Delta=1$, i.e.

$$C_jC_{j+1} = F(n\text{-}1,k). \tag{7.27}$$

The expression (7.27) is valid for any arbitrary line segment from the set (7.24). Then, by using (7.26), we obtain the following recursive relation for the calculation of the function $F(n,k)$:

$$F(n,k)=(k+1)F(n\text{-}1,k). \tag{7.28}$$

The recursive function (7.28) can be expressed in explicit form. For this purpose, we can use the recursive relation

$$F(n\text{-}1,k)=(k+1)F(n\text{-}2,k)$$

for the representation of (7.28) in the following form:

$$F(n,k)=(k+1)(k+1)F(n\text{-}2,k).$$

By continuing this process, we can represent the function (7.28) in the following form:

$$F(n,k)=(k+1)^{n-1}F(1,k). \tag{7.29}$$

Thus, according to (7.29) the solution to the problem is reduced to obtaining the expression for the function $F(1,k)$. The latter is the expression for the $(1,k)$-exactness of the optimal $(1,k,0)$-algorithm. Remember that the $(1,k,0)$-algorithm consists of 1 step. This step is to enclose the k IE's at the points of the line segment AB. It is easy to prove that in this case the optimal solution is to divide the line segment AB into $(k+1)$ equal parts. It follows from this consideration that the $(1,k)$-exactness of the optimal $(1,k,0)$-algorithm is equal to

$$F(1,k)=k+1. \tag{7.30}$$

By substituting (7.30) into (7.29), we obtain the expression for the function (7.28) in the explicit form:

$$F(n,k)=(k+1)^n . \tag{7.31}$$

The expression (7.31) allows one to formulate the essence of the optimal $(n,k,0)$-algorithm. Each step of the algorithm is to divide the initial "indeterminacy interval" (the line segment AB) and all the next "indeterminacy intervals" into $(k+1)$ equal parts. As a result of the application of the algorithm, we obtain the formula (7.31), which characterizes the effectiveness of the optimal $(n,k,0)$-algorithm.

7.7.3. *Special Cases of the Optimal (n,k,0)-algorithm*

Consider some special cases of the optimal $(n,k,0)$-algorithm. For the case $k=1$ the formula (7.31) is reduced to the formula (7.17), which expresses the $(n,1)$-exactness of the "binary" algorithm. For the case $n=1$ the formula (7.31) is reduced to the formula (7.19), which expresses the $(1,k)$-exactness of the "ruler" algorithm. It follows from this examination that the well-known measurement algorithms, "binary" algorithm and "ruler" algorithm, are partial cases of the optimal $(n,k,0)$-algorithm.

Similar to the "binary" algorithm, which generates the "binary" number system (7.20), the optimal $(n,k,0)$-algorithms generate the positional number systems with radix $R=k+1$, i.e.

$$A=a_{n-1}R^{n-1}+ a_{n-2}R^{n-2}+...+a_iR^i+...+ a_1R^1+ a_0R^0, \tag{7.32}$$

where $a_i\in \{0,1,2,3,...,k\}$ is a numeral of the ith digit.

The expression (7.32) has the following "measurement" interpretation. The numeral a_{n-1} in (7.32) encodes the "indications" of the IE's on the first step of the optimal $(n,k,0)$-algorithm according to the rule:

$$\begin{aligned} 0&=000...00\\ 1&=100...00\\ 2&=110...00\\ &...\\ k-1&=111...10\\ k&=111...11 \end{aligned} \tag{7.33}$$

The numeral a_{n-2} encodes the "indications" of the IE's on the second step of the algorithm and the numerals a_1 and a_0 encode the "indications" of the k IE's according to the rule (7.33) on the $(n-1)$-th and n-th steps of the optimal $(n,k,0)$-algorithm, respectively.

It is clear that for $k=9$ the optimal $(n,9,0)$-algorithm generates the decimal system and for $k=59$ the Babylonian sexagesimal system. It follows from this consideration that the optimal $(n,k,0)$-algorithms generate all historically well-known positional systems.

7.8. Optimal Measurement Algorithms Based on the Arithmetical Square

7.8.1. *The Optimal (n,k,1)-algorithms*

Consider the class of the $(n,k,1)$-algorithms. Remember that the restriction $S\equiv1$ means that the indicator elements (IE) are moving along the line segment AB only in the direction from point A to point B. This means, if any IE on any step is found to the right of the "unknown" point X, then this IE will "leave the field," i.e. this IE cannot be used for further measurement. Note that the above "counting" algorithm (Fig. 7.6) is the simplest example of the restriction $S\equiv 1$.

Let the first step of the optimal $(n,k,1)$-algorithm be to enclose the k IE's to the certain points $C_1, C_2, ..., C_j, C_{j+1}$, ..., C_k of the line segment AB (Fig. 7.8). Consider the situations (7.24) that can appear after the first step of the algorithm. Consider the situation

$$X \in C_j C_{j+1}. \tag{7.34}$$

It is clear for the situation (7.34) that the j IE's are on the left of the "unknown" point X and the rest, the $(k-j)$-th IE's, are on the right of point X.

In accordance with the restriction $S\equiv 1$, the $(k - j)$ IE's, which are on the right of the "unknown" point X, cannot be used for further measuring. This means that the second step of the algorithm is to enclose to the line segment $C_j C_{j+1}$ only those j IE's which are found on the left of the "unknown" point X after the first step of the algorithm. That is why we can only use the optimal $(n-1,j,1)$-algorithm for the situation (7.34). In accordance with the inductive assumption, the optimal $(n-1,j,1)$-algorithm can divide the line segment $C_j C_{j+1}$ into $F(n-1,j)$ equal parts $\Delta=1$, that is,

$$C_j C_{j+1} = F(n-1,j). \tag{7.35}$$

Consider the situations

$$X \in AC_1 \tag{7.36}$$

and

$$X \in C_k B. \tag{7.37}$$

For the situation (7.36) all IE's are to the right of the point X after the first step. This means that all IE's "leave the field" and the process of the measurement is over. It is clear from this examination that the line segment AC_1 must be the line segment of a single length, i.e.

$$AC_1 = 1. \tag{7.38}$$

For the situation (7.37) all IE's are to the left of point X after the first step of the algorithm and they can be enclosed to the points of the line segment $C_k B$ on the second step of the algorithm. Using the optimal $(n-1,k,1)$-algorithm, we can divide the line segment $C_k B$ into $F(n-1,k)$ equal parts $\Delta=1$, i.e.

$$C_k B = F(n-1,k). \tag{7.39}$$

Using the expressions (7.26), (7.35), (7.38), (7.39), we obtain the following recursive relation for the calculation of the function $F(n,k)$, which gives the (n,k)-exactness of the optimal $(n,k,1)$-algorithm:

$$F(n,k) = 1 + F(n-1,1) + F(n-1,1) + F(n-1,2) + ... + F(n-1,j) + ... + F(n-1,k-1) + F(n-1,k). \tag{7.40}$$

Using the recursive relation (7.40), we can write the following recursive formula for the calculation of the function $F(n, k\text{-}1)$:

$$F(n,k\text{-}1)=1+F(n\text{-}1,1)+F(n\text{-}1,1)+F(n\text{-}1,2)+...+F(n\text{-}1,j)+...+F(n\text{-}1,k\text{-}1). \quad (7.41)$$

Comparing the expressions (7.40) and (7.41), we can rewrite the recursive relation (7.40) in the following form:

$$F(n,k)= F(n,k\text{-}1)+ F(n\text{-}1,k\text{-}1). \quad (7.42)$$

Calculating the values of the function $F(n, k)$ for the special cases n=0 and k=0, we clearly have:

$$F(0,k)=1 \quad (7.43)$$

$$F(n,0)=1. \quad (7.44)$$

For an explanation of the "physical sense" of the expressions (7.43) and (7.44) we can examine the optimal $(n, k,1)$-algorithm for cases n=0 and k=0. The case n=0 means that we do not have any step for the measurement and, hence, the "indeterminacy interval" (the line segment AB) cannot be decreased. Thus, the expression (7.43) is valid.

Now, suppose that k=0. This means that we do not have any "Indicator Elements" for the measurement and the "indeterminacy interval" (the line segment AB) cannot be decreased. Thus, the expression (7.44) is valid.

7.8.2. *Arithmetical Square*

Using the recursive relation (7.42) with the initial terms (7.43) and (7.44), we can construct a table of the values of the function $F(n,k)$ (Table 7.1). We can see that Table 7.1 coincides with the well-known *Arithmetical Square* or *Pascal Triangle* and that the terms $F(n,k)$ are *Binomial Coefficients,* that is,

$$F(n,k)=C_{n+k}^{k}=C_{n+k}^{n}. \quad (7.45)$$

Table 7.1. Arithmetical Square

k/n	0	1	2	3	4	5	...	n
0	1	1	1	1	1	1	...	1
1	1	2	3	4	5	6	...	$F(n,1)$
2	1	3	6	10	15	21	...	$F(n,2)$
3	1	4	10	20	35	56	...	$F(n,3)$
4	1	5	15	35	70	126	...	$F(n,4)$
5	1	6	21	56	126	252	...	$F(n,5)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$
k	1	$F(1,k)$	$F(2,k)$	$F(3,k)$	$F(4,k)$	$F(5,k)$	...	$F(n,k)$

7.8.3. *The Optimal Measurement Algorithms Based on Arithmetical Square*

The arithmetical square (Table 7.1) allows to demonstrate the optimal $(n,k,1)$-algorithm. In fact, for the given n and k, the value of the function $F(n,k)$ is at the intersection of the n-th column and k-th row of the arithmetical square. The coordinates of the k IE's, which should be enclosed to the points $C_1, C_2, \ldots, C_j, C_{j+1}, \ldots, C_k$ of the initial line segment AB, are in the n-th column of the arithmetical square, that is,

$$AC_1 = 1, AC_2 = F(n,1), \ldots, AC_j = F(n, j-1), AC_{j+1} = F(n,j), \ldots, AC_k = F(n,k-1). \tag{7.46}$$

If after the first step of the algorithm the j IE's are to the left of point X and the rest $(k\text{-}j)$ IE's are to the right of X, the "indeterminacy interval" about X is diminished up to the line segment C_jC_{j+1}. The length of the latter is equal to the binomial coefficient $F(n,j)$. This binomial coefficient is at the intersection of the $(n\text{-}1)$-th column and the j-th row of the arithmetical square. To obtain the binomial coefficient $F(n,j)$ we have to move from the initial coefficient $F(n,k)$ by one column to the left and by $(k\text{-}j)$ rows upwards. On the second step we have to accept the point C_j as the new beginning of coordinates. Here the second step is to enclose j IE's to the points $D_1, D_2, \ldots, D_j$ of the new "indeterminacy interval" C_jC_{j+1}. The coordinates of the points $D_1, D_2, \ldots, D_j$ with respect to the point C_j - the new beginning of coordinates - are in the $(n\text{-}1)$-th column of the arithmetical square, that is,

$$C_jD_1 = 1, C_jD_2 = F(n-1,1), C_jD_3 = F(n-1,2), \ldots, C_jD_j = F(n-1, j-1). \tag{7.47}$$

This process continues up to the exhaustion either of all measurement steps or all IE's.

7.8.4. *The Example of the Optimal (n, k, 1)-algorithm*

Consider the optimal (3, 3, 1)-algorithm (Fig. 7.9). The algorithm consists of 3 steps (n steps in the general case) and uses 3 IE's (k IE's in the general case). The value of the initial "indeterminacy interval" is the binomial coefficient $F(3, 3)$=20, which is at the intersection of the 3-d column and 3-d row of the arithmetical square.

The first step of the algorithm at line segment [0,20] is to enclose 3 IE's to the points 1, 4, 10 (Fig. 7.9). In accordance with the IE-"indications," four situations may appear after the first step: [0,1], [1,4], [4,10], [10,20].

The second step:

(1-a) For the situation [0,1] the process of measurement is over because all IE's are on the right of the point X.

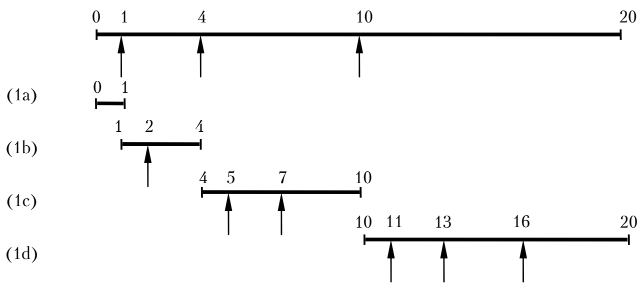


Figure 7.9. The optimal (3, 3, 1)-algorithm

(1-b) For the situation [1,4] we have the only 1 IE, which can be enclosed at point 2.

(1-c) For the situation [4,10] we have 2 IE's, which can be enclosed to points 5 and 7.

(1-d) For the situation [10,20] we have 3 IE's. For this case the coordinates of three points of the IE's enclosing line segment [10,20] with respect to the point 10, the new beginning of coordinates, are in the second column of the arithmetical square (Table 7.1) above binomial coefficient 10. Those are the binomial coefficients 1, 3, 6. By summing these numbers with the number 10, we obtain the coordinates of the points of the IE's enclosed on the second step, namely: 11=10+1, 13=10+3, and 16=10+6.

The third step. After the second step, the following situations will appear: [1,2], [2,4], [4,5], [5,7], [7,10], [10,11], [11,13], [13,16], [16,20]. Note that for the situations [1,2], [4,5], [10,11] the measurement process is over on the second step.

It is clear from Fig. 7.9 that the optimal (3,3,1)-algorithm can divide the initial line segment [0,20] into 20 equal parts.

7.9. Fibonacci Measurement Algorithms

7.9.1. *The Optimal Measurement Algorithms Based on the Fibonacci p-numbers*

Return to the "indicator" measurement model (Fig. 7.4) and try to introduce the above *Asymmetry Principle of Measurement* into this model. The following "restriction" to the movement of the IE along the line segment *AB* follows from the *Asymmetry Principle of Measurement.*

Suppose that the IE is enclosed at the point *C* on the first step of the *n*-step measurement algorithm (Fig. 7.10).

Thereafter, two situations (a) and (b) appear after the first step, as shown in Fig. 7.10. It is clear that for the situation (a) we can enclose the IE at any point of the "indeterminacy interval" *CB* on the next step. For the situation (b) we do not have the right to enclose the IE to the points of the "indeterminacy interval" AC, because on the first step the balance moves to the opposite position and we need p units of discrete time for returning the balance into the initial position. Thus, the restriction S, which follows from the *Asymmetry Principle of Measurement*, consists of the fact that for the situation (b) it is forbidden to enclose the IE to the points of the line segments AC during the next p steps of the algorithm.

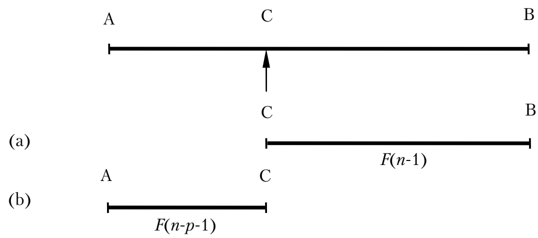


Figure 7.10. The first step of the Fibonacci measurement algorithm

We can use the above restriction for synthesis of the optimal measurement algorithm. Here we introduce the following "inductive assumption." Suppose that for the given $n\geq 0$ and $p\geq 0$ there is the optimal n-step measurement algorithm, which divides the initial line segment AB into $F_p(n)$ equal parts of $\Delta=1$, that is,

$$AB=F_p(n). \tag{7.48}$$

Suppose that the first step of the algorithm is to enclose the IE at the point C (Fig. 7.10). Then, after the first step two situations can appear. Line segment CB is the "indeterminacy interval" for situation (a). According to the restriction S we can enclose the IE at any point of CB on the second step. By using the optimal $(n-1)$-step algorithm in this situation we can divide the line segment CB into the $F_p(n-1)$ equal parts $\Delta=1$ and, hence,

$$CB = F_p(n-1). \tag{7.49}$$

Line segment AC is the "indeterminacy interval" for the situation (b). According to the restriction S, we cannot enclose the IE to the points of the line segment AC during the next p steps of the algorithm. Let us consider the situation (b) for two possible cases: (1) $p\geq n-1$ (2) $p<n-1$.

It is clear that all steps of the algorithm for case (1) will be "exhausted" much earlier before the "prohibition" to enclose the IE to the points of the line segment AC will be taken off. This means that for situation (1) the measurement is over after the first step. That is why line segment AC should be the line segment of unit length, that is,

$$AC=1. \tag{7.50}$$

Consider the situation (b) for case (2). For this case we have the right to enclose the IE to the points of the line segment AC only after p steps. Then, acting by the optimal $(n-p-1)$-algorithm we can divide the line segment AC into the $F_p(n-p-1)$ equal parts D=1, that is,

$AC=F_p(n-p-1)$. (7.51)

Taking into consideration that $AB=AC+CB$, we can write the following expression for the function $F_p(n)$ for cases (1) and (2), respectively:

$F_p(n)= F_p(n-1)+1$ with $p\geq n-1$ (7.52)

$F_p(n)= F_p(n-1)+ F_p(n-p-1)$ with $p<n-1$. (7.53)

Note that the recursive formula (7.53) coincides with the recursive formula for the Fibonacci p-numbers introduced in Chapter 4. That is why the measurement algorithms that correspond to the mathematical formulas (7.52) and (7.53) were called *Fibonacci Measurement Algorithms* [20]. Remember that Fibonacci p-numbers is a numerical sequence that is given by the following recursive relation

$F_p(n)=F_p(n-1)+ F_p(n-p-1)$ (7.54)

at the seeds:

$F_p(1)= F_p(2)=...= F_p(p+1)=1$. (7.55)

Consider in greater detail the mathematical formulas (7.52) and (7.53). For the case $p\geq n-1$ the Fibonacci measurement algorithms are described by the formula (7.53). The recursive relation (7.52) may be expressed in the explicit form. In fact, by decomposing the function $F_p(n-1)$ in (7.52) and all the following functions $F_p(n-2)$, $F_p(n-3)$, $F_p(1)$ in accordance with the same recursive relation (7.52), we obtain the following expression:

$F_p(n)=n+F_p(0)$. (7.56)

It follows from the "physical" reasoning that $F_p(0)=1$ and then the expression (7.56) may be represented in the following "explicit" form:

$F_p(n)=n+1$. (7.57)

Note that the formula (7.57) generates the natural numbers.

By joining the formulas (7.57) and (7.53), we can write the following expression for the function $F_p(n)$

$$F_p(n)=\begin{cases} n=1 & \text{with} \quad p\geq n-1 \\ F_p(n-1)+F_p(n-p-1) & \text{with} \quad p<n-1 \end{cases}. \quad (7.58)$$

Consider some special cases of the formula (7.58) for the different values of p. Let $p=0$. For this case the formula (7.58) is reduced to the following:

$F_0(n)=2\ F_0(n-1)$ (7.59)

$F_0(0)=1$. (7.60)

It is clear that the recursive formula (7.59) at seed (7.60) generates the "binary" sequence:

1, 2. 4. 8, 16, ..., $F_0(n)=2^{n-1}$.

It is clear that the Fibonacci measurement algorithm that corresponds to this case is reduced to the classical "binary" algorithm.

Let $p=\infty$. This means that the IE "leaves the field" when it is on the right of the point X. For this case the formula (7.58) takes the form of the expression (7.57) that generates natural numbers. It is clear that the classical "counting" algorithm is the optimal Fibonacci measurement algorithm corresponding to this case. Table 7.2 gives the values of the $(n,1)$-exactness (7.58) of the optimal Fibonacci algorithms for the different values p.

Table 7.2. The $(n,1)$-exactness of the Fibonacci measurement algorithms

$F_p(n)/n$	1	2	3	4	5	6	7	8	9
$F_0(n)$	2	4	8	16	32	64	128	256	512
$F_1(n)$	2	3	5	8	13	21	34	55	89
$F_2(n)$	2	3	4	6	9	13	19	28	41
$F_3(n)$	2	3	4	5	7	10	14	19	26
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F_\infty(n)$	2	3	4	5	6	7	8	9	10

The expression (7.58) gives the $(n,1)$-exactness of the optimal n-step Fibonacci algorithm for the restriction S that follows from the *Asymmetry Principle of Measurement*. We find the expression for the standard weights for Fibonacci algorithms in the form:

$\{W_p(1), W_p(2), ..., W_p(i), ..., W_p(n)\}$,

where $W_p(i)$ is the i-th standard weight. It is easily proven that the standard weight $W_p(n)$ is given by the following expression:

$$W_p(n)= W_p(n\text{-}1)+W_p(n\text{-}p\text{-}1) \text{ with } \quad n>p+1 \tag{7.61}$$

$$W_p(1)= W_p(2)= ... = W_p(p+1)=1. \tag{7.62}$$

Table 7.3 gives different variants of the standard weight systems corresponding to different values of p.

A comparison of the expressions (7.61) and (7.62) with expressions (7.52) and (7.53) shows that the standard weights $W_p(n)$ for Fibonacci's measurement algorithms coincide with the p-Fibonacci numbers [20], in particular, with the classical Fibonacci numbers for the case $p=1$.

Table 7.3. Standard weights for the Fibonacci measurement algorithms

$W_p(n)/n$	1	2	3	4	5	6	7	8	9
$W_0(n)$	1	2	4	8	16	32	64	128	256
$W_1(n)$	1	1	2	3	5	8	13	21	34
$W_2(n)$	1	1	1	2	3	4	6	9	13
$W_3(n)$	1	1	1	1	2	3	4	5	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$W_\infty(n)$	1	1	1	1	1	1	1	1	1

7.9.2. *The Example of the Fibonacci Measurement Algorithm*

Here we examine the optimal Fibonacci's measurement algorithm given by the expression (7.58) for the case p=1. Let n=5. Consider the 5-step Fibonacci algorithm (Fig. 7.11) corresponding to the case p=1. It follows from Table 7.2 that the above 5-step Fibonacci algorithm provides the (5,1)-exactness $F_1(5)$=13. This means that the optimal 5-step Fibonacci algorithm divides the line segment [0,13] into 13 equal parts. Thus, the given Fibonacci measurement algorithm consists of 5 steps and uses 5 standard weights {1, 1, 2, 3, 5} (see Table 7.3).

Let us consider the first 3 steps of the given algorithm.

The first step. We can use the first standard weight 5 for comparison with the measurable weight. This means that in the terms of the "indicator" model we have to enclose the IE at the point 5 of the line segment [0,13] (Fig. 7.11). We can see that the first step consists in the division of the line segment [0,13] in the Fibonacci ratio: 13=5+8. Two situations (a) and (b) (Fig. 7.11) can appear after the first step.

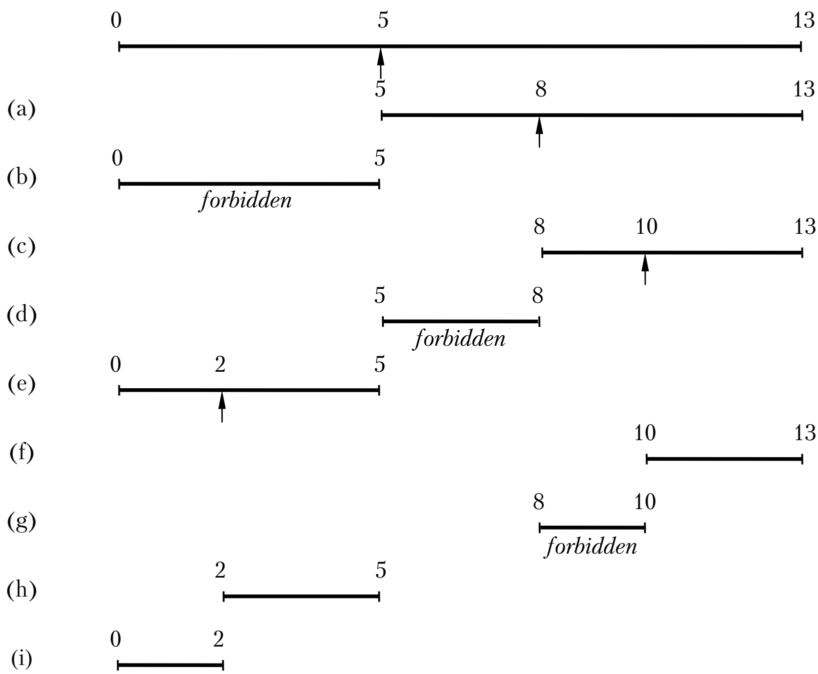


Figure 7.11. Example of the Fibonacci measurement algorithm

The second step. For situation (a) we can use the next standard weight 3 and divide the line segment [5,13] by the IE in the Fibonacci ratio: 8=3+5. Two new situations (c) and (d) (Fig. 7.11) appear after the second step.

For the situation (b) the second step is "empty" because in accordance with restriction S it is forbidden to enclose the IE to the points of the line segment [0, 5] on the second step.

The third step. For the situation (c) we can use the next standard weight 2 to divide the line segment [8,13] in the Fibonacci ratio: 5=3+2. Two new situations (f) and (g) appear after the third step.

For the situation (e) we can return to the situation (b) on the third step. In accordance with the restriction S we can enclose the IE at any point of the line segment [0,5] on the third step. We can use the standard weight 2 to divide the line segment [0,5] in the Fibonacci ratio: 5=2+3. Two new situations (h) and (i) appear after the third step.

It is easy to trace the actions of the Fibonacci algorithm for the next two steps (Fig. 7.11).

We can see from this example that the essence of the Fibonacci measurement algorithm is to divide the "indeterminacy interval" obtained on the preceding step in the Fibonacci ratio. It is easy to show that this general principle is valid for any arbitrary p. The division of the "indeterminacy interval" for this case is affected according to the recursive relation for Fibonacci p-numbers.

7.10. The Main Result of Algorithmic Measurement Theory

7.10.1. *A Further Generalization of the Bachet-Mendeleev Problem*

We can give a further generalization of the Bachet-Mendeleev problem [20]. Suppose that we use k balances simultaneously (k is a natural number, 1, 2, 3 ...). We can imagine that one and the same measurable weight Q is on the left-hand cups of all balances. Such a situation arises for the case of the "parallel measurement" of one and the same measured magnitude Q, when we compare the measurable magnitude Q with the standard weights by means of k "comparators" (such situation is widely used in the measurement of electric magnitudes). In this case the generalized Bachet-Mendeleev problem can be formulated as follows: it is necessary to synthesize the optimal n-step measurement algorithm to determine the value of the measurable magnitude Q with the help of k balances ("comparators"), which have the "inertness" p, for the condition that on each step the standard weights can be placed on the free cups of those balances, which are in the initial state.

Suppose that all balances, which participate in measuring, have the "inertness" p (p=0, 1, 2, 3, ...). This results in the problem to find the "parallel" optimal (n,k,S)-algorithm, which uses k balances having the "inertness" p. For the simulation of the "inertness" of the balances (or "Indicator Elements") we can introduce the notion of the state of the j-th IE on the l-th step (j=1, 2, 3, ..., k; l=1, 2, 3, ..., n). Denote the latter by $p_j(l)$. It follows from the physical examination of the Bachet-Mendeleev problem that the integer function $p_j(l)$ has the following properties:

$$0 \le p_j(l) \le p \tag{7.63}$$

$$p_j(l+1)=p_j(l)-1. \tag{7.64}$$

We can clarify the "physical sense" of the expressions (7.63) and (7.64). Note that the case $p_j(l)=0$ means that the j-th balance is in the initial position (Fig. 7.3-a) and the case $p_j(l)=p$ means that the balance is in the opposite position (Fig. 7.3-b). The expression (7.64) depicts the process of returning the j-th balance from the opposite position (Fig. 7.3-b) to the initial position (Fig. 7.3-a). According to (7.64) the state of the IE is decreased by 1 on the next step. Thus, if the j-th IE on the l-th step of the algorithm turns out to be in the state $p_j(l)=p$, then its states on the next steps will decrease successively according to the following:

$$p_j(l)=p \to p_j(l+1)=p-1 \to p_j(l+2)=p-2 \to \dots \to p_j(l+p-1)=1 \to p_j(l+p)=0.$$

Note that the case of $p_j(l)=0$ means that the j-th IE on the l-th step of the algorithm may be enclosed to the points of the "indeterminacy interval."

After such preliminary remarks we can try to synthesize the optimal (n,k,S)-algorithm. Before the l-th step is carried out, we renumber all indicator elements so that their states would be arranged according to the rule:

$$p_k(l) \ge p_{k-1}(l) \ge p_{k-2}(l) \ge \dots \ge p_2(l) \ge p_1(l), \tag{7.65}$$

where $p_j(l)$ is the state of the j-th IE on the l-th step (j=1,2,3, ..., k; l=1,2,3, ..., n).

We denote the IE-states on the first step of the (n,k,S)-algorithm by $p_1, p_2, p_3, \dots, p_k$. The initial IE-states are disposed in accordance with (7.65), that is,

$$p_k \ge p_{k-1} \ge p_{k-2} \ge \dots \ge p_2 \ge p_1. \tag{7.66}$$

7.10.2. *The Main Recursive Relation of the Algorithmic Measurement Theory*

We denote the (n,k)-exactness of the optimal (n,k,S)-algorithm for this case by $F(n,k)=F_p(n; p_1, p_2, p_3, \dots, p_k)$.

Let the initial IE-states $p_1, p_2, p_3, \dots, p_k$ be disposed according to (57) so that the first t states $p_1, p_2, p_3, \dots, p_t$ are equal to 0, that is,

$p_1 = p_2 = p_3 = \ldots = p_t = 0.$ (7.67)

This means that only the first t IE's, which satisfy (7.66), can be enclosed to the points of the line segment AB on the first step of the (n, k, S)-algorithm.

Suppose there is the optimal (n, k, S)-algorithm, which for the above restriction divides the line segment AB into $F_p\left(n; \underbrace{0, 0, \ldots, 0}_{t}, p_{t+1}, p_{t+2}, \ldots, p_k\right)$ equal parts of $\Delta=1$. Let the first step of the algorithm be to enclose the t IE's to the points C_1, C_2, ..., C_t (Fig. 7.12).

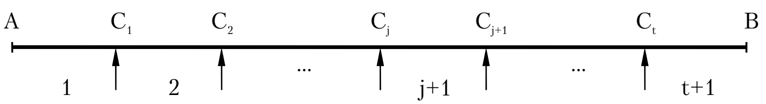


Figure 7.12. Synthesis of the optimal (n, k, S)-algorithm for the general case

After the first step of the algorithm, the $(t+1)$ situations

$X \in AC_1$, $X \in C_1C_2$, ..., $X \in C_jC_{j+1}$, ..., $X \in C_tB$

appear dependent upon the "indications" of the t IE's.

Consider the situation $X \in AC_1$. For this situation all t IE's are on the right of the "unknown" point X and hence they turn into the states of p after the first step. In accordance with property (7.64) the rest of the $(k\text{-}t)$ IE's decrease their states by 1 on the second step, that is, their states are:

$p_{t+1}\text{-}1, p_{t+2}\text{-}1, \ldots, p_k\text{-}1.$

If we arrange all k IE's in accordance with (7.65), we obtain the following IE-states on the second step of the algorithm:

$$p_{t+1}-1,\ p_{t+2}-1, \ldots,\ p_k-1,\ \underbrace{p,\ p, \ldots,\ p}_{t}. \tag{7.68}$$

If we use the optimal $(n\text{-}1,k,S)$-algorithm with the initial IE-states (7.68) for this situation, then in accordance with the inductive assumption we can divide the line segment AC_1 into

$$F_p\left(n-1;\ p_{t+1}-1,\ p_{t+2}-1, \ldots,\ p_k-1,\ \underbrace{p,\ p, \ldots,\ p}_{t}\right)$$

equal parts of $\Delta = 1$.

Consider the situation $X \in C_1C_2$. It follows from Fig. 7.12 that for this case the j IE's are on the left of the point X and hence remain in the 0-states, the $(t\text{-}j)$ IE's turn into the p states and the rest of the $(k\text{-}t)$ IE's decrease their states by 1, that is,

$p_{t+1}\text{-}1, p_{t+2}\text{-}1, \ldots, p_k\text{-}1.$

If we arrange all k IE's in accordance with (7.65), we obtain the following sequence of the IE-states on the second step of the algorithm:

$$\underbrace{0,0,...,0}_{j},\ p_{t+1}-1,\ p_{t+2}-1,...,\ p_k-1,\ \underbrace{p,\ p,...,\ p}_{t-j}. \tag{7.69}$$

If we use the optimal (n-1,k,S)-algorithm with the initial IE-states (7.69) for this situation, then in accordance with the inductive assumption we divide the line segment C_jC_{j+1} into

$$F_p\left(n-1;\underbrace{0,0,...,0}_{j},p_{t+1}-1,\ p_{t+2}-1,...,\ p_k-1,\ \underbrace{p,\ p,...,\ p}_{t-j}\right)$$

equal parts of $\Delta = 1$.

At least, it is easy to show that for situation $X\in C_tB$ the optimal (n-1,k,S)-algorithm divides the line segment C_tB into

$$F_p\left(n-1;\underbrace{0,0,...,0}_{t},p_{t+1}-1,\ p_{t+2}-1,...,\ p_k-1\right)$$

equal parts of $\Delta = 1$.

Taking into consideration the evidence equality

$$AB = AC_1 + C_1C_2 + ... + C_jC_{j+1} + ... + C_tB,$$

we obtain the following recursive relation for the calculation of (n,k)-exactness of the optimal (n,k,S)-algorithm:

$$F(n,k) = F_p\left(n;\underbrace{0,0,...,0}_{t},\ p_{t+1},\ p_{t+2},...,\ p_k\right)$$

$$=\sum_{j=0}^{t} F_p\left(n-1;\underbrace{0,0,...,0}_{j},\ p_{t+1}-1,\ p_{t+2}-1,...,\ p_k-1,\ \underbrace{p,\ p,...,\ p}_{t-j}\right). \tag{7.70}$$

Define the initial conditions for the calculation of $F_p(n,k)$ according to the recursive relation (7.70). We take into consideration that for the cases n=1 and $p_1=p_2=p_3=...=p_t=0, p_{t+1}>0$ the optimal (1,k,S)-algorithm divides the line segment AB into t=1 equal parts, that is,

$$F_p\left(1;\underbrace{0,0,...,0}_{t},p_{t+1},\ p_{t+2},...,\ p_k\right)=t+1. \tag{7.71}$$

Let us introduce the following definition:

$$F_p(n,k)=\begin{cases}0, & \text{with } n<0\\ 1, & \text{with } n=0\end{cases}. \tag{7.72}$$

It follows from the "physical sense" of the function $F(n,k)=F_p(n;p_1,p_2,p_3,...,p_k)$ that for the case $n\geq p_1>0$ we have:

$F_p(n; p_1, p_2, p_3, ..., p_k)=F_p(n\text{-}p_1; 0, p_2\text{-} p_1, p_3\text{-}p_1, ..., p_k\text{-}p_1).$ (7.73)

The recursive relation (7.70) at the initial terms (7.71) and (7.72) is the main result of the algorithmic measurement theory. This recursive relation gives a theoretically infinite set of optimal measurement algorithms for the given *n, k* and *p*.

7.10.3. *The Unusual Results*

The recursive relation (7.70) at the initial terms (7.71) and (7.72) contains a number of remarkable formulas of discrete mathematics.

Consider now the case of p=0. This means that the IE-states are equal to 0 on each step of the algorithm, that is,

$p_k(l)= p_{k\text{-}1}(l)= ... = p_2(l)= p_1(l)=0.$ (7.74)

Taking into consideration (7.74), we can prove that the recursive relation (7.70) and the initial condition (7.71) are reduced to the following:

$$F_p\left(n;\underbrace{0,0,...,0}_{k}\right)=(k+1)F_p\left(n-1;\underbrace{0,0,...,0}_{k}\right) \qquad (7.75)$$

$$F_p\left(1;\underbrace{0,0,...,0}_{k}\right)=(k+1). \qquad (7.76)$$

If we use the following definition

$$F_p\left(n;\underbrace{0,0,...,0}_{k}\right)=F(n,k),$$

we can then represent the expressions (7.75) and (7.76) as follows:

$F(n,k)=(k+1)F(n\text{-}1,k)$ (7.77)

$F(1,k)=k+1.$ (7.78)

We can see that the expressions (7.77) and (7.78) coincide with the formulas (7.28) and (7.30) for the optimal (*n,k,*0)-algorithm. As is shown above, the recursive function *F*(*n,k*) that is defined by the recursive relation (7.77) at the seeds (7.78) can be expressed in the explicit form given by the formula (7.31).

It follows from this consideration that for the case p=0 the general optimal (*n,k,S*)-algorithm given by the formulas (7.70) and (7.71) is reduced to the optimal (*n, k,* 0)-algorithm.

Now, let us consider the case when

$p=\infty$ (7.79)

and

$p_1=p_2= p_3= ...= p_k =0.$ (7.80)

The "physical sense" of the conditions (7.78) and (7.80) are as follows. The condition (7.80) means that all k IE's are in the 0-states on the first step of the algorithm. According to the condition (7.79) every IE that is found on the right above the "unknown" point X at any step of the algorithm "leaves the field," that is, cannot participate in further measurement.

Let us consider the function

$$F_p\left(n;\underbrace{0,0,...,0}_{j},\underbrace{p,p,...,p}_{k-j}\right). \tag{7.81}$$

It follows from the "physical sense" that for the cases (7.79) and (7.80) the function (7.81) cannot depend on the $(k\text{-}j)$ IE's that are found in the state $p=\infty$. Hence, for this case the function (7.81) depends only on the number n of the algorithm steps and the number j of the IE's that are found to the right of the point X on the first step of the algorithm. These "physical" considerations can be expressed in the following form:

$$F_p\left(n;\underbrace{0,0,...,0}_{j},\underbrace{p,p,...,p}_{k-j}\right)=F(n,j). \tag{7.82}$$

Note that

$$F_p\left(n;\underbrace{p,p,...,p}_{k}\right)=F(n,0)=1. \tag{7.83}$$

The "physical sense" of the expression (7.83) consists of the fact that all IE's are found in the state $p=\infty$ and we therefore do not have the possibility of decreasing the "indeterminacy interval."

Now let us return to formulas (7.70) and (7.71). It is clear that for the conditions (7.79) and (7.80) the formulas (7.82) and (7.83) take the following forms, respectively:

$$F_p\left(n;\underbrace{0,0,...,0}_{k}\right)=F(n,k) \tag{7.84}$$

$$F_p\left(1;\underbrace{0,0,...,0}_{k}\right)=k+1. \tag{7.85}$$

Taking into consideration definitions (7.82) and (7.83), we can represent the expressions (7.70) and (7.71) as follows:

$$F(n,k)=1+F(n\text{-}1,k)+F(n\text{-}1,1)+F(n\text{-}1,2)+...+F(n\text{-}1,j)+...+F(n\text{-}1,k) \tag{7.86}$$

$F(1,k)=k+1.$ (7.87)

Comparing the expressions (7.86) and (7.87) with the corresponding recursive relations for the optimal $(n,k,1)$-algorithm, we can see that these expressions coincide and hence,

$$F(n,k)=C_{n+k}^{k}=C_{n+k}^{n}.$$

It follows from this consideration that for conditions (7.79) and (7.80) the general optimal (n,k,S)-algorithm given by formulas (7.70) and (7.71) is reduced to the optimal $(n,k,1)$-algorithm based on the arithmetical square.

At least, consider the case

$k=1, p\geq 0$ and $p_1=0.$ (7.88)

This means that we deal with the $(n,1,S)$-algorithm $(k=1)$ and the IE-state on the first step of the algorithm equal to 0. Then, the expressions (7.70) and (7.71) for this case take the following form:

$F_p(n;0)= F_p(n\text{-}1;0)+F_p(n\text{-}1;p)$ (7.89)

$F_p(1;0)=2.$ (7.90)

Let us consider the function $F_p(n\text{-}1;p)$ in (7.89). Using (7.73), we can write the following expression:

$F_p(n\text{-}1;p)= F_p(n\text{-}p\text{-}1;0).$ (7.91)

Then, the expression (7.89) can be written as follows:

$F_p(n;0)= F_p(n\text{-}1;0)+F_p(n\text{-}p\text{-}1;0).$ (7.92)

Next let us introduce the following definition:

$F_p(n;0)= F_p(n).$

Then, the expressions (7.92) and (7.90) can be written as follows:

$F_p(n)= F_p(n\text{-}1)+F_p(n\text{-}p\text{-}1)$ (7.93)

$F_p(1)=2.$ (7.94)

Comparing the expressions (7.93) and (7.94) with the corresponding recursive relations for the optimal Fibonacci measurement algorithms, we can see that these expressions coincide and, hence, for the case (7.88) the general optimal (n,k,S)-algorithm given by the expressions (7.70) and (7.71) are reduced to the Fibonacci measurement algorithm.

The main result of the algorithmic measurement theory together with all the unusual results is demonstrated in Fig. 7.13.

Thus, the "unexpectedness" of the main result of the algorithmic measurement theory (Fig. 7.13) consists of the following. The general recursive relation (7.70) at the seed (7.71) sets in general form an infinite number of new,

	$p=0$		$0 \le p \le \infty$		$p=\infty$	
$k \ge 1$	$(k+1)^n$	$\leftarrow$	$F_p(n,k)$	$\rightarrow$	$C_{n+k}^k = C_{n+k}^n$	*Binomial coefficients*
	$\downarrow$		$\downarrow$		$\downarrow$	
$k=1$	2^n	$\leftarrow$	$F_p(n)=F_p(n-1)+F_p(n-2)$	$\rightarrow$	$n+1$	
	Binary numbers		*Fibonacci p – numbers*		*Natural numbers*	

Figure 7.13. The main result of the algorithmic measurement theory

until now unknown, optimal measurement algorithms. All well-known classical measurement algorithms that are used in measurement practice (the "binary" algorithm, the "counting" algorithm, the "ruler" algorithm) are special limiting cases of the optimal measurement algorithms. The main recursive relation (7.70) at the seed (7.71) includes a number of the well-known combinatorial formulas as special cases, in particular, the formula $(k+1)^n$, the formula $C_{n+k}^k = C_{n+k}^n$ that gives binomial coefficients, the Fibonacci recursive relation, and finally the formulas for the "binary" (2^n) and natural ($n+1$) numbers.

7.11. Mathematical Theories Isomorphic to Algorithmic Measurement Theory

7.11.1. *What is an Isomorphism?*

The concept of *Isomorphism* [from the Greek words *isos* (equal), and *morphe* (shape)] is used widely in mathematics. Informally, isomorphism is a kind of correspondence between objects that show a relationship between two properties or operations. If there is isomorphism between two structures, we call these structures isomorphic. The word "isomorphism" applies when two complex structures can be reflected onto each other in such a way that each part of one structure fits a corresponding part of the other structure.

We can give the following physical analogies of isomorphism:

1. A solid wood cube and a solid metallic cube are both solid cubes; although their physical nature differs, their geometric structures are isomorphic.
2. The Big Ben Clock in London and a wristwatch; although the clocks differ greatly in size, their time-counting mechanisms are isomorphic.

We can give different examples of isomorphism in mathematics. In abstract algebra, two basic isomorphisms are defined: *Group Isomorphism,* the isomorphism between groups, and *Ring Isomorphism,* the isomorphism between rings. There is also *Graph Isomorphism* in graph theory. In linear algebra, the isomorphism can also be defined as a *Linear Map* between two vector spaces.

7.11.2. *Isomorphism between the "Balance" and "Rabbits"*

In the above we formulated the *Asymmetry Principle of Measurement,* the main methodological principle of algorithmic measurement theory. Note that this principle has practical origin because it reflects some essential properties of the comparators used in *Analog-to-Digit Converters.* For the first time, the isomorphism between balances and comparators was stated in this author's Doctor of Science dissertation *Synthesis of Optimal Algorithms of Analog-to-Digit Conversion* (1972).

The above *Asymmetry Principle of Measurement* is based on the concept of the "inertial balances" used for measurement. If we define by *I* the initial position of the balance (Fig. 7.3-a) and by *O* the opposite position (Fig. 7.3-b), then the functioning of the balance can be described by using two transitions:

$$I \rightarrow \begin{cases} I \\ O \end{cases} \tag{7.95}$$

$$O \rightarrow I. \tag{7.96}$$

The transition (7.95) means that the balance can be in one of two extreme positions, *I* or *O,* after we place the next standard weight on the free cup of the balance. The transition (7.96) means that if the balance is in the position *O,* then during a certain time the balance is coming back into the initial position *I.*

In Chapter 2, we described Fibonacci's problem of rabbit reproduction. Let us recall that the "Law of rabbit reproduction" boils down to the following rule. Each mature rabbit's pair *A* gives birth to a newborn rabbit pair *B* during one month. The newborn rabbit's pair becomes mature during one month and then in the following month said pair starts to give birth to one rabbit pair each month. Thus, the maturing of the newborn rabbits, that is, their transformation into a mature pair is performed in 1 month. We can model the process of "rabbit reproduction" by using two transitions:

$$A \rightarrow AB \tag{7.97}$$

$$B \rightarrow A. \tag{7.98}$$

Note that the transition (7.97) simulates the process of the birth of the newborn rabbit pair *B* and the transition (7.98) simulates the process of the maturing of the newborn rabbit pair B. The transition (7.97) reflects an asymmetry of rabbit reproduction because the mature rabbit pair A is transformed into two non-identical pairs, the mature rabbit pair A and the newborn rabbit pair B.

Comparing the transitions (7.95) and (7.96) with the transitions (7.97) and (7.98), we can see analogies or isomorphism between them. Moreover, this isomorphism is confirmed by the fact that the solutions of these problems come to one and the same recursive numerical sequence, the Fibonacci numbers!

7.11.3. *The Generalized "Asymmetry Principle" of Organic Nature*

Using the model of "rabbit reproduction," which is described by the transitions (7.97) and (7.98), we can generalize the problem of rabbit reproduction in the following manner. Let us give a non-negative integer $p \geq 0$ and formulate the following problem:

"Let us suppose that in the enclosed place one pair of rabbits (female and male) is in the first day of January. This rabbit couple gives birth to a new pair of rabbits in the first day of February and then in the first day of each following month. The newborn rabbit pair becomes mature in $\boldsymbol{p}$ months and then gives birth to a new rabbit pair each month thereafter. The question is: how many rabbit pairs will be in the enclosed place in one year, that is, in 12 months from the beginning of reproduction?"

It is clear that for the case $p=1$ the generalized variant of the "rabbit reproduction" problem coincides with the classical "rabbit reproduction" problem described in Chapter 2.

Note that the case $p=0$ corresponds to the idealized situation when the rabbits become mature at once after birth. One may model this case using the transition:

$$A \rightarrow AA. \tag{7.99}$$

It is clear that the transition (7.99) reflects symmetry of "rabbit reproduction" when the mature rabbit pair A turns into two identical mature rabbit pairs AA. It is easy to show that for this case the rabbits are reproduced according to the "Dichotomy principle," that is, the amount of rabbits doubles each month: 1, 2, 4, 8, 16, 32, ...

Now, let us consider the case $p>0$. We can analyze the above "rabbit reproduction" problem in greater detail, taking into consideration new conditions of "rabbit reproduction." It is clear that the reproduction process is described by a more complex system of transitions. Really, let A and B be the pairs of mature and newborn rabbits, respectively. Then the transition (7.97) simulates a process of the monthly appearance of the newborn pair B from each mature couple A.

Let us examine the process of transformation of the newborn pair B into the mature pair A. It is evident that during the maturation process the newborn pair B passes via the intermediate stages corresponding to each month:

$$\begin{aligned} &B \to B_1 \\ &B_1 \to B_2 \\ &B_2 \to B_3 \\ &\dots \\ &B_{p-1} \to A \end{aligned} \tag{7.100}$$

For example, for the case p=2 a process of transformation of the newborn pair into the mature pair is described by the following system of transitions:

$$B \to B_1 \tag{7.101}$$

$$B \to A. \tag{7.102}$$

Then, taking into consideration (7.97), (7.101) and (7.102), the process of "rabbit reproduction" for the case p=2 can be represented in Table 7.4.

Table 7.4 The process of "rabbit reproduction" for the case p=2

Date	Rabbit's pairs	A	B	B_1	$A+B+B_1$
January, 1	A	1	0	0	1
February, 1	AB	1	1	0	2
March, 1	ABB_1	1	1	1	3
April, 1	ABB_1A	2	1	1	4
May, 1	ABB_1AAB	3	2	1	6
June, 1	$ABB_1AABABB_1$	4	3	2	9
July, 1	$ABB_1AABABB_1ABB_1A$	6	4	3	13

Note that column A gives the number of the mature pairs for each stage of reproduction, column B gives the number of the newborn pairs, column B_1 gives the number of the newborn pairs in stage B_1, column A+B+B_1 gives the general number of rabbit pairs for each stage of reproduction.

The analysis of the numerical sequences in each column

$A: 1,1,1,2,3,4,6,9,13,19,\dots$

$B: 0,1,1,1,2,3,4,6,9,13,19,\dots$

$B_1: 0,0,1,1,1,2,3,4,6,9,13,19,\dots$

$A+B+B_1: 1,2,3,4,6,9,13,19,\dots$

demonstrates that they are subordinated to one and the same regularity: each number of the sequence is equal to the sum of the preceding number and the number distant from the latter in 2 positions. However, as we know the Fibonacci 2-numbers are subordinated to this regularity!

Studying this process for the general case of p we come to the conclusion that the Fibonacci p-numbers are a solution to the generalized variant of the "rabbit reproduction" problem! They reflect the "Generalized Asymmetry Principle" of "Organic Nature."

At first appearance the above formulation of the generalized problem of "rabbit reproduction" appears to have no real physical sense. However, we should not hurry to such a conclusion! The article [171] is devoted to the application of the generalized Fibonacci p-numbers for simulation of biological cell growth. The article affirms, "In kinetic analysis of cell growth, the assumption is usually made that cell division yields two daughter cells symmetrically. The essence of the semi-conservative replication of chromosomal DNA implies complete identity between daughter cells. Nonetheless, in bacteria, insects, nematodes, and plants, cell division is regularly asymmetric, with spatial and functional differences between the two products of division.... Mechanism of asymmetric division includes cytoplasmic and membrane localization of specific proteins or of messenger RNA, differential methylation of the two strands of DNA in a chromosome, asymmetric segregation of centrioles and mitochondria, and bipolar differences in the spindle apparatus in mitosis." In the models of cell growth based on the Fibonacci 2- and 3-numbers are analyzed [171].

The authors of [171] made the following important conclusion: "Binary cell division is regularly asymmetric in most species. Growth by asymmetric binary division may be represented by the generalized Fibonacci equation .... Our models, for the first time at the single cell level, provide rational bases for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology, founded on the occurrence of regular asymmetry of binary division."

Now we return to the *Asymmetry Principle of Measurement*. We can see that the transitions (7.100) for the generalized problem of "rabbit reproduction" are similar to the return of the balance to the initial position in the Bashet-Mendeleev problem. This means that the generalized problem of "rabbit reproduction" is isomorphic to the Fibonacci measurement algorithm!

We can generalize this idea. As the *Asymmetry Principle of Measurement* is the main idea of the algorithmic measurement theory, we can put forward the following hypothesis. The algorithmic measurement theory described by the very general recursive relation (7.70) and (7.71) is isomorphic to a *General Theory of Biological Populations*. This means that we can use the main result of the algorithmic measurement theory in the scientific field, which is very far from measurement theory.

7.11.4. *Isomorphism between Measurement Algorithms and Positional Number Systems*

In the above we found that each measurement algorithm generates some positional number system. For example, the binary measuring algorithm generates the binary system, the optimal $(n,k,0)$-algorithm generates all well-known positional number systems, including the Babylonian sexagesimal system and the decimal system. This means that the algorithmic measurement theory is isomorphic to the theory of positional number systems and we can interpret all optimal measurement algorithms as fundamentally new positional number systems.

It is very important to emphasize, that the *Algorithmic Measurement Theory* [20, 21] as though precede a positional principle of number representation and positional numeral systems. This means that the creation of a general theory of positional numeral systems is reduced to a synthesis of the optimal measurement algorithms. This remark is very important in order to determine the role and place of the *Algorithmic Measurement Theory* in the development of mathematics. In the Introduction, we attributed the Babylonian positional principle of number representation to the greatest mathematical discovery, which preceded number theory and mathematics. The *Algorithmic Measurement Theory* infinitely enlarges a number of new, until now unknown positional numeral systems, turning mathematics back to the period of its origin. It is important to emphasize the “great practicability” of the *Algorithmic Measurement Theory*, which could become the basis of a new computer arithmetic that is of fundamental interest to computer science.

7.12. Conclusion

1. It is well known that the “problem of measurement” played a fundamental role in mathematics. It stimulated the development of two basic theories of mathematics – geometry and number theory. The famous Bulgarian mathematician and academician L. Iliev wrote, “During the first epoch of mathematics development, from antiquity to the discovery of differential and integral calculus, mathematics, investigating first of all the measurement problems, created Euclidean geometry and number theory” [5]. That is why, since antiquity mathematical measurement theory together with number theory have been considered to be the fundamental mathematical theories underlying mathematics.

2. Let us compare the *Classical Mathematical Measurement Theory* [3, 4] with the *Algorithmic Measurement Theory* [20]. The creation of the classical measurement theory is connected with internal problems that appeared in mathematics after the discovery of *Incommensurable Line Segments*. This discovery staggered the Pythagoreans and caused the first crisis in the foundations of mathematics. The resulting *Irrational Numbers* became one of the fundamental concepts of mathematics. In order to overcome the first crisis in the foundations of mathematics, the great mathematician Eudoxus (408 - 355 BC) suggested the *Method of Exhaustion*, used by him for the creation of a *Theory of Magnitudes*, which preceded the *Mathematical Measurement Theory* completed in the 19th century. The interest in mathematical measurement theory in the 19th century increased again. The measurement theory received a new impulse thanks to Cantor's theory of infinite sets. The mathematical measurement theory was constructed on the basis of the *Continuity Axioms - Eudoxus-Archimedes' Axiom* and *Cantor's Axiom*, which underlie classical mathematical measurement theory. Thanks to Cantor's axiom, Cantor's idea of *Actual Infinity* was introduced into mathematical measurement theory that allowed for the proof of the *Basic Measurement Equality* $Q=qV$. This equality for the given measurement unit V sets one-to-one correspondence between geometric magnitudes Q and real numbers q. However, a constructive approach to mathematics foundations, based on the *Potential Infinity* concept, demands the elimination of Cantor's axiom from the *Continuity Axioms*, and puts forward a problem to prove the equality $Q=qV$ from the constructive point of view (*Constructive Measurement Theory*).

3. In contrast to the classical mathematical measurement theory, the creation of the *Algorithmic Measurement Theory* is connected with the practical need for new algorithms of measurement to appear in theoretical metrology, in particular, in the theory of analog-to-digital conversion [20]. It was found, that this practical problem is deeply connected with the ancient "*Problem about the Choice of the Best Weights System*." This problem appeared in the 1202 work *Liber Abaci* by Fibonacci and is the first optimization problem in measurement theory. In the Russian mathematical literature, this problem is called the *Bashet-Mendeleev Problem* [169]. The analysis of the Bashet-Mendeleev problem from point of view of analog-to-digital conversion resulted in the discovery of the very unusual measurement property known as the *Asymmetry Principle of Measurement,* which in turn underlies *Algorithmic Measurement Theory* [20, 21, 88, 90, 91]. The algorithmic measurement theory is an absolutely new and original mathematical theory. The synthesis of optimal measurement algorithms is its main topic. This resulted in the discovery

of the infinite number of new, unknown until now, optimal measurement algorithms, which have great theoretical and practical interest. These optimal measurement algorithms contain all the classical measurement algorithms ("binary," "counting" and ruler" algorithms) and generate a wide class of new original measurement algorithms based on the Fibonacci p-numbers, binomial coefficients, Pascal's triangle, and so on. The main recursive relation of the algorithmic measurement theory (7.70), which generates an infinite number of different numerical sequences including natural numbers, binary numbers, Fibonacci p-numbers and binomial coefficients, is of fundamental interest for combinatorial analysis.

4. The idea of isomorphism resulted in an expansion of the applications of the algorithmic measurement theory. There is an isomorphism between measurement algorithms, the Fibonacci "rabbit reproduction," and positional number systems. This idea gives us the right to put forward a new hypothesis that the new theory of measurement is the source of the three isomorphic theories, namely *Algorithmic Measurement Theory, New Theory of Positional Number Systems* and *New Theory of Biological Populations.*

5. The new theory of positional number systems returns mathematics to the Babylonian "positional principle of number representation," which is considered to be the greatest mathematical achievement of Babylonian mathematics. We can emphasize that positional number systems were never considered in mathematics as a subject of serious mathematical research. That is why, a new theory of positional number systems, which follows from the algorithmic measurement theory, fills this gap, and a new theory of positional number systems should become a fundamental part of number theory.

6. We have a new theory of biological populations based upon the isomorphism between measurement algorithms and rabbit reproduction. In particular, as formulated above, the *Generalized "Asymmetry Principle" of Organic Nature* can play fundamental role here. This idea has been confirmed by contemporary research in the field of cell division [171].

Chapter 8

Fibonacci Computers

8.1. A History of Computers

8.1.1. *Era of Information*

An interesting view of the history of computer science is given in the book [172]. The time when people started creating instruments for hunting and manual labor, is usually considered to be the beginning of human civilization. The secret of the promethean-like "fire abduction" got lost in antiquity. However, the subsequent history of technical progress – from the "fire abduction" up to the discovery of nuclear energy – is the history of more and more powerful forces of nature being subjugated by humans: taming of animals, windmills, water-mills, thermal engines, and atomic energy. During several millennia the main problem of material culture of humanity was to increase the muscular force of humans by various tools and machines. On the other hand, from abacus stones up to modern computers, the efforts to create the tools strengthening human information processing mark the way for the accumulation of ideas in the general stream of scientific and technical progress, evident in the thin trickle of facts and museum pieces.

Already in the earliest stages of the collective work development, a person required some encoded signals for communication in order to synchronize labor operations with others. Human speech, as the predecessor of information technology and as the first transmitter of human knowledge, arose as a result of the complication of these information signals. At first, human knowledge was accumulated in the form of oral stories and legends transmitted from one generation to the next.

For the first time, the natural potential of a person accumulating and transmitting knowledge obtained "technological support" with the creation of *Written Language*. Written language became the first historical stage in the development of information technology.

We can find the proof of this stage imbedded deep in human history: such as, cave paintings (pictographs) and carvings (petroglyphs) in the form of people and animal images in stone, created twenty to twenty-five thousand years ago; a lunar calendar, engraved on bone more than twenty thousand years ago. However, according to modern archaeological data, the time between the use of the first work tools (axe, trap, etc.) and tools for representing images (in stone, bone, etc.) is roughly a million years. That is, nearly 99% of human history deals only with the development of material labor tools. It appears that only the last (roughly) 1% of human history involves the development of information representation through images.

The wheel, a recognized symbol of contemporary technical progress, was supposedly invented in the Ancient East about 4 millennia BC, but the most ancient written code of laws, the first monument of the written language, arose only one thousand years later. About four millennia passed before the creation of the first *Printing Machine* in the middle of 15th century proclaimed to the world the onset of the *Era of Printed Knowledge*. The printing press rapidly increased the circulation of books as a passive means of information transmission. Printing machines for the first time created information accelerating the growth of productive forces. The creation of *Computers* became the next stage in the development of human civilization. These new machines for information processing appeared in the middle of the 20th century, when the growing load of information problems became one of the most noticeable factors, limiting economic growth in industrially developed countries.

Thus, during the history of human civilization before the 20th century, material objects were the main subject of labor, and the economic power of countries was measured by material resources. At the end of the 20th century for the first time in the history of humanity, information became the main subject of labor in industrially developed countries. The tendency for a steady increase in the information sphere in comparison to the material sphere is the most noticeable symptom of the "information era" approach.

8.1.2. *The Basic Milestones in Computer Progress*

The history of computers, that is, the machines created for automation of information processing, is considered one of the least developed themes in the history of science. In the present book we will use a classification of periods of the computer development suggested in the book [173]. According to [173], the history of computers is divided into two main periods:

1. From abacus up to electronic computers
2. Electronic computers onward

8.1.2.1. *Abacus*

The necessity for calculation arose in the early stages of human civilization: notches on bones and stones representing calculations were found beginning in the Paleolithic and Neolithic periods. The *abacus* was probably the first calculating device. The word "abacus" originates from the Greek word *abax*, which in turn comes from the Hebrew word *avak* (dust). The history of science testifies to the fact that many different cultures used the abacus – Babylon, China, India, Egypt, Arabian countries, Japan, Pakistan, and others.

In medieval times, the word "abacus" was used as a synonym for mathematics. This was the meaning contained in the title of the mathematical book, *Liber Abaci,* written in 1202 by the famous Italian mathematician Leonardo from Pisa (Fibonacci). His book could be considered the original mathematical encyclopaedia of the Middle Ages.

It is important to emphasize that in this early period of scientific development, mathematical history is inseparably linked to the history of primitive computers (abaci), because the problem of counting and calculation hastened the development of many important mathematical concepts. Note that at this stage, the Babylonian mathematicians discovered the positional principle of number representation underlying modern computers. The majority of the historians of mathematics emphasize that the creation of the first primitive abacus played a large role in the development of mathematics, and resulted in the development of a concept of natural number – one of our fundamental mathematical concepts.

8.1.2.2. *Mechanical Calculation Machines*

Already philosophers in the Middle Ages advanced the problem of replacing human brain functions with various mechanisms. Several scientists were interested in the idea of the construction of "thinking" gear, and this idea stimulated the designs of the first mechanical calculating machines. In 1623, English professor of mathematics Schickard, a friend of Kepler, made such an attempt. In Kepler's archive we find Schickard's letter addressed to him. Schickard informed Kepler about a calculation machine he had constructed. Schickard's machine was known, apparently, by a very small group of people. Because of this limited exposure, for a long time it was thought that the first mechanical calculator was invented in 1642 by the famous French mathematician and physicist Pascal. About 50 copies of Pascal's calculator were purportedly produced. Some copies of Pascal's calculator are still available today. Five copies are in the Parisian Museum and one is in the Dresden physical and mathematical salon, which this author personally saw in

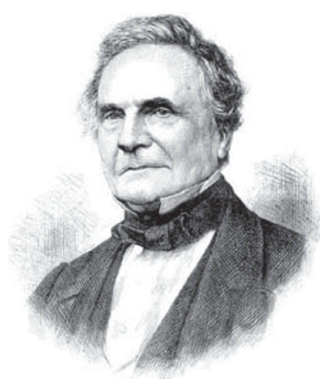
1988. In addition, the German mathematician Gottfried Leibniz, Wurttemberg's pastor Gan, and the famous Russian mathematician and mechanician Chebyshev were inventors of original mechanical calculating machines.

Production of calculating machines for the first time was started by Karl Thomas, the founder and chief of two Parisian insurance companies ("Fenix" and "Soleil"). In 1820, he built the calculation machine called *Calculator*. In 1821, 15 calculators were made in Thomas' workshop. Later their production was increased to 100 per year. These calculators had been produced during 100 years and had (certainly for that time) very impressive technical characteristics – two 8-digit numbers could be multiplied in 15 seconds, and a 16-digit number could be divided by the 8-digit number in 25 seconds.

A further improvement of calculators is connected with the name of Russian engineer Odner. Beginning in the 1890s, the triumphal procession of Odner's calculators began with the serial production of 500 calculators per year in St. Petersburg, Russia. In the 20th century, Odner's calculators were produced under a variety of names in different countries. In the first quarter of the 20th century, Odner's calculators were the main mathematical machines used in many fields of human activity. In 1914, in Russia alone more than twenty-two thousand Odner-calculators were produced.

8.1.2.3. *Babbage's Analytical Machine*

Charles Babbage was the mathematician, mechanical engineer and computer scientist who originated the idea of a *Programmable Computer*. Babbage's analytical machine was the best of the mechanical calculation machines. For the first time this machine embodies the idea of a completely automatic computer with program control. This idea was a revolutionary discovery in computer science. It started to be used only in the middle of the 20th century in the design of the first electronic computers. The first copy of his analytical machine was made by Babbage in September 1834.



Charles Babbage (1791 –1871)

From 1842 to 1848, Babbage paid significant attention to the design of his analytical machine. According to his idea, the machine should include three main blocks:

1. A device where digital information could be stored on sprocket registers. In modern computers a similar block is named the *Storage Device* or *Memory*. Babbage named this part of the machine the *Storehouse*.

2. A device where various operations with numbers, taken from the storehouse, can be carried out. Today this part of the machine is called the *Arithmetical Device*. Babbage named this part of the machine the *Factory*.
3. A device for control of the operating sequence. Babbage didn't have a special name for this part of the machine. Today a similar computer block is called the *Control Unit*.

In addition, his analytical machine had both *Input* and *Output Devices*.

Babbage was the creator of the greatest invention in computer history, the *Programmable Computer*. Babbage's main idea was only embodied in modern computers. A simple list of problems that was advanced by Babbage in his analytical machine is astonishing for its depth and foresight regarding the progress of modern computers.

8.1.2.4. *Hollerith's Tabulators*

The development of the theory of electricity and the theory of weak electrical currents naturally resulted in the idea of using these currents in calculation devices. At first, the electric energy in calculation machines was used only as a motion force for the actuation of the calculator mechanism in place of the hand. Such machines were called *Electromechanical Machines*.

At the end of the 19th century, the need for population census processing arose. In 1888, an employee of the U.S. Census Bureau, Herman Hollerith, designed a calculation machine that was used for processing the population census. Hollerith named his machine the *Tabulator*. His tabulator was intended to automatically process punch cards for the population census. In his machine Hollerith applied the benefits of weak current engineering and his machine was based on the electromechanical principle. The same electromechanical principle was used for improvement of the technical parameters of calculators. Thus, at the beginning of the 20th century, digital computer engineering developed in two directions. The first direction was the small *Calculators* intended for the mechanization of the elementary arithmetical operations. The second direction was that of *Tabulators* intended for the processing of statistical information.

8.1.2.5. *Electromechanical Computers by Zuse, Aiken and Stibitz*

In the 1930s and 1940s the development of tabulators, as well as, the achievements in the field of the electromechanical relay applications (for example, in automatic telephone stations) resulted in the development of computer machines similar to Babbage's machine by their structure. The first universal electromechanical computers with programmed control were designed in Germany and the U.S. in the early 1940s. German engineer Konrad Zuse, along with American scientists Howard Aiken from Harvard University and George Robert Stibitz from Bell Labs, played significant roles in these projects.

Zuse developed several models of electromechanical computers. The first model Z-1 was constructed over a period of two years (1936-1938). It was based completely on mechanical elements, rendering it somewhat unsatisfactory. The next models Z-2, Z-3, Z-4 used electromagnetic relays as their basis. An elaboration of the Z-3 machine was completed in 1941. This machine was the first universal computer with programmed control. The computer performed eight commands including four arithmetical operations, multiplying by negative numbers, and calculating square roots. All calculations were fulfilled in the binary system using a floating point.

Aiken's universal digital computer used standard punch cards produced by IBM (USA). In August, 1944 the project was completed, and the machine called MARC-1 was installed at Harvard University, where the machine was used thereafter for over 15 years. If we compare the machine MARC-1 with the machine Z-3, we should note that Zuse's machine exceeded Aiken's machine from the point of view of its circuit and structural solutions.

Also the works of American scientist Stibitz played an essential role in the creation of the first computers, which were intended for the fulfilment of complicated scientific and technical calculations. In 1938, working for Bell Laboratory, Stibitz developed the computer "Bell-1" capable of operating with complex numbers. In 1947, Stibitz developed a high-power universal computer with programmed control (the machine "Bell-5") based on electromagnetic relays. At the same time, the computing laboratory of Harvard University developed the large program-controlled computer MARC-2 which is also based on electromagnetic relays.

8.1.2.6. *ENIAC*

One of the first attempts to use electronic elements in computers was launched in the U.S. in 1939-1941 at the State College of Iowa (now the Iowa State University). Physics professor John Atanasov was the principal developer of this project. Unfortunately, the project was not completed, but in 1973 the U.S. Federal Court confirmed Atanasov's priority in the design of the first electronic computer project.

Work on the creation of the electronic computer ENIAC was started in 1942, at Pennsylvania University under John Mauchly and John Eckert's leadership. This works was completed at the end of 1945, and in February 1946 the machine's first public demonstration was held. The important role of ENIAC in the development of computer science is clear from the fact that it was the first machine in which electronic elements were used for the realization of arithmetical and logical operations, and also the storage of information. The use of new electronic technology in this machine allowed for an increase of computer operation speed by approximately 1000 times greater in comparison with electromechanical computers.

8.1.2.7. *John von Neumann's Principles*

The ENIAC confirmed in practice the high efficiency of electronic technology in computers. The problem of maximal realization of the large advantages of electronic technology arose for computer designers. It was necessary to analyze the strengths and weaknesses of the ENIAC project and to give appropriate recommendations. A brilliant solution to this problem emerged in the famous report *Preliminary Discussion on Logic Designing the Electronic Computing Device* (1946). This report, written by famed mathematician *John von Neumann* and his colleagues from Princeton University, *Goldstein* and *Berks*, presented the project of developing the new electronic computer.

John von Neumann
(1903-1957)

The essence of the primary recommendations of the report consisted of the following:

1. The machines based on electronic elements should work in *Binary System* instead of the decimal system.
2. The program should be stored in the machine block called the *Storage Device*, which should have sufficient capacity and appropriate speeds for access and entry of a variety of program commands.
3. Programs, as well as numbers, with which the machine operates, should be represented in *Binary Code*. Thus, *Commands and Numbers Should Have One and the Same Form of Representation*. This meant that all programs and all intermediate outcomes of calculations, constants and other numbers should be stored in one and the same storage device.
4. Difficulties of physical realization of the storage device, speed of which should correspond to the speed of logical elements, demanded a *Hierarchical Organization of Memory*.
5. The arithmetical device of the machine should be constructed on the basis of the *Logical Summation Element*; it is not recommended to design special devices for the realization of other arithmetical operations.

6. The machine should use the *Parallel Principle of Organization of Computing Processes*, that is, the operations on binary words should be performed on all digits simultaneously.

John von Neumann's Principles, in general, were some of the most important contributions to the development of universal electronic computers. The further development of electronic computers was based upon these principles. The changes concerned only the technology of the element base. Depending on the electronic base used, electronic computers started developing in the following three directions:

1. Computers based on the electronic lamps.
2. Computers based on the discrete semi-conducting and magnetic elements.
3. Computers based on the integrated schemes.

8.1.2.8. *The Phenomenon of Personal Calculators*

The capacity of the present book, which is not a specialty textbook on the history of the computer, does not allow us to refer to many achievements of modern computers. We can only discuss one phenomenon of modern computer history – that of personal computers (PC), which were the first mass tools for the active formalization of professional knowledge. Varied influence of the PC on progress of industrial society is comparable to book printing, which began the information era. Continuing this analogy: while a book was and remains the tool for mass duplication and passive storage of knowledge, the PC is the first mass produced tool for direct transformation of professional knowledge into an active industrial product. The discovery of the PC phenomenon in the U.S. was connected with the name Steve Jobs, founder of *Apple Computer*.

In 1980, **Steve Jobs** defined this type of computer as the tool for increasing the natural potentialities of human intellect. The PC, more than any other tool for information processing, brings forward the information era. According to the opinion of many competent scientists, we can now identify a few of the symptoms, which testify to the beginning of the transition within the industrially developed countries to a qualitatively new stage of technological development – the information era:

1. Information becomes the main subject of labor in public production of the industrially developed countries.
2. The time of doubling accumulated scientific knowledge is something like 2-3 years.
3. The material expenses for storage, transmission and processing of information begin to exceed similar costs for energy expenses.
4. Humanity for the first time in its history becomes the really observed astronomical "cosmic factor," as the level of Earth radio emissions in the separate parts of the radio diapason approaches the level of the Sun's radio emissions.

8.2. Basic Stages in the History of Numeral Systems

The creation of the first numeral systems is attributed to the period of the mathematics origin, when the necessity to count things, and measure time, land and product quantities, resulted in the development of the basic principles of the arithmetic of natural numbers. In the history of numeral systems we can identify several stages: the *Initial Stage of Counting, Non-Positional Numeral Systems*, *Alphabetic Notations,* and *Positional Numeral Systems*. Initially people used body parts, fingers, sticks, knots, etc. for representation of counted sets. Article [2] emphasizes that "despite the extreme primitiveness of this way of number representation, it played an exclusive role in the development of the number concept." This statement confirms that numeral systems played an important role in the formation of the concept of natural numbers – one of mathematics' most fundamental concepts.

8.2.1. *Babylonian Sexagecimal Numeral System*

At an early stage in the history of mathematics, one of the greatest mathematical discoveries was made. This was the *Positional Principle* of number representation. "The Babylonian sexagecimal numeral system that arose in approximately 2000 B.C. was the first numeral system based on the positional principle" [2]. There were a number of competing hypotheses explaining its origin. M. Cantor originally assumed that Sumerians (the primary population of the Euphrates valley) considered that one year consisted of 360 days and therefore that base 60 had an astronomical origin. According to Kevitch's hypothesis, in the Euphrates valley two persons met. The first person knew the decimal system, and the other one knew the base 6 system (Kevitch explains the occurrence of such base by finger counting, in which the compressed hand meant the number 6). After the two systems were merged the compromise base of 60=6×10 arose.

We should note that Cantor's and Kevitch's hypotheses concerned the question of the origin of base 60, but they do not explain the origin of the positional principle of number representation. Neugebauer's hypothesis presented in the book [174] gives an answer to this last question. According to his hypothesis, the positional principle has a measurement origin.

Neugebauer asserts [174] that "the basic stages of the origin of the positional principle in Babylon were the following: (1) determination of a quantitative ratio between two independent existing systems of measures and (2) omitting the names of numerals in the writing." According to his opinion, these stages of the origin of the positional numeral systems had a common foundation. He emphasizes that "the positional sexagecimal number system was the natural outcome of a long historical development, which does not differ from similar processes in other cultures."

However, we can explain the choice of the number 60 as the base of the Babylonian system from the point of view of the Universal Harmony concept. As is well known, the dodecahedron was the main geometric figure expressing the Universal Harmony in ancient science. As is shown in Chapter 3, the numerical parameters of the dodecahedron (12, 30, 60) were connected to the basic cycles of the Solar system (Jupiter's 12-year cycle, Saturn's 30-year cycle and the Solar system's 60-year main cycle). The ancient scientists chose the numbers 12, 30, 60 and derived from them the number 360=12×30 as the "main numbers" of the calendar system and systems of time and angular values measurement. Taking into consideration the deep connection between the Babylonian and Egyptian cultures, we may advance the hypothesis that the base 60 of the Babylonian number system was chosen by the Babylonians from astronomical considerations. This hypothesis coincides with Cantor's hypothesis about the astronomical origin of base 60.

8.2.2. *Alphabetic and Roman Systems of Numeration*

Many people used the so-called alphabetic system of numeration, where numerical values were attributed to the letters of the alphabet. For instance, ancient Greek, Hebrew, Sanskrit, Slavic and Arabic alphabets employed this idea.

The Roman system of numeration is widely known. It uses special numerals for the representation of the "nodal" numbers (I, V, X, L, C, D, M). The greatest shortcoming of Roman numeration was the fact that it was not adapted to perform arithmetical operations in written form.

8.2.3. *Mayan System*

The origin of positional numeral systems is considered to be one of the main events in the history of material culture. Many people took part in its creation. In the 6th century AD a similar numeral system arose within the Maya people. It is widely accepted that the number 20 is the base of the Mayan number system and, therefore, may have a "finger/toe" origin (10+10=20). It is well known that the number system with base 20 used the following set of the "nodal" numbers for number representation: $\{1, 20^1, 20^2, 20^3\}$. However, the Mayan number system used another set of "nodal" numbers: $\{1, 20, 360, 20\times360, 20^2\times360,...\}$. This means that the next "nodal" number, which follows after 20, is equal to 360 (instead of $20^2=400$) and all subsequent "nodal" numbers are derivable from the numbers 20 and 360. As is emphasized in the article [2], this somewhat unusual numbering system can be explained by the fact that "the Maya divided one year into 18 months, with 20 days in every month, plus five more days." Thus, like the base of the Babylonian system, the "nodal" numbers of Mayan notation have an astronomical origin. It is essential to emphasize, that the structure of the Maya calendar year (360+5=18×20+5) is similar to the structure of the Egyptian calendar year (360+5=12×30+5).

8.2.4. *Decimal System*

We use a decimal system for daily calculations. In mathematics this numeral system is called the *Hindu-Arabic* or *Arabic* numeral system. This number system was invented by Hindu mathematicians in approximately the 6th-8th centuries AD and it is the predecessor of the traditional decimal system. This number system uses the number 10 as the base and requires 10 different numerals, the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 for the representation of numbers. From India, the decimal system penetrated into the Arabic world. Arabic mathematicians extended the system to decimal fractions. In 9th century Muhammad ibn Musa al-Khwarizmi wrote an important work about the decimal system. Thanks to a translation of al-Khwarizmi's work, the decimal system began to be used in Europe.

There are different opinions concerning the choice of the number 10 as the base of the decimal system; the most widespread opinion is that the base of 10 has a "finger" origin. However, it is necessary to remind oneself that the number 10 always had special significance in ancient science. Pythagoreans considered this number to be the "Sacred Number" and named it the *Tetractys*. The number 10=1+2+3+4 was considered by Pythagoreans to be one of the greatest values, being "a symbol of the Universe," because it comprised four "basic elements": the *One* or *Monad* symbolizes spirit, out of which all the visible world appears; the *Two* or *Dyad* (2=1+1), symbolizes the material atom; the *Three* or *Triad* (3=2+1), symbolizes the living world. The *Four* or *Tetrad* (4=3+1) connected the living world with the *Monad* and consequently symbolizes the *Whole*, that is, the *Visible and Invisible World together*. As the Tetractys 10=1+2+3+4, which meant that the *Tetractys,* 10, expressed *Everything*. Thus, the hypothesis about the "harmonic" origin of base 10 is as tenable as the hypothesis of the "finger" origin of the decimal system.

8.2.5. *Binary System*

The binary system uses two symbols, 1 and 0, for number representation. The number 2 is its base or *radix*. In connection with the development of computer technology, the binary system was introduced into modern science. Discussing the history of the binary system, we should note that the ancient Indian writer and mathematician *Pingala* presented in 2nd century AD the first known description of a binary system, more than 1500 years before its discovery by German mathematician Gottfried Leibniz in 1695. Pingala's work also contains elements of combinatorics (Pascal triangle) and the basic ideas of Fibonacci numbers.

Gottfried Wilhelm von Leibniz
(1646-1716)

The modern binary system was fully documented by Gottfried Leibniz in 17th century in his article *Explication de l'Arithmŭtique Binaire*. Leibniz's system used 0 and 1 in a manner similar to the modern binary system. Gottfried Wilhelm von Leibniz (1646 -1716) was a German polymath who wrote mostly in French and Latin. He invented calculus independently of Newton. Leibniz also made major contributions to physics and technology, and anticipated notions that surfaced much later in biology, medicine, geology, probability theory, psychology, and information science. His contributions to this vast array of subjects are scattered in journals and in numerous letters and unpublished manuscripts. To date, there is no complete edition of Leibniz's writings, and a full list of his accomplishments is not yet known.

The binary arithmetic developed by Leibnitz is much similar to arithmetic in other numeral systems. Summation, subtraction, multiplication, and division can be performed on binary numerals. All arithmetical operations in the binary system are reduced to binary summation. The summation of two single-digit binary numbers is relatively simple:

$0 + 0 = 0$
$0 + 1 = 1$
$1 + 0 = 1$
$1 + 1 = 10$ (*carry of* 1)

In 1854, British mathematician George Boole published an important paper detailing a system of logic that became known as Boolean Algebra. His logic system became the instrumental tool for the development of the binary system, particularly in its implementation in electronic circuits.

In 1937, Claude Shannon presented his master's thesis at MIT that initiated the implementation of Boolean algebra and binary arithmetic using electronic relays and switches for the first time in history. Entitled *A Symbolic Analysis of Relay and Switching Circuits*, Shannon's thesis essentially developed the practical digital circuit design.

In 1946, John von Neumann, Goldstein and Berks in their *Preliminary Discussion on Logic Designing the Electronic Computing Device* gave a decisive preference for the use of the binary system in electronic computers.

8.2.6. *Exotic Number Systems*

However, we should note that the binary system suffers from some serious disadvantages [175, 176]:

1. *The problem of a number sign.* In the binary system we can represent in the "direct" code only positive numbers. That is why, for the representation of negative numbers we need to use special codes - *additional* and *inverse codes*. This complicates the arithmetical structures of computers and influences the internal performance of a computer.
2. *The problem of error detection.* As is well known, all computer devices are subjected to severe external and internal influences that are a cause of various errors, which can appear in the information transmission and processing. Unfortunately, such errors cannot be detected because all binary code combinations are permissible.
3. *The problem of long carry-over.* If, for example, we sum the two binary numbers 011111111+000000001=100000000, we should take into consideration the "long carry-over" from the lowest digit to the highest digit appearing in such a summation. This "long carry-over" fundamentally influences the internal performance of computers.

The intention to overcome these disadvantages of the binary system led to the appearance of different numeral systems with "exotic" titles and properties: *System for Residual Classes*, *Ternary Symmetrical Numeral System*, *Numeral System with Complex Radix, Nega-positional, Factorial, Binomial Numeral Systems* [175, 176], etc. All of them had certain advantages in comparison with the binary system, and were directed at the improvement of computer characteristics; some of them became the basis for the creation of new computer projects (e.g. the ternary computer "Setun," the computer based on system for residual classes, and so on).

However, there is also another interesting aspect to this problem. Four millennia after the invention by Babylonians of the positional principle of number representation, we observe some kind of "Renaissance" in the field of numeral systems. Thanks to the efforts of the experts in computers and mathematics, it is as if we once again have returned to the origin of computation, when the numeral systems defined the subject-matter and were the heart of mathematics (Babylon, Ancient Egypt, India, and China).

The main purpose of chapters 8, 9 and 10 is to demonstrate that new methods of number representation based on the golden mean and Fibonacci numbers can become the source of a new and unusual computer arithmetic and from this will follow new and exciting computer projects.

8.3. Fibonacci *p*-Codes

8.3.1. *Zeckendorf Sums*

Many number-theorists know about Zeckendorf sums [16], however, few know about the man for whom these sums are named. The Fibonacci Association considers Edouard Zeckendorf to be one of the famous Fibonacci mathematicians of the 20th century although he did not have mathematical education. Zeckendorf was a medical doctor and was licensed for dental surgery. Mathematics was Zeckendorf's hobby. Through the years, he published several mathematical articles. The most important article, on Zeckendorf sums, was published in 1939. In it, Zeckendorf proved that each positive integer can be represented as the unique sum of non-adjacent Fibonacci numbers, as exemplified below:

38=34+3+1; 39=34+5; 40=34+5+1; 41=34+5+2; 42=34+8.

The numerous articles published in *The Fibonacci Quarterly* discussed Zeckendorf sums and their generalizations.

Let us write Zeckendorf representation in the following form:

$$N=a_nF_n+a_{n-1}F_{n-1}+...+a_iF_i+...+a_1F_1, \tag{8.1}$$

where $a_i \in \{0,1\}$ is the binary numeral of the i-th digit of the representation (8.1); n is the number of digits of the representation (8.1); F_i is a Fibonacci number given by the recursive relation

$$F_n= F_{n-1}+F_{n+2} \tag{8.2}$$

at the seeds

$$F_1= F_2=1. \tag{8.3}$$

It is important to emphasize that the Fibonacci numbers F_i (i=1,2,3,...,n) are used as the digit weights in Zekendorf's representation (8.1).

The binary representation of Zeckendorf sum (8.1) has the following form:

$$N = a_n a_{n-1} ... a_i ... a_1, \tag{8.4}$$

where $a_i \in \{0,1\}$ is a binary numeral of the i-th digit of the binary representation

(8.4). Thus, Zeckendorf sum (8.1) can be considered as the binary representation (8.4) with "Fibonacci weights."

It is clear that the minimal number $N_{\min}$ that can be represented by using Zeckendorf sum (8.1) is equal to 0 and has the following binary representation:

$$N_{\min} = 0 = \underbrace{00...0}_{n}. \tag{8.5}$$

The maximal number $N_{\max}$ that can be represented by using Zeckendorf sum (8.1) has the following binary representation:

$$N_{\max} = \underbrace{11...1}_{n}. \tag{8.6}$$

The binary representation (8.6) is an abridged representation of the following sum:

$$N_{\max}=F_n+F_{n-1}+...+F_i+...+F_1 \tag{8.7}$$

Using the formula (2.20), we can write:

$$N_{\max}=F_{n+2}-1. \tag{8.8}$$

This means that we have proved the following theorem.

Theorem 8.1. Using the n-digit Zeckendorf sum (8.1), we can represent F_{n+2} integers in the range from 0 to F_{n+2}-1.

8.3.2. *Definition of the Fibonacci p-Code*

In Chapter 7 we developed the so-called *Algorithmic Measurement Theory* [20, 21] that can be a source of new ideas in numeral systems field. Fibonacci codes and Fibonacci arithmetic, which are a wide generalization of the binary system, are amongst the most interesting applications of the algorithmic measurement theory.

The Fibonacci measurement algorithms, which were examined in Chapter 7, are isomorphic to the following sum:

$$N=a_nF_p(n)+a_{n-1}F_p(n-1)+...+a_iF_p(i)+...+a_1F_p(1), \tag{8.9}$$

where N is a natural number, $a_i \in \{0,1\}$ is a binary numeral of the i-th digit of the code (8.9); n is the digit number of the code (8.9); the Fibonacci number $F_p(i)$ is the i-th digit weight calculated in accordance with the recursive relation (4.18) at the seeds (4.19).

The positional representation of the natural number N in the form (8.9) is called a *Fibonacci p-code* [20]. The abridged representation of the Fibonacci p-code (8.9) has the same form (8.4) as the abridged representation of Zeckendorf sum (8.1).

Note that the notion of a Fibonacci p-code (8.9) includes an infinite number of different positional "binary" representations of natural numbers because every p produces its own Fibonacci p-code (p=0,1,2,3,...). In particular, for the case p=0 the Fibonacci p-code (8.9) is reduced to the classical binary code:

$$N=a_n 2^{n-1}+a_{n-1}2^{n-2}+...+a_i 2^{i-1}+...+a_1 2^0 . \qquad (8.10)$$

For the case p=1 the Fibonacci p-code (8.9) is reduced to Zeckendorf sum (8.1).

Now, let us consider the partial case $p=\infty$. For this case each Fibonacci p-number is equal to 1, that is, for any integer i we have:

$$F_p(i)=1.$$

Then the sum (8.9) takes the form of the "unitary code":

$$N=\underbrace{1+1+...+1}_{N}. \qquad (8.11)$$

Thus, the Fibonacci p-code given by (8.9) is a very wide generalization of the binary system (8.10) and Zeckendorf sum (8.1) that are partial cases of the Fibonacci p-code (8.9) for the cases p=0 and p=1, respectively. On the other hand, the Fibonacci p-code (8.9) includes the so-called "unitary code" (8.11) as another extreme case for $p=\infty$.

8.3.3. *The Range of Number Representation in the Fibonacci p-Code*

Consider the set of n-digit binary words. The number of them is equal to 2^n. For the classical binary code (8.10) (p=0) the mapping of the set of n-digit binary words onto the set of natural numbers has the following peculiarities:

(a) *Uniqueness of mapping*. This means that for the infinite n there are one-to-one correspondences between natural numbers and sums (8.10), that is, every integer N has only one representation in the form (8.10).

(b) For a given n with the help of the binary code (8.10) we can represent all integers in the range from 0 to 2^n -1, that is, the range of number representation is equal to 2^n.

(c) The minimal number 0 and the maximal number 2^n-1 have the following binary representations in the binary code (8.10), respectively:

$$N_{\min}=0=\underbrace{00...0}_{n}$$

$$N_{\max}=2^n-1=\underbrace{11...1}_{n}.$$

For the Fibonacci p-code (8.9) the mapping of the n-digit binary words onto natural numbers has distinct peculiarities for the case p>0.

Let n=5. Then, for the cases p=1 and p=2 the mappings of the 5-digit Fibonacci p-code (8.9) onto the natural numbers have the forms, represented in Tables 8.1 and 8.2, respectively.

Table 8.1. Mapping the Fibonacci 1-code onto natural numbers

CC	5	3	2	1	1	*N*	*CC*	5	3	2	1	1	*N*
A_0	0	0	0	0	0	0	A_{16}	1	0	0	0	0	5
A_1	0	0	0	0	1	1	A_{17}	1	0	0	0	1	6
A_2	0	0	0	1	0	1	A_{18}	1	0	0	1	0	6
A_3	0	0	0	1	1	2	A_{19}	1	0	0	1	1	7
A_4	0	0	1	0	0	2	A_{20}	1	0	1	0	0	7
A_5	0	0	1	0	1	3	A_{21}	1	0	1	0	1	8
A_6	0	0	1	1	0	3	A_{22}	1	0	1	1	0	8
A_7	0	0	1	1	1	4	A_{23}	1	0	1	1	1	9
A_8	0	1	0	0	0	3	A_{24}	1	1	0	0	0	8
A_9	0	1	0	0	1	4	A_{25}	1	1	0	0	1	9
A_{10}	0	1	0	1	0	4	A_{26}	1	1	0	1	0	9
A_{11}	0	1	0	1	1	5	A_{27}	1	1	0	1	1	10
A_{12}	0	1	1	0	0	5	A_{28}	1	1	1	0	0	10
A_{13}	0	1	1	0	1	6	A_{29}	1	1	1	0	1	11
A_{14}	0	1	1	1	0	6	A_{30}	1	1	1	1	0	11
A_{15}	0	1	1	1	1	7	A_{31}	1	1	1	1	1	12

An analysis of Tables 8.1 and 8.2 leads to the following peculiarities of binary representations of natural numbers in Fibonacci p-codes (8.9). By using the 5-digit Fibonacci 1-code (Table 8.1), we can represent 13 integers in the range from 0 to 12, inclusively. Note that the number 13 is the Fibonacci 1-number with index 7, that is, $F_1(7)= F_7=13$. We can see from Table 8.2 that by using the 5-digit Fibonacci 2-code (Table 8.2) we can represent 9 integers in the range from 0 to 8, inclusively. Here, the number 9 is the Fibonacci 2-number with index 8, that is, $F_2(8)=9$. The results of this consideration are partial cases of the following theorem.

Theorem 8.2. For the given integers $n>0$ and $p>0$, with the help of the n-digit Fibonacci p-code, we can represent $F_p(n+p+1)$ integers in the range from 0 to $F_p(n+p+1)$-1, inclusively.

Proof. It is clear that the minimal number $N_{\min}$ can be represented in the Fibonacci p-code (8.9) as follows:

$$N_{\min} = 0 = \underbrace{00...0}_{n}. \qquad (8.12)$$

The binary representation of the maximal number $N_{\max}$ has the following form:

$$N_{\max} = \underbrace{11...1}_{n}. \qquad (8.13)$$

Table 8.2. Mapping the Fibonacci 2-code onto natural numbers

CC	3	2	1	1	1	*N*	*CC*	3	2	1	1	1	*N*
A_0	0	0	0	0	0	0	A_{16}	1	0	0	0	0	3
A_1	0	0	0	0	1	1	A_{17}	1	0	0	0	1	4
A_2	0	0	0	1	0	1	A_{18}	1	0	0	1	0	4
A_3	0	0	0	1	1	2	A_{19}	1	0	0	1	1	5
A_4	0	0	1	0	0	1	A_{20}	1	0	1	0	0	4
A_5	0	0	1	0	1	2	A_{21}	1	0	1	0	1	5
A_6	0	0	1	1	0	2	A_{22}	1	0	1	1	0	5
A_7	0	0	1	1	1	3	A_{23}	1	0	1	1	1	6
A_8	0	1	0	0	0	2	A_{24}	1	1	0	0	0	5
A_9	0	1	0	0	1	3	A_{25}	1	1	0	0	1	6
A_{10}	0	1	0	1	0	3	A_{26}	1	1	0	1	0	6
A_{11}	0	1	0	1	1	4	A_{27}	1	1	0	1	1	7
A_{12}	0	1	1	0	0	3	A_{28}	1	1	1	0	0	6
A_{13}	0	1	1	0	1	4	A_{29}	1	1	1	0	1	7
A_{14}	0	1	1	1	0	4	A_{30}	1	1	1	1	0	7
A_{15}	0	1	1	1	1	5	A_{31}	1	1	1	1	1	8

The binary combination (8.13) is an abridged representation of the following sum:

$$N_{max}=F_p(n)+ F_p(n-1)+...+F_p(i)+...+F_p(1). \qquad (8.14)$$

Using the formula (4.39), we can write:

$$N_{max}=F_p(n+p+1)-1. \qquad (8.15)$$

The theorem is proved.

Note that for the case p=0, $F_0(n+1)=2^n$ and Theorem 8.2 is reduced to the well-known theorem about the range of number representation for the classical binary system. This range is equal to 2^n for the n-digit binary code (8.10).

8.3.4. *Multiplicity of Number Representation*

The *Multiplicity* of number representation in the form (8.9) is the next peculiarity of the Fibonacci p-code (8.9) for the case p>0. By accepting the minimal number 0 given by (8.12) and the maximal number given by (8.13), the remaining integers from the range [0, $F_p(n+p+1)-1$] have more than one representation in the form (8.9). This means that all integers in the range [1, $F_p(n+p+1)-2$] have multiple representations in the Fibonacci p-code (8.9) for the case p>0.

Table 8.3. Mapping natural numbers to binary combinations in the Fibonacci 1 and 2 codes

$p=1$	$p=2$
$0=\{A_0\}$	
$1=\{A_1,A_2\}$	
$2=\{A_3,A_4\}$	$0=\{A_0\}$
$3=\{A_5,A_6,A_8\}$	$1=\{A_1,A_2,A_4\}$
$4=\{A_7,A_9,A_{10}\}$	$2=\{A_3,A_5,A_6,A_8\}$
$5=\{A_{11},A_{12},A_{16}\}$	$3=\{A_7,A_9,A_{10},A_{12},A_{16}\}$
$6=\{A_{13},A_{14},A_{17},A_{18}\}$	$4=\{A_{11},A_{13},A_{14},A_{17},A_{18},A_{20}\}$
$7=\{A_{15},A_{19},A_{20}\}$	$5=\{A_{15},A_{19},A_{21},A_{22},A_{24}\}$
$8=\{A_{21},A_{22},A_{24}\}$	$6=\{A_{23},A_{25},A_{26},A_{28}\}$
$9=\{A_{23},A_{25},A_{26}\}$	$7=\{A_{27},A_{29},A_{30}\}$
$10=\{A_{27},A_{28}\}$	$8=\{A_{31}\}$
$11=\{A_{29},A_{30}\}$	
$12=\{A_{31}\}$	

Now, let us examine the mapping of integers onto the 5-digit binary code combinations A in accordance with Table 8.1 (p=1) and Table 8.2 (p=2). This mapping is represented in Table 8.3.

Note that for the arbitrary p the minimal and the maximal numbers have only binary representations in the n-digit Fibonacci p-code:

0 = 0 0 ... 0 (n digits)

and

$F_p(n+p)$ - 1 = 1 1 ... 1 (n digits).

8.4. Minimal Form and Redundancy of the Fibonacci p-Code

8.4.1. *Convolution and Devolution*

The different binary representations of one and the same integer N in the Fibonacci p-code (8.9) for the case $p>0$ may be obtained from one another by means of the peculiar code transformations called *Convolution* and *Devolution* of the binary digits. These code transformations are carried out within the scope of one binary combination and follow from the basic recursive relation (4.18) that connects adjacent digit weights of the Fibonacci p-code (8.9). The idea of such code transformations consists in the following.

Let us consider the abridged representation (8.4) of integer N in the Fibonacci p-code (8.9). Suppose that the binary numerals of the l-th, (l-1)-th and (l-p-1)-th digits in (8.4) are equal to 0, 1, 1, respectively, that is,

$$N=a_n\, a_{n-1} \dots a_{l+1}\; 01\; a_{l-2} \dots a_{l-p}\; 1\; a_{l-p-2} \dots a_1. \tag{8.16}$$

This binary representation (8.16) may be transformed into another binary representation of the same number N, if, in accordance with the recursive relation (4.18), we replace the binary numerals 1 in the (l-1)-th and (l-p-1)-th digits with the binary numerals 0 and the binary numeral 0 in the l-th digit with the binary numeral 1, that is,

$$N=\begin{cases} a_n\ \dots\ a_{l+1}\ \ 0\ \ 1\ \ a_{l-2}\ \dots\ a_{l-p}\ \ 1\ \ a_{l-p-2}\ \dots\ a_1 \\ a_n\ \dots\ a_{l+1}\ \ 1\ \ 0\ \ a_{l-2}\ \dots\ a_{l-p}\ \ 0\ \ a_{l-p-2}\ \dots\ a_1 \end{cases}. \tag{8.17}$$

Such transformation of the binary representation (8.17) is named *Convolution* of the (l-1)-th and (l-p-1)-th digits into the l-th digit.

The initial binary representation (8.17) can be restored if we fulfill the following code transformation over the binary representation of the number N:

$$N=\begin{cases} a_n\ \dots\ a_{l+1}\ \ 1\ \ 0\ \ a_{l-2}\ \dots\ a_{l-p}\ \ 0\ \ a_{l-p-2}\ \dots\ a_1 \\ a_n\ \dots\ a_{l+1}\ \ 0\ \ 1\ \ a_{l-2}\ \dots\ a_{l-p}\ \ 1\ \ a_{l-p-2}\ \dots\ a_1 \end{cases}. \tag{8.18}$$

Such transformation of the code representation is named *Devolution* of the l-th digit into the (l-1)-th and (l-p-1)-th digits.

Note that in accordance with (4.18) the fulfillment of the convolution and the devolution in the binary representation (8.16) does not change the initial number N, represented by the binary combination (8.16).

It is simple to fulfill convolutions and devolutions for the Fibonacci p-code corresponding to the case p=1. In this case operations are carried out over three adjacent digits, namely over the l-th , $(l$-1)-th and $(l$-2)-th digits. Let us consider the fulfillment of these operations for the Fibonacci 1-code ("Zeckendorf sum"):

(a) *Convolution*

$$7=\begin{cases}0\ 1\ 1\ 1\ 1\\1\ 0\ 0\ 1\ 1\\1\ 0\ 1\ 0\ 0\end{cases} \tag{8.19}$$

(b) *Devolution*

$$5=\begin{cases}1\ 0\ 0\ 0\ 0\\0\ 1\ 1\ 0\ 0\\0\ 1\ 0\ 1\ 1\end{cases} \tag{8.20}$$

The procedure that consists in the fulfillment of all possible convolutions or devolutions in the initial binary combination (8.16) is named *Code Convolution* and *Code Devolution,* respectively. It is easy to prove that the fulfillment of the code convolution and the code devolution result in the so-called C*onvolute* and *Devolute* binary representations of the number N.

For the case p=1, the convolute and devolute binary representations of the number N have peculiar indications. In particular, in the convolute binary representation, two binary numerals of 1 together do not meet and in the devolute binary representation two binary numerals of 0 together do not meet since the highest 1 of the binary representation (8.16).

The rule of the "convolution-devolution inversion" is of great importance for technical applications. This rule consists in the following: the "convolution" of the initial binary combination is equivalent to the "devolution" of the inverse binary combination, and conversely. By using this rule, we can fulfill the reduction to the "convolute" form for the example (8.19) as follows:

(a) Inversion of the initial binary combination 0 1 1 1 1:

$$\overline{01111}=10000$$

(b) "Code devolution" of the inverse binary combination:

$$10000=01100=01011$$

(c) Inversion of the obtained code combination:

$$\overline{01011}=10100$$

Now, let us consider the peculiarities of the "convolution" and "devolution" for the lowest digits of the Fibonacci 1-code. As is well-known in the case p=1, the weights of the two lowest digits of the Fibonacci 1-code equal 1, that is, F_1=F_2=1. And then the operations of the "devolution" and "convolution" for these digits are fulfilled as follows:

10=01 ("devolution") and 01=10 ("convolution").

8.4.2. *Radix of the Fibonacci p-Code*

What is the radix of the p-Fibonacci code? For the case $p=0$, the radix of the binary system (8.10) is calculated as the ratio of the adjacent digit weights, that is, $2^k/2^{k-1}=2$.

Apply this principle to the Fibonacci p-code (8.9) and examine the ratio

$$F_p(k)/F_p(k\text{-}1). \tag{8.21}$$

The radix of the Fibonacci p-code (8.9) is the limit of the ratio (8.21). From Chapter 4 we know that

$$\lim_{k\to\infty}\frac{F_p(k)}{F_p(k-1)}=\tau_p,$$

where τ_p is the golden p-proportion.

This means that the radix of the Fibonacci p-code (8.9) is an irrational number τ_p, the positive root of the characteristic equation $x^{p+1}\text{-}x^p\text{-}1=0$.

8.4.3. *A Minimal Form of the Fibonacci p-Code*

The following theorem is of great importance for the theory of Fibonacci p-codes.

Theorem 8.3. For the given integers $p\geq 0$ and $n\geq p+1$, the arbitrary integer N can be represented in the only form:

$$N=F_p(n)+R_1, \tag{8.22}$$

where

$$0\leq R_1<F_p(n\text{-}p). \tag{8.23}$$

Proof. Note that the sequence of the Fibonacci p-numbers that are generated by the recursive relation (4.18) at the seeds (4.19) is strictly an increasing sequence starting with $n=p+1$. Consider the following number sequence:

$$F_p(p+1), F_p(p+2),..., F_p(n), F_p(n+1),... . \tag{8.24}$$

We can choose in this sequence the pair of adjacent Fibonacci p-numbers $F_p(n)$ and $F_p(n+1)$ so that

$$F_p(n)\leq N< F_p(n+1). \tag{8.25}$$

By subtracting the Fibonacci p-number $F_p(n)$ from all terms of the inequality (8.25), we obtain:

$$0\leq N\text{-}F_p(n)<F_p(n+1)\text{-}F_p(n). \tag{8.26}$$

The formulas (8.22) and (8.23) follow directly from (8.26) because

$$F_p(n+1)\text{-}F_p(n)=F_p(n\text{-}p).$$

The theorem is proved.

Note that for the case p=0 we have: $F_0(n)=2^{n-1}$ and therefore the expressions (8.22) and (8.23) take the following well-known (in "binary" arithmetic) form:

$$N=2^{n-1}+R_1, 0\le R_1<2^{n-1}. \tag{8.27}$$

Let us represent the integer N according to the formula (8.22) and then let us represent all the remainders R_1, R_2, R_k , which can arise in the process of such representation, according to one and the same formula (8.22). We will continue this process up to obtaining the remainder equal to 0. As a result of this decomposition of the number N, we obtain a peculiar representation of integer N in the Fibonacci p-code (8.9). Its peculiarity is that in the binary representation of the integer N given by (8.16) no less than p binary numerals 0 follow after every digit a_l=1 from the left to the right, that is,

$$a_{l-1}= a_{l-2}= a_{l-p}=0. \tag{8.28}$$

Such representation of the integer N is called the *Minimal Form* or *Minimal Representation* of the integer N in the Fibonacci p-code. This name reflects the fact that for the case p=1 the minimal form of the integer N has a minimal number of the binary numerals 1 in the binary representation of the Fibonacci 1-code among all binary representations of the same integer N.

For example, by using the above algorithm, we can obtain the following minimal forms of the number 25 in the Fibonacci 1- and 2-codes (Table 8.4).

Table 8.4. Minimal forms of the Fibonacci1- and 2-codes

$p=1$	$F_1(i)$	55	34	21	13	8	5	3	2	1	1
	25=	0	0	**1**	**0**	0	0	**1**	**0**	**1**	**0**
$p=2$	$F_2(i)$	19	13	9	6	4	3	2	1	1	1
	25=	**1**	**0**	**0**	**1**	**0**	**0**	0	0	0	0

A peculiarity of the binary representations of number 25 given by Table 8.4 consists in the following. For the case p=1, not less than one binary numeral 0 follows after every binary numeral 1 from left to right in the binary representation of number 25; for the case p=2 not less than two binary numerals 0 follow after every binary numeral 1 from left to right in the code representation of the same number 25.

Corollary from Theorem 8.3. For a given p (p=0,1,2,3,...) every integer N has the only minimal form in the Fibonacci p-code.

This means that there is a one-to-one mapping of natural numbers onto the minimal forms of the Fibonacci p-code (8.9).

Note that for the case p=0 (the classical binary code) every integer N has the only representation in the form (8.10). This means that every code representation (8.10) is its "minimal form."

Now, let us prove the following theorem.

Theorem 8.4. For a given integer $p\geq 0$ by using the n-digit Fibonacci p-code in the minimal form we can represent $F_p(n+1)$ integers in the range from 0 to $F_p(n+1)$-1, inclusively.

Proof. To prove the theorem, recall that there is a one-to-one mapping between natural numbers and their minimal forms in the Fibonacci p-code (8.9). Suppose that the number of different n-digit minimal forms of the Fibonacci p-code (8.9) is equal to $T_p(n)$. Let us find a recursive relation for $T_p(n)$. First of all, we can write the following evidence equalities:

$$T_p(1)=T_p(2)=...=T_p(p)=1. \tag{8.29}$$

It is easy to prove that for the case $n=p+1$ we can represent only two numbers in the minimal form, 0 and 1, as follows:

$$0=\underbrace{00...0}_{p+1};\quad 1=1\underbrace{00...0}_{p}.$$

This means that

$$T_p(p+2)=2. \tag{8.30}$$

Let $n>p+1$. Suppose that for this case the set A of n-digit minimal forms in the Fibonacci p-code (8.9) consists of $T_p(n)$ elements, that is, we can represent $T_p(n)$ different numbers in the minimal form. To obtain a recursive relation for $T_p(n)$, we divide the set A into two non-crossing subsets A_0 and A_1. All minimal n-digit forms of the subset A_0 begin from 0 and all minimal n-digit forms of the subset A_1 begin from 1. By taking away the first binary numeral 0 at the beginning of all minimal forms of the subset A_0, we obtain the subset of (n-1)-digit minimal forms. According to the inductive hypothesis, the number of elements in it is equal to $T_p(n-1)$.

Next let us examine the subset A_1. In accordance with the condition for minimal form, the p binary numerals 0 follow after the binary numeral 1. This means that all binary combinations of the subset A_1 begin from the code combination 100...0 (p+1 digits). By taking away these (p+1) digits standing at the beginning of all minimal forms of the subset A_1, we obtain the set of all (n-p-1)-digit minimal forms. According to the inductive hypothesis, the number of elements of this subset is equal to $T_p(n-p-1)$.

The next recursive relation for $T_p(n)$ follows from this consideration:

$$T_p(n)= T_p(n-1)+T_p(n-p-1). \tag{8.31}$$

If we compare the recursive relation (8.31) that is given at the seeds (8.29) and (8.30) with the recursive relation (4.18) at the seeds (4.19), we can conclude:

$T_p(n)=F_p(n+1)$.

The theorem is proved.

For the case $p=1$, the minimal form has a very simple indicator: in the minimal form two binary numerals 1 together do not meet. But the "convolute" form, considered above, has the same property. This means that for the Fibonacci 1-code the "convolute" form coincides with the minimal form and the reduction of the Fibonacci 1-code combination to the minimal form can be performed by using "convolutions." The example (8.19) demonstrates the process of reduction of the Fibonacci 1-code combination to the minimal form. Also the notion of the maximal form is very important for the Fibonacci 1-code. The maximal form can be obtained from the initial binary combination by means of "devolutions" and it coincides with the "devolute" form. The example (8.20) demonstrates the process of reduction of the Fibonacci 1-code combination from the minimal form of the number 5 to its maximal form. Note that the operations of the reduction of the Fibonacci 1-code combinations to the minimal and maximal forms are the most important operations of Fibonacci arithmetic.

8.4.4. *Comparison of Numbers in the Fibonacci p-Codes*

Now, let us deduce a rule for the number comparison of Fibonacci p-codes. With this purpose we compare the n-digit numbers corresponding to the subsets A_0 and A_1. As all numbers of the subset A_1 have the binary numeral 1 in the highest, that is, the n-th digit, the minimal number of the subset A_1 is equal to $F_p(n)$, the weight of the n-th digit of the Fibonacci p-code (8.9). On the other hand, all numbers of the subset A_0 have the binary numeral 0 in the highest, that is, the n-th digit. Then according to Theorem 8.4, the maximal number, which may be represented in the minimal form by using $(n-1)$ digits is equal to $F_p(n)-1$. Hence, the minimal number of the subset A_1 is always greater than the maximal number of the subset A_0.

From this examination, the simple rule of comparison follows. The comparison of two numbers A and B represented in the minimal form of the Fibonacci p-code (8.9) can be carried out digit-by-digit starting with the highest digit until the first pair of non-coincident digits of the comparable codes A and B is found. If the number A has the binary numeral 1 and the number B has the binary numeral 0 in the first pair of the non-coincident digits, we can then conclude that $A>B$. If the code combinations of the comparable numbers coincide for all digits, then the numbers are equal to each other.

Thus, the comparison of numbers in the Fibonacci p-codes is carried out in a manner similar to the classical binary code, provided before the comparison we reduce the comparable codes to their minimal form. This property (simplicity of number comparison) is one of the important arithmetical advantages of the Fibonacci p-codes (8.9).

For example, we need to compare two numbers A=00111101101 and B=00111110110 that are represented in the Fibonacci 1-code. The number comparison is carried out in two steps:

1. Reduce the comparable codes to their minimal form:

$$A = 00111101101 = 0100111001 = 0101001010$$

$$B = 00111110110 = 01001111000 = 01010011000 = 01010100000.$$

2. Compare digit-by-digit the minimal forms of the number A and B beginning with the highest digit on the left:

$$A = 01010[0]10010$$

$$B = 01010[1]00000.$$

We can see that the first non-coincident pair of the comparable combinations (from left to right) contains the binary numeral 0 in the minimal form of the first number A and the binary numeral 1 in the minimal form of the second number B. This means that $B>A$.

8.4.5. *Redundancy of the Fibonacci p-Codes*

For the case p=0, the Fibonacci 0-code (classical binary code) is non-redundant. However, for $p>0$, all Fibonacci p-codes (8.9) are redundant. And their redundancy shows in the *Multiplicity* of the Fibonacci p-code representation of one and the same integer N. Theorems 8.2 and 8.4 allow one to calculate the redundancy of the Fibonacci p-codes for $p>0$ in comparison with the classical binary code (p=0).

We can calculate the relative code redundancy r by the following formula [177]:

$$r = \frac{n-m}{m} = \frac{n}{m} - 1, \tag{8.32}$$

where n and m are the code length of the redundant and non-redundant codes, respectively, for the representation of one and the same number range.

Note that the redundancy definition given by (8.32) characterizes a relative increase of the code length of the redundant code relative to the non-redundant code for the representation of one and the same number range.

Now, let us examine the range of number representation for the n-digit Fibonacci p-codes ($p>0$). The problem has two solutions. It follows from Theorem 8.2 that we can represent $F_p(n+p+1)$ integers by using the n-digit Fibonacci p-code (8.9) if we represent integers $N\geq 0$ without the restrictions on the form of the code representation. However, it follows from Theorem 8.4 that we can represent $F_p(n+1)$ integers N by using the n-digit Fibonacci p-code (8.9) if we use only the minimal forms for the number representation.

For the code representation of the numbers that are in the number range $F_p(n+p+1)$ or $F_p(n+1)$, it is necessary to use either $m_1\approx\log_2 F_p(n+p+1)$ or $m_2\approx\log_2 F_p(n+1)$ binary digits of the non-redundant code, respectively. Using (8.32) we obtain the following formulas for the calculation of the relative code redundancy of the Fibonacci p-code (8.9):

$$r_1 = \frac{n}{\log_2 F_p(n+p+1)} - 1 \tag{8.33}$$

$$r_2 = \frac{n}{\log_2 F_p(n+1)} - 1. \tag{8.34}$$

The simplest redundant Fibonacci p-code is the code corresponding to the case $p=1$. Let us calculate the limiting value of the relative redundancy for this code. The formulas (8.33) and (8.34) for the Fibonacci 1-code take the following forms, respectively:

$$r_1 = \frac{n}{\log_2 F_{n+2}} - 1 \tag{8.35}$$

$$r_2 = \frac{n}{\log_2 F_{n+1}} - 1, \tag{8.36}$$

where F_{n+1}, F_{n+2} are the classical Fibonacci numbers.

We can represent the Fibonacci numbers F_{n+1}, F_{n+2} by using the Binet formulas (2.68). For any large n we can write the Binet formulas (2.68) in the following approximate form:

$$F_n \approx \frac{\tau^n}{\sqrt{5}}. \tag{8.37}$$

Using (8.37) and substituting the approximate values for F_{n+2} and F_{n+1} into the formulas (8.35) and (8.36), we obtain the following formulas:

$$r_1 = \frac{n}{(n+2)\log_2\tau - \log_2\sqrt{5}} - 1 \tag{8.38}$$

$$r_2 = \frac{n}{(n+1)\log_2\tau - \log_2\sqrt{5}} - 1. \tag{8.39}$$

If we direct n to infinity in the expressions (8.38) and (8.39), we find that they coincide for this case. Here, the utmost value of the relative redundancy for the Fibonacci 1-code is determined by the following expression:

$$r = \frac{1}{\log_2 \tau} - 1 = 0.44.$$

Thus, the utmost value of the redundancy of the Fibonacci 1-code (Zeckendorf representation) is a constant value equal to 0.44 (44%).

8.4.6. *Surprising Analogies Between the Fibonacci and Genetic Codes*

Among the biological concepts that have a level of general scientific significance and are well formalized, the genetic code plays a special role. The discovery of the now well-known fact of the striking simplicity of the basic principles of the genetic code is one of the major modern discoveries of modern science. This simplicity consists of the fact that the inheritable information is encoded by the texts of the three-alphabetic words - *Triplets* or *Codonums* - compounded on the base of the alphabet that consists of the four characters - nitrogen bases: ***A*** (adenine), ***C*** (cytosine), ***G*** (guanine), ***T*** (thiamine). The given recording system is unique for all boundless set of miscellaneous living organisms and is called the *Genetic Code* [141].

It is known [141] that by using the three-alphabetic triplets or codonums, we can encode 21 items that include 20 amino acids and one more item called the *Stop-codonum* (sign of the punctuation). It is clear that 4^3=64 different combinations (from four by three nitrogen bases) are used for encoding 21 items. In this connection some of the 21 items are encoded by several triplets. It is called the *Degeneracy of the Genetic Code*. Finding the conformity between triplets and amino acids (or signs of the punctuation) is interpreted as *Decryption of the Genetic Code*. Now, let us examine the 6-digit Fibonacci 1-code (Zeckendorf representation) (8.1) that uses 6 Fibonacci numbers 1, 1, 2, 3, 5, 8 as digit weights:

$$N = a_6 \times 8 + a_5 \times 5 + a_4 \times 3 + a_3 \times 2 + a_2 \times 1 + a_1 \times 1. \tag{8.40}$$

There are the following surprising analogies between the 6-digit Fibonacci code and the genetic code:

1. **The first analogy.** For the representation of numbers the 6-digit binary Fibonacci code uses 2^6=64 binary combinations from 000000 up to 111111 which coincide with the number of triplets of the genetic code (4^3=64).
2. **The second analogy**. By using the 6-digit Fibonacci code (8.40), we can represent 21 integers including the minimal number 0 that is encoded by the 6-digit binary combination 000000 and the maximal number 20 that is encoded by the 6-digit binary combination 111111. Note that by using the

triplet coding we can also represent 21 items including 20 amino acids and one additional object, the stop-codonum or the sign of the punctuation that indicates a termination of protein synthesis.

3. **The third analogy**. The main feature of the Fibonacci code is the *Multiplicity* of number representation. Except for the minimal number 0 and the maximal number 20 that have the only code representations 000000 and 111111, respectively, all the rest of numbers from 1 up to 19 have multiple representations in the Fibonacci code, that is, they use no less than two code combinations for their representation. It is necessary to note that the genetic code has the similar property called the *Degeneracy* of the genetic code.

Thus, between the Fibonacci code (8.40) and the genetic code based on the triplet representation of amino acids, there are very interesting analogies that allow us to consider the Fibonacci code to be a peculiar class of redundant codes among various ways of redundant coding. Thus we suggest that the study of the Fibonacci code may be of great interest for the genetic code. It is very possible that the similar analogies can become rather useful in the designing of DNA-based bio-computers.

8.5. Fibonacci Arithmetic: The Classical Approach

8.5.1. *Fibonacci Addition*

Let us begin with developing the Fibonacci arithmetic. There are two methods for its development. The first way is to use an analogy between classical binary arithmetic and Fibonacci arithmetic (a “classical approach”). The second way is to develop the “original Fibonacci arithmetic” based on some peculiarities of the Fibonacci p-codes.

Let us first start the development with the classical approach. As the notion of the Fibonacci p-code given by (8.9) is a generalization of the notion of the classical binary code (8.10), we can use the following method for obtaining the arithmetical rules of the Fibonacci p-arithmetic. We begin by analyzing the corresponding rule for the classical binary arithmetic and then by analogy we find a similar rule for the Fibonacci p-arithmetic.

We start from the addition rule for the Fibonacci p-code (8.9). It is well known that the classical binary addition is based on the following identity for the binary numbers:

$$2^{k-1}+2^{k-1}=2^k, \tag{8.41}$$

where 2^{k-1} and 2^k are the weights of the $(k-1)$-th and k-th digits of the binary code (8.10), respectively.

To obtain the rule of number addition for the Fibonacci p-code, we analyze the following sum:

$$F_p(k)+F_p(k), \tag{8.42}$$

where $F_p(k)$ is the weight of the k-th digit of the Fibonacci p-code (8.9).

Let $p=1$. For this case we have

$$F_p(k)=F_k, \tag{8.43}$$

where F_k is the classical Fibonacci number given by the recursive relation (8.2) at the seeds (8.3). If we take into consideration (8.2) and (8.3), we can represent the sum (8.42) as follows:

$$F_k+F_k=F_k+F_{k-1}+F_{k-2} \tag{8.44}$$

$$F_k+F_k=F_{k+1}+F_{k-2}. \tag{8.45}$$

The following addition rule of two binary digits $a_k + b_k$ with the same index k for the Fibonacci 1-code follows from (8.44) and (8.45) represented in Table 8.5.

The following rules of the Fibonacci 1-addition of the binary numerals of the k-th digits a_k+b_k follow from Table 8.5:

Table 8.5. A rule of summation for the Fibonacci 1-code

a_k	+	b_k	=	c_{k+1}	s_k	c_{k-1}	c_{k-2}	
0	+	0	=	0	0	0	0	
0	+	1	=	0	1	0	0	
1	+	0	=	0	1	0	0	
1	+	1	=	0	1	1	1	(a)
1	+	1	=	1	0	0	1	(b)

Rule 8.1. At the addition of the binary numerals 1+1 of the k-th digits the carry-over of the binary numeral 1 from the k-th digit to the next two digits arises.

Rule 8.2. There are two ways of carrying-over the formation. In method (a) (see Table 8.5) the binary numeral 1 is attributed to the k-th digit of the intermediate sum s_k and the carry-over of the binary numeral 1 to the next two lower digits, that is, to the $(k-1)$th and $(k-2)$th digits, c_{k-1} and c_{k-2}, occurs. Method (b) (see Table 8.5) assumes another rule of the Fibonacci 1-summation of binary digits. The binary numeral 0 is attributed to the k-th digit of the intermediate sum s_k and the binary numeral 1 is carried-over to the next two digits, that is, to the $(k+1)$-th and $(k-2)$-th digits, c_{k+1} and c_{k-2}, appear.

The summation of the multi-digit numbers in the Fibonacci 1-code is fulfilled in accordance with the Fibonacci 1-summation according to Table 8.5. However, we should adopt the following rules:

Rule 8.3. Before summation, the addends are reduced to the minimal form.

Rule 8.4. In accordance with Table 8.5, it is necessary to form the multi-digit intermediate sum and multi-digit carry-over.

Rule 8.5. The multi-digit intermediate sum is reduced to the minimal form and is then summarized with the multi-digit carry-over.

Rule 8.6. The addition process continues in accordance with rules 8.4 and 8.5 until the multi-digit carry-over equal to 0 is obtained. The last intermediate sum, which is reduced to the minimal form, is the result of the addition.

For the Fibonacci 1-addition it is necessary to fulfill the additional rule 8.7.

Rule 8.7. Let us examine the case, where we have two binary numerals 1 in the k-th digits of the addends. It follows from the property of the minimal form that the binary numerals of the (k+1)-th and (k-1)-th digits of both of the addends are always equal to 0. It is clear that for this case the intermediate sum that appears at the addition of the (k+1)-th and (k-1)-th digits of both of the addends are also always equal to 0. This means that we can place one of the carry-overs that appear at the addition of the k-th significant digits (1+1) at once to the (k-1)-th digit of the intermediate sum (for the (a)-method) or to the (k+1)-th digit of the intermediate sum (for the (b)-method).

We can demonstrate the above Fibonacci addition rules in the example below.

Example 8.1. Summarize the numbers 31=10011011 and 22=01011010 represented in the Fibonacci 1-code.

The first step is to reduce the binary representations of the numbers 31 and 22 to the minimal form:

31=10100100; 22=10000010.

The second step is the formation of the multi-digit intermediate sum S_1 and multi-digit carry-over C_1 in accordance with method (a) of Table 8.5:

$$+\begin{array}{l} 31=10100100 \\ 22=10000010 \\ \hline S_1 = 11100110 \\ C_1 = 00100000 \end{array}$$

The third step is to reduce the intermediate sum S_1 to the minimal form:

S_1=11100110=100101000.

The fourth step is the addition of S_1 and C_1:

$$+\begin{array}{l} S_1=100101000 \\ C_1=000100000 \\ \hline S_2 = 100111000 \\ C_2 = 000001000 \end{array}$$

The fifth step is to reduce the intermediate sum S_2 to the minimal form:

S_2=100111000=101001000.

The sixth step is the addition of S_2 and C_2:

$$\begin{array}{l} S_2 = 101001000 \\ + \\ C_2 = 000001000 \\ \hline S_3 = 101001100 \\ C_3 = 000000010 \end{array}$$

The seventh step is to reduce the intermediate sum S_3 to the minimal form:

S_3=101001100=101010000.

The eighth step is the addition of S_3 and C_3:

$$\begin{array}{l} S_3 = 101010000 \\ + \\ C_3 = 000000010 \\ \hline S_4 = 101010010 \\ C_4 = 000000000 \end{array}$$

The addition is over because the carry-over C_4=0 appears in the eighth step.

8.5.2. *Direct Fibonacci Subtraction*

The well-known "direct" method of number subtraction in classical binary arithmetic (p=0) is based on the following property of binary numbers:

$$2^{n+k}-2^n=2^{n+k-1}+2^{n+k-2}+...+2^n \; . \tag{8.46}$$

Now, let us write a similar identity for the Fibonacci 1-numbers:

$$F_{n+k} - F_n = F_{n+k-2}+F_{n+k-3}+...+F_{n-1} \; . \tag{8.47}$$

Using the identity (8.47) and the Fibonacci recursive relation (8.2), we obtain the following Fibonacci 1-subtraction table represented in Table 8.6.

Table 8.6. The rule of subtraction for the Fibonacci 1-code

0 – 0 = 0
1 – 1 = 0
1 – 0 = 0 1 1
1 0 – 1 = 0 1
1 0 0 – 1 = 1 1
1 0 0 0 – 1 = 1 1 1

The reader of the present book can "deduce" the rules of the "direct" Fibonacci p-subtraction by using the recursive relation for the Fibonacci p-numbers.

The "direct" Fibonacci 1-subtraction of multi-digit numbers is based on the following rules:

Rule 8.8. The numbers are reduced to the minimal form before subtraction.

Rule 8.9. After reduction to the minimal form, the numbers are compared by their value and then in accordance with Table 8.6, the smaller number being subtracted from the larger one.

8.5.3. *Fibonacci Inverse and Additional Codes*

It is well-known that the classical binary arithmetic employs the notions of "inverse" and "additional" codes for the representation of negative numbers. This approach allows one to reduce the binary subtraction to binary addition. For the case of integer representation, the following mathematical properties underlie the "inverse" code (G_{inv}) and the "additional" code (G_{ad}):

$$G_{inv}=2^n-1-|G| \tag{8.48}$$

$$G_{ad}=2^n-|G|, \tag{8.49}$$

where $|G|$ is the module of the initial integer G.

Note that the "-" ("minus") sign is encoded by the binary numeral 1 and the "+" ("plus") sign by the binary numeral 0. They are placed at the beginning of the code combination. The digital part coincides with its direct code for the positive number G, and is determined by (8.48) and (8.49) for the negative number G.

The notions of inverse and additional codes have the following generalization for the Fibonacci 1-code. By analogy to (8.48) and (8.49) we can introduce the notions of the "Fibonacci inverse" and "Fibonacci additional" n-digit 1-codes:

$$G_{inv}=F_{n+1}-1-|G| \tag{8.50}$$

$$G_{ad}=F_{n+1}-|G|, \tag{8.51}$$

where F_{n+1} is the Fibonacci 1-number, and G is any integer.

The expressions (8.50) and (8.51) have the following code interpretation. Suppose that the initial integer G is represented in the minimal form of the n-digit Fibonacci 1-code (8.1). It follows from Theorem 8.4 that the integer G is in the following number range:

$$0 \le G \le F_{n-1}. \tag{8.52}$$

On the other hand, the following important expressions that connect the Fibonacci inverse and additional codes follow from (8.50) and (8.51):

$$G_{inv}+|G|=F_{n-1}-1 \tag{8.53}$$

$$G_{ad}=G_{inv}+1. \tag{8.54}$$

Using the general property (2.20), we can represent the number $F_{n+1}-1$ as follows:

$$F_{n+1}-1= F_{n-1}+F_{n-2}+...+F_2+F_1. \tag{8.55}$$

Note that the sum (8.55) has the following code interpretation:

$$F_{n+1}-1=0\underbrace{11...1}_{n-1}=G_{inv}+|G|. \tag{8.56}$$

The formula (8.56) can be used for the introduction of the following algorithm for obtaining the Fibonacci inverse and Fibonacci additional 1-codes from the initial code G (for example, from the initial Fibonacci 1-code G=10010100):

1. By using the operation of “devolution” for the initial code G:

G=10010100=01110011=01101111.

2. By using the operation of “inversion” for all binary digits of the “devolute” form of the number G, except for the higher digit:

$0\overline{1101111} = 00010000.$

The resulting binary combination is the Fibonacci inverse 1-code of G:

G_{inv}=00010000.

3. By adding the binary numeral 1 to the lowest digit of the Fibonacci inverse 1-code in accordance to (8.54) and by reducing the obtained code combination to the minimal form, we obtain the Fibonacci additional 1-code G:

G_{ad}=00010001=00010010.

Note that in the general case $p\geq 0$, the expressions for the Fibonacci inverse and additional p-codes have the following forms:

$$G_{inv} = F_p(n+1) - 1 - |G| \tag{8.57}$$

$$G_{ad} = F_p(n+1) - |G|. \tag{8.58}$$

We can see that the expressions (8.48), (8.49) and (8.53), (8.54) are partial cases of expressions (8.57) and (8.58) because for the case p=0, $F_0(n+1)=2^n$ and for the case p=1, $F_1(n+1)=F_{n+1}$.

Now, let us introduce the following rule of the sign encoding for the Fibonacci inverse and additional 1-codes. We will be encoding the “–” sign with the binary numeral 1 and the “+” sign with the binary numeral 0. The sign of the code combinations is put at the beginning of the numerical part of the code. Hence, the Fibonacci inverse and additional 1-codes of the negative number

G=-10010100

have the following forms, respectively:

G_{inv}=1.00010000

G_{inv}=1.00010010.

Note that the numerical part of the Fibonacci inverse and additional 1-codes of the positive number G, coincide with the Fibonacci 1-code of the same number G. For example, the Fibonacci inverse and additional 1-codes of the positive number G=+10010100 have the following form:

G_{ad}=0.10010100.

Let us formulate the Fibonacci subtraction algorithm based on the notions of the Fibonacci inverse and additional 1-codes:

1. Represent the initial numbers in the Fibonacci inverse or additional 1-codes.
2. Sum the numerical parts in accordance with the rule of the Fibonacci 1-addition and the signs in accordance with the rule of classical binary addition. The addition result is represented in the Fibonacci inverse or additional 1-codes. If the sign of the code combination is equal to 1, it means that the addition result is a negative number. For obtaining the absolute value of the addition result, it is necessary to use the above algorithm of code transformation for the Fibonacci inverse or additional 1-codes for the numerical part of the addition result.

8.6. Fibonacci Arithmetic: An Original Approach

8.6.1. *Basic Micro Operations*

As mentioned above the multiplicity of number representation is the main peculiarity of the Fibonacci p-code (8.9) for the case $p>0$ in comparison to the classical binary code (8.10). By using the above operations of "devolution" and "convolution", we can change the form of representation of one and the same number. This means that the binary 1's can move in the code combination to the left or to the right. This fact allows us to develop an original approach to the Fibonacci arithmetic based upon the so-called *Basic Micro-operations*.

We can introduce the following four "basic micro-operations" that are used in the Fibonacci processor for fulfillment of the logical and arithmetical operations: (a) *Convolution;* (b) *Devolution;* (c) *Replacement;* (d) *Absorption.*

Consider these micro-operations for the case of the Fibonacci 1-code (Zeckendorf representation). We showed above that for the case $p=1$ the *Convolution* and *Devolution* are the following code transformations that are carried out for all the adjacent triple digits of one and the same Fibonacci 1-code combination.

Convolution:

$[011\rightarrow100]$

Devolution:

$[100\rightarrow011]$

The micro-operation of *Replacement*

$$\begin{bmatrix} 1 & & 0 \\ \downarrow & = & \\ 0 & & 1 \end{bmatrix}$$

is a two-placed micro-operation that is fulfilled at the same digit of two registers, the top register A and the bottom register B. Examine the case when the register A has the numeral 1 in the k-th digit and the register B has the numeral 0 in the same digit. For this condition we can fulfill the following micro-operation. We shift the above binary numeral 1 of the top register A to the bottom register B. This micro-operation is named *Replacement*. Note that this operation can be carried out only for the condition if the k-th digits of the registers A and B are equal to 1 and 0, respectively.

The micro-operation of *Absorption*

$$\begin{bmatrix} 1 & & 0 \\ \updownarrow & = & \\ 1 & & 0 \end{bmatrix}$$

is a two-placed micro-operation that consists in mutually annihilating two 1's in the k-th digit of two registers A and B and replacing them with the binary numerals 0.

It is necessary to pay attention to the following "technical" peculiarity of the above "basic micro-operations." At register interpretation of these micro-operations, each micro-operation may be considered to be the inversion of the triggers (flip-flops) that are involved in the micro-operation. This means that each micro-operation is carried out by trigger switching.

8.6.2. *Logical Operations*

We can demonstrate the possibility of carrying out all logical operations by using the above four "basic micro-operations."

Let us construct all the possible "replacements" from top register A to bottom register B:

$$\begin{array}{ccccccccc} A & = & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ & & \downarrow & & & \downarrow & & & \\ B & = & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline A' & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ B' & = & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

We obtain two new code combinations A' and B' as the outcome of the "replacement." We can see that the binary combination A' is a logical "conjunction" ($\wedge$) of the initial binary combinations A and B, that is,

$A' = A \wedge B$

and the binary combination B' is a logical "disjunction" ($\vee$) of the initial code combinations A and B, that is,

$B' = A \vee B \cdot$

The logical operation of "module 2 addition" is performed by means of the simultaneous fulfillment of all possible micro-operations of "replacement" and "absorption." For example,

$$
\begin{array}{lcccccccccccl}
A & = & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & & \\
 & & \updownarrow & & \downarrow & & & \downarrow & \downarrow & & \updownarrow & & \\
B & = & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & & \\
\hline
A' & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & = & const\,0 \\
B' & = & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & = & A \oplus B
\end{array}
$$

We can see that the two new code combinations $A' = \text{const } 0$ and $B' = A \oplus B$ are the result of this code transformation. The logical operation of "code A inversion" is reduced to fulfillment of the operation of "absorption" at the initial code combination A and the special binary combination B=*const* 1 (see below):

$$
\begin{array}{lcccccccccccl}
A & = & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & & \\
 & & \updownarrow & & \updownarrow & & & \updownarrow & \updownarrow & & \updownarrow & & \\
B & = & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
\hline
A' & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & = & const\,0 \\
B' & = & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & = & \overline{A}
\end{array}
$$

8.6.3. *The Counting and Subtracting of Binary 1*

Let us demonstrate the possibility of carrying out the simplest arithmetical operations by using the "basic micro-operations." We start with the operations of "counting" and "subtracting" of the binary numerals 1.

The "counting" of the binary numerals 1 in the Fibonacci 1-code (a "summing counter") is carried out by using "convolution." For example, the transformation of the initial Fibonacci 1-code combination 01010 - the minimal form of the number 4

F_i	=	5	3	2	1	1
4	=	0	1	0	1	0

to the next Fibonacci code combination of the number 5 is carried out in the "summing Fibonacci counter" according to the rule:

F_i	=	5	3	2	1	1		
4	=	0	1	0	1	0	+	1
5	=	0	1	0	1	1		
5	=	0	1	1	0	0		
5	=	1	0	0	0	0		

Here, in the top row we see Fibonacci numbers 5, 3, 2, 1, 1 that are the weights of the Fibonacci 1-code. The second row is a representation of the number 4 in the minimal form (01010). We can see that the binary numeral 1 is added (+1) to the lowest digit of the Fibonacci binary combination 01010. Then, the initial binary combination 4=01010 is transformed into the binary combination 5=01011 that is the Fibonacci binary representation of the next number 5 (the third row). After that the Fibonacci binary combination 01011 is reduced to the minimal form. This transformation is fulfilled in 2 steps by using the "convolutions." The first step is to carry out the "convolution" over the three lowest digits of the binary combination 01011 (the third row). Using the "convolution" [011→100], we transform the Fibonacci binary representation of the number 5=01011 to another Fibonacci binary representation of the same number 5=01100 (the fourth row). Then, we carry out the "convolution" over the next group 011 of the Fibonacci binary combination 5=01100 (the fourth row). In the fifth row we see the minimal form of the number 5=10000. We can continue the "counting" of the binary numerals 1 as follows:

$$6 = 10000 + 1 = 10001 = 10010$$

$$7 = 10010 + 1 = 10011 = 10100.$$

If we add the binary numeral 1 to the lowest digit of the number 7=10100, we obtain:

10100+1=10101=10110=11000=00000.

This situation is well known in computer engineering and is named "overfilling" of the "summing counter."

The "subtracting" of the binary numerals 1 (a "subtracting counter") is fulfilled by using "devolution." For this purpose the initial Fibonacci 1-code combinations are reduced to the "devolute" or "maximal" form, and then the binary numeral 1 is subtracted sequentially from the lowest digit:

F_i	=	5	3	2	1	1		
5	=	1	0	0	0	0		
5	=	0	1	1				
5	=	0	1	0	1	1	–	1
4	=	0	1	0	1	0		

Here, in the second row we see the representation of the number 5 in the minimal form (5=10000). In the third row we carry out the "devolution" over the highest three digits [100→011] of the binary representation of the initial binary combination 5=10000. In the fourth row we carry out the next "devolution" over the next three digits of the binary representation of the number 5=01100 (see the third row). As the outcome we obtain the maximal form of the number 5=01011 (the fourth row). Then, we subtract the binary numeral 1 from the lowest digit of the binary combination 5=01011. A result of this transformation 4=01010 is represented in the fifth row. Then, we can continue the "subtracting" of the 1's as follows:

$$4 = 01010 = 01001\,(\text{"}\textit{devolutions}\text{"})$$
$$3 = 4 - 1 = 01001 - 1 = 01000$$
$$3 = 01000 = 00110 = 00101\,(\text{"}\textit{devolutions}\text{"})$$
$$2 = 3 - 1 = 00101 - 1 = 00100$$
$$2 = 00100 = 00011\,(\text{"}\textit{devolution}\text{"})$$
$$1 = 2 - 1 = 00011 - 1 = 00010$$
$$1 = 00010 = 00001\,(\text{"}\textit{devolution}\text{"})$$
$$0 = 1 - 1 = 00001 - 1 = 00000.$$

8.6.4. *Fibonacci Summation*

The idea of summation of two numbers A and B is based on the "basic micro-operations." The first step is to shift all binary numerals 1 from the top register A to the bottom register B. With this purpose we use "replacement," "devolution," and "convolution." The sum is formed in register B.

For example, let us sum the following numbers

A_0=010100100 and B_0=001010100.

The first step of the Fibonacci addition consists in the "replacement" of all possible binary numerals 1 from register A to register B:

$$\begin{array}{lcccccccccc} A_0 & = & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ & & & \downarrow & & \downarrow & & & & & \\ B_0 & = & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline A_1 & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ B_1 & = & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}$$

The second step is the fulfillment of all possible "devolutions" in the binary combination A_1 and all possible "convolutions" in the binary combination B_1:

A_1=000000100→A_2=000000011

B_1=011110100→B_2=100110100.

The third step is the "replacement" of all possible binary numerals 1 from register A to register B:

$$\begin{array}{lcccccccccc}
A_2 & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & & & & & & \downarrow & \downarrow \\
B_2 & = & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
\hline
A_3 & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_3 & = & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1
\end{array}$$

The addition is over because all binary numerals 1 are shifted from the register A to the register B. After reducing the binary combination B_3 to the minimal form, we obtain the sum $B_3=A+B$, which is represented in the minimal form:

B_3=100110111=101001001=101001010=$A+B$.

Thus, the addition is reduced to a sequential fulfillment of the micro-operations of "replacement" over two binary combinations A and B and the micro-operations of both "convolution" over the binary combination B and "devolution" over the binary combination A.

8.6.5. *Fibonacci Subtraction*

The idea of direct subtraction of number B from number A which is based on the "basic micro-operations," consists of the mutual "absorption" of the binary numerals 1 in the binary combinations of numbers A and B until one of them becomes equal to 0. To perform this process, we have to sequentially carry out the aforementioned micro-operations of "absorption" and "devolution" over the code combinations A and B. The subtraction result is always formed in the register that contains the larger number. If the subtraction result is formed in the top register A, it follows from this fact that the sign of the subtraction result is "+" ("plus"); in the opposite case the subtraction result has the "–" ("minus") sign.

Let us now demonstrate this idea for the following example. Subtract the number B_0=101010010 from the number A_0=101001000 in the Fibonacci 1-code.

The first step is "absorption" of all possible binary 1's in the numbers A and B:

$$\begin{array}{lcccccccccc}
A_0 & = & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & \updownarrow & & \updownarrow & & & & & & \\
B_0 & = & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
A_1 & = & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
B_1 & = & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}$$

The second step is "devolution" of the code combinations A_1 and B_1:

A_1=000001000→ A_2=000000110

B_1=000010010→ B_2=000001101.

The third step is "absorption" of A_2 and B_2:

$$\begin{array}{lcccccccccc} A_2 & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ & & & & & & & & \updownarrow & & \\ B_2 & = & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ \hline A_3 & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ B_3 & = & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

The fourth step is "devolution" of A_3 and B_3:

A_3=000000010→A_4=000000001

B_3=000001001→B_4=000000111.

The fifth step is "absorption" of A_4 and B_4:

$$\begin{array}{lcccccccccc} A_4 & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & & & & & \updownarrow \\ B_4 & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline A_5 & = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_5 & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

The subtraction is finished. After reducing the code combination B_5 to the minimal form we obtain:

B_5=000001000.

The subtraction result is in the register B. This means that the sign of the subtraction result is "–" ("minus"), that is, the difference of the numbers A-B is equal to:

R=A-B=-000001000.

8.7. Fibonacci Multiplication and Division

8.7.1. *The Egyptian Method of Multiplication*

In order to find the algorithms for Fibonacci multiplication and division, we use an analogy with classical binary multiplication and division. We start with multiplication. To multiply two numbers A and B in the classical binary code

(8.10) we should represent the multiplier B in the form of the n-digit binary code. The product $P=A\times B$ can then be written in the following form:

$$P=A\times b_n 2^{n-1}+A\times b_{n-1}2^{n-2}+...+A\times b_i 2^{i-1}+...+A\times b_1 2^0, \tag{8.59}$$

where b_i is the binary numeral of the multiplier B. It follows from (8.59) that the binary multiplication is reduced to form the partial products of the kind $A\times b_i 2^{i-1}$ and their addition. The partial product $A\times b_i 2^{i-1}$ is made up by shifting the binary code of number A to the left by $(i-1)$ digits.

The binary multiplication algorithm based on (8.59) has a long history and goes back in its origin to the Egyptian *Doubling Method* [169].

As is known, the Egyptian number system was decimal, but non-positional. The Egyptians represented the "nodal" numbers of their number system 1, 10, 100, 1000, 10 000, etc. by using special hieroglyphs. Then, in the record of any number, for example, 325, they used the 5 hieroglyphs representing the "nodal" number 1, the 2 hieroglyphs representing the "nodal" number 10, and the 3 hieroglyphs representing the "nodal" number 100.

The invention of the *Doubling Method* that underlies the arithmetical operations of multiplication and division is the main achievement of Egyptian arithmetic. In order to multiply the number 35 by the number 12, the Egyptian mathematician acted as follows. He built up a special table of numbers (see Table 8.7). The binary numbers 2^k (k=0,1,2,3,...) were placed in the first column of Table 8.7. In the second column of Table 8.7 we can see the numbers 35, 70, 140, 280, that is, the numbers formed from the multiplier 35 according to the *Doubling Method*.

Table 8.7. Egyptian method of multiplication

1	35		
2	70		
/4	**140**	→	**140**
/8	**280**	→	**280**
			420

After that the Egyptian mathematician used the inclined line to mark those binary numbers of the first column whose sum is equal to the second multiplier (12=8+4). Then he selected those numbers of the second column corresponding to the marked "binary" numbers of the first column. The result of multiplication is equal to the sum of the selected numbers of the second column (140+280=420).

The analysis of the Egyptian method of multiplication based on the *Doubling Method* results in a rather unexpected conclusion. The representation of the multiplier 12 in the form 12=8+4 corresponds to its representation in the binary system (12=1100). On the other hand, if we represent the second multiplier 35 in the binary system (35=100011), then the "doubling" of the number 35 corresponds to the displacement of the binary code of the multiplier 35=100011 to the left (70=1000110, 140=10001100, etc.). In other words, the Egyptian multiplication method based on the *Doubling Method* coincides with the basic multiplication algorithm used in modern computers!

This basic algorithm follows from the representation of the product in the form of (8.56).

8.7.2. *The Egyptian Method of Division*

The Egyptians also used the *Doubling Method* for the division of numbers. If, for example, we need to divide the number 30 by the number 6, the Egyptian mathematician acted as follows (see Table 8.8).

The first stage of the division consists of the following. Let us write in the first column of Table 8.8 the binary numbers 2^k (k=0, 1, 2, 3,...). Then, let us write in the second column the divisor 6 and its "doubling" numbers 12, 24, and 48. In every "doubling" step we will compare the dividend 30 with the numbers of the second column until the next "doubling" number (the number 48 in this example) becomes more than the dividend (48>30). After that we shall subtract the "doubling" number of the previous row (30-24=6) from the dividend 30 and mark the corresponding binary number of the first column (the number 4) with the inclined line. After that we shall perform the same procedure with the remainder 6 (see the third column) until the "doubling" number becomes strictly more than the difference 6 (12>6). By marking the number 1 of the first column with the inclined line and then by subtracting the divisor 6 from the remainder 6, we obtain the number 0 (6-6=0). This means that the division is over. After that we can see that the sum of the binary numbers of the first column marked by the inclined line (4+1=5) is equal to the result of the division.

Table 8.8. Egyptian method of division

/1	$6 \le 30$	$6 \le 6$
2	$12 \le 30$	$\mathbf{12 > 6}$
/4	$24 \le 30$	
8	$\mathbf{48 > 30}$	
$4+1=5$	$30-24=6$	$6-6=0$

Thus, after due consideration one is astonished by the genius of the Egyptian mathematicians who several millennia ago invented the methods of multiplication and division that are today used in our modern computers!

8.7.3. *Fibonacci Multiplication*

Analysis of the Egyptian *Doubling Method* allows us to develop the following method of "Fibonacci p-multiplication."

Let us consider the product $P=A\times B$, where the numbers A and B are represented in the Fibonacci p-code (8.9). By using the representation of the multiplier B in the Fibonacci p-code (8.9), we can write the product $P=A\times B$ in the following form:

$$P=A\times b_n F_p(n)+A\times b_{n-1} F_p(n-1)+...+A\times b_i F_p(i)+...+A\times b_1 F_p(1), \qquad (8.60)$$

where $F_p(i)$ is the Fibonacci p-number.

Note that the expression (8.60) is a generalization of the expression (8.59) that underlies the algorithm of the "binary" multiplication. The following algorithm of the Fibonacci p-multiplication follows directly from the expression (8.60). The multiplication is reduced to the addition of the partial products of the kind $A\times b_i F_p(i)$. They are formed from the multiplier A according to the special procedure that reminds one of the Egyptian *Doubling Method*. Let us demonstrate the "Fibonacci multiplication" for the case of the simplest Fibonacci 1-code (p=1).

Example 8.2. Find the following product: 41×305.

Solution:

1. Construct a table consisting of three columns marked by F, G and P (see Table 8.9).
2. Insert the Fibonacci 1-sequence (the classical Fibonacci numbers) 1, 1, 2, 3, 5, 8, 13, 21, 34 into the F-column of Table 8.9.
3. Insert the generalized Fibonacci 1-sequence: 305, 305, 610, 915, 1525, 2440, 3965, 6505, 10370, which is built up from the first multiplier 305 according to the "Fibonacci recursive relation" into the G-column.
4. Mark by inclined line (/) and bold type all numbers of the F-column, which make up the second multiplier 41 in the sum (41=34+5+2).
5. Mark by bold type all G-numbers corresponding to the marked F-numbers and rewrite them in the P-column.
6. Summing all P-numbers, we obtain the product: 41×305=12505.

Table 8.9. Fibonacci multiplication

F	G		P
1	305		
1	305		
/2	**610**	→	**610**
3	915		
/5	**1525**	→	**1525**
8	2440		
13	3965		
21	6505		
/34	**10370**	→	**10370**
41=34+5+2	41×305	=	12505

The multiplication algorithm is easily generalized for the case of Fibonacci p-codes.

8.7.5. *Fibonacci Division*

We can apply the above Egyptian method of division to construct the algorithm of Fibonacci p-division. Consider this method for the following specific example.

Example 8.3. Divide the number 481 (the dividend) by the number 13 (the divisor) in the Fibonacci 1-code.

Solution has two steps as shown below.

8.7.5.1. *The first stage*

1. Construct the table, which consists of three columns marked by *F, G* and *D* (see Table 8.10).
2. Insert the Fibonacci 1-sequence (the classical Fibonacci numbers) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 to the *F*-column of Table 8.10.
3. Insert the generalized Fibonacci 1-sequence: 13, 13, 26, 39, 65, 104, 169, 273, 442, 615 formed from the divisor 13 according to the "Fibonacci recursive relation" into the *G*-column.
4. Compare sequentially every *G*-number with the dividend 481, inscribed into the *D*-column, and fix the result of comparison ($\leq$ or >) until we obtain the first comparison result of the kind: 615>481.
5. Mark with the inclined line (/) and bold type the Fibonacci number 34, which corresponds to the preceding *G*-number 442, and mark the latter with bold type.
6. Calculate the difference: R_1=481-442=39.

Table 8.10. The first stage of the Fibonacci division

F	*G*		*D*
1	13	$\leq$	481
1	13	$\leq$	481
2	26	$\leq$	481
3	39	$\leq$	481
5	65	$\leq$	481
8	104	$\leq$	481
13	169	$\leq$	481
21	273	$\leq$	481
/34	**442**	$\leq$	481
55	**615**	**>**	**481**
R_1	=	$481-442$	$=39$

8.7.5.2. *The second stage*

The second stage of the Fibonacci 1-division is a repetition of the first stage when we instead use the dividend 481, the remainder R_1=39 (see Table 8.11).

The second remainder R_2=39–39=0, which means the Fibonacci 1-division is complete. The result of the division is equal to the sum of all the marked *F*-numbers obtained throughout all stages (see Tables 8.10 and 8.11), that is, 34+3=37.

Table 8.11. The second stage of the Fibonacci division

F	*G*		*D*
1	13	$\leq$	39
1	13	$\leq$	39
2	26	$\leq$	39
/3	**39**	$\leq$	39
5	**65**	**>**	**39**
R_2	=	$39-39$	$=0$

8.8. Hardware Realization of the Fibonacci Processor

8.8.1. *A Device for Reduction of the Fibonacci Code to Minimal Form*

The devices for the "convolution" and "devolution" play an important role in the technical realization of arithmetical operations in the Fibonacci code. They can be designed on the basis of the binary register, which has special logical circuits for the "convolutions" and "devolutions." Each digit of the register contains a binary "flip-flop" (trigger) and logical elements. The operations of "convolution" [011→100] and "devolution" [100→011] can be carried out by means of the inversion of the "flip-flops" (triggers).

One of the possible variants of the "convolution" register or "the device for the reduction of the Fibonacci code to the minimal form" is shown in Fig. 8.1. This device consists of the five *R*-*S*-triggers and the logical elements *AND, OR* which are used to carry out the "convolution." The "convolution" is carried out by means of the logical elements AND_1 - AND_5 and *OR* which stand before the *R*- and *S*-inputs of the triggers. The logical element AND_1 fulfills the "convolution" of the 1st digit to the 2nd digit. Its two inputs are connected with the direct output of trigger T_1 and the inverse output of trigger T_2. The 3rd input is connected with the synchronization input *C*. The logical element AND_1 analyzes the states Q_1 and Q_2 of the triggers T_1 and T_2. If Q_1=1 and Q_2=0, it means that the convolution condition is satisfied for the 1st and 2nd digits. The synchronization signal C=1 is the cause of the appearance of the logical 1 at the output of the element AND_1. The latter causes the switching of triggers T_1 and T_2. This results in the "convolution" [01→10].

The logical element AND_k of the *k*-th digit (k=2,3,4,5) fulfills the "convolution" of the (k-1)-th and *k*-th digits to the (k+1)-th digit. Its three inputs are connected with the direct outputs of triggers T_{k-1} and T_k and the inverse output of trigger T_{k+1}. The 4-th input is connected with the synchronization input *C*. The logical element AND_k analyzes the states Q_{k-1}, Q_k and Q_{k+1} of the triggers T_{k-1}, T_k, and T_{k+1}. If Q_{k-1}=1, Q_k=1, and Q_{k+1}=0, this means that the "convolution" condition is satisfied. The synchronization signal C=1 results in switching triggers T_{k-1}, T_k, and T_{k+1}. The "convolution" of the corresponding digits [011→100] is complete.

Note that all elements AND_1 - AND_5 are connected through the common element OR_c with the check output of the "convolution" register.

The device for the reduction of the Fibonacci code to the minimal form in Fig. 8.1 operates in the following manner. The input code information is sent to

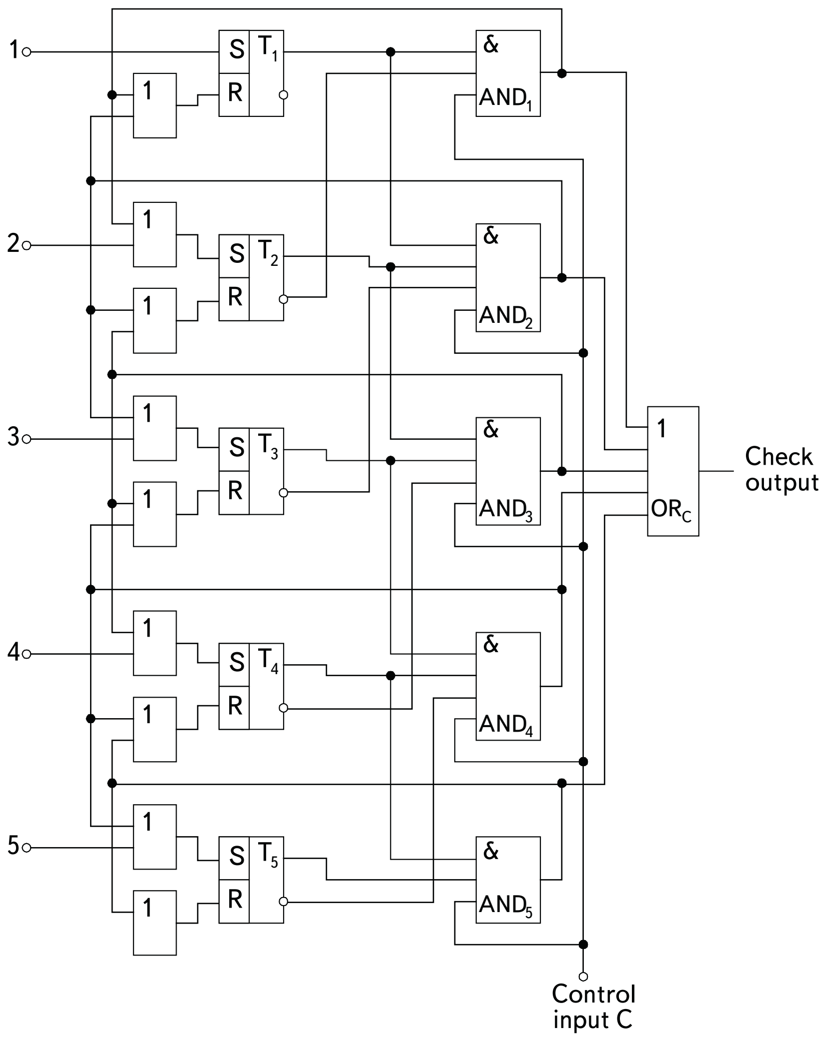


Figure 8.1. A device for reducing the Fibonacci code to minimal form

the information inputs 1-5 of the "convolution" register and enters the *S*-inputs of the triggers through the corresponding logical elements *OR* of the register. Let the initial condition of the convolution register be in the following state:

i	=	5	4	3	2	1
N	=	0	1	0	1	1

It is clear that the "convolution" condition is satisfied only for the 1st, 2nd and 3rd digits. The first synchronization signal C=1 results in the transition of the "convolution" register into the following state:

i	=	5	4	3	2	1
N	=	0	1	1	0	0

Here the "convolution" condition is satisfied for the 3rd, 4th and 5th digits. The next synchronization signal C=1 results in the transition of the "convolution" register into the following state:

i	=	5	4	3	2	1
N	=	1	0	0	0	0

The "convolution" is complete.

One may estimate the maximal delay time of the "convolution" operation for the n-digit "convolution" register. It is clear that we have to estimate the maximal time of the "convolution" delay for the following situation:

0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	0	0	1
1	0	1	0	1	0

We can see from this example that for the even n, the maximal number of the sequential "convolutions" is equal to $n/2$.

The analysis of the logical circuit in Fig. 8.1 shows that the delay time Δ of one "convolution" is defined by the sum of the (R-S)-trigger delay time Δ_T and the delay time Δ_e of the two sequential logical elements *AND*, *OR*, that is,

$$\Delta = \Delta_T + 2\Delta_e.$$

It follows from this consideration that the maximal "convolution" delay time for the n-digit "convolution" register is equal to:

$$\Delta_C = \frac{n}{2}(\Delta_T + 2\Delta_e).$$

8.8.2. *"Convolution" Register as a Self-checking Device*

The outputs of the logical elements AND_1-AND_5 of the "convolution" register in Fig. 8.1 are connected with the register checking output through the common element *OR*. This output plays an important role as the check output of the "convolution" register.

It follows from the functioning principle of the "convolution" register that the logical 1 appears on the check output in only two situations:

(1) The binary combination, written into the "convolution" register, is not in "minimal form." This means that the "convolution" condition is satisfied at

least for one triple of the adjacent triggers of the “convolution” register. This causes the appearance of the logical 1 at the output of the corresponding element *AND*. Hence, in this case the appearance of the logical 1 at the check output of the “convolution” register indicates the fact that the “convolution” process is not over. This means that we have the possibility of determining the termination of the “convolution” process by means of observing the check output of the “convolution” register.

(2) The appearance of the constant logical 1 at the check output is an indication of a fault in the “convolution” register. Hence, the “convolution” register is a natural self-checking device.

8.8.3. *Combinative Logical “Convolution” Circuits*

The further increase in speed of the “convolution” device is connected with its realization as a combinative logical circuit. The simplest solution is the application of the constant electronic memory (Fig. 8.2).

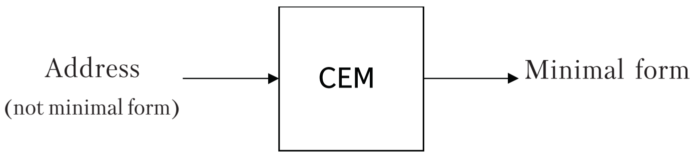


Figure 8.2. The constant electronic memory as the “convolution” circuit.

For this case the initial Fibonacci code representation is the address of the constant electronic memory(*CEM*). Its output is the minimal form of the initial code combination.

The other variant is the combinative logical convolution circuit that consists of $n/2$ identical logical circuits (Fig. 8.3), where n is the digit number of the initial code combination.

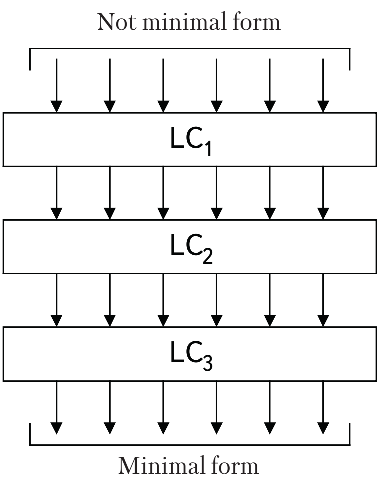


Figure 8.3. The combinative “convolution” circuit.

Let the initial code combination be the following:

i	=	6	5	4	3	2	1
N	=	0	1	0	1	1	1

We can see that the “convolution” condition (011) is satisfied for the 2nd, 3rd and 4th digits.

The logical circuit LC_1 transforms the initial code combination by the combinative process into the next code combination of the kind:

i	=	6	5	4	3	2	1
N	=	0	1	1	0	0	1

,

where the values of the 2nd, 3rd and 4th digits are replaced by their inverse values. We can see that the convolution conditions are satisfied for two triples of the digits, namely for the 6th, 5th and 4th digits and for the 2nd and 1st digits. The logical circuit LC_2 transforms the latter code combination into the next code combination through the combinative process

i	=	6	5	4	3	2	1
N	=	1	0	0	0	1	0

We can see that the obtained code combination is the minimal form, that is, the "convolution" process is complete. The logical circuit LC_3 passes this code combination onto the output without any change.

In the conclusion we consider the combinative logical circuit for checking the minimal form (Fig. 8.4). The logical circuit consists of n logical elements *AND*. Their outputs are connected with the inputs of the common element *OR*. If the initial code combination has two adjacent binary 1's or a binary 1 in the lower digit, the logical 1 appears at least at one input of the logical elements *AND*. It results in the appearance of the logical 1 at the output of the common element *OR* and this logical 1 is an indication of error.

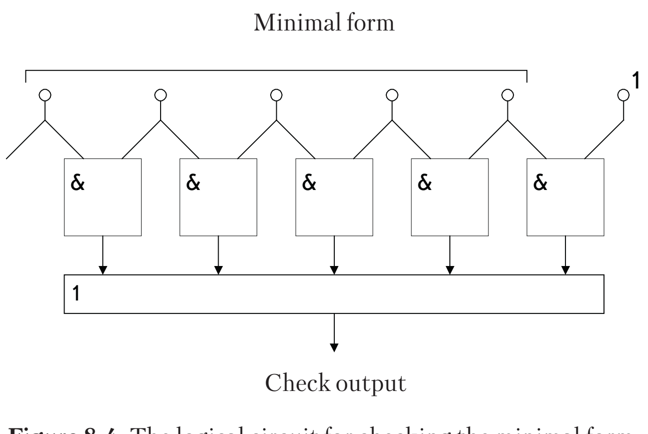


Figure 8.4. The logical circuit for checking the minimal form.

8.8.4. *"Devolution" devices*

One may synthesize the "devolution" devices similar to the above "convolution" device on Fig. 8.1. However, we can design a special "devolution" device based on the "convolution" device (Fig. 8.5). Remember that "devolution" is reduced to "convolution" in the corresponding inverse code. Figure 8.5 demonstrates the application of this idea.

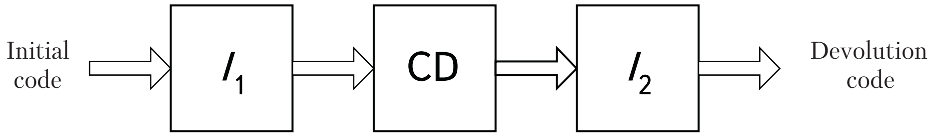


Figure 8.5. "Devolution" device based on the "convolution" device *CD*.

The "devolution" device consists of the convolution device *CD* and two inverters I_1 and I_2. Consider how we can fulfill the "devolution" of the code combination 101000. The input inverter I_1 transforms the initial code combination into the inverse code combination 010111. The "convolution" device *CD* transforms the latter into the minimal form 100010. The output inverter I_2 transforms the minimal form into the inverse code 011101. It is the "devolution" form of the initial code combination.

8.9. Fibonacci Processor for Noise-tolerant Computations

8.9.1. *Noise-tolerant Computations*

In modern science the need exists for fault-tolerant and noise-tolerant computers. What distinction is there between these two important directions of highly-reliable computers? As is well-known, a computer program is carried out with the help of a processor. The processor consists of flip-flops (triggers) that are connected with combinative logic. In this case the fulfillment of the program is reduced to the flip-flop switching. Unfortunately, it is impossible to eliminate computer errors caused by the malfunctioning of computer elements. Nevertheless, it is necessary to distinguish two types of malfunctioning of computer elements. The first type is the so-called *Constant Failure* of the elements, when the latter constantly "falls out of order". The second type is the so-called *Alternating Failure* of the elements when the element temporarily "falls outside the order," that is, in the accidental time moments, while at other times the computer elements work correctly. The second type of failure is called a *Malfunction*. Processor malfunctions appear under the influence of different internal and external noises in computer elements. Thus, the fault-tolerant computers are intended to eliminate the "constant failures" that may appear in processors and other computer parts during their work. The noise-tolerant computers are intended to eliminate the "malfunctions" that may appear in computer elements during their work.

It is clear that the problem of designing noise-tolerant computers and processors is absolutely vital to modern computer science. For example, many modern crypto-systems are based on computations in very large finite fields. The hardware realization of such computational tools or processors requires thousands of logical gates. It is very difficult and costly to design these kinds of processors giving error-free results. It means that the problem of designing processors for noise-tolerant computations is extremely important for reliable crypto-system design.

It has been experimentally demonstrated that the intensity of the "malfunctions" in computer elements in the switching regime is greater than the elements in the stable states. It follows from this consideration that the flip-flop (trigger) "malfunctions" in the switching regime is the most probable reason for unreliable computer processor functioning. This is why designing self-checking digital automatons, which can guarantee effective checking for trigger "malfunctions," is one of the most important issues surrounding noise-tolerant computers.

8.9.2. *Checking the Basic Micro-Operations*

The basic idea behind designing the self-checking Fibonacci processor consists of the following. It is necessary to choose a certain set of micro-operations called the *Basic Micro-operations* and to introduce an effective system for checking these micro-operations.

Let us demonstrate the possibility of realizing this idea on the basis of the "basic micro-operations" of *Convolution*, *Devolution, Replacement,* and *Absorption* as used in Fibonacci arithmetic.

We must pay attention to the following "technical" peculiarity of the aforementioned "basic micro-operations." For the register interpretation of these micro-operations, each micro-operation may be considered the inversion of triggers involved in the micro-operation. This means that each micro-operation is fulfilled by means of trigger switching.

Let us now evaluate the potential ability of the "basic micro-operations" to detect errors arising during micro-operation fulfillment.

It is well-known that the potential detection ability is determined by the ratio between the number of detectable errors and the general number of all possible errors [177]. Let us explain the essence of our approach to the detection of errors with an example of the micro-operation of "convolution":

$$[011\rightarrow100]. \tag{8.61}$$

The "convolution" can be fulfilled for the 3-digit binary code combination (8.61). It is clear that there are $2^3=8$ possible transitions, which can arise at the fulfillment of the micro-operation (8.61). Note that only one of them given by (8.61) is a *correct* transition. The code combinations

{011, 100} (8.62)

which make up the correct transition (8.61), are called the *Allowed Code Combinations* for the "convolution." The remaining code combinations that can appear during the "convolution" (8.61)

{000,011,010,101,110,111} (8.63)

are called *Prohibited Code Combinations*.

The idea behind error detection is the following. If during the fulfillment of the micro-operation (8.61) one of the "prohibited" code combinations (8.63) appears, it is an indication of error. Note that the transition

011→011, (8.64)

is an erroneous transition. However, the transition (8.64) can be interpreted as *Non-detectable Error* because code combination 011 is an allowed one.

Let us consider the different erroneous situations that can arise during the performance of the micro-operation (8.61):

011→{011,000,001,010,101,111}. (8.65)

Among them only the erroneous transition (8.64) cannot be detected, because the code combination 011 is an allowed code combination. All the other erroneous transitions of (8.65) are detectable. Study the erroneous transition (8.64) from an arithmetical point of view. It is clear that the essence of the erroneous transition (8.64) consists of the repetition of the preceding code combination 011. If we analyze this transition from arithmetical point of view, we may conclude that this transition neither destroys numerical information nor influences the outcome of arithmetical operations. Hence, the erroneous transition (8.64) does not belong to the class of numerical errors of "catastrophic character." This error only delays data processing. All other erroneous transitions of (8.65) destroy numerical information and therefore results in errors of a "catastrophic nature."

The main conclusion that follows from this examination is that the set of "catastrophic" code combinations from (8.65) coincides with the set of detectable code combinations (8.63). This means that all "catastrophic" transitions for "convolution" are detectable. It thus follows from this study that we can design a computer device for fulfillment of the "convolution" with the "absolute" (i.e. 100%) potential ability to detect all "catastrophic" transitions that may appear at the fulfillment of the "convolution."

We can draw an analogous conclusion for the other basic micro-operations. However, the fulfillment of any algorithm for data processing in the Fibonacci processor is fulfilled by the use of certain basic micro-operations on each computation step. As the checking method for each micro-operation has "absolute" detecting ability regarding "catastrophic" errors, it means that we can design the Fibonacci arithmetical self-checking processor. It has "absolute" detection ability regarding the "catastrophic" errors that can appear in the Fibonacci processor during trigger switching.

8.9.3. *The Hardware Realization of a Noise-tolerant Fibonacci Processor*

The noise-tolerant Fibonacci-processor is based on the principle of "cause - effect" [30]. The essence of this principle consists in the following. The initial information (the "cause") that is subjected to data processing can be transformed back to the "result" by means of the inverse micro-operation. After that, we transform the "result" (the "effect") into the initial information (the "cause") and then check that the "effect" fits with its "cause."

For example, at the fulfillment of the "convolution" for the code combination 011 (the "cause") we obtain a new code combination 100 (the "effect"), which is a necessary condition for the fulfillment of the "devolution" (the inverse micro-operation). This means that the correct fulfillment of the "convolution" results in the condition necessary for the "devolution." Analogously, the correct fulfillment of the "devolution" [100→011] results in the condition necessary for the "convolution." It follows from this observation that the micro-operations of "convolution" and "devolution" are mutually checkable micro-operations.

Now, let us consider other "basic micro-operations" from this point of view. The indication of the correct fulfillment of the "replacement" $\begin{bmatrix} 1 & & 0 \\ \downarrow & = & \\ 0 & & 1 \end{bmatrix}$ is the code combination $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The combinations $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are erroneous for the "replacement." The micro-operation of "absorption" $\begin{bmatrix} 1 & & 0 \\ \updownarrow & = & \\ 1 & & 0 \end{bmatrix}$ is correct if we obtain the code combination $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The code combinations $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are erroneous for the "absorption."

For the "register interpretation," the finding is that of correspondence between the "cause" and the "effect" carried out by means of the "checking" flip-flop. The "cause" sets up the "checking" flip-flop in the state 1, and then the correct fulfillment of the "inverse" micro-operation (the "effect" fits the "cause") switches the "checking" flip-flop at the state 0. If the "effect" does not fit the "cause" (i.e. the micro-operation has been carried out incorrectly), then the "checking" flip-flop remains in the state 1 which is an indication of the error.

If we analyze the "causes" and the "effects" for every basic micro-operation, we may conclude that every "effect" is the inversion of its "cause," that is, all micro-operations can be fulfilled by the inversion of digits involved to the micro-operation.

The block diagram of the Fibonacci device for the realization of the principle of "cause-effect" is shown in Fig. 8.6. The device in Fig. 8.6 consists of two registers, the information register and the checking register, which are connected with the help of the logical circuits of "cause" and "effect." The code information, which enters into the information register through the "Input," is analyzed by the logical circuit of "cause."

Suppose that we need to fulfill "convolution" for the code combination in the information register. Let some flip-flops T_{k-1}, T_k, T_{k+1} of the information register be in the state 011, that is, the condition for "convolution" is satisfied for this group of flip-flops. Then the logical "cause" circuit (the logical circuit for "convolution" for the considered example) results in recording the logical 1 into the corresponding flip-flop T_k of the checking register. The logical 1 results in the inversion of the flip-flops T_{k-1}, T_k, T_{k+1} of the information register by using the back connection, that is, their new states are 100. This means that the condition for "devolution" is satisfied for this group of flip-flops. Then, the logical "effect" circuit (the logical circuit for "devolution" for the above example) analyzes the states of the flip-flops T_{k-1}, T_k, T_{k+1} of the information register and switches the same flip-flop T_k of the checking register to the initial state 0. This switching of the flip-flop T_k of the checking register into the initial state 0

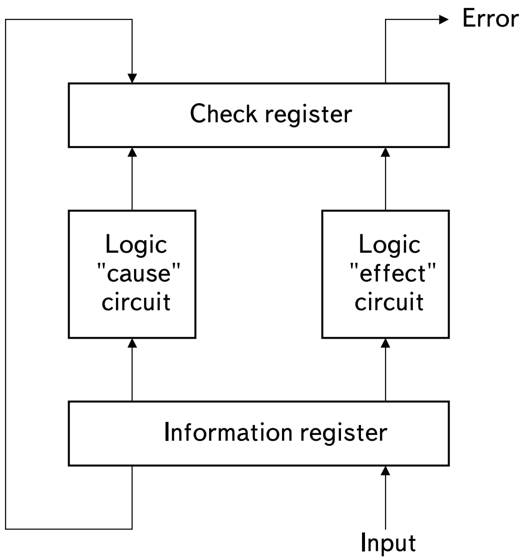


Figure 8.6. A block diagram of the Fibonacci device for realization of the principle of "cause-effect."

confirms that the "cause" (011) fits its "effect" (100), that is, the micro-operation of "convolution" is carried out correctly.

Hence, if we have the code combination 00…0 in the checking register after ending all micro-operations, it means that all "causes" fit their "effects," that is, all micro-operations are carried out correctly. If the checking register contains at least one logical 1 in some flip-flop, this means that at least one basic micro operation is carried out incorrectly. The logical signals 1 in the flip-flops of the checking register cause the error signal at the output "Error" of the device in Fig. 7.6.

The most important advantage of the principle of "cause-effect," which is fulfilled in the Fibonacci device in Fig. 8.6, is the possibility of detecting an error at the moment of its appearance. The correction of error is fulfilled by repetition of this micro-operation.

Hence, the above approach based on the principle of "cause-effect" allows one to detect and then to correct, by means of repetition, all errors that can arise at the moment of the flip-flop's switching over with 100% guaranteed success.

8.10. The Dramatic History of the Fibonacci Computer Project

8.10.1. *First Steps in the Development of Fibonacci Arithmetic*

Recall that it was the Dutch amateur of mathematics Eduardo Zeckendorf who invented in 1939 the Fibonacci representation (or Zeckendorf sum) given by (8.1). However, Zeckendorf did not try to develop "Fibonacci arithmetic," that is, the rules for the Fibonacci arithmetical operations. This did not interest him.

In 1972, the author of this book Alexey Stakhov, defended his DrSci dissertation [19], in which for the first time a theory of Fibonacci measurement algorithms and Fibonacci p-codes (8.9) were developed. After this, Stakhov began to develop Fibonacci arithmetic. The invention of the rule of Fibonacci summation given by Table 8.5 was the first step in the development of this arithmetic. Later the idea of Fibonacci subtraction, multiplication and division were also developed. These results were presented by Stakhov in the articles [87, 89] as new computer arithmetic – *Fibonacci Arithmetic*. Though the initial steps in the development of Fibonacci arithmetic were made personally by Alexey Stakhov [87, 89], the further development of this arithmetic was the result of a creative collective scientific effort. Many talented students of the Taganrog Radio Engineering Institute participated in this work. In 1977, in the Taganrog Radio Engineering Institute, the first Ph.D. dissertation in the field of Fibonacci computers, *Development of Principles of Construction and Research of Counting Devices in the Fibonacci p-codes*, was defended by Yuri Wishnjakov under Stakhov's scientific supervision.

8.10.2. *Stakhov's Lecture in Austria (1976)*

International recognition of the Fibonacci arithmetic and its offshoot of *Fibonacci Computers* began after Stakhov's lecture in Vienna on the joint meeting of the Austrian Computer and Cybernetic Societies in 1976. On the evening of March 3, 1976, the seminar room at the Austrian Cybernetic society (Vienna, Shottengasse, 3) was filled with people. Well-known Austrian scientists, members of the Austrian Cybernetic and Computer societies, scientific employees of the IBM computer laboratory in Vienna, and representatives of the Soviet Embassy in Austria gathered together in the seminar room. Stakhov's lecture, *Algorithmic Measurement Theory and Foundations of Computer Arithmetic*, was the main reason for this unusual meeting. However, there were other reasons, why the well-known scientists in the field of cybernetics and computer science and officers of the Soviet Embassy (basically KGB representatives) gathered together to listen to Alexey's lecture.

The advertisement of Stakhov's lecture proclaimed the following:

"Methods of representation of numbers can be considered to be special measurement algorithms. Such an interpretation is the main idea of the present lecture."

The basic scientific results are:

· *Asymmetry Principle of Measurement as a new scientific principle*

· *Algorithmic measurement theory*

· *The generalization of Fibonacci numbers*

· *Fibonacci arithmetic as a novel way to increase the informational reliability of computers*

Stakhov's main objective was to present a new measurement theory – *Algorithmic Measurement Theory* – and the new methods of positional representation of numbers, and new computer arithmetic – *Fibonacci Arithmetic*– that can be used for the development of *Fibonacci Computers,* as a new direction in computer science. These scientific results were the main focus of interest.

However, the interest of the Soviet Embassy in Stakhov's lecture had a political aspect. On March 5, 1976, two days after Stakhov's lecture, the "historical" 25th Congress of the CPSU should have been opened in Moscow. A couple of months prior to the Congress, the well-known West German journal, *Spiegel*, published a scathing article "*If Lenin had known this!*" The article was devoted to the facts of corruption in the highest levels of the CPSU that was in sharp contrast with the representation about the CPSU as "Intelligence, Honor and Mind of our Epoch." In connection with this publication the Central Committee of the CPSU sent a confidential memorandum to all Soviet embassies in the Western countries. In it the Central Committee recommended providing assistance to Soviet scientists, athletes, and actors in Western countries that could smooth over any negative impression the world community had as a result of the *Spiegel* article. Thus, Stakhov's lecture was one of the political measures of the Soviet Embassy in Austria before the 25th Congress of the CPSU.

The very positive reaction to Stakhov's lecture by the Austrian scientists, including Professor **Aigner,** Director of the Mathematics Institute of the Graz Technical University, Professor **Trappel**, President of the Austrian Cybernetic society, Professor **Eier,** Director of the Institute of Data Processing of the Vienna Technical University, and also Professor **Adam** the representative of the Faculty of Statistics and Computer Science of Johannes Kepler Linz University, caused the decision of the Soviet Embassy in Austria to assist in the development of Stakhov's scientific pursuits within the U.S.S.R. With this purpose the Soviet Ambassador in Austria **Ivan Efremov** sent a letter to the Soviet State Committee on Science and Engineering. In this letter the Ambassador positively evaluated Stakhov's 2-month scientific work in the Austrian Universities and offered to begin the needed foreign patenting with the intention of "protecting the priority of the Soviet science."

8.10.3. *Patenting the Soviet Fibonacci Computer Inventions*

The decision by the Soviet State Committee on Inventions and Patenting to patent Stakhov's inventions on Fibonacci computers in all leading computer producing countries, including U.S., Japan, Germany, England, France, Canada, and other countries became the major consequence of the Austrian Soviet Embassy letter. Later it was found that this was the U.S.S.R's first effort at extensively patenting Soviet computer inventions.

The protection of Soviet scientific priority in Fibonacci computers was the main purpose of this patenting. New computer arithmetic - *Fibonacci Arithmetic* - was its main object. However, according to the patent laws of the majority of the countries, it is impossible to procure a patent for mathematical inventions, in particular, Fibonacci arithmetic. Therefore, the idea arose to protect Fibonacci arithmetic by means of computer devices for the performance of arithmetical operations in Fibonacci code. The different arithmetical and other devices based on Fibonacci code such as counters, adders, devices for multiplication and division, analog-to-digit and digit-to-analog converters, and so on, became the subjects of patenting. However, from a financial point of view, it was desirable to invent an original device, which could be recognized as the "pioneer invention" in the Fibonacci computer field. On the basis of such a "pioneer invention," other computer devices could be developed. Out of this consideration the idea arose to create a multistage invention formula with the first term being the protection of the Fibonacci computer "pioneer invention."

What should the pioneer Fibonacci computer invention be? The analysis of the Fibonacci arithmetic indicates that the basic micro-operations of Fibonacci arithmetic are *Convolution, Devolution* and the *Reduction of the Fibonacci p-code to the Minimal Form* based upon "convolution" and "devolution." For this reason, the *Device for Reduction of the Fibonacci p-code to Minimal Form* became the main subject of patent protection. An example of this device is shown in Fig. 8.1.

The first patent application on the invention *Reduction Method of Fibonacci p-Code to the Minimal Form and Device for Its Realization* contained over 200 pages of text material, about 100 figures (operational devices and their elements), and its multistage invention formula consisted of 85 points. This meant that the application on invention, offered for patenting, contained 85 technical decisions, that is, 85 inventions.

The general outcome of the Fibonacci invention patenting surpassed all expectations. 65 foreign patents on various devices for the Fibonacci computer were given by the State Patent Offices of the U.S., Japan, England, France, Germany, Canada, Poland and GDR, including 12 patents on the invention of the *Reduction Method of Fibonacci p-Code to the Minimal Form and Device for Its Realization* [120-131]. These patents testify to the fact that the Fibonacci computer was a world class innovation, as the Western experts could not challenge the Soviet Fibonacci computer inventions. This means, as a result, the Fibonacci patents [120-131] are the official legal documents, which confirm Soviet priority in this computer direction.

The Vinnitsa Technical University, where Alexey Stakhov worked during 1977-1995 as Head of the Computer Engineering Department, became the main Soviet scientific and engineering center for engineering developments in the Fibonacci computer field. The projects of the Fibonacci computers and Fibonacci measurement and information systems were of great interest to Soviet military organizations, which gave about $15,000,000 for the realization of these projects. The most important engineering developments of this project included special micro-elements for the Fibonacci processors, the self-correcting highly stable 18-digit analog-to-digital converters, and so on. The brochure [30] is recommended to all readers interested in engineering developments in the Fibonacci computer field.

The cutbacks on scientific and engineering research became one of the negative results of the Gorbachev's so-called "perestroika," which eventually resulted in the breakdown of the Soviet Union. In 1989, the Soviet Ministry of General Mechanical Engineering (the Soviet Rocket Ministry), which financed the Fibonacci computer project, informed Professor Stakhov about the financial cutbacks to the Fibonacci computer project. Unfortunately, this fact practically halted all main research in the Fibonacci computer field. So, Gorbachev's "perestroika" led to a full stop in this dramatic history of the Soviet Fibonacci computer project.

8.10.4. *The Fibonacci Computer Developments in the U.S.*

Any expert, who is interested in the Fibonacci computer project, will ask the question: what Fibonacci computer research is done in other countries? Some publications of American scientists on the Fibonacci arithmetic and applications in the Fibonacci computer field are presented in [132-135]. One concludes from these publications that the concept of the Fibonacci computer is widely used in American computer science. In particular, work on Fibonacci computers was done at the University of Maryland under scientific supervision of Professor Newcomb, during roughly the same period that Professor Stakhov was supervising the Fibonacci computer research in the Taganrog Radio Engineering Institute (1971-1977) and at the Vinnitsa Technical University (1977-1995). It is important to emphasize that the first Ph.D. dissertation on Fibonacci computers [135] was defended by V.D. Hoang at the University of Maryland in 1979. This was two years after the Ph.D. dissertation was defended by Yuri Wishnjakov at the Taganrog Radio Engineering Institute [136].

8.10.5. *Fibonacci Digital Signal Processing*

It is important to note the recent applications of the Fibonacci codes to "Digital Signal Processing." In Russian science the idea of the use of Fibonacci p-numbers for the design of super-fast algorithms of digital processing were actively developed by Professor Vladimir Chernov, DrSci in Physics and Mathematics at Samara the Images Processing Institute of the Russian Academy of Science [178]. Also Fibonacci p-numbers for the development of super-fast algorithms of digital signal processing [179] are widely used by the research group from the Tampere International Center for Signal Processing (Finland). As is shown in the book [179], the super fast algorithms of digital signal processing requires a processing of numerical data represented in the Fibonacci p-codes. This means that for the realization of such super-fast transformations require the specialized Fibonacci signal processors! This is why the problem of Fibonacci processor development is of vital concern today!

8.11. Conclusion

1. A new and unique view of computer theory was developed in this chapter. We discussed *Fibonacci Computers* based on a new positional method of number representation - *Fibonacci p-codes*. The Fibonacci p-codes are a wide generalization of the classical binary code and use the Fibonacci p-numbers as the weights of binary digits. In contrast to the classical binary code, the Fibonacci p-codes have code redundancy, which shows this feature in the plural representation of one and the same number. This leads to the creation of a very special computer arithmetic, which allows one to detect errors appearing during arithmetical transformations. The main thrust of this Chapter was to show that computers can be designed on other positional number systems. Chapters 9 and 10 further develop this idea.

2. The first attempt to design computer and measurement systems based on Fibonacci representation was undertaken in the Soviet Union during the 1970s and 1980s. Soviet computer inventions were awarded 65 patents by the Patenting Offices of the U.S., Japan, England, Germany, France, Canada and several other countries. This appears to support the Soviet claim to priority in this important computer field. Scientific research and engineering developments [30] demonstrated the high efficiency of Fibonacci codes and Fibonacci arithmetic for designing self-correcting analog-to-digital and digital-to-analog converters and noise-tolerant processors. In addition, the Fibonacci p-codes provide a solid foundation for developments in the new super-fast transformations for digital signal processing.

Chapter 9

Codes of the Golden Proportion

9.1. Numeral Systems with Irrational Bases

9.1.1. *Bergman's Numeral System*

In this Section we discuss the mathematical work of American mathematician George Bergman. At present, **George Mark Bergman** is a Professor of Mathematics Department at the University of California (USA). He has authored many articles in the field of discrete mathematics and co-authored the books *An Invitation to General Algebra and Universal Constructions* (1998) and *Co-groups and Co-rings in Categories of Associative Rings* (1996).

In 1957 George Bergman published his first article *A number system with an irrational base* [86] in the authoritative journal *Mathematics Magazine*. The numeral system with irrational base developed by George Bergman in 1957 is possibly the most important mathematical discovery in the field of numeral systems after the discoveries of the positional principle of number representation (Babylon, c. 2000 B.C.) and the decimal system (India, 5th century B.C.). However, it is most surprising that George Bergman made his mathematical discovery at the age of 12!

The following sum is called *Bergman's Numeral System*:

$$A=\sum_i a_i\tau^i, \tag{9.1}$$

where A is any real number, a_i is a binary numeral {0,1} of the i-th digit, i=0,±1,±2,±3,..., τ^i is the weight of the i-th digit, and $\tau=\left(1+\sqrt{5}\right)/2$ is the base or radix of the numeral system (9.1).

On the face of it, there appears to be no distinction between the formula (9.1) for Bergman's system and the formulas for the canonic positional numeral system, for example, the binary system. However, it is only on the face of it.

The principal distinction of the numeral system (9.1) from the canonical positional numeral systems is the fact that the irrational number $\tau=\left(1+\sqrt{5}\right)/2$ (the golden mean) is used as the radix of the numeral system (9.1). That is why, Bergman called it the *Number System with an Irrational Base* or *Tau System*. Although Bergman's article [86] contained a result of fundamental importance for number theory, mathematicians and engineers of the period did not take notice. In the conclusion of his paper [86] George Bergman wrote: *"I do not know of any useful application for systems such as this, except as a mental exercise and pastime, though it may be of some service in algebraic number theory."*

9.1.2. *A Definition of the Golden p-Proportion Code*

However, progress in computer science rejected Bergman's pessimistic assessment of the practical application of his numeral system. During the 1970's and 1980's scientific and engineering developments based on Bergman's numeral system were developed in the former Soviet Union [24, 30, 93]. Researchers there showed the high efficiency of Bergman's system (9.1) for designing self-correcting analog-to-digit converters (ADC) and noise-tolerant processors. The theoretical substantiation of this research is given in this author's book [24] and the engineering elaborations are described in [30].

It follows from the Bashet-Mendeleev problem studied in Chapter 7 that the classical binary system has the following "measuring" interpretation. Consider the infinite set of the binary standard line segments:

$$\{2^i\}\ (i=0,1,\pm2,\pm3,...). \tag{9.2}$$

Using (9.2), we can represent every real number A in the form:

$$A=\sum_i a_i 2^i\ \left(i=0,\pm1,\pm2,\pm3,...\right), \tag{9.3}$$

where a_i is the binary numeral of the i-th digit; 2^i is the weight of the i-th digit.

The binary representation of the real number A in the form (9.3) can be generalized as follows. Consider the set of the following standard line segments:

$$\left\{\tau_p^i\right\},\left(i=0,\pm1,\pm2,\pm3,..\right), \tag{9.4}$$

where τ_p is the golden p-proportion, a real root of the characteristic equation (4.42). Let us remember that the powers of the golden p-proportion are connected by the remarkable identity (4.48).

Using (9.4), we obtain this positional representation of real number A:

$$A=\sum_i a_i\tau_p^i\ \left(i=0,\pm1,\pm2,\pm3,...\right), \tag{9.5}$$

where a_i is the binary numeral of the i-th digit; τ_p^i is the weight of the i-th digit; and τ_p is the radix of the numeral system (9.5).

Let us examine in greater detail the aforementioned method of the positional representation of real numbers given by (9.5). First of all, we note that the formula (9.5) gives a theoretically infinite number of binary representations of real numbers because every p=0,1,2,3,... generates its own method of binary representation in the form (9.5).

The radix of a numeral system is one of the fundamental notions of positional numeral systems. An analysis of the sum (9.5) shows that the golden p-proportion τ_p is the radix of the numeral system (9.5). That is why the representation of the real number A in the form (9.5) was called the *Code of the Golden p-Proportion* [24].

Note that, except for the case $p=0$ ($\tau_0=2$), all other golden p-proportions τ_p ($p>0$) are irrational numbers. It follows from this fact that the codes of the golden p-proportion given by the sum (9.5) are binary numeral systems with irrational radices for the case $p>0$.

Now, let us consider the particular cases of the golden p-proportion codes (9.5). For the case $p=0$ we have: $\tau_p=\tau_0=2$ and, therefore, the golden p-proportion code (9.5) becomes the classical binary system (9.3). For the case $p=1$, the golden p-proportion τ_p coincides with the classical golden mean $\tau_1=\tau=\left(1+\sqrt{5}\right)/2$ and the golden p-proportion code (9.5) becomes Bergman's system (9.1).

The abridged representation of the sums (9.1) and (9.5) has the following form:

$$A=a_n\,a_{n-1}\,\ldots\,a_1\,a_0,\,a_{-1}\,a_{-2}\,\ldots\,a_{-k}. \tag{9.6}$$

Here the comma divides the abridged representation (9.6) into two parts. The left-hand part corresponds to the binary digits with nonnegative indices; the right-hand part corresponds to the binary digits with negative indices.

9.1.3. *Representation of the Golden p-Proportion Powers*

One may prove that all real numbers, in particular, natural numbers can be represented in the form (9.5). All properties of such representation are determined by the fundamental identity (4.48) that connects the powers of the golden p-proportions.

Note that the identity (4.48) is the cause of the redundancy of the golden p-proportion codes (9.5) for the case $p>0$ in comparison to the binary system (9.3). The redundancy of the code (9.5) proves itself in the plurality of the code representation of numbers in the form (9.5). Similar to the situation with Fibonacci p-codes (8.9), the different code representations of the same real number A can be obtained from each other by means of the fulfillment of the operations of "convolution" and "devolution." These operations are based on the fundamental identity (4.48).

Consider some peculiarities of the representation of numbers in the form (9.5). The powers of the golden p-proportions are represented in the form (9.5) very simply. In particular, the radix of the numeral system (9.5) is represented traditionally, that is,

$$\tau_p=10. \tag{9.7}$$

Note that the expression (9.7) is a generalization of the representation of the radix 2 in the binary system (9.3), that is,

$$2=10. \tag{9.8}$$

Also note that for case $p>0$, the radix τ_p of the numeral system (9.5) is an irrational number. This means that the expression (9.7) represents the irrational number τ_p by the finite number of the binary numerals 1 and 0, which is impossible for traditional numeral systems.

The number $\tau_p^0=1$ has the following code representation in (9.5):

$$\tau_p^0=1.0. \tag{9.9}$$

The positive and negative powers of the golden p-proportion are represented as follows:

$$\begin{aligned} \tau_p^1 &= 10; & \tau_p^{-1} &= 0.1 \\ \tau_p^2 &= 100; & \tau_p^{-2} &= 0.01 \\ \tau_p^3 &= 1000; & \tau_p^{-3} &= 0.001. \end{aligned} \tag{9.10}$$

For the case $p>0$, every power of the golden p-proportion has an infinite number of code representations in the form (9.5). For example, we can find the representation of the radix τ_p for the case $p=1$. Using "devolution," we can represent the radix $\tau_p=\tau_1=\tau$ as follows:

$$\tau=\frac{1+\sqrt{5}}{2}=\begin{cases} 1\ 0.\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 1.\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 1.\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 1.\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0 \\ 0\ 1.\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0 \\ 0\ 1.\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1 \end{cases} \tag{9.11}$$

Let us consider the "golden" representations of the sums of the golden p-proportion powers. For example, the sum

$$A=\tau^4+\tau^3+\tau^0+\tau^{-1}+\tau^{-2}+\tau^{-5} \tag{9.12}$$

has the following binary representation:

$$A=11001.11001. \tag{9.13}$$

Using the Binet formula (2.60), we represent the number (9.12) as follows:

$$A=\frac{L_4+F_4\sqrt{5}}{2}+\frac{L_3+F_3\sqrt{5}}{2}+\frac{L_0+F_0\sqrt{5}}{2}+\frac{L_{-1}+F_{-1}\sqrt{5}}{2}+\frac{L_{-2}+F_{-2}\sqrt{5}}{2}+\frac{L_{-5}+F_{-5}\sqrt{5}}{2}. \quad (9.14)$$

Substituting the values of the Fibonacci and Lucas numbers from Table 2.7 L_4=7, L_3=4, L_0=2, L_{-1}=-1, L_{-2}=3, L_{-5}=11, F_4=3, F_3=2, F_0=0, F_{-1}=1, F_{-2}=-1, F_{-5}=5, into the expression (9.14), we obtain the number A in the explicit form:

$$A=\frac{4+10\sqrt{5}}{2}=2+5\sqrt{5}. \quad (9.15)$$

Note that all real numbers given by the expressions (9.7), (9.10), (9.12) and (9.15) are irrational numbers! However, according to (9.7), (9.10), and (9.13) they are all represented by finite combinations of binary numerals. This means that the Bergman system (9.1) and its generalization (9.5) allow us to represent some irrational numbers (in particular, the powers of the golden p-proportions and their sums) by finite combinations of binary numerals what is absolutely impossible for classical positional numeral systems! This is the first unusual property of the numeral systems (9.1) and (9.5) and their fundamental difference from the traditional positional numeral systems with integer radices (i.e. binary system, decimal system, and so on).

9.2. Some Mathematical Properties of the Golden p-Proportion Codes

9.2.1. *A Minimal Form of the Golden p-Proportion Code*

The codes of golden p-proportions (9.5) are connected closely with Fibonacci p-codes (8.9). This follows from the similarity of the mathematical relations (4.18) and (4.48) that connect the digit weights of the Fibonacci p-codes and the golden p-proportion codes. This allows one to use the same code transformations of "convolution" and "devolution" for the representations of numbers in the Fibonacci p-code (8.9) and the code of "golden" p-proportion (9.5).

We can prove the following theorem for the golden p-proportion codes.

Theorem 9.1. For a given $p\geq 0$, there is the unique representation of any real number A in the following form:

$$A=\tau_p^n+r, \quad (9.16)$$

where

$$0\leq r<\tau_p^{n-p}. \quad (9.17)$$

Proof. The sequence of the golden p-proportion powers is the monotonically increasing numerical sequence. That is why we can always find in this sequence the only pair of adjacent golden p-proportion powers τ_p^n and τ_p^{n+1} so that

$$\tau_p^n \le A < \tau_p^{n+1}. \tag{9.18}$$

Subtracting the number τ_p^n from all terms of the non-equality (9.18), we obtain:

$$0 \le r = A - \tau_p^n < \tau_p^{n+1} - \tau_p^n = \tau_p^{n-p}.$$

The theorem is proved.

Note that for the case $p=0$, Theorem 9.1 becomes the well-known representation of real number A in the following form:

$$A=2^n+r,$$

where $0 \le r < 2^n$.

By making the decomposition of any real number A and all remainders r in accordance with (9.17) and (9.18), we obtain the peculiar representation of number A in the golden p-proportion code (9.5). Let us examine the abridged representation of number A in the golden p-proportion code given by (9.6). It is clear that in accordance with (9.16) and (9.17) not less than p binary numerals 0 follow after every binary numeral 1 in the abridged representation (9.6) from left to right. Such a representation of number A is called the *Minimal Form of the Number A in the Golden p-Proportion Code*. According to Theorem 9.1, the representation of number A in the form of (9.16) and (9.17) is unique; the uniqueness of the minimal form of representation of number A in the golden p-proportion code (9.5) follows from this fact.

9.2.2. *Comparison of Numbers*

Similar to the Fibonacci p-codes the golden p-proportion codes are found within the class of positional numeral systems. As is well known, the simplicity of comparing numbers represented in positional numeral systems is one of their fundamental advantages over non-positional numeral systems. Let us now prove that this important advantage is maintained for the codes of the golden p-proportion.

A very simple rule for the comparison of the values of two "golden" numbers A and B follows from (9.16) and (9.17), if beforehand the numbers are represented in their minimal form. Let's examine the set of all n-digit minimal forms of the golden p-proportion code (9.5) of the kind

$$A(n)=a_n a_{n-1} \ldots a_1 a_0, a_{-1} a_{-2} \ldots a_{-m}, \tag{9.19}$$

where $a_n, a_{n-1},\ldots, a_{-m}$ are binary numerals $\{0,1\}$ and a_n is the higher digit of the code combination (9.19).

Compare two n-digit minimal forms of the kind (9.19) with a_n=1 and a_n=0, that is,

$$A(n;1)=1\ a_{n-1}\ ...\ a_1\ a_0,\ a_{-1}\ a_{-2}\ ...\ a_{-m} \tag{9.20}$$

and

$$A(n;0)=0\ a_{n-1}\ ...\ a_1\ a_0,\ a_{-1}\ a_{-2}\ ...\ a_{-m}. \tag{9.21}$$

Note that the representation $A(n;1)$ is the minimal n-digit form (9.19) with a_n=1 and $A(n;0)$ being the minimal n-digit form (9.19) with a_n=0. Further note that binary numerals $a_{n-1}, a_{n-2},..., a_{-m}$ in (9.20) and (9.21) do not coincide in the general case.

Every minimal form (9.20) and (9.21) represents some real numbers of the kind $A(n;1)$ and $A(n;0)$ given by the sum (9.5). Let us prove the following non-equality:

$$A(n;1)>A(n;0), \tag{9.22}$$

which is valid for the general case.

It is important to note that the non-equality (9.22) depends only on the values of the higher digits a_n of the comparable minimal forms (9.20) and (9.21).

The non-equality (9.22) is valid in the general case if we can prove that

$$A_{\min}(n;1)>A_{\max}(n;0), \tag{9.23}$$

where $A_{\min}(n;1)$ is a minimal number among the numbers of the kind (9.20) and $A_{\max}(n;0)$ is a maximal number among the numbers of the kind (9.21).

Let us examine the numbers $A_{\min}(n;1)$ and $A_{\max}(n;0)$. It is clear that a number of the kind (9.20) takes its minimal value $A_{\min}(n;1)$, when all digits $a_{n-1}, a_{n-2},..., a_{-m}$ in (9.21) are equal to 0, that is,

$$A_{\min}(n;1)=100...0,000...=\tau_p^n. \tag{9.24}$$

Now, let us find the value of the number $A_{\max}(n;0)$. According to Theorem 9.1, the number $A_{\max}(n;0)$ can be represented in the form:

$$A_{\max}(n;0)=\tau_p^{n-1}+r_{\max}, \tag{9.25}$$

where

$$r_{\max}<\tau_p^{n-p-1}. \tag{9.26}$$

As $\tau_p^n=\tau_p^{n-1}+\tau_p^{n-p-1}$, it then follows from the comparison of the expressions (9.24), (9.25), and (9.26) that the non-equality (9.23) is valid and, therefore, the non-equality (9.22) is valid for general case.

A very simple rule for the comparison of numbers A and B represented in the golden p-proportion code (9.5) follows from (9.22). Before the comparison we must represent the compared codes A and B in the minimal form. Comparison of the minimal forms of numbers A and B begins with the highest digits and continues until obtaining the first pair of non-coincident digits 0 and 1. The number, which contains the binary numeral 1 in the first pair of the non-coincident digits, is greater. If all the compared digits coincide, the numbers A and B are equal.

It is important to emphasize that the simplicity of number comparison is one of the important advantages of the golden p-proportion codes.

9.2.3. *Representation of Numbers with a Floating Comma*

Next let us compare the golden p-proportion codes with the Fibonacci p-codes to show the differences:

(1) The Fibonacci p-codes are intended only for the representation of integers and require fewer digits (for number representation) than do the golden p-proportion codes. For example, the decimal number 10 is represented in the golden 1-proportion code by the 9-digit binary combination:
10=10100.0101.

However, we need only 6 binary digits for the representation of the same number 10 in the Fibonacci 1-code:
10=100100.

Thus, here we can give a preference to the Fibonacci p-codes for code representation of integers.

(2) Another difference between Fibonacci p-codes and golden p-proportion codes is connected with digit weights. The digit weights of the golden p-proportion codes (9.5) build up the geometric progression (9.4). This fact raises the possibility of shifting the golden p-proportion code combination to the left or to the right. This corresponds to multiplication and division of the initial number by the radix τ_p (the golden p-proportion). The "shift-property" of the golden p-proportion codes raises the possibility of representing numbers with the floating comma. In fact, the decimal number 10=10100.0101 can be represented with the floating comma as follows:

$$10=0.101000101\times\tau^5, \qquad (9.27)$$

where τ is the radix of the number system (the golden mean).

The representation (9.27) consists of two parts. The first part is the *Mantissa* of the number 10

$$m(10)= 0.101000101$$

and the second part is the golden mean power τ^5. The value 5 is called the *Exponent* of the number 10.

9.3. Conversion of Numbers from Traditional Numeral Systems into the Golden *p*-Proportion Codes

9.3.1. *The Table Method*

There are two general methods of number conversion from one numeral system to another. One method is called the *Table Method* and the other method is called the *Analytic Method*. The table method is based on the preliminary design of a special table for golden p-proportion codes. This method can be performed with a computer by means of a special constant electronic memory. In this case the golden p-proportion code of the number N is kept in memory at the address that is the classical binary representation of the number N. For the case p=1 such a table has the following form (Table 9.1).

Table 9.1. A table of the golden mean code

Address N	τ^4	τ^3	τ^2	τ^1	τ^0	τ^{-1}	τ^{-2}	τ^{-3}	τ^{-4}
0 = 0000	0	0	0	0	0.	0	0	0	0
1 = 0001	0	0	0	0	1.	0	0	0	0
2 = 0010	0	0	0	1	0.	0	1	0	0
3 = 0011	0	0	1	0	0.	0	1	0	0
4 = 0100	0	0	1	0	1.	0	1	0	0
5 = 0101	0	1	0	0	0.	1	0	0	1
6 = 0110	0	1	0	1	0.	0	0	0	1
7 = 0111	1	0	0	0	0.	0	0	0	1
8 = 1000	1	0	0	0	1.	0	0	0	1
9 = 1001	1	0	0	1	0.	0	1	0	1
10 = 1010	1	0	1	0	0.	0	1	0	1

9.3.2. *The Conversion of Fractional Numbers*

Now let us consider the analytic method of the "golden" number conversion. This method is widely used in classical numeral systems. Its essence consists of the fulfillment of any arithmetical operation in the initial numeral system for obtaining numerals of the unknown code.

Consider the case of the conversion of a given fractional number. Suppose that the representation of the fractional number A in the golden mean code (p=1) is as follows:

$$A = a_{-1}\tau^{-1} + a_{-2}\tau^{-2} + \ldots + a_{-n}\tau^{-n} = 0.a_{-1}a_{-2}\ldots a_{-n} \tag{9.28}$$

Suppose that the fractional number A is represented in the minimal form. Then, by the multiplication of the fractional number (9.28) by the radix τ, we obtain the following result:

$$A \times \tau = a_{-1} + a_{-2}\tau^{-1} + \ldots + a_{-n}\tau^{-n+1} = a_{-1}.a_{-2}\ldots a_{-n} \tag{9.29}$$

where a_{-1} is the integral part of the product $A\times\tau$ and the sum

$$A_1 = a_{-2}\tau^{-1} + \ldots + a_{-n}\tau^{-n+1} = 0.a_{-2}\ldots a_{-n} \tag{9.30}$$

is the fractional part of the product $A\times\tau$.

Thus, it follows from this examination that after the first multiplication of the initial fraction (9.28) by the radix τ, the integral part of the product $A\times\tau$ is the binary numeral a_{-1} of the golden mean code of the fractional number (9.29).

By multiplying the fractional number (9.30) by the radix τ, we obtain the following result:

$$A_1\times\tau = a_{-2} + a_{-3}\tau^{-1} + \ldots + a_{-n}\tau^{-n+2} = a_{-2}.a_{-3}\ldots a_{-n}\,. \tag{9.31}$$

The analysis of (9.31) shows that the second multiplication results in the binary numeral a_{-2} of the golden mean code of the fractional number (9.28).

By continuing the multiplication process n times, we obtain the representation of fractional number A in the golden mean code.

Example 9.1. Convert the decimal fraction 1/2 into the golden mean code (Bergman's system).

Solution:

9.3.2.1. *The first multiplication:*

$$\frac{1}{2}\times\tau = \frac{1}{2}\times\frac{1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{4} = 0.809. \tag{9.32}$$

As the integral part of the fractional number (9.32) is equal to 0, it follows from this examination that the first "golden" binary numeral of the decimal fraction 1/2 is equal to a_{-1}=0.

9.3.2.2. *The second multiplication:*

$$\left(\frac{1}{2}\times\tau\right)\times\tau = \frac{1}{2}\times\tau^2 = \frac{1}{2}\times\frac{3+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{4} = 1.309. \tag{9.33}$$

As the integral part of the resulting product (9.33) is equal to 1, this means that a_{-2}=1. It follows from this result that prior to the third multiplication, it is necessary to subtract the number 1 from the number (9.33):

$$\frac{3+\sqrt{5}}{4} - 1 = \frac{\sqrt{5}-1}{4} = \frac{1}{2}\times\tau^{-1}.$$

9.3.2.3. *The third multiplication:*

$$\left(\frac{1}{2}\times\tau^{-1}\right)\times\tau = \frac{1}{2} = 0.5.$$

We have obtained the fractional number 1/2 as the result of the third multiplication. This means that the "golden" binary numeral a_{-3} of the decimal fraction 1/2 is equal to a_{-3}=0.

After the third multiplication we have arrived at the initial fractional number 1/2. It follows from this fact that the further multiplications result in the repetition of the obtained binary numerals, namely, $a_{-4}=a_{-1}=0$; $a_{-5}=a_{-2}=1$; $a_{-6}=a_{-3}=0$ and so forth. Hence, the decimal fraction 1/2 is represented in the golden mean code as a periodic fraction:

$$\frac{1}{2}=0.010010010....$$

9.3.3. *A Conversion of Integers*

The analytic method of the conversion of integers into the golden p-proportion code is based on the relations (9.16) and (9.17). It follows from (9.16) and (9.17) that the process of the integer N conversion into the golden p-proportion code becomes a sequential comparison of the initial number N and the remainders r with the powers of the golden p-proportion.

Example 9.2. Convert integer number 4 into the golden mean code (p=1).

Solution:

Using the Binet formula (2.60), we can obtain analytical expressions for the powers of the golden mean (see Table 9.2).

Table 9.2. The powers of the golden mean

n	3	2	1	0	−1	−2	−3
τ^n	$\frac{4+2\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\tau^0=1$	$\frac{-1+\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$	$\frac{-4+2\sqrt{5}}{2}$
D.E.	4.236	2.618	1.618	1	0.618	0.382	0.236

Here D.E. stands for a "decimal equivalent."

9.3.3.1. *The first step of conversion*

Comparing the integer 4 with the golden mean powers in Table 9.2, we can find the pair of the powers $\tau^3=\left(4+2\sqrt{5}\right)/2=4.236$ and $\tau^2=\left(3+\sqrt{5}\right)/2=2.618$ that are connected with the number 4 by the following non-equality:

$$\tau^2=\left(3+\sqrt{5}\right)/2=2.618\le 4<\left(4+2\sqrt{5}\right)/2=4.236. \tag{9.34}$$

It follows from (9.34) that the binary numeral of the second digit of the "golden" representation of number 4 is equal to a_2=1.

9.3.3.2. *The second step of conversion*

Represent the number 4 as follows:

$$4=\left[\left(3+\sqrt{5}\right)/2\right]+r_1. \tag{9.35}$$

We can calculate the reminder r_1 as follows:

$$r_1 = 4 - \left[\left(3+\sqrt{5}\right)/2\right] = \left(5-\sqrt{5}\right)/2 = 1.382. \qquad (9.36)$$

9.3.3.3. *The third step of conversion*

Comparing the difference (9.36) with the golden ratio powers in Table 9.2, we can find the next pair of the golden ratio powers, $\tau^1 = \left(1+\sqrt{5}\right)/2$ = 1.618 and τ^0=1 that are connected with the reminder r_1=1.382 by the following non-equality:

$$\tau^0 = 1 \le 1.382 < \left(1+\sqrt{5}\right)/2 = 1.618. \qquad (9.37)$$

It follows from (9.37) that the binary numeral of the 0-th digit of the "golden" representation of the number 4 is equal to a_0=1.

9.3.3.4. *The fourth step of conversion*

Represent the remainder r_1 as follows:

$$r_1 = \left(5-\sqrt{5}\right)/2 = 1 + r_2, \qquad (9.38)$$

where the second reminder r_2 is equal:

$$r_2 = r_1 - 1 = \left[\left(5-\sqrt{5}\right)/2\right] - 1 = \left(3-\sqrt{5}\right)/2 = 0.382. \qquad (9.39)$$

9.3.3.5. *The fifth step of conversion*

Comparing the remainder (9.39) with the golden mean powers of Table 9.2, we can find the next pair of the golden mean powers, $\tau^{-1} = \left(-1+\sqrt{5}\right)/2 = 0.618$ and $\tau^{-2} = \left(3-\sqrt{5}\right)/2 = 0.382$ that are connected with the remainder r_2=0.382 by the following non-equality:

$$\left(3-\sqrt{5}\right)/2 = 0.382 \le r_2 = 0.382 < \left(-1+\sqrt{5}\right)/2 = 0.618. \qquad (9.40)$$

It follows from (9.40) that the binary numeral a_{-2} of the "golden" representation of number 4 is equal to a_{-2}=1.

9.3.3.6. *The sixth step of conversion*

Represent the remainder $r_2 = \left(3-\sqrt{5}\right)/2 = 0.382$ as follows:

$$r_2 = \left(3-\sqrt{5}\right)/2 = \left[\left(3-\sqrt{5}\right)/2\right] + r_3, \qquad (9.41)$$

where

$$r_3 = r_2 - \left(3-\sqrt{5}\right)/2 = 0. \qquad (9.42)$$

As the remainder r_3=0, this means that the conversion process is over and the conversion result is the following:

$$4=101.01. \quad (9.43)$$

Note that the above numerical examples 9.1 and 9.2 are the basis for the development of computer algorithm of number conversion into the golden mean code.

It is important to emphasize that the algorithms of number conversion are similar to the algorithms of number conversion of the classical binary system.

9.4. Golden Arithmetic

9.4.1. *Golden Summation and Subtraction*

Recall the fundamental identity that connects the digit weights of the golden p-proportion codes (9.5):

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}. \quad (9.44)$$

Now let us compare the identity (9.44) with the recursive relation for the Fibonacci p-numbers:

$$F_p(n)=F_p(n\text{-}1)+F_p(n\text{-}p\text{-}1). \quad (9.45)$$

We can see from this comparison that for the given p the digit weights of the golden p-proportion code (9.5) and the Fibonacci p-code (8.9) are subordinated to the same mathematical regularity. This means that the above micro-operations "convolution" and "devolution" based on (9.44) can be used in the golden p-proportion code (9.5) for the case $p>0$. These operations are the basic micro-operations underlying Fibonacci summation and subtraction, this means that the rules of "golden" summation and subtraction coincide with the rules of Fibonacci summation and subtraction that are reduced to "basic micro-operations" (see Section 8.6).

Example 9.3. Summarize the numbers 5+4 in the golden mean code (p=1).

Solution:

(1) Represent the numbers 5 and 4 in the golden mean code (see Table 9.1):

5=1000.1001 and 4=101.0100.

(2) Carry out the micro-operation "replacement" over the "golden" representations of numbers 5 and 4:

$$\begin{array}{rl} 5= & 1000.1001 \\ + & \downarrow \quad\ \downarrow\ \downarrow \\ 4= & 0101.0100 \\ 5+4= & 1101.1101 \end{array}$$

(3) Reduce the addition result 5+4 to its minimal form:

$$5+4=1101.1101=10010.0101. \qquad (9.46)$$

Example 9.4. Subtract number 11 from number 3 in the golden mean code (p=1).

Solution:

(1) Represent numbers 3 and 11 in the golden mean code (see Table 9.1):

3=100.01 and 11=10101.0101.

(2) Carry out the micro-operation "absorption":

$$\begin{array}{rl} 3= & 00100.0100 \\ - & \quad\ \ \updownarrow \quad \updownarrow \\ 11= & 10101.0101 \\ \hline R= & 10001.0001 \end{array}$$

That the subtraction result R=3-11 is in the bottom register, means that the result R has the sign "minus," that is,

3 -11= -10001.0001.

9.4.2. *Golden Multiplication*

"Golden" multiplication is based on the following trivial identity of the golden p-proportion powers:

$$\tau_p^n \times \tau_p^m = \tau_p^{n+m}. \qquad (9.47)$$

The following table of "golden" multiplication that is true for all golden p-proportion codes (9.5) follows from (9.47).

Table 9.3. Golden multiplication

0×0	=	0
0×1	=	0
1×0	=	0
1×1	=	1

We can see that the given table coincides with the multiplication table for classical binary arithmetic. This means that "golden" multiplication is reduced to classical binary multiplication, that is, to the following rules:

(1) Make up the partial products in accordance with Table 9.3.

(2) Summarize the partial products in accordance with the rule for "golden" addition.

Example 9.5. Multiply the fractions A=0.010010 and B=0.001010 in the golden mean code (p=1).

Solution:

1. Represent the "golden" fractional numbers A=0.010010 and B=0.001010 in the form with the floating comma:

$A=010010\times\tau^{-6}$; $B=001010\times\tau^{-6}$.

This means that the mantissas and exponents of the numbers A and B are equal, respectively:

$m(A)$= 010010; $e(A)$=-6 and $m(B)$= 001010; $e(B)$=-6.

2. Multiply the mantissas:

$$
\begin{array}{ccccccccccc}
 & & & & & 0 & 1 & 0 & 0 & 1 & 0 \\
 & & & & & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 1 & 0 & 0 & 1 & 0 & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & & \\
 & & 0 & 1 & 0 & 0 & 1 & 0 & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
\hline
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}
$$

3. Reduce the product to the minimal form:

00010110100=00100000100.

4. Summarize the exponents:

$e(A)+e(b)=(-6)+(-6)=-12$.

Then we can represent the product $A\times B$ in the form with the floating comma:

$A\times B=00100000100\times\tau^{-12}$.

9.4.3. *Golden Division*

In order to formulate the rules of the "golden" division, we can use an analogy between classical binary arithmetic and "golden" arithmetic. As is well known, the classical binary division consists of the shift of the divisor and comparison of the shifted divisor with the dividend or with the partial remainder. If the dividend or the intermediate remainder is greater than the shifted divisor, then the binary numeral 1 is written down corresponding to the digit of the quotient, whereas in the opposite case, the binary numeral 0 is written down here. For the former case the shifted divisor is subtracted from the dividend or the intermediate remainder. One may show that these operations also underlie “golden” division. As the “golden” comparison of numbers is fulfilled with the numbers, represented in minimal form, it follows from this examination that the dividend and all intermediate remainders should be reduced to minimal form at every step of “golden” division. Let us demonstrate “golden” division with the following example.

Example 9.6. Divide the number 5=1000.1001 (the dividend) by the number 10=10100.0101 (the divisor) in the golden mean code (p=1).

Solution:

We can represent the numbers 5 and 10 in the form with the floating comma as follows:

$$m(5)=10001001; e(5)=4 \tag{9.48}$$

$$m(10)=101000101; e(10)=4. \tag{9.49}$$

As their exponents are equal, that is, $e(5)= e(10)=4$, we can write:

$5:10=m(5):m(10)$,

that is, the division of the initial numbers 5:10 is reduced to the division of their mantissas $m(5):m(10)$.

Let us consider the division $m(5):m(10)$:

1. If we compare $m(5)$ with $m(10)$, we find that $m(5)<m(10)$. This means that the result of the division is a proper fraction, that is, the binary numeral of the 0-th digit of the quotient Q is equal to $a_0=0$.

2. If we shift $m(5)=10001001$ by one digit to the left, we obtain:

$$A_1=100010010. \tag{9.50}$$

3. If we compare the number (9.50) with the mantissa $m(10)$ given in (9.49), we find that $A_1<m(10)$. This means that the binary numeral of the next digit of the quotient Q is equal to

$a_{-1}= 0$.

4. By shifting the number (9.50) by one digit to the left, we obtain:

$$A_2=1000100100. \tag{9.51}$$

5. If we compare the number (9.51) with the mantissa $m(10)$, we find that $A_2\geq m(10)$. This means that the next binary numeral of the quotient Q is equal to $a_{-2}=1$.

6. Since we obtained the first significant digit of the quotient equal to $a_{-2}=1$, we should subtract the mantissa $m(10)$ from the number A_2. We find the next intermediate result as the outcome

$A_3=A_2-m(10)=1000100.1.$ (9.52)

A peculiarity of the number (9.52) is the fact that it has the numeral 1 in its fractional part.

7. By shifting the number of (9.52) by one digit to the left, we obtain a new intermediate result:

$A_4=10001001.$ (9.53)

Note that the number (9.53) is equal to the mantissa $m(10)$, meaning that the division process will be repeated. Hence, the next binary numerals of the quotient are equal to $a_{-3}=0$, $a_{-4}=0$, $a_{-5}=1$, $a_{-6}=0$, and so on.

It follows from this examination that the quotient has the following "golden" binary representation:

$Q=0.010010010...,$

that is, for this case the quotient $Q=5:10=1:2$ is represented in the golden mean code as a periodic fraction.

If we compare "golden" arithmetic with Fibonacci arithmetic, and with classical binary arithmetic, we discover two unique properties of "golden" arithmetic:

1. The rules of "golden" summation and subtraction coincide with the corresponding rules for Fibonacci arithmetic.
2. Similar to the classical binary code, the golden p-proportion code (9.5) possesses an important arithmetical property to represent numbers with a floating comma.
3. The rules of "golden" multiplication and division coincide with analogous rules for classical binary arithmetic.

Thus, "golden" arithmetic is a unique synthesis of Fibonacci and classical binary arithmetic.

9.5. A New Approach to the Geometric Definition of a Number

9.5.1. *The Geometric Definition of Number*

It is well known, that *Number* is the most important notion of mathematics and *Number Theory* is one of the ancient mathematical theories called *Tsarina of Mathematics*. However, we can ask: what is a number? On the face of it, it seems that mathematicians have a common answer to this question. But all is not quite so simple. There are various definitions of number. *Euclid's Definition* of natural numbers is the most simple of them.

Euclid considered all numbers as geometric line segments (geometric algebra) and such an approach resulted in the following definition of natural number. Suppose that we have an infinite number of "standard line segments" of length 1. Euclid named them *Monads* but did not consider the *Monad* a number. It was simply the "beginning of all numbers." It is clear that for the construction of natural numbers we need to have the infinite set S of the *Monads,* that is,

$$S=\{1,1,1,...\}. \quad (9.54)$$

Then we can define a natural number N as some geometric line segment that can be represented as the sum of the *Monads* taken from (9.54), that is,

$$N=\underbrace{1+1+1+...+1}_{N}. \quad (9.55)$$

Despite the utmost simplicity of the *Euclidean Definition* (9.55), it nevertheless played a major role in mathematics, particularly in number theory. This definition underlies many important mathematical concepts, for example, concepts of *Prime* and *Composite* numbers, and also of the concept of *Divisibility,* one of the main concepts of number theory.

But there are also other definitions of number. According to the "constructive" approach [167], the real number A is some mathematical object that can be represented by using the binary system in the form (9.3). The representation of the real number A in the form (9.3) has the following geometric interpretation. Let us consider an infinite set of "binary" line segments of the length 2^n, that is,

$$B=\{2^n\}\ (n=0,\pm 1,\pm 2,\pm 3,...). \quad (9.56)$$

Then, all real numbers can be represented in the form (9.3), that is, in the form of the sum that is formed from the "binary" line segments taken from (9.56).

Note that the number of the terms in the sum (9.3) is always finite but potentially unlimited, that is, the definition (9.3) is a brilliant example of the potential infinity concept used in the "constructive" mathematics [167].

Clearly, the definition (9.3) determines on the numerical axis only a part of real numbers that can be represented exactly by the sum (9.3). We name such numbers *Constructive Real Numbers*. All other real numbers that cannot be represented exactly by the sum (9.3) are named *Non-constructive Real Numbers*.

We can ask: what numbers can be classified as non-constructive numbers by the definition (9.3)? Clearly, all irrational numbers, in particular, the main mathematical constants π, e, the number $\sqrt{2}$, and the golden mean are classified as non-constructive numbers. But within the definition (9.3) some "rational" numbers (for example, 2/3, 3/7, etc.) that cannot be represented by the finite sum (9.3) are also classified as non-constructive real numbers. Note that this example shows a distinction between the two approaches to real numbers. The first approach - *Classical Approach* - supposes that all real numbers are separated into two nonoverlapping sets: *Rational* and *Irrational* numbers; the second approach - *Constructive Approach* - supposes that all real numbers are separated into two nonoverlapping sets: *Constructive* and *Non-constructive* numbers. Regarding this, the set of constructive numbers does not coincide with the set of rational numbers and the set of the non-constructive numbers does not coincide with the set of irrational numbers. We should also note that the set of constructive numbers depends directly upon the use of the positional method of number representation.

Note that the definition (9.3) considerably limits the set of real numbers. However, this fact does not detract from the "practical" significance of its computational point of view. It is easy to prove that any non-constructive real number can be represented approximately in the form (9.3); where the approximation error Δ is decreasing as we are increasing the number of the terms in (9.3). However, $\Delta>0$ for all non-constructive real numbers. Note that in modern computers we use only constructive numbers given by (9.3). However, we do not have any problem with the non-constructive numbers because they can be represented in the form (9.3) with an approximation error that potentially aims for 0. This means that the concept of non-constructive numbers has only theoretical interest.

9.5.2. *Newton's Definition of Real Numbers*

Over the centuries, mathematicians developed and defined with greater precision the concept of a number. In the 17th century, in the period of the origin of new science, in particular, new mathematics, several new methods of the study of "continuous" processes were developed; in particular, the concept of a real number stands out in the foreground. Most clearly, Isaac Newton, one of the founders of mathematical analysis, gives a new definition of this concept in his *Arithmetica Universalis* (1707):

"We understand a number not as a set of units, but as the abstract ratio of one magnitude to another magnitude of the same kind taken for that unit."

This formulation gives us a general definition of real numbers, rational and irrational. If we consider now the *Euclidean Definition* (9.50) from the point of view of *Newton's Definition*, we can see that in (9.50) the monad here plays the role of a unit. In the binary system (9.3) the number 2, the radix of the binary system, plays the role of a unit.

9.5.3. *Numeral Systems with Irrational Radices as a New Definition of Real Numbers*

We have developed the so-called "constructive" approach to the definition of real number based on the binary system (9.3). This idea allows us to give the following generalized definition of real number. We can develop Newton's definition of real numbers on the basis of numeral systems with irrational radices (9.1) and (9.5). In fact, these systems with irrational radices are fundamentally new numeral systems that are of major theoretical importance for modern mathematics. They revolve around our ideas about real numbers, in general. Historically, natural numbers were the first class of real numbers; the irrational numbers were introduced into mathematics much later, after the discovery of "incommensurable segments." In the traditional numeral systems (Babylonian sexagecimal, decimal, binary, and so on), the natural numbers 60, 10, and 2 are used as the "beginning of numbers." All real numbers can be represented by using the bases 60, 10 or 2. In the systems with irrational bases, some irrational numbers of the kind τ_p named the golden p-proportions are the "beginning of numbers." All other real numbers (including natural numbers) can be represented by using the irrational numbers τ_p.

Following this general reasoning, we can develop Newton's definition of real numbers, which is based on the numeral systems with irrational radices.

Thus, in our conceptual scheme we may consider the sum (9.5) to be a new definition of real number. Above we have considered the particular extreme cases of number representation (9.5). For the case p=0, the formula (9.5) becomes (9.3), and for the case p=1, the system (9.5) becomes Bergman's system (9.1). Finally for the case p=∞, the radix $\tau_p \to 1$; this means that the positional representation (9.5) approaches the Euclidean definition (9.55). It follows from this examination that the positional representation (9.5) is broad generalization of the classical binary system (9.3), Bergman's system (9.1) and Euclid's definition (9.50) that underlie number theory.

Note that the above definition of real numbers based on their representation in the form of a sum (9.5) gives us an infinite number of variants within the *Constructive Number Theory* because every p (p=0, 1, 2, 3, ...) generates its variant of the constructive number theory. Indeed, for the case p=0, we have the variant of constructive number theory developed in constructive mathematics [167]. For the case p=1, we get the variant of constructive number theory based on Bergman's system (9.1). For the cases p>1 we get the variants of constructive number theory based upon the general definition (9.5). Below we will only develop the beginning of this new theory of real numbers.

9.5.4. *The Golden Representations of Natural Numbers*

A new definition of real numbers based on (9.5) can become the source of a new number-theoretical result. We begin our study with the "golden" representations of natural numbers in the form (9.5), that is,

$$N = \sum_i a_i \tau_p^i. \tag{9.57}$$

We name the sum (9.57) the τ_p-*code* of natural number N. The abridged notation of the τ_p-code of natural number N has the following form:

$$N= a_n a_{n-1} \ldots a_1 a_0 , a_{-1} a_{-2} \ldots a_{-m} . \tag{9.58}$$

We use the rule

$$N'=N+1 \tag{9.59}$$

for obtaining all "golden" representations of natural numbers in the form (9.57).

In order to apply rule (9.59) for obtaining all the "golden" representations of natural numbers in the form (9.57), we need to transform the code representation (9.58) of the initial number N into such a form, when the binary numeral of the 0-th digit becomes equal to 0, that is, $a_0=0$. We can always carry out such a transformation by using the operations of "convolution" and "devolution" based on the fundamental property (9.44). If we then add the binary 1 to the 0-th digit of the code representation (9.57), we carry out rule (9.59).

Let us demonstrate this method for the case $p=1$ (Bergman's system). We begin the demonstration of this method with the transformation of the "golden" representation of number 1 to the "golden" representation of number 2. The "golden" representation of number 1 in the form (9.57) has the following form:

$$1=\tau^0=1.00.$$

Using the micro-operation "devolution" [100→011], we obtain another "golden" representation of number 1 in the following form:

$$1=0.11=\tau^{-1}+\tau^{-2}. \tag{9.60}$$

Now we apply rule (9.59) to the "golden" representation (9.60). In order to carry out the transformation of number 1 into number 2, we have to add the binary numeral 1 to the 0-th digit of the "golden" representation (9.60). As a result, we obtain the "golden" representation of number 2:

$$2=1.11. \tag{9.61}$$

If we carry out the operation of "convolution" [011→100] on the "golden" representation (9.61), we obtain another "golden" representation of number 2 (the minimal form):

$$2=10.01=\tau^1+\tau^{-2}. \tag{9.62}$$

By adding the binary 1 to the 0-th digit of the "golden" representation (9.62) and by carrying out the "convolution" [011→100], we obtain the following "golden" representation of the number 3:

$$3=11.01=100.01=\tau^2+\tau^{-2}. \tag{9.63}$$

The "golden" representation of number 4 is the following:

$$4=101.01=\tau^2+\tau^0+\tau^{-2}. \quad (9.64)$$

We can obtain the "golden" representation of number 5 from the representation (9.64), if we carry out the following transformation of (9.64) by using "devolution" [100→011]:

$$4=101.01=101.0011=100.1111. \quad (9.65)$$

By adding the binary numeral 1 to the 0-th digit of the right-hand "golden" combination (9.65), we obtain the following "golden" representation of number 5:

$$5=101.1111. \quad (9.66)$$

By carrying out the "convolutions" [011→100] in the "golden" combination (9.66), we obtain new "golden" representations of number 5:

$$5=101.1111=110.0111=1000.1001=\tau^3+\tau^{-1}+\tau^{-4}. \quad (9.67)$$

Continuing this process, we obtain "golden" representations of all natural numbers. Thus, this study results in the following unexpected result that can be formulated as the following theorem.

Theorem 9.2. All natural numbers can be represented in Bergman's system (9.1) by using a finite number of binary numerals.

This result could be generalized for the golden p-codes [105].

Theorem 9.3. For a given $p>0$ all natural numbers can be represented in the golden p-proportion code (9.5) by using a finite number of binary numerals.

Note that this unusual property of the "golden" representation (9.5) selects natural numbers from the remaining rational numbers because only natural numbers have this unique property. And we regard this unique property of natural numbers as a first confirmation of the fruitfulness of the constructive approach to number theory based on (9.5).

9.6. New Mathematical Properties of Natural Numbers (*Z*- and *D*-properties)

Bergman's system (9.1) is a source of new number-theoretical results. The *Z*-property of natural numbers is one of these results. This property is based upon the following very simple reasoning.

Let us study the representation of natural number N in Bergman's system (9.1):

$$N=\sum_i a_i\tau^i \ \left(i=0,\pm1,\pm2,\pm3,\ldots\right), \quad (9.68)$$

where a_i is a binary numeral, $\{0,1\}$, τ^i is the weight of the i-th digit, $\tau = \left(1+\sqrt{5}\right)/2$ is the golden mean – the radix of the number system (9.68).

The representation of the natural number N in the form (9.68) is called the τ-*code* of natural number N.

Note that according to Theorem 9.2, the sum (9.68) is always a finite sum for the arbitrary natural number N.

If we use the Binet formula (2.60), we can represent the τ-code of N as follows:

$$N = \frac{1}{2}\left(A + B\sqrt{5}\right), \tag{9.69}$$

where

$$A = \sum_i a_i L_i \tag{9.70}$$

$$B = \sum_i a_i F_i. \tag{9.71}$$

Note that all binary numerals $\{0,1\}$ in the sums (9.70) and (9.71) coincide with the corresponding binary numerals of the τ-code (9.68) of natural number N.

Let us represent the expression (9.69) as follows:

$$2N = A + B\sqrt{5}. \tag{9.72}$$

Note that the expression (9.72) has a general character and is valid for any arbitrary natural number N.

Let us study the "strange" expression (9.72). It is clear that the number $2N$ that stands on the left of the expression (9.72) is always an even number. The right-hand part of the expression (9.72) is the sum of the number A and the product of the number B multiplied by the irrational number $\sqrt{5}$. However, according to (9.70) and (9.71) the numbers A and B are always integers because the Fibonacci and Lucas numbers are integers. Then, it follows from (9.72) that for every natural number N, the even number $2N$ is equal to the sum of the integer A and the product of the integer B multiplied by $\sqrt{5}$. This assertion is valid for all natural numbers N! We can ask the question: when is the identity (9.72) valid in the general case? The answer to this question is very simple: the identity (9.72) will be valid for any natural number N only if the sum (9.71) is equal to 0 ("zero"), and the sum (9.70) is equal to the double of N, that is,

$$B = \sum_i a_i F_i = 0 \tag{9.73}$$

$$A = \sum_i a_i L_i = 2N. \tag{9.74}$$

Next let us compare the sums (9.68) and (9.71). Since the binary numerals a_i in these sums coincide, it follows that the expression (9.71) can be obtained from the expression (9.68) if we substitute the Fibonacci number F_i for every power of the golden mean τ^i in the expression (9.68), where the index i takes its values from the set {0,±1,±2,±3,...}. However, according to (9.73) the sum (9.71) is equal to 0 independently of the initial natural number N in the expression (9.68). Thus, we have discovered a new fundamental property of natural numbers, which can be formulated through the following theorem.

Theorem 9.4 (*Z*-property of natural numbers). If we represent an arbitrary natural number N in Bergman's system (9.1) and then substitute the Fibonacci number F_i for the power of the golden ratio τ^i in the expression (9.68), where the index i takes its values from the set {0,±1,±2,±3,...}, then the sum that appears as a result of such substitution is equal to 0 independently of the initial natural number N, that is,

$$\sum_i a_i F_i = 0.$$

Let's compare the sum (9.68) and (9.70). Since the binary numerals a_i in these sums coincide, the expression (9.70) can be obtained from the expression (9.68) if we substitute the Lucas number L_i for the power of the golden mean τ^i in the expression (9.68), where the index i takes its values from the set {0,±1,±2,±3,...}. However, according to (9.74) the sum (9.70) is equal to $2N$ independently of the initial natural number N in the expression (9.68). Thus, we have discovered one more fundamental property of Bergman's system (9.1) that can be formulated as the following theorem.

Theorem 9.5 (*D*-property). If we represent an arbitrary natural number N in Bergman's system (9.68) and then substitute the Lucas number L_i for the power of the golden mean τ^i in the expression (9.68), where the index i takes its values from the set {0,±1,±2,±3,...}, then the sum that appears as a result of such substitution is equal to $2N$ independently of the initial natural number N, that is,

$$\sum_i a_i L_i = 2N.$$

Thus, Theorems 9.4 and 9.5 provide new fundamental properties of natural numbers. For the first time the Z- and D-properties of natural numbers are described in this author's article [105] published in the *Ukrainian Mathematical Journal*. It is suprising for many mathematicians to find that the new mathematical properties of natural numbers were only discovered at the end of the 20th century, 2.5 millennia after the beginning of their theoretical study. The golden mean and the extended Fibonacci and Lucas numbers play a fundamental role in this discovery. This discovery connects together two outstanding mathematical concepts of Greek mathematics - *Natural Numbers* and the *Golden Section*. This discovery is the second confirmation of the fruitfulness of the constructive approach to the number theory based upon Bergman's system (9.1).

9.7. The *F*- and *L*-Codes

9.7.1. *Definition of the F- and L-Codes*

The above Z- and D-properties of natural numbers given by Theorems 9.4 and 9.5 allow us to create new and very unusual codes for the representation of natural numbers.

Taking the Z-property (9.73) into consideration, we can write the expression (9.69) in the following form:

$$N=\frac{1}{2}(A+B), \tag{9.75}$$

where A is defined by the expression (9.70) and B by the expression (9.71).

By using the expressions (9.70) and (9.71), we can rewrite the expression (9.69) as follows:

$$N=\sum_i a_i \frac{L_i+F_i}{2}\ (i=0,\pm1,\pm2,\pm3,\ldots). \tag{9.76}$$

Taking into consideration the following well-known identity [16]

$$\frac{L_i+F_i}{2}=F_{i+1},$$

we obtain from (9.76) the following representation of the same natural number N:

$$N=\sum_i a_i F_{i+1}\ (i=0,\pm1,\pm2,\pm3,\ldots). \tag{9.77}$$

The expression (9.77) is named the *F-code of natural number N* [105].

As the binary numerals of the expressions (9.68) and (9.77) coincide, it follows from this fact that the F-code of the natural number N can be obtained from the τ-code (9.68) of the same natural number N by means of substitution of the Fibonacci number F_{i+1} for the golden mean power τ^i, where $i=0,\pm1,\pm2,\pm3,\ldots$.

Let us now represent the F-code of N (9.77) in the following form:

$$N=\sum_i a_i F_{i+1}+2B, \tag{9.78}$$

where the term B is defined by the expression (9.71). Note that according to (9.73) the expression (9.71) is equal to 0. Then, the expression (9.78) can be represented in the following form:

$$N=\sum_i a_i (F_{i+1}+2F_i). \tag{9.79}$$

Taking into consideration the following well-known identity [16]

$$L_{i+1}=F_{i+1}+2F_i,$$

the expression (9.79) can be represented in the following form:

$$N=\sum_i a_i L_{i+1}\left(i=0,\pm1,\pm2,\pm3,\ldots\right). \tag{9.80}$$

The expression (9.80) is named the *L-code of natural number N* [105].

As the binary numerals of the expressions (9.68) and (9.80) coincide, it follows that the L-code of N can be obtained from the τ-code of N (9.68) by means of substituting the Lucas numbers L_{i+1} for the golden mean powers τ^i, where i=0,±1,±2,±3,... . It is clear that the L-code of N (9.80) can also be obtained from the F-code (9.77) of the same number N by means of substituting the Lucas number L_{i+1} for the Fibonacci number F_{i+1} in the formula (9.80).

Let us represent the sums (9.68), (9.77) and (9.80) in the abridged form (9.58). It is clear that the expressions (9.68), (9.77) and (9.80) give three different methods for the binary representation of one and the same natural number N. The τ-code (9.68) is a representation of the number N as the sum of the golden mean powers, the F-code (9.77) is a representation of the same number N as the sum of Fibonacci numbers and the L-code (9.80) is a representation of the same number N as the sum of Lucas number. As we mentioned above, all sums (9.68), (9.77) and (9.80), which represent one and the same natural number N, have one and the same abridged representation (9.58).

9.7.2. *A Numerical Example*

Once again let us consider the abridged representation (9.58). We can see that the abridged representation (9.58) is divided by the comma into two parts, namely the left-hand part, which consists of the digits with non-negative indices, and the right-hand part, which consists of the digits with negative indices. For example, we can consider the "golden" representation of the decimal number 10 in Bergman's system:

$$10=10100.0101. \tag{9.81}$$

For the τ-code (9.68) the "golden" representation (9.81) has the following algebraic interpretation:

$$10=\tau^4+\tau^2+\tau^{-2}+\tau^{-4}. \tag{9.82}$$

Using Binet's formula (2.60), we can represent the sum (9.82) as follows:

$$10=\frac{L_4+F_4\sqrt{5}}{2}+\frac{L_2+F_2\sqrt{5}}{2}+\frac{L_{-2}+F_{-2}\sqrt{5}}{2}+\frac{L_{-4}+F_{-4}\sqrt{5}}{2}. \tag{9.83}$$

If we take into consideration the following correlations that connect the Fibonacci and Lucas numbers

$L_{-2}=L_2$; $L_{-4}=L_4$; $F_{-2}=-F_2$; $F_{-4}=-F_4$, we can reduce the expression (9.83) to the following:

$$10 = \frac{2(L_4 + L_2)}{2} = L_4 + L_2 = 7 + 3.$$

Now, let us consider the interpretation of the "golden" notation (9.81) as the F- and L-codes:

$10 = F_5 + F_3 + F_{-1} + F_{-3} = 5+2+1+2;$

$10 = L_5 + L_3 + L_{-1} + L_{-3} = 11+4-1-4.$

Also we can check the sum (9.82) according to the Z- and D-properties. If we substitute in (9.82) the Fibonacci numbers F_i and the Lucas numbers L_i for the powers τ^i, we obtain the following sums:

$F_4 + F_2 + F_{-2} + F_{-4} = 3+1+(-1)+(-3) = 0$ (Z-property)

$L_4 + L_2 + L_{-2} + L_{-4} = 7+3+3+7 = 20 = 2\times 10$ (D-property).

9.7.3. *Some Properties of the F- and L-Codes*

Once again, we note that the "golden" combination (9.58) represents one and the same natural number N in the τ-, F- and L-codes given by (9.68), (9.77) and (9.80), respectively. However, a difference between them appears when we start to shift the binary code combination (9.58) to the right or to the left.

Let us denote by $N_{(k)}$ and $N_{(-k)}$ the results of the shift of the binary combination (9.58) on the k digits to the left and to the right, respectively.

If we interpret the binary combination (9.58) as the τ-code of natural number N given by (9.68), then its shift to the left (that is, to the side of the highest digits) by one digit corresponds to the multiplication of the number N by the radix τ, and its shift to the right (that is, to the side of the lowest digits) by one digit corresponds to the division of the number N by the radix τ, that is,

$$N_{(1)} = N \times \tau = \sum_i a_i \tau^{i+1} \tag{9.84}$$

$$N_{(-1)} = N \times \tau^{-1} = \sum_i a_i \tau^{i-1}. \tag{9.85}$$

It is clear that the shift of the code combination (9.58) on the k digits to the left corresponds to the multiplication of the number N by τ^k and the shift on the k digits to the right corresponds to the division by τ^{-k}, that is,

$$N_{(k)} = N \times \tau^k = \sum_i a_i \tau^{i+k} \tag{9.86}$$

$$N_{(-k)} = N \times \tau^{-k} = \sum_i a_i \tau^{i-k}. \tag{9.87}$$

Consider the shift of the code combination (9.58) to the left and to the right when we interpret it as the F- or L-codes. If we interpret the code combination (9.58) as the F-code (9.77), then by its shifting to the left on k digits we obtain the following sum, which expresses this code transformation:

$$N_{(k)} = \sum_i a_i F_{i+k+1}. \tag{9.88}$$

Apply to the expression (9.88) the following property of the generalized Fibonacci numbers G_k [28]:

$$G_{n+m} = F_{m-1}G_n + F_m G_{n+1}. \tag{9.89}$$

For the case $G_k=F_k$ the identity (9.89) takes the following form:

$$F_{n+m}=F_{m-1}F_n+F_mF_{n+1}. \tag{9.90}$$

It follows from (9.90) that for $n=i$ and $m=k+1$ the identity (9.90) amounts to the following:

$$F_{i+1+k}=F_kF_i+F_{k+1}F_{i+1}. \tag{9.91}$$

Substituting (9.91) into the expression (9.88), we obtain:

$$N_{(k)} = \sum_i a_i F_{i+k+1} = \sum_i a_i \left(F_k F_i + F_{k+1}F_{i+1}\right) = F_k \sum_i a_i F_i + F_{k+1} \sum_i a_i F_{i+1}. \tag{9.92}$$

Taking into consideration (9.73) and (9.77), we can simplify the expression (9.92) as follows:

$$N_{(k)}=F_{k+1}\times N. \tag{9.93}$$

Let us consider the shift of the code combination (9.58), which is interpreted as the F-code (9.77), to the right. Then shifting it to the right *on* k digits we obtain the following sum, which expresses this code transformation:

$$N_{(-k)} = \sum_i a_i F_{i-k+1}. \tag{9.94}$$

If we take $n=i$ and $m=-k+1$, we can then write the identity (9.90) as follows:

$$F_{i-k+1}=F_{-k}F_i+F_{-k+1}F_{i+1}. \tag{9.95}$$

Substituting (9.95) into the expression (9.94), and after simple transformations with regard to (9.73) and (9.77), we obtain:

$$N_{(-k)} = \sum_i a_i F_{i-k+1} = F_{-k+1} \times N. \tag{9.96}$$

Let us formulate the results (9.93) and (9.96) as the following theorem.

Theorem 9.6. The shift of the code combination (9.58), that is interpreted as the F-code, on the k digits to the left (that is, to the side of the highest digit) corresponds to the multiplication of the number N by the Fibonacci number F_{k+1}. However, its shift on the k digits to the right (that is, to the side of the lowest digit) corresponds to the multiplication of the number N by the Fibonacci number F_{-k+1}.

Let us next examine the formula (9.96). For the case $k=1$ (the shift to the right of one digit) the formula (9.96) takes the following form:

$$\sum_i a_i F_i = F_0 \times N. \tag{9.97}$$

However, the Fibonacci number $F_0=0$ and therefore the formula (9.97) becomes the formula (9.73), which sets the Z-property. This examination is another proof of the relevancy of the Z-property given by Theorem 9.4.

Note that the shift of the code combination (9.58), that is interpreted as the F-code, of three digits to the right corresponds to the multiplication of the number N by the Fibonacci number $F_{-2}=-1$. This means that such a shift can result in the number $(-N)=(-1)\times N$. This property of the F-code together with the Z-property has a number of interesting applications in computer and measurement systems.

Now, let us consider the shift of the code combination (9.58), which is interpreted as the L-code (9.80) of number N. Shifting it k digits to the right and k digits to the left results in the following respective sums:

$$N_{(k)} = \sum_i a_i L_{i+k+1} \tag{9.98}$$

$$N_{(-k)} = \sum_i a_i L_{i-k+1}. \tag{9.99}$$

Using the identity (9.89) and taking $G_k=L_k$, we can express the Lucas numbers L_{i+k+1} and L_{i-k+1} as follows:

$$L_{i+k+1}=L_k F_i+ L_{k+1} F_{i+1} \tag{9.100}$$

$$L_{i-k+1}=L_{-k} F_i+L_{-k+1} F_{i+1}. \tag{9.101}$$

Then, the expressions (9.98) and (9.99) can be represented in the following forms, respectively:

$$N_{(k)} = \sum_i a_i L_{i+k+1} = L_k \sum_i a_i F_i + L_{k+1} \sum_i a_i F_{i+1} \tag{9.102}$$

$$N_{(-k)} = \sum_i a_i L_{i-k+1} = L_{-k} \sum_i a_i F_i + L_{-k+1} \sum_i a_i F_{i+1}. \tag{9.103}$$

With regard to the correlations (9.73) and (9.77), we obtain from (9.102) and (9.103) the following results:

$$N_{(k)} = \sum_i a_i L_{i+k+1} = L_{k+1} \times N \tag{9.104}$$

$$N_{(-k)} = \sum_i a_i L_{i-k+1} = L_{-k+1} \times N. \tag{9.105}$$

We can formulate the results (9.104) and (9.105) as the following theorem.

Theorem 9.7. The shift of the code combination (9.58), that is interpreted as the L-code, on the k digits to the left (that is, to the side of the highest digit) corresponds to the multiplication of the number N by the Lucas number L_{k+1}. However, its shift on the k digits to the right (that is, to the side of the lowest digit) corresponds to the multiplication of the number N by the Lucas number L_{-k+1}.

Let's consider the formula (9.105). For the case k=1, (the shift to the right on one digit) the formula (9.105) takes the following form:

$$\sum_i a_i L_i = L_0 \times N.$$

As the Lucas number L_0=2, then this formula emerges as the formula (9.74), which sets the D-property. This examination is another proof of the relevancy of the D-property given by Theorem 9.5.

Note that the F- and L-code given by (9.77) and (9.80) and their surprising properties are additional mathematical properties of natural numbers which confirm the fruitfulness of the constructive approach to the number theory based on Bergman's system (9.1).

9.7.4. *Algebraic Summation of Integers*

We can use the Z-property of natural numbers to check arithmetical operations in computers. For example, let us consider the operation of the algebraic summation of two integers $N_1 \pm N_2$. We always obtain a new integer as the outcome of this operation. This means that the "golden" algebraic summation of two integers, which are represented in the τ- , F- or L-codes, results in a new code representation of the algebraic sum $N_1 \pm N_2$ in the τ- , F- or L-codes. It follows from this consideration that the Z-property is invariant regarding the "golden" algebraic summation. A similar conclusion is valid for "golden" multiplication. The outcome of "golden" division of two integers is always another two integers, quotient Q and remainder R. It follows from this consideration that the results of "golden" division, the integers Q and R, retain the Z-property.

Hence, we have obtained some new fundamental properties of natural numbers that can be represented in the τ-, F- and L-codes given by (9.68), (9.77), and (9.80). These properties (for example, the Z-property) are invariant to arithmetical operations and may be used for checking arithmetical operations in computers.

9.8. Number-theoretical Properties of the Golden *p*-Proportion Codes

9.8.1. *The Z_p-property of Natural Numbers*

Above we found the interesting mathematical properties of natural numbers (Z-property, D-property, F- and L-codes) that appear in the representation of natural numbers in Bergman's system (9.1). We can ask further question: whether or not a similar property is inherent in golden p-proportion codes corresponding to the case $p>1$?

Consider the representation of a natural number N in the golden p-proportion code given by (9.57). Remember that for a given $p>0$ all radices τ_p of the codes (8.57) are irrational numbers, the roots of the "golden" algebraic equation (4.42). According to Theorem 9.3, all natural numbers can be represented in the golden p-proportion code (9.57) by a finite number of binary numerals.

Above we have introduced a number of interesting properties of Bergman's system (9.1) that appear in the representation of natural numbers in Bergman's system given by (9.68), in particular, the Z- and D-properties of natural numbers. We can try to generalize the Z- and D-properties for the general case of the golden p-proportion code (9.57).

Let us prove the following theorem.

Theorem 9.8 (Z_p-property of natural numbers). If we represent some natural number N in the τ_p-code (9.57) and substitute into this representation the Fibonacci p-number $F_p(i)$ for the golden p-proportion τ_p^i, where $p>0$ and $i=0,\pm1,\pm2,\pm3,...$, then the sum $\sum_i a_i F_p(i)$, which appears at this replacement, is equal to 0 independent of the initial natural number N, that is,

$$\sum_i a_i F_p(i) = 0. \tag{9.106}$$

Proof. Let us prove this theorem by induction on N. For the case $N=1$, the sum (9.57) becomes:

$$1 = 1.0 = 1\times\tau_p^0. \tag{9.107}$$

If we substitute the Fibonacci p-number $F_p(0)$ for the power τ_p^0 in (9.107), we obtain the following expression:

$$1\times F_p(0). \tag{9.108}$$

Above we proved that for any given $p>0$ the Fibonacci p-number $F_p(0)=0$. It follows from this consideration that

$$1\times F_p(0)=0 \tag{9.109}$$

for any τ_p-code (9.57). The basis of the induction is proved.

Suppose that the statement of the theorem is valid for some natural number K ("the inductive hypothesis"). Let us prove the validity of the theorem for the next natural number $K+1$ by this "inductive hypothesis."

Represent the natural number K in the τ_p-code (9.57) as follows:

$$K = \sum_{i=1}^{+\infty} a_i\tau_p^i + a_0\tau_p^0 + \sum_{i=-1}^{-\infty} a_i\tau_p^i. \tag{9.110}$$

Then, the sum $\sum_i a_i F_p(i)$, which is formed from (9.110) by the substitution of $F_p(i)$ for τ_p^i, takes the following form:

$$\sum_{i=1}^{+\infty} a_i F_p(i) + a_0 F_p(0) + \sum_{i=-1}^{-\infty} a_i F_p(i). \tag{9.111}$$

According to the "inductive hypothesis," the sum (9.111) is equal to 0 for the given natural number K, that is,

$$\sum_{i=1}^{+\infty} a_i F_p(i) + a_0 F_p(0) + \sum_{i=-1}^{-\infty} a_i F_p(i) = 0. \tag{9.112}$$

In order to obtain the next natural number $K+1$ from (9.110), it is necessary to add 1 to the 0-th digit of (9.110). Here two cases are possible: (a) $a_0=0$ and (b) $a_0=1$.

(a) If $a_0=0$ in (9.110), then the τ_p-code of the number $K+1$ can be obtained from (9.110) by the replacement of $a_0=1$ for $a_0=0$. Let us study how this replacement can influence the value of the sum (9.111). As $F_p(0)=0$ for every $p>0$, the value of the sum (9.111) does not depend on the value of the digit a_0, that is, the identity (9.112) is valid for the next natural number $K+1$.

(b) Now, let us consider the case $a_0=1$ in (9.110). In this case, as shown above, by using "devolutions" and "convolutions," we can always carry out such transformation of (9.110), when the digit a_0 in (9.110) becomes equal to 0. However, then the case (b) becomes case (a) and this means that Theorem 9.8 is proven.

9.8.2. *F_p-code*

As is shown above, the F-code of the natural number N (9.77) can be obtained from the τ-code of the same natural number N (9.68) by means of the simple substitution of all Fibonacci numbers F_{i+1} for the powers τ^i in (9.63), where $i=0,\pm1,\pm2,\pm3,...$. Also the L-code of natural number N (9.80) can be obtained from the τ-code of the same natural number N (9.68) by means of the simple substitution of the Lucas number L_{i+1} for the powers τ^i in (9.68), where $i=0,\pm1,\pm2,\pm3,...$. We can generalize these results for the case of the golden p-proportion code (9.57).

Let us prove the following theorem.

Theorem 9.9 (F_p-code). If we represent some natural number N in the τ_p-code (9.57) and substitute in this representation the corresponding Fibonacci p-number $F_p(i+1)$ for the golden p-proportion power τ_p^i ($i=0,\pm1,\pm2,\pm3,...$), then the sum $\sum_i a_i F_p(i+1)$, which appears at this substitution, is equal to the initial natural number N, that is,

$$N=\sum_i a_i F_p(i+1). \tag{9.113}$$

We name the sum (9.113) the *F_p-code of the natural number N.*

Proof. Let us prove the theorem by the induction on N. For the case N=1, the sum (9.57) becomes:

$$1=1.0=1\times\tau_p^0. \tag{9.114}$$

If we substitute the Fibonacci p-number $F_p(1)$ for the power τ_p^0, we obtain the following expression:

$$1\times F_p(1). \tag{9.115}$$

Above we proved that the Fibonacci p-number $F_p(1)=1$ for any given $p>0$. It follows from this consideration that

$$1\times F_p(1)=1 \tag{9.116}$$

for any τ_p-code (9.57). The basis of the induction has been proved.

Suppose that the assertion of the theorem is valid for some natural number K ("the inductive hypothesis"). We prove the validity of the theorem for the next natural number K+1 with this "inductive hypothesis."

Represent the natural number K in the τ_p-code (9.57) as follows:

$$K=\sum_{i=1}^{+\infty}a_i\tau_p^i+a_0\tau_p^0+\sum_{i=-1}^{-\infty}a_i\tau_p^i. \tag{9.117}$$

Then, the sum $\sum_i a_iF_p(i+1)$, which is formed from (9.117) by the substitution of $F_p(i+1)$ for τ_p^i, where $i=0,\pm1,\pm2,\pm3,\ldots$, takes the following form:

$$\sum_{i=1}^{+\infty}a_iF_p(i+1)+a_0F_p(1)+\sum_{i=-1}^{-\infty}a_iF_p(i+1). \tag{9.118}$$

According to the "inductive hypothesis," the sum (9.118) is equal to natural number K, that is,

$$\sum_{i=1}^{+\infty}a_iF_p(i+1)+a_0F_p(1)+\sum_{i=-1}^{-\infty}a_iF_p(i+1)=K. \tag{9.119}$$

In order to obtain the next natural number K+1 from (9.119), it is necessary to add 1 to the 0-th digit of (9.119). Here, two cases are possible: (a) a_0=0 and (b) a_0=1.

(a) If a_0=0 in (9.119), then the F_p-code of the number K looks as follows:

$$\sum_{i=1}^{\infty}a_iF_p(i+1)+0\times F_p(1)+\sum_{i=-1}^{-\infty}a_iF_p(i+1)=K. \tag{9.120}$$

If we substitute $a_0 = 1$ for a_0=0 in (9.120), it takes the following form:

$$\sum_{i=1}^{\infty} a_i F_p(i+1) + 1\times F_p(1) + \sum_{i=-1}^{-\infty} a_i F_p(i+1) = K+1 \tag{9.121}$$

because $F_p(1)=1$ for any given $p>0$.

(b) Now, let us consider the case $a_0=1$ in (9.119). In this case, as is shown above, by using "devolutions" and "convolutions," we can always carry out such transformation of (9.119), when the digit a_0 in (9.119) becomes equal to 0. However, then the case (b) becomes case (a) and this means that Theorem 9.9 is proven.

Theorem 9.9 is a partial case of the following general theorem.

Theorem 9.10 (the shift of the F_p-code). The shift of the F_p-code of the natural number N on k digits to the left (that is, towards the higher digits) corresponds to the multiplication of the number N by the Fibonacci p-number $F_p(k+1)$ and its shift on the k digits to the right (that is, towards the lower digits) corresponds to the multiplication of the natural number N by the Fibonacci p-number $F_p(-k+1)$.

Proof. Consider the formula (9.113) for the F_p-code of the natural number N. The proof of the theorem consists of two parts:

(a) If we shift the code combination, which corresponds to the sum (9.113), to the left (that is, towards the higher digits) on the k digits, we obtain the shifted code combination that corresponds to the following sum:

$$\sum_i a_i F_p(i+k). \tag{9.122}$$

Let us prove that this sum is equal to $N\times F_p(k)$, that is,

$$\sum_i a_i F_p(i+k) = N\times F_p(k). \tag{9.123}$$

It is clear that for the case $k=1$, the formula (9.123) is valid because $F_p(1)=1$ and the formula (9.123) becomes (9.113). Suppose that the formula (9.123) is valid for the case $k=m$ and prove that this formula is valid for the case $k=m+1$. Thus, our "inductive hypothesis" is the following:

$$\sum_i a_i F_p(i+m) = N\times F_p(m). \tag{9.124}$$

Let us consider the sum

$$\sum_i a_i F_p(i+m+1). \tag{9.125}$$

Using the recursive relation

$$F_p(i+m+1)= F_p(i+m)+F_p(i+m-p), \tag{9.126}$$

we can write the sum (9.125) as follows:

$$\sum_i a_i F_p(i+m+1) = \sum_i a_i F_p(i+m) + \sum_i a_i F_p(i+m-p). \tag{9.127}$$

According to the "inductive hypothesis" (9.124), that is valid for all $k<m+1$, we can write the sum (9.127) as follows:

$$\sum_i a_i F_p(i+m+1) = N\times F_p(m) + N\times F_p(m-p). \tag{9.128}$$

By using the recursive relation (9.126), we can rewrite the sum (9.128) as follows:

$$\sum_i a_i F_p(i+m+1) = N\times F_p(m) + N\times F_p(m-p). \tag{9.129}$$

(b) If we shift the code combination, which corresponds to the sum (9.113), to the right (that is, towards the lower digits) on the k digits, we obtain the shifted code combination that corresponds to the following sum:

$$\sum_i a_i F_p(i-k+1). \tag{9.130}$$

By analogy to the case (a) we can prove the following identity:

$$\sum_i a_i F_p(i-k+1) = N\times F_p(-k+1). \tag{9.131}$$

The theorem is proved.

Let us consider the formula (9.131) for the case $k=1$:

$$\sum_i a_i F_p(i) = N\times F_p(0). \tag{9.132}$$

As $F_p(0)=0$ for every given $p>0$, we can write the following identity:

$$\sum_i a_i F_p(i) = 0. \tag{9.133}$$

Then we can formulate the following theorem that is a partial case of Theorem 9.10.

Theorem 9.11. For a given $p>0$, and for any natural number N, which is represented in F_p-code (9.113), the shift of the F_p-code of natural number N on the one digit to the right (that is, towards the lower digits) results in the code combination corresponding to the sum $\sum_i a_i F_p(i)$, which is equal to 0 independent of the initial natural number N, that is,

$$\sum_i a_i F_p(i) = 0 \quad (i = 0, \pm1, \pm2, \pm3, \ldots).$$

9.8.3. L_p-code

The Lucas p-numbers given by the recursive relation (4.162) at the seeds (4.163) and (4.164) are a source of the following number-theoretical results that are given by Theorems 9.12 and 9.13.

Theorem 9.12 (L_p-code). If for a given $p>0$ we represent some natural number N in the τ_p-code (9.57) and substitute into this representation the Lucas p-

number $L_p(i+1)$ for the golden p-proportion power τ_p^i, where i=0,±1,±2,±3,..., then the sum $\sum_i a_i L_p(i+1)$, which appears at this substitution, is also equal to the initial natural number N, that is,

$$N = \sum_i a_i L_p(i+1) \quad (i = 0, \pm1, \pm2, \pm3, \ldots). \tag{9.134}$$

We name the sum (9.134) the L_p*-code of the natural number N*.

Theorem 9.12 is proved by analogy with Theorem 9.9.

Theorem 9.13 (the shift of the L_p-code). The shift of the L_p-code of the natural number N on k digits to the left (that is, towards the higher digits) corresponds to the multiplication of the number N by the Lucas p-number $L_p(k+1)$ and its shift on k digits to the right (that is, towards the lower digits) corresponds to the multiplication of the natural number N by the Lucas p-number $L_p(-k+1)$.

Theorem 9.13 is proved by analogy to Theorem 9.10.

9.9. The Golden Resistor Dividers

9.9.1. *The Binary Resistor Divider*

In engineering practice the so-called resistor dividers, which are intended for the current and voltage division in the given ratio, are widely used. One of the variants of such a divider is shown in Fig. 9.1.

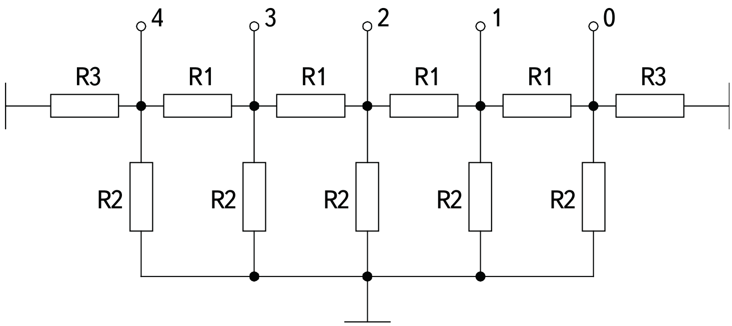


Figure 9.1. Resistor dividers

The resistor divider in Fig. 9.1 consists of the "horizontal" resistors of the kind $R1$ and $R3$ and the "vertical" resistors $R2$. The resistors of the divider are connected between themselves by the "connecting points" 0, 1, 2, 3, 4. Each point connects three resistors, which form together the resistor section. Note that Fig. 9.1 shows the resistor divider, which consists of 5 resistor sections. In general, the number of resistor sections can be equal to n (n=1,2,3,...).

First of all, we note that the parallel connection of the resistors $R2$ and $R3$ to the right of the "connecting point" 0 and to the left of the "connecting point" 4 can be replaced by the equivalent resistor with a resistance that can be calculated according to the law of the resistor parallel connection:

$$R_{e1}=\frac{R2\times R3}{R2+R3}. \tag{9.135}$$

Taking into consideration (9.135), it is easy to find the equivalent resistance of the resistor section to the right of the "connecting point" 1 and to the left of the "connecting point" 3:

$$R_{e2}=R1+R_{e1}. \tag{9.136}$$

Depending upon the choice of the resistance values of the resistors $R1$, $R2$, $R3$, we will obtain different coefficients of current or voltage division. Let us consider the so-called "binary" divider that consists of the following resistors: $R1=R$; $R2=R3=2R$, where R is some standard resistance value. For this case the expressions (9.135) and (9.136) take the following values:

$$R_{e1}=R;\quad R_{e2}=2R. \tag{9.137}$$

Then taking into consideration (9.137), we discover that the equivalent resistance of the resistor circuit to the left or to the right of any "connecting point" 0, 1, 2, 3, 4 is equal to $2R$. This means that the equivalent resistance of the divider in the "connecting points" 0, 1, 2, 3, 4 can be calculated as the resistance of the parallel connection of three resistors $2R$. By using the electrical circuit laws, we can calculate the equivalent resistance of the divider in each "connecting point" 0, 1, 2, 3, 4:

$$R_{e3}=\frac{2}{3}R. \tag{9.138}$$

Now, let us connect the generator of electric current I to one of the "connecting points," for example, to point 2. Then, according to Ohm's law, the following electric voltage appears at this point:

$$U=\frac{2}{3}RI. \tag{9.139}$$

Let us next find the electrical voltages in the "connecting points" 3 and 1 that are adjacent to point 2. It is easy to show that the voltage transmission coefficient between the adjacent "connecting points" is equal to 1/2. This means that the "binary" divider fits the binary system very well. This results in the wide use of "binary" dividers in modern digit-to-analog and analog-to-digit converters.

9.9.2. *The Golden Resistor Dividers*

We can take the values of the resistors in Fig. 9.1 as follows:

$$R1=\tau_p^{-p}R;\ R2=\tau_p^{p+1}R;\ R3=\tau_p R, \tag{9.140}$$

where τ_p is the golden p-proportion, p=0,1,2,3,... .

It is clear that the divider in Fig. 9.1 gives an infinite number of different resistor dividers because every p produces a new divider. In particular, for the case p=0, the value of the golden 0-proportion τ_0=2 and the divider is reduced to the classical "binary" divider.

For the case p=1, the resistors $R1$, $R2$, $R3$ take the following values:

$$R1=\tau^{-1}R;\ R2=\tau^2 R;\ R3=\tau R, \tag{9.141}$$

where $\tau=\left(1+\sqrt{5}\right)/2$ is the golden mean.

Let us examine the basic electrical properties of the "golden" resistor divider in Fig. 9.1 that is given by (9.140). For this purpose we use the following properties of the golden p-proportion:

$$\tau_p=1+\tau_p^{-p} \tag{9.142}$$

$$\tau_p^{p+2}=\tau_p^{p+1}+\tau_p. \tag{9.143}$$

Now, let us find the equivalent resistance of the resistor circuit of the divider to the left and to the right with respect to the "connecting points" 0 and 4. By using the expression (9.143), we can write:

$$R_{e1}=\frac{R2\times R3}{R2+R3}=\frac{\tau_p^{p+1}R\times\tau_p R}{\tau_p^{p+1}R+\tau_p R}=R. \tag{9.144}$$

Note that we have simplified the expression (9.144), using the mathematical identity (9.143).

Using (9.136) and (9.142), we can find the equivalent resistance of R_{e2}:

$$R_{e2}=\tau_p^{-p}R+R=\tau_p R. \tag{9.145}$$

Thus, according to (9.145) the equivalent resistance of the resistor circuit of the divider to the left or to the right of the "connecting points" 0, 1, 2, 3, 4 is equal to $\tau_p R$, where τ_p is the golden p-proportion. This fact can be used for the calculation of the equivalent resistance R_{e3} of the divider in the "connecting points" 0, 1, 2, 3, 4. In fact, the equivalent resistance R_{e3} can be calculated as the resistance of the electrical circuit that consists of the parallel connection of the "vertical" resistor $R2=\tau_p^{p+1}$ and two "lateral" resistors with the resistance $\tau_p R$. However, as the equivalent resistance of the parallel connection of the resistors $R2=\tau_p^{p+1}$ and $R3=\tau_p R$ is equal to R, then the equivalent resistance R_{e3} of the divider in each "connecting point" can be calculated by the formula:

$$R_{e3}=\frac{\tau_p R\times R}{\tau_p R+R}=\frac{\tau_p}{\tau_p+1}R=\frac{1}{1+\tau_p^{-1}}R. \tag{9.146}$$

Note that for the case p=0 (the "binary" divider), τ_p=τ_0=2 and the expression (9.146) is reduced to (9.138).

Now, let us find the coefficient of voltage transmission between the adjacent "connecting points" of the "golden" divider. For this purpose we connect the generator of the electric current I to one of the "connecting points," for example, to point 2. Then, according to Ohm's law the following electrical voltage appears at this point:

$$U=\frac{1}{1+\tau_p^{-1}}RI. \tag{9.147}$$

Now, let us calculate the electrical voltage in the adjacent "connecting points" 3 and 1. The voltages at the points 3 and 1 can be calculated as a result of linking the voltage U given by (9.147) to the resistor circuit that consists of the sequential connection of the "horizontal" resistor $R1=\tau_p^{-p}R$ and the resistor circuit with the equivalent resistance R. Then for this case, the electrical current, which appears in the resistor circuit to the left and to the right of the "connecting point" 2, is equal to

$$\frac{U}{R1+R}=\frac{U}{\left(\tau_p^{-p}+1\right)R}=\frac{U}{\tau_p R}. \tag{9.148}$$

If we multiply the electrical current (9.148) by the equivalent resistance R, we obtain the following value of the electrical voltage in the adjacent "connecting points" 3 and 1:

$$\frac{U}{\tau_p}. \tag{9.149}$$

This means that, in the general case, the coefficient of voltage transmission between the adjacent "connecting points" of the "golden" divider in Fig. 9.1 is equal to the reciprocal of the golden p-proportion!

Thus, the "golden" resistor divider, that is based on the golden p-proportions τ_p, are quite real electrical circuits. It is clear that the above theory of the "golden" dividers could become a new source for the development of the "digital metrology" and analog-to-digital and digital-to-analog converters.

9.10. Application of the Fibonacci and Golden Proportion Codes to Digital-to-Analog and Analog-to-Digital Conversion

9.10.1. *The "Golden" Digital-to-Analog Converters*

The electrical circuit of the "golden" DAC, that is based on the "golden" resistor divider in Fig. 9.1, is shown in Fig. 9.2.

Note that the "golden" DAC in Fig. 9.2 consists of 5 digits. However, the number of DAC digits may be increased to some arbitrary n by extending the resistor divider to the left and to the right.

The "golden" DAC contains 5 (n in the general case) generators of the standard electrical current I_0 and 5 (n in the general case) electrical current keys K_0-K_4. The key states are controlled by the binary digits of the golden p-proportion code $a_4\ a_3\ a_2\ a_1\ a_0$. For the case a_i=1, the key K_i is closed, for the case a_i=0, it is open (i=0,1,2,3,...,n). One can show that the closed key K_i results in the following voltage in the i-th point of the resistor divider: $U_i=\beta_p I_0 R$, where

$$\beta_p = 1/\left(1+\tau_p^{-1}\right).$$

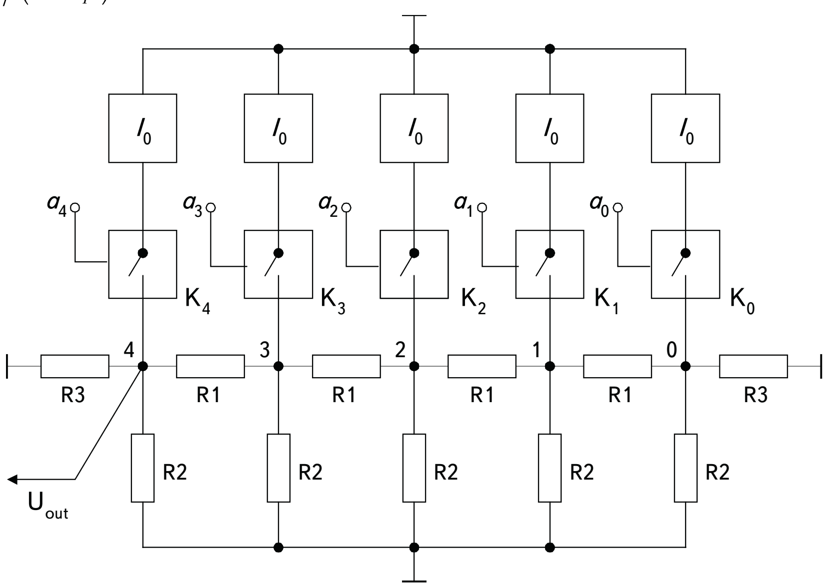


Figure 9.2. The "golden" DAC

As the potential U_i is transferred from the i-th point to the (i-1)-th point with the transmission coefficient $1/\tau_p$, the following voltage appears at the DAC output:

$$U_{out} = \frac{\beta_p I_0 R}{\tau_p^{n-i-1}} = \frac{\beta_p I_0 R}{\tau_p^{n-1} \times \tau_p^{-i}} = \frac{\beta_p I_0 R}{\tau_p^{n-1}} \times \tau_p^i .$$

Using the superposition principle, it is easy to show that the golden p-proportion code $a_n a_{n-1} a_{n-2} \dots a_1 a_0$ results in the following voltage U_{out}:

$$U_{out} = B_p \sum_{i=0}^{n-1} a_i \tau_p^i , \tag{9.150}$$

where $B_p = \beta_p I_0 R / \tau_p^{n-1}$.

It follows from (9.150) that the electrical circuit in Fig. 9.2 converts the golden p-proportion code into the electrical voltage U_{out} with regard to the constant coefficient B_p.

9.10.2. *Checking the "Golden" DAC*

In measurement practice there is a necessity to check the *DAC* for linearity in the process of its production and operation. For the classical binary *DAC* the following correlation for checking the *DAC* linearity is used:

$$2^n = \sum_{i=0}^{n-1} 2^i + 1.$$

The mathematical properties of the golden p-proportion give a very wide possibility for checking the *DAC* linearity. In particular, checking the linearity of the "golden" *DAC*, which is based on the classical golden mean τ is reduced to checking the following relations:

$$\tau^n = \tau^{n-1} + \tau^{n-2} = \tau^{n-1} + \tau^{n-3} + \tau^{n-4} = \tau^{n-1} + \tau^{n-3} + \tau^{n-5} + \tau^{n-6} = \dots \tag{9.151}$$

This check up is performed in the following manner. We may check that the output voltage of the "golden" *DAC* in Fig. 9.2 would be unchanged for the following input code combinations:

1 0 0 0 0 0 0
0 1 1 0 0 0 0
0 1 0 1 1 0 0
0 1 0 1 0 1 1.

Note that the different input code combinations are obtained from the top code combination 1000000 by means of the "devolutions" $[100 \to 011]$.

9.10.3. *The "Golden" ADC*

The functioning algorithms and structural scheme of the "golden" *ADC* coincide with the functioning algorithms and the structural scheme of the classical "binary" *ADC*. However, the golden mean relations, that connect the adjacent digit weights of the golden mean code, provide a number of interesting technical advantages for the "golden" *ADC*.

Now we need to convert the analogous magnitude of the value X, which is in the range $0 \le X < \tau^n$ into the n-digit golden mean code by using the digit-by-digit algorithm of the analog-to-digit conversion. Then, the "golden" digit-by-digit algorithm may be considered as a process of the sequential presentation of the value X and all remainders $r_1, r_2, \dots, r_n$ as follows.

9.10.3.1. *The first step*

$X = \tau^{n-1} + r_1$,
where $0 \le r_1 < \tau^{n-2}$.

9.10.3.2. *The second step*

$r_1 = \tau^{n-3} + r_2$,
where $0 \le r_2 < \tau^{n-4}$.

9.10.3.3. *The third step*

$r_2 = \tau^{n-5} + r_3$,
where $0 \le r_3 < \tau^{n-6}$.

It follows from this consideration that the output code of the "golden" *ADC* is represented in the minimal form. This means that the process of the "golden" analog-to-digital conversion is checked in accordance with the minimal form.

9.10.4. *The Self-correcting "Golden" ADC*

A guarantee of the high long-time and temperature stability of the technical parameters of *ADC* and *DAC* is one of the most important problems of designing such systems.

Designing the self-correcting *ADC* and *DAC* is one of the most effective fields of the Fibonacci and golden proportion codes applications [30].

As is well known, for the evaluation of the reliability of measurement systems and devices, in particular, *ADC* and *DAC*, a notion of *Metrological Stability* is widely used. While faults and failures of the digital components are the basic cause of non-reliability of the digital systems, the deviations of parameters of analogous elements from their standard values are the main cause of instability of measurement systems. The problem of the diminution of technological precision of the analogous elements arises in designing measurement systems of high accuracy. The solution to this problem is based upon the use of the *Principle of Self-correction*.

The Fibonacci and golden proportion codes allow one to use the principle of self-correction to increase the accuracy and metrological stability of *ADC* and *DAC*. Let us demonstrate the application of this principle on the example of the 8-digit Fibonacci *ADC*. The *ADC* uses the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21 as the digit weights. Let us suppose that the Fibonacci weight 21 has a deviation from its standard value by 20% that equals 4 units. Thus the "real" weights of the Fibonacci *ADC* are equal to: 1, 2, 3, 5, 8, 13, **25**. At the stage of self-checking, we give the *ADC* the input value 26 that exceeds the weight 21 taking into consideration all its possible deviations from the standard value. Let us carry out the analog-to-digit conversion of the input value 26 twice: the first time with the use of the higher weight **25** and without this weight the second time. As a result, we obtain two code combinations:

	25	13	8	5	3	2	1	1
$A =$	1	0	0	0	0	0	1	0
$B =$	0	1	1	1	0	0	0	0

If we interpret code combinations A and B as Fibonacci code combinations with the "ideal" weights, we find that their difference is $D=B-A=4$. Hence, we obtain the deviation of the higher digit weight 25-21=4 from the standard value.

The above method of measurement of deviation of Fibonacci weights from their standard values underlies the high-precision self-correcting *ADC* and *DAC* [30] that possess high metrological stability. By using this procedure, we can measure deviations of all the higher digit Fibonacci number weights. After that, we can measure the input value X by using "real" weights that allow us to increase the precision of *ADC*.

9.10.5. *An Application of the Z-property for Checking DAC*

There is a problem when *Checking the Last Cascade* in a reliable control system. Usually the control system uses *DAC* as its "last cascade" and therefore the problem is reduced to checking the *DAC*.

Let us demonstrate the possibility of solving this problem with the use of the golden mean code Z-property. The Z-property is fundamental to all natural numbers N that appear in the representation of number N in the F-code (9.77). Consider the technical application of the F-code for checking the *DAC* (Fig. 9.3).

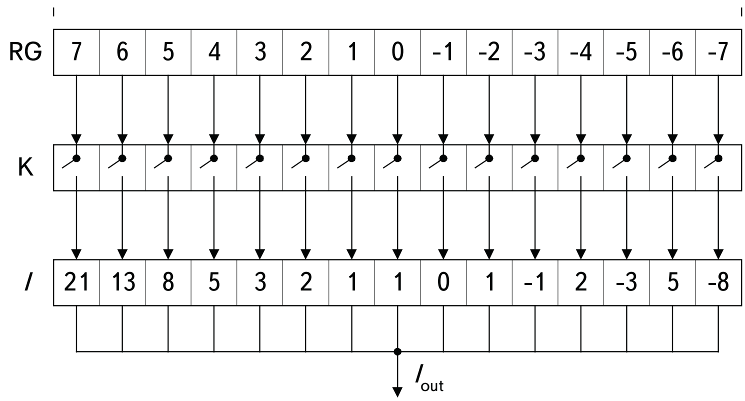


Figure 9.3. *The DAC* for the *F*-code

The *DAC* for the *F*-code consists of the register *RG,* used for memorizing the initial *F*-code, the set of electric current keys *K*, and the set of standard electric current generators *I* with electric currents proportional to the Fibonacci numbers 21,13,8,5,2,1,1,0,1,-1,2,-3,5,-8, ... If we send the *F*-code of the number *N* to the *ADC* input, we form an electric current proportional to *N* at its output:

$$I_{out} = \sum_{i=-n}^{n} a_i F_{i+1}. \tag{9.152}$$

We can see in Fig. 9.3 that we need to send the *F*-code to the input of the register *RG*. This results in switching the corresponding keys *K* and then switching the corresponding standard electric currents *I* proportional to Fibonacci numbers. It is clear that each item (register *RG*, keys *K* and the current generators *I*) may be a source for *DAC* faults. That's why there is a problem checking all the electronic components that compose the *DAC*.

We can see that the *DAC* for the *F*-code consists of a discrete part (register *RG*) and an analogous part, that include the electric current keys *K* and the current generators *I*. It is clear that the following opportunities arise for checking the discrete part of the *F*-code *DAC*:

1. Checking the *F*-code according to minimal form.
2. Checking the *F*-code according to the *Z*-property.

The following opportunities arise for checking the analogous part of the *DAC*:

3. The application of "convolution" and "devolution" micro-operations. For example, if we write the binary numeral 1 to the highest digit of the register *RG* and then carry out all possible "devolutions," we can check that the value of the output current remains unchanged, i.e.

$I_{out}=F_{n+1}=$const.

4. Checking the electric current output in accordance with the Z-property. By shifting the F-code in the register RG to the right by one digit, we can check that

$$I_{out}=0. \tag{9.153}$$

It is essential for highly-reliable control systems that the checking of the Fibonacci DAC is carried out without disconnection of the DAC output. This problem is solved in the following manner. The initial F-code combination that is shifted by one digit to the right is written into the register. Then, the equality (9.153) is checked in accordance with the Z-property. Having 0 at the ADC output means that the initial F-code is valid and then the F-code is shifted one digit to the left causing the electric current (9.152) on the output of the DAC in accordance with the F-code.

9.11. Conclusion

1. A rather unusual approach to the fundamental principles of *Number Theory* is developed in this Chapter. The traditional approach is that number theory as a mathematical discipline arose in ancient Greece. However, a study of the history of mathematics shows that long before Greek science, quite a number of outstanding discoveries in number theory and arithmetic had occurred in antiquity. The discovery of the positional principle of number representation was the most important achievement of the "pre-Greek" stage in mathematical history. This discovery was made by Babylonian mathematicians and was used in their sexagecimal positional number system. All the best known positional number systems, including the decimal system, discovered by Hindu mathematicians (somewhere in the 8th to 5th century B.C.), and the binary system, underlying modern computers, are based upon this principle.

2. Some historical sources assert that the golden section was also discovered by Babylonian mathematicians and Pythagoras simply borrowed this mathematical discovery from the Babylonians. Two outstanding Babylonian mathematical discoveries, the positional principle of number representation and the golden section, developed independently of each other over the ages. Their unification occurred in 1957, when a young 12-year-old American mathematician, George Bergman, published the article *A number system with an irrational base* [144]. In Bergman's number system [32] the golden mean, an irra-

tional number, plays the role of progenitor of all numbers, because all numbers, including natural, rational and irrational, can be represented in Bergman's number system. This result is of great methodological importance for all of mathematics and science generally. Since all numbers can be represented in Bergman's number system, that is, all numbers can be represented by the golden mean, it follows that we can formulate the new scientific doctrine that "Everything is the golden mean" in place of the Pythagorean doctrine, "Everything is a number."

3. There is a "strange tradition" in mathematics. Mathematical history shows that many mathematicians are unable to properly assess the outstanding quality of their contemporaries' mathematical discoveries. As a rule, many outstanding mathematical discoveries meet incomprehension, rejection and even gibes at the moment of their appearance. As a rule, their general recognition begins only 40-50 years after their emergence. The 19th century became especially saturated with such blunders. Failure of 19th century Russian academic science to recognize the signifigance of Lobachevsky's geometry, the sad fate of the mathematical discoveries of French mathematician Evariste Galois, killed in a duel at only 21 years of age, and Norwegian mathematician Niels Abel, who died in poverty and obscurity at 27 years of age, are sad examples of this "strange tradition." Unfortunately, we must note that George Bergman's mathematical discovery did not receive proper recognition by his contemporary mathematicians. Taking into consideration that 50 years have passed since Bergman's discovery, and following the "strange tradition" rule, it would seem that we now finally have the right to properly assess his discovery. This discovery changes the relationship between rational and irrational numbers, pushing the golden proportion into the forefront of mathematics! The discovery of Bergman's number system [86] with its generalizations, and the codes of golden p-proportions [24, 93], are significant events in the realm of knowledge, if not wisdom. Numeral systems with irrational radices [24, 93, 86] are amongst the most important mathematical discoveries in numeral systems, possibly secondary only to discoveries of the positional number representation principle (Babylon, 2000 B.C.) and the decimal system (India, 8th to 5th century B.C.).

4. Bergman's system and its generalization – the codes of the golden p-proportions - are of great importance for theoretical arithmetic. As shown here in this Chapter, it is clear that the codes of the golden p-proportions are a bountiful source of new ideas for the development of number theory. They form a new constructive definition for real numbers, which can become a significant source of new number-theoretical results (including, for example, the Z-property of natural numbers, and F- and L-codes). This new theory of real numbers has roots in ancient Babylonian mathematics (both in the positional principle of number

representation and the golden section) with an origin that is two millennia older than classical number theory. In addition, Bergman's system and the golden p-proportion codes are of great importance in computer and measurement practice because the numeral systems with irrational radices reveal the way to highly original computer and measurement projects. Thus, this new theory of real numbers, developed in this Chapter, both extends the theory of real numbers and returns number theory to its proper applications in computer and measurement systems (the "golden" computers and the "golden" analog-to-digital and digital-to-analog converters).

Chapter 10

Ternary Mirror-Symmetrical Arithmetic

10.1. A Ternary Computer "Setun"

10.1.1. *Brousentsov's Ternary Principle*

As is well known, computer process design begins with the choice of numeral system that determines many technical characteristics of computers. At the beginning of the computer era, the problem of choosing the "optimal" number system for electronic computers was brilliantly solved by American physicist and mathematician John von Neumann, who forcefully argued his preference for the binary system in electronic computers. The famous *John von Neumann Principles* include three basic ideas for electronic computer design: the *Binary System, Binary (Boolean) Logic,* and *the Binary Memory Element ("Flip-Flop")*.

Even though the binary system is the most popular one in contemporary computers, the study and development of new numeral systems continued. The desire to overcome a number of significant shortcomings in the classical binary system is the primary motivation for this ongoing study. Two shortcomings of the binary system are certainly well known. The first of them involves the fact that it is impossible to represent negative numbers (the *Sign Problem*) and perform arithmetical operations on them in "direct" binary code what complicates arithmetical computer structures. The second shortcoming of the binary system is the *Problem of "Zero" Redundancy*. The fact that all binary combinations are allowed complicates errors detection during information transmission, processing, and storage.

The initial attempt to overcome the *Sign Problem* was made in the Soviet Union during the very dawn of the computer era. The original computer project – the ternary computer "Setun" [180] – was designed in 1958 at Moscow University, and became a brilliant example for an "optimal" solution of the *Sign Problem.* A new principle of construction for computers was implemented in the "Setun." This principle was based on the concepts of *Ternary Logic, Ternary Symmetrical Numeral System,* and *Ternary Memory Element ("Flip-Flap-Flop")*. This principle is called *Brousentsov's Ternary Principle* [104] in honor of the Soviet scientist Nikolay Brousentsov, the principal designer of the "Setun" computer.

10.1.2. *The Dramatic History of the "Setun" Computer*

Nikolay Brousentsov was born in 1925 in Kamenskoe, Ukraine. In 1953 he graduated from the Moscow Energy University and started to work as a computer engineer at the Special Designing Bureau (SDB) of Moscow University.

Nikolay Brousentsov (born in 1925)

In the beginning of his career Brousentsov participated in the modification of the M-2 computer that was constructed in a special computer laboratory at the Soviet Academy of Sciences under the scientific supervision of Soviet computer specialist M. Kartcev. The M-2 computer, which was one of the best Soviet computers of that period, literally overwhelmed Brousentsov's heart, and he began dreaming about designing his own computer. His intention coincided with the decision of famous Soviet mathematician Sergey Sobolev, who headed the Computational Mathematics Department at Moscow University, to create a new computer for educational purposes. Sergey Sobolev organized the scientific seminar, in which the Moscow mathematicians and programmers Shura-Bura, Semendyaev, Jogolev and Brousentsov participated. They discussed the shortcomings of today's binary computers, and considered their architecture and various technical methods of design. After heated discussions, the seminar participants decided to give the preference to magnetic elements as the basis of new computer. The magnetic cores and diodes were the preferred basic elements at that period, because transistors did not yet exist, and vacuum tubes were excluded because of large size and low reliability.

In April 1956, when academician Sobolev formulated the new computer project at the Moscow seminar, intensive work began. Brousentsov was appointed principal developer of the project. To realize this project, Sobolev organized the *Problem Computer Laboratory* in the Mechanics and Mathematics Department of Moscow University, where in 1962 Brousentsov was appointed its head. The first idea was to design a traditional binary computer based upon magnetic elements. After studying the binary magnetic computer projects, Brousentsov found many disadvantages. Therefore, he decided to develop the ternary magnetic computer. Brousentsov wrote in his journal: "Of course, I knew about the advantages of the ternary code from the special books devoted to this problem. Later I found that American scientist Grosh ("Grosh's law") was interested in the ternary number system. However, the American scientists could not develop a ternary computer."

In 1957, Brousentsov developed the basic components of ternary computers: the summator, counter and other devices. In 1958, engineers at the Laboratory produced the first prototype ternary computer which began functioning 10 days after the start of its debugging! It was called "Setun" after the small river Setun near Moscow University.

The "Setun" was a one-address computer of sequential functioning with a fixed comma. From the functional point of view, the computer was divided into six components: arithmetic, control, operative memory, input, output, and magnetic drum memory. From the mathematical point of view a special feature of the "Setun" computer was its use of a ternary symmetrical numeral system with ternary numerals {-1,0,1}. From the engineering perspective, a special feature of the computer was its use of a magnetic amplifier as its basic component. Such an amplifier consisted of a non-linear transformer with a miniature magnetic core and a germanium diode. The three stable states that are necessary for carrying out ternary representation were obtained by using one pair of such amplifiers. The 18-digit ternary code combination was the "ternary word." In "Setun's" arithmetic device the 18-digit ternary word was considered to be a number, in which the comma was located between the second and third digits. The commands were encoded by the 9-digit ternary half-word.

According to a resolution of the Ministerial Council of the Soviet Union, the Kazan Computer Plant (now known as ICL-KPO) was ordered to produce 50 prototypes of "Setun." Thirty computers were distributed amongst the Soviet universities. These computers functioned efficiently in all climate zones of the former U.S.S.R. – from Kalinigrad to Magadan, and from Odessa and Ashchabad to Novosibirsk. These computers functioned beautifully, practically without any service or repairs.

During 1961-1968, Brousentsov designed the architecture of the new ternary computer, "Setun-70." Its functional algorithm was described in the programming language "ALGOL." Unfortunately, due to a negative opinion from the Moscow University administration regarding his new computer idea, Brousentsov's laboratory was not able to continue further developments after "Setun-70". Brousnetsov's laboratory was moved to the attic of a student dormitory where daylight was absent and the overall conditions were unsuitable. Based upon a decision by the University administration, the prototype of the "Setun" computer (having operated without failure for 17 years), was subjected to barbaric destruction. The computer was chopped into pieces and thrown into the garbage.

10.1.3. *Ternary Technology*

Unfortunately, the history of the ternary "Setun" computer ended dramatically. The main reason was the fact that "Setun" was designed on magnetic elements, and could not compete with binary electronic computers. However, "Setun" is of great importance in the development of computer science and its relevance is preserved today. In contrast with computers based on *von Neumann's Binary Principle* (the binary system, Boolean logic and the binary memory element, flip-flop), "Setun" was the first computer designed on *Brousentsov's Ternary Principle* [104]. According to Brousentsov's ternary principle, computers can be designed on the "ternary" base: the ternary symmetrical numeral system, ternary logic, and the ternary memory element, flip-flap-flop.

Next let us compare the *Binary Digital Technology* based on *von Neumann's Binary Principle* with the *Ternary Digital Technology* based on *Brousentsov's Ternary Principle*. The binary digital technology is based on two-valued signals (*bits*) and two-state memory elements (*flip-flop*). The discrete objects that have more than two states are represented by combinations of bits or bytes (8 bits). For example, the decimal numerals are represented by four bits; the symbols of the alphabet are represented by bytes, etc. Accordingly, all operations over multivalued objects are carried out as sequences of operations of two-valued logic.

The ternary digital technology is based on three-valued signals (*trits*) and three-state memory elements (*flip-flap-flop*). The objects that have more than 3 states are represented as combinations of trits. The operations over these objects are carried out as sequences of operations of three-valued logic. The analog of a *byte* (8 bits) is the combination of 6 trits, called a *trite*.

One of the barriers that restrains the development and spread of ternary digital technology is disbelief and the inability to comprehend the extraordinary nature of three-valued logic. Actually three-valued logic is not only reasonable and related to reality; but is more convenient and comprehendible for people than two-valued logic. In real life three-valued relations occur quite frequently. For example, "increase - does not change - decrease," "forward - stop - backward," "victory - a draw - defeat," "surplus - the norm - shortage," "friendly - neutrality - hostility," "earlier - now - later," "to the left - center - to the right," etc.

Many modern computer experts have come to the conclusion that the ternary computer design principle may become an alternative in the future of computer progress. In this connection, it is important to recall the opinion of well-known Russian scientist, Prof. D. Pospelov, about the ternary-symmetrical number system. In his book [175] he wrote: "The barriers, which stand in the way of application of ternary-symmetrical number systems in computers, are of a technical character.

Until now, economical and effective elements with three stable states have not been developed. As soon as such elements will be designed, a majority of computers of the universal kind and many special computers will most likely be re-designed so that they will operate on the ternary-symmetrical number system."

Also, American scientist Donald Knuth expressed the opinion [181] that one day the replacement of "flip-flop" by "flip-flap-flop" will occur.

10.2. Ternary Symmetrical Numeral System

10.2.1. *Ternary Symmetrical Representation*

The key idea of "Brousentsov's Ternary Principle" is the "ternary symmetrical numeral system" [175, 176, 180]. This uses the following positional method of number representation:

$$N=\sum_{i=1}^{n} b_i 3^{i-1}, \tag{10.1}$$

where b_i (i=1,2,3,...,n) is the ternary numeral {-1,0,1} of the i-th digit; 3^{i-1} is the "weight" of the i-th digit; and the number 3 is the base of the numeral system.

We can explain the essence of the ternary symmetrical representation of numbers with the example of the Bashet-Mendeleev problem considered in Chapter 7. This numeral system appears as an outcome of the solution to the Bashet-Mendeleev problem, when we have the right to place the standard weights on both cups of the balance, the "free" cup and the "weight" cup. It was proved that the ternary numbers

$$1, 3, 27, ..., 3^{n-1} \tag{10.2}$$

are the optimal solution to the Bashet-Mendeleev problem for this condition.

Using the standard weights (10.2), we can weigh the different unknown weights according to the following rule. The weighing of the unknown weight Q=1kg is carried out by using the first "standard weight" 1kg that is put on the "free" cup of the balance. The weighing result can be represented as follows:

$1=0001_3$.

The weighing of the unknown weight Q=2kg can be carried out by using two "standard weights": 1kg and 3kg. The standard weight of 3kg is put on the "free" cup of the balance and the standard weight of 1kg on the "weight" cup of the balance. The weighing result can be represented as follows:

$2=001\bar{1}_3$.

The negative unit $\bar{1}=-1$, which appeared in this ternary representation, has the following "measurement interpretation." It means that the first "standard weight" of 1kg is on the "weight" cup of the balance and this weight is "subtracted" from the second "standard weight" of 3kg, which is on the "free" cup of the balance.

The weighing resulting in the unknown weights of Q=3kg and Q=4kg can be written as follows:

$$3=0010_3,\quad 4=0011_3.$$

However, the ternary representation of the unknown weight of Q=5kg has the following form:

$$5=01\bar{1}\bar{1}_3.$$

This ternary representation means that the "standard weight" of 9kg is on the "free" cup and the "standard weights" of 3kg and 1kg are on the "weight" cup of the balance.

From the above numerical representation, a new interpretation of the notion of "numeral" follows. The positive unit 1 means that the corresponding standard weight is on the "free" cup of the balance, the negative unit $\bar{1}$ means that the corresponding standard weight is on the "weight" cup of the balance, and the numeral 0 in the ternary representation means that the corresponding standard weight is not involved in the weighing.

Hence, the ternary representation $01\bar{1}\bar{1}_3$ has the following numerical interpretation:

$$01\bar{1}\bar{1}_3=0\times3^3+1\times3^2-1\times3^1-1\times3^0=5_{10}.$$

Now, let us consider some other examples:

$$01\bar{1}0_3=0\times3^3+1\times3^2-1\times3^1+0\times3^0=6_{10}$$

$$01\bar{1}1_3=0\times3^3+1\times3^2-1\times3^1+1\times3^0=7_{10}$$

$$010\bar{1}_3=0\times3^3+1\times3^2-0\times3^1-1\times3^0=8_{10}.$$

10.2.2. *Ternary Inversion*

The basic advantage of this numeral system (10.1) in comparison to the classical binary system is the graceful solution to the "sign problem." A sign of the number is determined by the highest significant digit of the ternary symmetrical representation (10.1). For example, the number $N_1=0\bar{1}110\bar{1}$ is negative because the highest significant digit has the ternary numeral $\bar{1}$, while the number $N_2=1\bar{1}0\bar{1}001$ is positive because the highest significant digit is positive. Both positive and negative numbers are represented in the "direct" code and all arithmetical operations are fulfilled in the “direct” code. It is easy to obtain a representation of the negative number (-N) from the ternary representation of positive number N by using the rule of “ternary inversion”:

$$1 \to \bar{1}, 0 \to 0, \bar{1} \to 1. \qquad (10.3)$$

For example, by applying the rule of ternary inversion (10.3) to the ternary representations $N_1 = 0\,\bar{1}\,1\,1\,0\,\bar{1}$ and $N_2 = 1\,\bar{1}\,0\,\bar{1}\,0\,0\,1$, we can obtain ternary representations for the following numbers:

$$(-N_1) = 0\,1\,\bar{1}\,\bar{1}\,0\,1; \quad (-N_2) = \bar{1}\,1\,0\,1\,0\,0\,\bar{1}.$$

10.2.3. *The Range of Number Representation*

It is clear that the maximal and minimal integer numbers in the ternary system (10.1) have the following “register representations”:

		$n-1$	$n-2$	...	i	...	1	0
A_{max}	=	1	1	...	1	...	1	1

(10.4)

		$n-1$	$n-2$	...	i	...	1	0
A_{min}	=	$\bar{1}$	$\bar{1}$	...	$\bar{1}$	...	$\bar{1}$	$\bar{1}$

(10.5)

The numeral notations (10.4) and (10.5) have the following numerical interpretations, respectively:

$$A_{max}=3^{n-1}+3^{n-2}+...+3^1+3^0 \qquad (10.6)$$

$$A_{min}=-3^{n-1}-3^{n-2}-...-3^1-3^0. \qquad (10.7)$$

It follows from the comparison of (10.6) and (10.7) that

$$A_{min}=-\,A_{max}. \qquad (10.8)$$

And it is easy to prove the following identity:

$$A_{max} = \sum_{i=0}^{n} 3^i = \frac{3^n - 1}{2}. \qquad (10.9)$$

We can formulate the results given by the expressions (10.8) and (10.9) as the following theorem.

Theorem 10.1. The ternary-symmetrical numeral system (10.1) with the radix 3, that uses the ternary numerals {-1,0,1}, allows one to represent, by using n ternary digits, 3^n integers (including positive, negative numbers, and the number 0) in the range from

$$A_{min} = -\frac{3^n - 1}{2} \text{ to } A_{max} = \frac{3^n - 1}{2}.$$

10.3. Ternary-Symmetrical Arithmetic

10.3.1. *Ternary-Symmetrical Summation and Subtraction*

The following elementary identity for the powers of the radix 3 underlies the ternary-symmetrical summation:

$$3^i+3^i=3^{i+1}-3^i. \tag{10.10}$$

The next table of the ternary-symmetrical summation follows from this identity (Table 10.1).

Table 10.1. Ternary-symmetrical summation

b_k / a_k	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}1$	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$1\bar{1}$

A number of peculiarities of the ternary-symmetrical summation follows from Table 10.1. These peculiarities appear in the summation of the ternary units of the same sign, namely:

$1+1=1\bar{1}$ and $\bar{1}+\bar{1}=\bar{1}1$.

We can see that there appears to be an intermediate sum and carry-over at the addition of the ternary units of the same sign. In this case the sign of the carry-over coincides with the sign of the summable numerals; however, the sign of the intermediate sum is opposite.

The ternary-symmetrical subtraction of the numbers A-B comes to the summation of the numbers A+(-B) if the rule of the ternary inversion (10.3) is applied to the subtrahend B.

Example 10.1. Sum up two ternary-symmetrical numbers $64_{10}=1\bar{1}101$ and $16_{10}=1\bar{1}\bar{1}1$.

Solution:

$$\begin{array}{r} 1\ \bar{1}\ 1\ 0\ 1 \\ +\quad\quad \\ 1\ \bar{1}\ \bar{1}\ 1 \\ \hline 1\ 0\ 0\ 0\ \bar{1} \end{array}$$

The summation result $1000\bar{1}=1\times 3^4+\bar{1}\times 3^0=80$ is a positive number because its ternary-symmetrical representation begins with the positive 1.

Example 10.2. Subtract the ternary-symmetrical number $16_{10}=1\bar{1}\bar{1}1$ from the ternary-symmetrical number $64_{10}=1\bar{1}101$.

Solution. Subtraction of two ternary numbers 64-16 amounts to the summation of the numbers 64+(-16), if we apply to the number 16 the rule of the ternary inversion (10.3):

$$(-16)=\bar{1}11\bar{1}.$$

Then, the subtraction amounts to the summation:

$$\begin{array}{r} 1\ \bar{1}\ 1\ 0\ 1 \\ + \\ \bar{1}\ 1\ 1\ \bar{1} \\ \hline 1\ \bar{1}\ \bar{1}\ 1\ 0 \end{array}$$

The subtraction result $1\bar{1}\bar{1}10 = 1\times3^4 + \bar{1}\times3^3 + \bar{1}\times3^2 + 1\times3^0 = 48$ is a positive number because its ternary-symmetrical representation begins with the positive 1.

10.3.2. *Ternary-Symmetrical Multiplication*

The ternary-symmetrical multiplication table (Table 10.2) is based on the following trivial mathematical identity for the number 3 powers:

$3^m \times 3^n = 3^{m+n}$. (10.11)

Example 10.3. Multiply two ternary-symmetrical numbers $(-10)_{10} = \bar{1}\,0\,\bar{1}$ and $2_{10} = 1\bar{1}$.

Solution:

$$\begin{array}{r} \bar{1}\ 0\ \bar{1} \\ \times \\ 1\ \bar{1} \\ \hline 1\ 0\ 1 \\ \bar{1}\ 0\ \bar{1}\ \\ \hline \bar{1}\ 1\ \bar{1}\ 1 \end{array}$$

You can see that the ternary-symmetrical multiplication amounts to the summation of two partial products, that are formed as the result of the multiplication of the first multiplier $\bar{1}\,0\,\bar{1}$ by the lowest ternary numeral $\bar{1}$ of the second multiplier and then by the highest ternary numeral 1 of the second multiplier.

Note that we have multiplied the negative number $(-10)_{10}$ by the positive number 2 in the “direct” code. After summation, we obtained the following result of multiplication:

$\bar{1}1\bar{1}1 = \bar{1}\times3^3 + 1\times3^2 + \bar{1}\times3^1 + 1\times3^0 = -20.$

By looking at the ternary representation $\bar{1}1\bar{1}1$, we can find that it represents a negative number because the ternary representation $\bar{1}1\bar{1}1$ begins with the negative unit $\bar{1}$.

Table 10.2. Ternary-symmetrical multiplication

b_k / a_k	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	0	1

10.3.3. *Ternary-Symmetrical Division*

The ternary-symmetrical division amounts to a sequential shift of the divisor to the left until the highest significant digit (1 or $\bar{1}$) of the shifted divisor coincides with the highest significant digit of the dividend *D*. Consider the case, when the divisor is shifted on the *k* digits to the left. Then, the shifted divisor is compared with the dividend *D*. If the signs of the highest significant digit of the shifted divisor and the dividend coincide, then the highest significant digit of the first partial quotient Q_1 is equal to 1 and the first partial quotient has the form:

$$Q_1 = 1\underbrace{0\,0\ldots 0}_{k}, \tag{10.12}$$

where the number of 0's after 1 is equal to *k*. For this case the shifted divisor is subtracted from the dividend.

If the highest significant digits of the shifted divisor and the dividend are opposite by sign, then the first partial quotient Q_1 is equal to $\bar{1}$ and the first partial quotient has the form:

$$Q_1 = \bar{1}\underbrace{0\,0\ldots 0}_{k}, \tag{10.13}$$

where the number of 0's after $\bar{1}$ is equal to *k*. For this case the shifted divisor is added to the dividend.

As a result of the first stage of division, we obtain the first partial quotient in the forms (10.12) or (10.13) and the first intermediate dividend D_1 as the result of summation or subtraction of the shifted divisor from the dividend.

The next stage of the ternary-symmetrical division consists of the comparison of the first intermediate dividend D_1 with the shifted divisor according to the rules described above.

The procedure of the comparison of the intermediate dividend and the shifted divisor continues until we obtain the intermediate dividend equal to 0 or when the exactness of the division becomes acceptable for us. Then, we have to sum up all partial quotients for obtaining the result of ternary-symmetrical division.

Example 10.4. Divide the ternary-symmetrical number $16_{10} = 1\,\bar{1}\,\bar{1}\,1$ (the dividend) by the ternary symmetrical number $2_{10} = 1\,\bar{1}$ (the divisor).

Solution. The *first stage* of the division shifts the divisor to the left two digits. Then we obtain the shifted divisor in the form:

$$1\,\bar{1}\,0\,0. \tag{10.14}$$

Comparing the shifted divisor (10.14) with the dividend $16_{10} = 1\,\bar{1}\,\bar{1}\,1$ we can see that their highest significant digits coincide in sign. This means that the first partial quotient is equal to:

$$Q_1 = 100. \tag{10.15}$$

Then, we have to subtract the shifted divisor (10.14) from the dividend $16_{10} = 1\bar{1}\bar{1}1$. The ternary-symmetrical subtraction comes to the ternary-symmetrical summation of the dividend with the number $\bar{1}100$ that is opposite in sign to (10.14):

$$\begin{array}{r} 1\;\bar{1}\;\bar{1}\;1 \\ +\quad \\ \bar{1}\;1\;0\;0 \\ \hline 0\;0\;\bar{1}\;1 \end{array}$$

As a result of the first stage of the division, we obtain two numbers, the first partial quotient (10.15) and the first intermediate dividend $D_1 = \bar{1}1$.

The *second stage* of the division consists in a comparison of the first intermediate dividend $D_1 = \bar{1}1$ with the divisor $1\bar{1}$. As the highest significant digits of the comparable ternary symmetrical numbers are opposite in sign, it follows that we have to write the second partial quotient as:

$$Q_2 = \bar{1}. \tag{10.16}$$

Then, we should sum up the first intermediate dividend $D_1 = \bar{1}1$ and the divisor $1\bar{1}$:

$$\begin{array}{r} \bar{1}\;1 \\ +\quad \\ 1\;\bar{1} \\ \hline 0\;0 \end{array}$$

Since the second intermediate dividend is equal to 0, the division is over. The result of the division is the sum $Q=Q_1+Q_2$:

$$\begin{array}{r} 1\;0\;0 \\ +\quad \\ \bar{1} \\ \hline 1\;0\;\bar{1} \end{array}$$

Hence, $Q = 10\bar{1} = 1\times 3^2 + 0\times 3^1 + \bar{1}\times 3^0 = 8_{10}$.

10.4. Ternary Logic

10.4.1. *Basic Functions of Ternary Logic*

Ternary logic is a special case of the so-called k-valued logic (k=2,3,4,5,...) for the case k=3. For coordination with the ternary-symmetrical number system we assume that the ternary logical variables take their values from the set $\{\bar{1}, 0, 1\}$.

Then, the basic logic functions of one ternary variable are determined in the following manner:

Inversion function

$$f(v) = \bar{v} = \begin{cases} \bar{1} \text{ } with \text{ } v = 1 \\ 0 \text{ } with \text{ } v = 0 \\ 1 \text{ } with \text{ } v = \bar{1} \end{cases}$$

Cyclic negation

$$f(v) = \tilde{v} = \begin{cases} \bar{1} \text{ } with \text{ } v = 0 \\ 0 \text{ } with \text{ } v = 1 \\ 1 \text{ } with \text{ } v = \bar{1} \end{cases}$$

Consider the following important functions of two ternary variables:

(1) Ternary conjunction $f(v_1, v_2) = \min(v_1, v_2) = v_1 \wedge v_2$

$\wedge$	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
0	$\bar{1}$	0	0
1	$\bar{1}$	0	1

(2) Ternary disjunction $f(v_1, v_2) = \max(v_1, v_2) = v_1 \vee v_2$

$\vee$	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}$	0	1
0	0	0	1
1	1	1	1

(3) Addition by modulo 3 $f(v_1, v_2) = v_1 \oplus v_2 \pmod 3$

$\oplus$	$\bar{1}$	0	1
$\bar{1}$	1	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$\bar{1}$

(4) Multiplication by modulo 3 $f(v_1, v_2) = v_1 \otimes v_2 \pmod 3$

$\otimes$	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	0	1

The following identities are for the above ternary logical functions:

$$\bar{\bar{v}} = v, v \wedge v = v, v \wedge \bar{1} = \bar{1}, v \wedge 1 = v, v \vee v = v, v \vee 1 = 1, v \vee \bar{1} = v,$$

$$v \oplus 0 = v, v \otimes 0 = 0, v \otimes 1 = v, v \otimes \bar{1} = \bar{v}.$$

The ternary functions of conjunction, disjunction and inversion are connected by De Morgan's formulas:

$\overline{v_1 \wedge v_2} = \overline{v_1} \vee \overline{v_2}, \overline{v_1 \vee v_2} = \overline{v_1} \wedge \overline{v_2}.$

Note that the three-valued logic used in the "Setun" [180] appears for engineers as the long known logic of the positive, negative and equal to zero electrical current, and for programmers as the logic of number signs: $+, -, 0,$ etc.

Similar to Boolean logic, there are different variants of the functionally-completed systems of ternary logical functions. We can use the functions of so-called "modular logic" to synthesize the ternary logic elements. The system of "modular logic" includes the following functions:

$$f(v_1, v_2) = v_1 \oplus v_2, \; f(v_1, v_2) = v_1 \otimes v_2. \tag{10.17}$$

We can add to the modular functions (10.17) a special function

$$f(v_1, v_2) = v_1 \odot v_2. \tag{10.18}$$

This function gives the rule for the carry-over formation of the addition of single-digit ternary numbers. The logic table for this function has the following form:

Θ	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}$	0	0
0	0	0	0
1	0	0	1

The set of the ternary functions (10.17) and (10.18) is a functionally-completed set of ternary logic functions that may be used for the synthesis of the ternary logic elements. It is easy to prove that the ternary inversion function $\bar{v}$ and the cyclic negation function $\tilde{v}$ are carried out by using the modulo 3 addition logic element (Fig. 10.1).

The ternary single-digit half-summator of the kind 2Σ is designed by using the logical elements $\oplus$ and $\odot$ (Fig. 10.2-a) and the ternary single-digit multiplier is based on the logical element $\otimes$ (Fig. 10.2-b).

If we take the ternary single-digit half- summator of the kind 2Σ as the basic logic element for designing the ternary-symmetrical

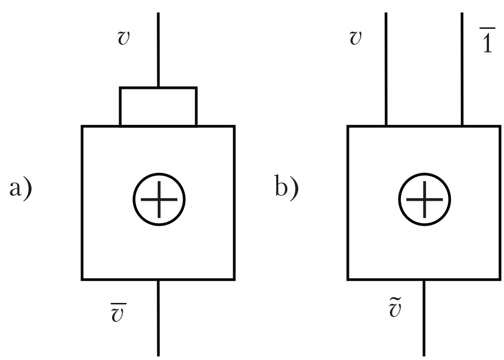


Figure 10.1. The logic elements of ternary inversion (a) and cyclic negation (b)

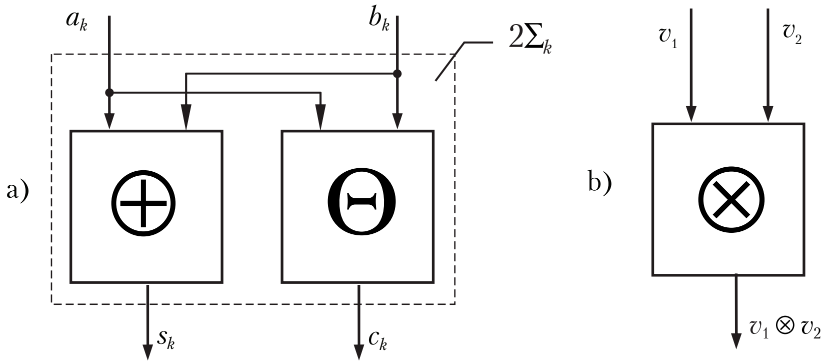


Figure 10.2. The ternary single-digit half-summator (a) and multiplier (b)

arithmetical devices, we can prove that the 5 single-digit half-summators build up the ternary single-digit full summator of the kind 4Σ (Fig. 10.3).

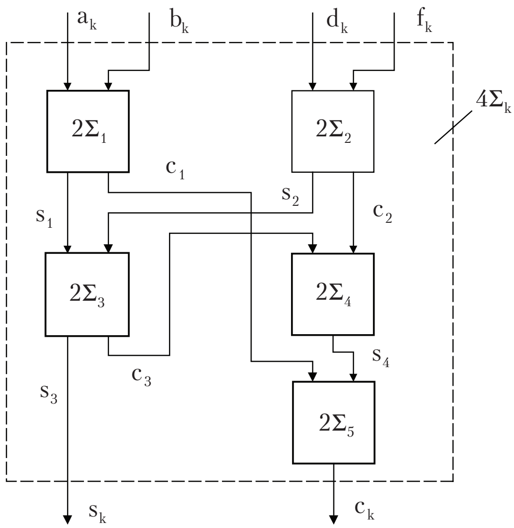


Figure 10.3. The full ternary single-digit summator

10.4.2. *Binary Realization of Ternary Logic Elements*

For micro-electronic realization by using *VLSI* we can use the binary encoding of ternary variables as shown in Table 10.3.

Using Table 10.3, every ternary element, for instance, the ternary summators and multipliers in Fig. 10.2 and Fig. 10.3, can be represented by means of *VLSI* with binary inputs and outputs. Then, the problem of designing the ternary elements comes to the design of the binary *VLSI*.

Note that some ternary functions are designed very simply in this manner. For example, the logic element of the ternary inversion $f(v)=\overline{v}$ ($v=x_1x_2$ and $\overline{v}=x_2x_1$) is represented as it is shown in Fig. 10.4.

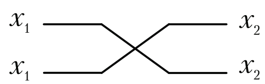


Figure 10.4. The binary representation of "ternary inversion"

Table 10.3. Binary encoding of ternary numerals

v	=	x_1	x_2
$\bar{1}$	=	1	0
0	=	0	0
1	=	0	1

10.4.3. *Flip-flap-flop*

The same "binary approach" can be used for designing the ternary memory element called *Flip-Flap-Flop*. As is well known, the classical binary *Flip-Flop* is based on the logical elements 1 and 2 of the kind *OR-NOT* (Fig. 10.5-a) that are connected by the back logic connections.

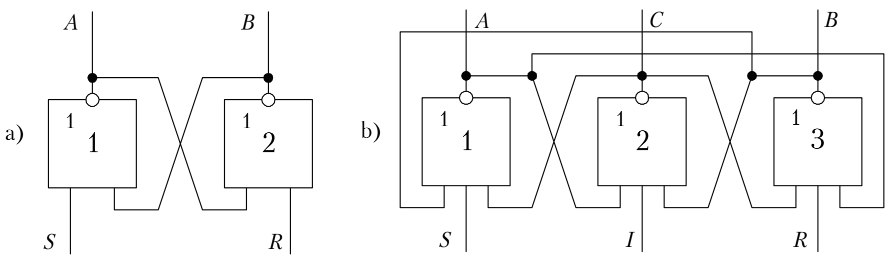


Figure 10.5. "Flip-flop" (a) and "flip-flap-flop" (b)

Now, let us consider the logical circuit that consists of three logic elements 1, 2, 3 of the kind *OR-NOT* (Fig. 10.5-b). Suppose that the logic elements 2 and 3 are adjacent to the logic element 1, the logic elements 3 and 1 are adjacent to the logic element 2, and the logic elements 1 and 2 are adjacent to the logic element 3. Every logic element *OR-NOT* is connected with its adjacent logic elements by the back logic connections. This is cause for three stable states of the logic circuit in Fig. 10.5-b. In fact, suppose that we have the logic 1 on the input *C* of the logic element 2. This logic signal 1 enters the inputs of the adjacent logic elements 2 and 3 and supports the logic signals 0 on their outputs *A* and *B*. These logic signals 0 enter the inputs of the logic element 2 and support the logic signal 1 on its output *C*. Hence, this state of the circuit in Fig. 10.5-b is the first stable state. This stable state corresponds to the code combination 0 1 0 on the outputs *A, C, B*. One may show that the circuit has one or two stable states that correspond to the code combinations 1 0 0 and 0 0 1 on the outputs *A, C, B*. Really, it is easy to prove that the logic signal 1 on the output *A* is a cause of the second stable state 100 of the logic circuit in Fig. 10.5-a. At least, the logic signal 1 on the output *B* is a cause of the third stable state of the logic circuit in Fig. 10.5-a. We can use the above-mentioned stable states of the circuit in Fig. 10.5-b for binary coding of the ternary numerals according to the following rule:

0	=	0	1	0
1	=	0	0	1
$\bar{1}$	=	1	0	0

If we eliminate the middle output *C,* we obtain the binary outputs *A* and *B* that correspond to the binary encoding of ternary variables according to Table 10.3.

Hence, the logic circuit in Fig. 10.5-b can be considered as the ternary-binary memory element called *Flip-Flap-Flop*. Let us consider the functioning of the "flip-flap-flop" in Fig. 10.5-b. It has three stable states $\bar{1}$, 0 and 1. Let the "flip-flap-flop" in Fig. 10.5-b be in the state Q=0. This means that the output C=1 and the other outputs A=B=0. If we need to set the "flip-flap-flop" into the state Q=1 (001), we have to send to the "flip-flap-flop" inputs *S, I, R* the following adjusting signals S=1, I=1, R=0. The signals S=1 and I=1 cause an appearance of the logic signals 0 on the outputs *A* and *C*. These logic signals 0 enter the inputs of the logic element 3 and together with the logic signal R=0 cause an appearance of the logic signal 1 on the output *B*.

By analogy one may show that installation's signals S=0, I=1, R=1 turn over the "flip-flap-flop" in Fig. 10.5-b into the state $\bar{1}\,(100)$.

10.5. Ternary Mirror-Symmetrical Representation

10.5.1. *A Conversion of the Binary "Golden" Representation to the Ternary "Golden" Representation*

We start our study from Bergman's system (9.1). Let us consider the τ-representation of natural number N given by (9.68). We use the minimal form of the τ-representation (9.68). This means that each binary unit a_k=1 in the binary "golden" representation (9.68) surrounded by two adjacent binary "zeros" a_{k-1}=a_{k+1}=0.

Let us consider the following well-known identity for the powers of the golden mean:

$$\tau^k = \tau^{k+1} - \tau^{k-1}. \tag{10.19}$$

The identity (10.19) has the following code interpretation:

$k+1$	k	$k-1$		$k+1$	k	$k-1$
0	1	0	=	1	0	$\bar{1}$

(10.20)

where $\bar{1}$ is the negative unit, that is, $\bar{1} = -1$. It follows from (10.20) that the positive binary 1 of the *k*-th digit in the binary "golden" representation (9.68) is transformed into two 1's, the positive unit 1 of the (k+1)-th digit and the negative unit $\bar{1}$ of the (k-1)-th digit.

The code transformation (10.20) can be used for conversion of the minimal form of the binary τ-representation (9.68) of the number *N* into a *Ternary τ-Representation* of the same integer *N*.

Let us consider the τ-representation of number 5 that is represented in minimal form:

		4	3	2	1	0	–1	–2	–3	–4
5	=	0	1	0	0	0.	1	0	0	1

(10.21)

Convert the binary τ-representation (10.21) into the ternary τ-representation of the same number 5. With this purpose we can apply the code transformation (10.20) simultaneously to all digits that are binary numerals 1 and have odd indices (k=2m+1). We can see that the transformation (10.20) can be applied in the situation (10.21) only to the 3rd and (-1)-th digits that are the binary numerals of 1. As the result of such a transformation, we obtain the following ternary representation of the number 5:

		4	3	2	1	0	–1	–2	–3	–4
5	=	1	0	$\bar{1}$	0	1.	0	$\bar{1}$	0	1

(10.22)

We can see from (10.22) that all digits with odd indices (k=2m+1) are identically equal to 0. However, the digits with even indices can take the ternary values from the set $\{\bar{1}, 0, 1\}$. This means that all digits with odd indices are "non-informative" because their values are identically equal to 0 and they do not influence the value of the number 5. Omitting in (10.22) all the "non-informative" digits, we obtain the following ternary representation of number 5:

		4	2	0	–2	–4
5	=	1	$\bar{1}$	1.	$\bar{1}$	1

(10.23)

This means that number 5 can be represented by using only the even digits in the form of the following sum:

$$5=1\times\tau^4+\bar{1}\times\tau^2+1\times\tau^0+\bar{1}\times\tau^{-2}+1\times\tau^{-4}=\tau^4-\tau^2+\tau^0-\tau^{-2}+\tau^{-4}. \quad (10.24)$$

If we transform the binary τ-representation (9.68), that is represented only in the minimal form, into the ternary τ-representation according to the above rule, after omitting the "non-informative" digits we obtain the following sum:

$$N=\sum_i b_{2i}\tau^{2i}, \quad (10.25)$$

where b_{2i} is the ternary numeral of the $(2i)$-th digit.

We can make the following digit enumeration for the ternary representation (10.25). Each ternary digit b_{2i} is replaced by the ternary digit c_i. As a result of such enumeration, we obtain the expression (10.25) in the following form:

$$N=\sum_i c_i\tau^{2i}, \quad (10.26)$$

where c_i is the ternary numeral of the i-th digit; τ^{2i} is the weight of the i-th digit; and τ^2 is the base or radix of the numeral system (10.26). With regard to the expression (10.26) the ternary representation (10.23) takes the following form:

		2	1	0	–1	–2
5	=	1	$\bar{1}$	1.	$\bar{1}$	1

(10.27)

This is the ternary τ-representation of the number 5.

The conversion of the binary τ-representation (9.68) of the natural number N into the ternary τ-representation (10.26) of the same natural number N can be carried out by using a simple combinative logic circuit that transforms the next three binary digits $a_{2i+1}\, a_{2i}\, a_{2i-1}$ of the initial binary τ-representation that is represented in the minimal form into a ternary informative digit $b_{2i}=c_i$ of the ternary τ-representation in accordance with Table 10.4.

Table 10.4. Conversion of the binary τ-representation to a ternary τ-representation

a_{2i+1}	a_{2i}	a_{2i-1}	$\to$	c_i
0	0	0	$\to$	0
0	0	1	$\to$	1
0	1	0	$\to$	1
1	0	0	$\to$	$\bar{1}$
1	0	1	$\to$	0

Note that Table 10.4 uses only 5 binary code combinations from the 8 possible binary code combinations because the initial binary τ-representation of the kind (10.21) is represented in minimal form and the code combinations 011, 110, 111 are prohibited for the minimal form.

The code transformations given by the 2nd and 4th rows of Table 10.4 are trivial. The code transformations given by the 3rd, 5th and 6th rows of Table 10.4 follow directly from the rule (10.20). For instance, the code transformation of the 6th row $101\to 0$ means that the negative unit of $\bar{1}$ that appears in accordance with (10.20) from the left-hand binary digit a_{2i+1}=1 to the digit a_{2i} is summed with the positive unit of 1 that appears from the right-hand binary digit a_{2i-1}=1 to the digit a_{2i}. It follows from this consideration that their sum is equal to the ternary numeral c_i=0.

10.5.2. *Ternary F- and L-Representations*

Above we have introduced the so-called *F*- and *L*-codes (9.77) and (9.80). Let us remember that these unusual representations are the equivalent of the τ-code (9.68) of the same natural number *N*. By using the ternary τ -representation of the natural number *N* given by (10.26), it is easy to write the ternary *F*- and *L*-representations of the same natural number *N* in the following forms:

$$N = \sum_i c_i F_{2i+1} \tag{10.28}$$

$$N = \sum_i c_i L_{2i+1}. \tag{10.29}$$

Note that the values of the ternary digits in the representations (10.26), (10.28) and (10.29) coincide. It follows from this consideration that the ternary τ- *F*-, *L*-representations of the number 5 given by the example (10.27) have three different numerical interpretations:

(a) The ternary τ-representation:

$$\begin{aligned}
5 &= 1\times\tau^4 + \bar{1}\times\tau^2 + 1\times\tau^0 + \bar{1}\times\tau^{-2} + 1\times\tau^{-4} \\
&= \frac{L_4 + F_4\sqrt{5}}{2} - \frac{L_2 + F_2\sqrt{5}}{2} + 1 - \frac{L_{-2} + F_{-2}\sqrt{5}}{2} + \frac{L_{-4} + F_{-4}\sqrt{5}}{2} \\
&= L_4 - L_2 + 1 = 7 - 3 + 1.
\end{aligned}$$

(b) The ternary *F*-representation:

$$5 = 1\times F_5 + \bar{1}\times F_3 + 1\times F_1 + \bar{1}\times F_{-1} + 1\times F_{-3} = 5 - 2 + 1 - 1 + 2.$$

(c) The ternary *L*-representation:

$$5 = 1\times L_5 + \bar{1}\times L_3 + 1\times L_1 + \bar{1}\times L_{-1} + 1\times L_{-3} = 11 - 4 + 1 + 1 - 4.$$

10.5.3. *The Representation of Negative Numbers*

Similar to the ternary-symmetrical numeral system (10.1), the possibility of representation of both positive and negative numbers in the “direct” code is a most important advantage of the numeral system (10.26). The ternary code representation of a negative number (-*N*) can be obtained from the ternary τ-representation (10.26) of the initial N by means of the application of the "ternary inversion" rule (10.3). By applying this rule to the ternary τ-representation (10.27) of the number 5, we obtain the ternary τ-representation of the negative number (-5):

		2	1	0	–1	–2
–5	=	$\bar{1}$	1	$\bar{1}$.	1	$\bar{1}$

10.5.4. *Mirror-Symmetrical Property of Integer Representation*

Considering the ternary τ-representation of the number 5 given by (10.27), we find one unusual property of the ternary representation (10.27). We see that the left-hand part $\left(1\bar{1}\right)$ of the ternary τ-representation (10.27) is mirror-symmetrical to its right-hand part $\left(\bar{1}1\right)$ with respect to the 0-th digit. This property of the "mirror symmetry" of the numeral system (10.26) is a general property of integers that appears at their representation in (10.26). Table 10.5 demonstrates this property for some initial natural numbers.

Thus, thanks to this simple observation, we have discovered one more fundamental property of integers called the *Mirror-Symmetrical Property* of integers. Based upon this fundamental property, the "ternary numeral system" given by (10.26) is named the *Ternary Mirror-Symmetrical Numeral System* [104].

Table 10.5. Mirror-symmetrical property

i	3	2	1	0	–1	–2	–3
τ^{2i}	τ^6	τ^4	τ^2	τ^0	τ^{-2}	τ^{-4}	τ^{-6}
F_{2i+1}	13	5	2	1	1	2	5
L_{2i+1}	29	11	4	1	–1	–4	–11
N	↓	↓	↓	↓	↓	↓	↓
0	0	0	0	0.	0	0	0
1	0	0	0	1.	0	0	0
2	0	0	1	$\bar{1}$.	1	0	0
3	0	0	1	0.	1	0	0
4	0	0	1	1.	1	0	0
5	0	1	$\bar{1}$	1.	$\bar{1}$	1	0
6	0	1	0	$\bar{1}$.	0	1	0
7	0	1	0	0.	0	1	0
8	0	1	0	1.	0	1	0
9	0	1	1	$\bar{1}$.	1	1	0
10	0	1	1	0.	1	1	0

10.5.5. *The Radix of the Ternary Mirror-Symmetrical Numeral System*

It follows from (10.26) that the radix of this ternary numeral system is the square of the golden mean, that is,

$\tau^2 = \left(3+\sqrt{5}\right)/2 = 2.618.$

This means that the numeral system (10.26) is a number system with an irrational radix. The radix of the numeral system (10.26) has the following traditional representation:

τ^2=10.

10.5.6. *Comparison of Number in the Ternary Mirror-Symmetrical Numeral System*

Let us consider the set of weights for the (2n-1)-th ternary mirror-symmetrical representation (10.26):

$$\left\{\tau^{2n}, \tau^{2(n-1)}, ..., \tau^2, \tau^0, \tau^{-2}, \tau^{-4}, ..., \tau^{-2(n-1)}, \tau^{-2n}\right\}.$$

It is easy to prove that the weight of the n-th digit of the representation (10.26) is always strictly more than the sum of the rest weights of the representation (10.26). It follows from this fact that the highest significant digit of the representation (10.26) contains in itself the information about the sign of the ternary mirror-symmetrical number. If the numeral of the highest significant digit of the ternary mirror-symmetrical representation is equal to 1, it follows that the ternary mirror-symmetrical number is positive. If the numeral of the highest significant digit of the ternary mirror-symmetrical representation is equal to $\bar{1}$, it follows that the ternary mirror-symmetrical number is negative.

The very simple method of comparison of the two ternary mirror-symmetrical numbers A and B follows from this consideration. The comparison begins with the highest digits of the comparable numbers and continues until obtaining the first pair of non-coincident ternary digits a_k and b_k. If $a_k > b_k \left(1>0, 1>\bar{1}, 0>\bar{1}\right)$, then $A>B$. In the opposite case: $A \le B$.

Hence, we have found two important advantages of the ternary mirror-symmetrical representation (10.26):

1. Similar to the classical ternary symmetrical representation (10.1), the highest significant digit of the representation (10.26) determines the sign of the ternary mirror-symmetrical number given by (10.26).

2. A comparison of the numbers is fulfilled by analogy with the classical ternary-symmetrical representation (10.1), that is, by beginning with the highest digits until obtaining the first pair of non-coincident ternary digits.

10.6. The Range of Number Representation and Redundancy of the Ternary Mirror-Symmetrical Numeral System

10.6.1. *The Range of Number Representation*

Let us now consider the range of number representation in the numeral system (10.26). Suppose that the ternary τ-representation (10.26) has $2m+1$ ternary digits. In this case, by using (10.26), we can represent all integers in the range from the maximal number

$$N_{\max} = \underbrace{11\ldots1}_{m}1.\underbrace{11\ldots1}_{m} \tag{10.30}$$

to the minimal number

$$N_{\min} = \underbrace{\bar{1}\,\bar{1}\ldots\bar{1}}_{m}\,\bar{1}.\underbrace{\bar{1}\,\bar{1}\ldots\bar{1}}_{m}. \tag{10.31}$$

It is clear that the minimal number $N_{\min}$ is the ternary inversion of the maximal number $N_{\max}$, that is, we have:

$$|N_{\min}| = N_{\max}.$$

It follows from this examination that by using $2m+1$ ternary digits, we can represent in the numeral system (10.26)

$$2N_{\max}+1 \tag{10.32}$$

integers from $N_{\min}$ to $N_{\max}$ including the number of 0.

For the calculation of $N_{\max}$ we can interpret (10.30) as the ternary L-code (10.29). Then we can interpret the ternary representation (10.30) as follows:

$$N_{\max}=L_{2m+1}+L_{2m-1}+\ldots+L_3+L_1+L_{-1}+L_{-3}+\ldots+L_{-2m+1}. \tag{10.33}$$

For the odd indices $i=2k-1$, we have the following property for Lucas numbers [28]:

$$L_{-2m+1}= L_{2m-1}. \tag{10.34}$$

Taking into consideration property (10.34), we obtain the following value of the sum (10.33):

$$N_{\max}=L_{2m+1}. \tag{10.35}$$

Taking into consideration (10.32) and (10.35), we can formulate the following theorem.

Theorem 10.2. By using $(2m+1)$ ternary digits, we can represent in the ternary mirror-symmetrical numeral system (10.26) $2L_{2m+1}+1$ integers in the range from $(-L_{2m+1})$ to $(+L_{2m+1})$, where L_{2m+1} is Lucas number.

10.6.2. *The Redundancy of Ternary Mirror-Symmetrical Representation*

Now, let us compare the ternary numeral system (10.1) with the ternary-symmetrical numeral system (10.26).

According to Theorem 10.1 we can represent in the ternary-symmetrical numeral system (10.1) 3^n integers in the range from $N_{\min} = -(3^n - 1)/2$ up to $N_{\max} = (3^n - 1)/2$.

For the representation of numbers in the range given by Theorem 10.2 in the n-digit ternary-symmetrical numeral system (10.1), we need to fulfill the following condition:

$$3^n \geq 2L_{2m+1} + 1 \approx 2L_{2m+1}. \tag{10.36}$$

Using the Binet formula for Lucas numbers given by (2.67), we can write the following approximate formula for the calculation of the Lucas number L_{2m+1}:

$$L_{2m+1} \approx \tau^{2m+1}. \tag{10.37}$$

Substituting (10.37) into (10.36) and taking the logarithm with the base 3 of both parts of the inequality (10.36), we obtain the following inequality:

$$n \geq (2m+1)\log_3 \tau + \log_3 2. \tag{10.38}$$

By increasing m, the inequality (10.38) transforms into the following approximate equality:

$$n \approx (2m+1)\log_3 \tau. \tag{10.39}$$

The expression (10.39) can be used for calculation of the code redundancy of the ternary mirror-symmetrical numeral system (10.26). The relative redundancy R is calculated according to the formula:

$$R = \frac{k-n}{n} = \frac{k}{n} - 1,$$

where k and n are the digit numbers of the redundant and non-redundant numeral systems for the representation of the same range of numbers.

If we choose $k=2m+1$ digits for the ternary mirror-symmetrical representation (10.26), then we need n digits for the representation of the same range of numbers in the non-redundant numeral system (10.1). Then, from this consideration we can write the following expression for the calculation of the relative code redundancy of the ternary mirror-symmetrical numeral system (10.26):

$$R = \frac{1 - \log_3 \tau}{\log_3 \tau} = 1.283 = 128.3\%. \tag{10.40}$$

Thus, the relative redundancy of the ternary mirror-symmetrical number system (10.40) is sufficiently large. However, we should take into consideration the number of essential advantages of this numeral system, in particular, the unique possibility to check arithmetical operations in the given numeral system.

10.7. Mirror-Symmetrical Summation and Subtraction

10.7.1. *Mirror-Symmetrical Summation*

The following identities for the golden mean powers underlie the mirror-symmetrical summation:

$$2\tau^{2k}=\tau^{2(k+1)}-\tau^{2k}+\tau^{2(k-1)} \tag{10.41}$$

$$3\tau^{2k}=\tau^{2(k+1)}+0+\tau^{2(k-1)} \tag{10.42}$$

$$4\tau^{2k}=\tau^{2(k+1)}+\tau^{2k}+\tau^{2(k-1)}, \tag{10.43}$$

where k=0,±1,±2,±3,... .

The identity (10.41) is a mathematical base for the mirror-symmetrical summation of two single-digit ternary digits and gives a rule of the carry-over formation (Table 10.6).

Table 10.6. Mirror-symmetric summation a_k+b_k

b_k / a_k	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}1\bar{1}$	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$1\bar{1}1$

The main peculiarity of Table 10.6 consists of the rule of summation of two ternary units with equal signs, i.e.

a_k+b_k	=	c_k	s_k	c_k
1+1	=	1	$\bar{1}$	1
$\bar{1}+\bar{1}$	=	$\bar{1}$	1	$\bar{1}$

We can see that at the mirror-symmetrical summation of ternary units with the same sign, the intermediate sum s_k with opposite sign and the carry-over c_k with the same sign appear. However, the carry-over from the k-th digit is spreading simultaneously to the adjacent two digits, namely to the adjacent left-hand, that is, (k+1)-th digit, and to the adjacent right-hand, that is, (k-1)-th digit.

Table 10.6 describes an operation of the simplest ternary mirror-symmetrical summator called the *Single-Digit Ternary Mirror-Symmetrical Half-*

Summator. This half-summator is a combinative logic circuit that has two ternary inputs a_k and b_k and two ternary outputs s_k and c_k. It operates in accordance with Table 10.6 (Fig. 10.6-a).

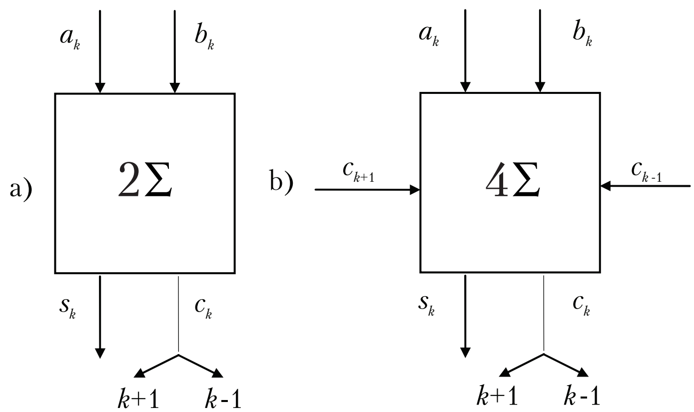


Figure 10.6. Mirror-Symmetric single-digit summators: (a) half-summator; (b) full summator.

As the carry-over from the k-th digit is spreading to the left-hand and to the right-hand digits, it means that the full mirror-symmetrical single-digit summator has to have two additional inputs for the carry-overs that come from the (k-1)-th and (k+1)-th digits to the k-th digit. Thus, the full mirror-symmetrical single-digit summator is a combinative logic circuit that has 4 ternary inputs and 2 ternary outputs (Fig. 10.6-b). Let us denote by 2Σ the mirror-symmetrical single-digit half-summator that has 2 inputs and by 4Σ the mirror-symmetrical single-digit full summator that has 4 inputs.

Now, let us describe the logical operation of the mirror-symmetrical full single-digit summator of the kind 4Σ. First of all, we note that the number of all possible 4-digit ternary input combinations of the mirror-symmetrical full summator in Fig. 10.6-b is equal to $3^4=81$. The values of the output variables s_k and c_k are some discrete functions of the algebraic sum S of the input ternary variables a_k, b_k, c_{k-1}, c_{k+1}, that is,

$$S=a_k+b_k+c_{k-1}+c_{k+1}. \tag{10.44}$$

The sum (10.44) takes the values from the set {-4,-3,-2,-1,0,1,2,3,4}. The operation rule of the mirror-symmetrical full summator of the kind 4Σ (Fig. 10.6-b) consists of the following. The summator forms the output ternary code combinations $c_k s_k$ in accordance with the value of the sum (10.44) as follows:

$$-4=\bar{1}\,\bar{1};\; -3=\bar{1}\,0;\; -2=\bar{1}\,1;\; -1=0\,\bar{1};\; 0=0\,0; 1=0\,1; 2=1\,\bar{1}; 3=1\,0; 4=1\,1.$$

The lower digit of such 2-digit ternary representations are values of the intermediate sum s_k and the higher digit are the values of the carry-over c_k that is spreading to the neighboring (the left-hand and the right-hand) digits.

Note that the functioning rule of the ternary mirror-symmetrical summator in Fig. 10.6-b fully coincides with the functioning rule of the classical ternary-symmetrical summation. This means that we can use the logical combinative circuit in Fig. 10.3 for designing the ternary mirror-symmetrical summator in Fig. 10.6-b.

10.7.2. *Ternary Mirror-Symmetrical Multi-Digit Summator*

The multi-digit combinative mirror-symmetrical summator (Fig. 10.7) that carries out the addition of two $(2m+1)$-digit mirror-symmetrical numbers is a combinative logic circuit that consists of $(2m+1)$ ternary mirror-symmetrical summators of the kind 4Σ (Fig. 10.6-b).

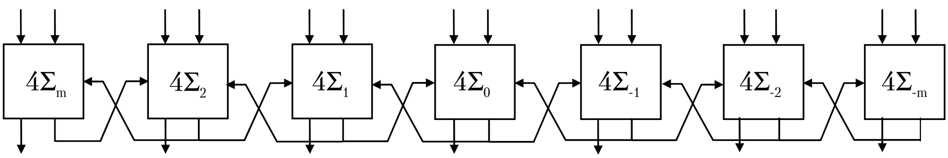


Figure 10.7. Ternary mirror-symmetric multi-digit summator

We can see from Fig. 10.7 that the main peculiarity of the multi-digit ternary mirror-symmetrical summator consists in the fact that the carry-over from each digit is spreading symmetrically to the adjacent digits to the left and to the right. Two mirror-symmetrical numbers A and B enter the multi-digit input of the summator. The single-digit adder $4\Sigma_0$ separates the summator into two parts: the single-digit summators $4\Sigma_1, 4\Sigma_2, 4\Sigma_3$ for the highest digits and the single-digit summators $4\Sigma_{-1}, 4\Sigma_{-2}, 4\Sigma_{-3}$ for the lowest digits.

Example 10.5. Sum up two ternary mirror-symmetrical numbers 5+10:

$$\begin{array}{lcccccccc}
5 & = & 0 & 1 & \bar{1} & 1. & \bar{1} & 1 & 0 \\
10 & = & 0 & 1 & 1 & 0. & 1 & 1 & 0 \\
S_1 & = & 0 & \bar{1} & 0 & 1. & 0 & \bar{1} & 0. \\
C_1 & & 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
\hline
15 & = & 1 & \bar{1} & 1 & 1. & 1 & \bar{1} & 1
\end{array}$$

Note that the symbol $\leftrightarrow$ marks the process of carry-over spreading.

We can see that the addition process for this example consists of two steps. The first step is forming the first multi-digit intermediate sum S_1 and the first multi-digit carry-over C_1 according to Table 10.6. The second step is the summation of the numbers S_1+C_1 according to Table 10.6. As for this case the second multi-digit intermediate carry-over $C_1=0$, meaning that the summation is over and the sum $S_1+C_1=15$ is the summation result. It is important to emphasize that the summation result

$$15=1\bar{1}11.1\bar{1}1 \tag{10.45}$$

is represented in mirror-symmetrical form.

As noted above, the possibility of summing up all integers (positive and negative) in the "direct" code is an important advantage of the ternary mirror-symmetrical numeral system (10.26), that is, we do not use the notions of inverse and additional codes.

Example 10.6. Sum up the negative mirror-symmetrical number (-24) and the positive mirror-symmetrical number 15:

$$\begin{array}{rccccccc}
-24= & \bar{1} & \bar{1} & 0 & 1. & 0 & \bar{1} & \bar{1} \\
15= & 1 & \bar{1} & 1 & 1. & 1 & \bar{1} & 1 \\
S_1 & 0 & 1 & 1 & \bar{1}. & 1 & 1 & 0 \\
C_1'= & & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
C_1''= & \bar{1} & \leftrightarrow & \bar{1} & & \bar{1} & \leftrightarrow & \bar{1} \\
-9= & \bar{1} & 1 & 1 & \bar{1}. & 1 & 1 & \bar{1}
\end{array}$$

We can see that the summation process consists of two steps. The first step is forming the first multi-digit intermediate sum S_1 and the first multi-digit carry-over $C_1=C_1'+C_1''$ according to Table 10.6. The second step is to sum up the numbers $S_1+C_1'+C_1''$. Here, we use the functioning rule of the ternary mirror-symmetrical single-digit summator in Fig. 10.6-b. As for this case the second multi-digit intermediate carry-over $S_1=0$, meaning that the summation is over and the sum $S_1+C_1'+C_1''=-9$ is the summation result. It is important to emphasize that the summation result

$$-9=\bar{1}11\bar{1}.11\bar{1} \tag{10.46}$$

is negative number because the ternary mirror-symmetrical representation (10.46) begins with the negative unit $\bar{1}$. In addition, the summation result (10.46) is represented in mirror-symmetrical form which allows one to check the process of summation.

10.7.3. *Mirror-Symmetrical Subtraction*

Subtraction of two mirror-symmetrical numbers N_1-N_2 transforms to summation if we represent their difference in the form of the following sum:

$$N_1-N_2=N_1+(-N_2). \tag{10.47}$$

It follows from (10.47) that until a subtraction we have to take the ternary inversion of the subtrahend N_2 according to (10.3). The above example of the summation of the numbers (-24)+15 can be considered as the subtraction of the numbers (-24)–(-15), where

$$-15=\bar{1}1\bar{1}\bar{1}.\bar{1}1\bar{1}. \tag{10.48}$$

It is clear that until the subtraction we have to apply the "ternary inversion" rule (10.3) to the number (10.48) to obtain the number 15:

$$15=1\bar{1}11.1\bar{1}1.$$

10.7.4. *The "Swing" Phenomenon*

Now, let us sum up two equal numbers 5+5 represented in the numeral system (10.26):

$$\begin{array}{ccccccccc}
5 & = & 0 & 1 & \bar{1} & 1. & \bar{1} & 1 & 0 \\
5 & = & 0 & 1 & \bar{1} & 1. & \bar{1} & 1 & 0 \\
\hline
 & & 0 & \bar{1} & 1 & \bar{1}. & 1 & \bar{1} & 0 \\
 & & & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
 & & 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
 & & & \bar{1} & \leftrightarrow & \bar{1} & \downarrow & & \\
 & & & & & \bar{1} & \leftrightarrow & \bar{1} & \\
\hline
 & & 1 & 1 & 0 & 0. & 0 & 1 & 1 \\
 & & & & \bar{1} & \leftrightarrow & \bar{1} & & \\
 & & & 1 & \leftrightarrow & 1 & \downarrow & & \\
 & & & \downarrow & & 1 & \leftrightarrow & 1 & \\
 & & \bar{1} & \leftrightarrow & \bar{1} & & \bar{1} & \leftrightarrow & \bar{1} \\
\hline
 & & 0 & \bar{1} & 1 & \bar{1}. & 1 & \bar{1} & 0 \\
 & & & & 1 & \leftrightarrow & 1 & & \\
 & & & \bar{1} & \leftrightarrow & \bar{1} & \downarrow & & \\
 & & & \downarrow & & \bar{1} & \leftrightarrow & \bar{1} & \\
 & & 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1
\end{array}$$

It follows from this example that we have found a special summation case called *Swing*. If the summation process goes on, then since at some step the process of the carry-over formation begins to reiterate, and it follows that the process of summation becomes infinite. The "swing"-phenomenon is a variety of the "races" that appear in digit automatons, when the electronic elements switch over.

In order to eliminate the "swing"-phenomenon, we use the following effective "technical" method [104]. The "swing"-phenomenon appears in the ternary mirror-symmetrical summator in Fig. 10.7 because the carry-overs come at the same time from two adjacent single-digits adders. A "technical" solution of this phenomenon is to delay the input signals of the single-digit summators with odd indices $(k=\pm1,\pm3,\pm5,\dots)$ by one summation step. For this situation at the first step of the summation only the summators with the even indices $(k=0,\pm2,\pm4,\pm6,\dots)$ operate and they form the intermediate sums and corresponding carry-overs to the single-digit summators with the odd indices. Then, at the second summation step the carry-overs that were formed at the first step are summarized with the corresponding ternary variables of the odd digits of the summable numbers. Thanks to such an approach, the "swing"-phenomenon is eliminated.

Now, let us demonstrate the above method to eliminate the "swing"-phenomenon at the summation of the numbers 5+5:

$$
\begin{array}{lcccccccc}
5 & = & 0 & 1 & \bar{1} & 1. & \bar{1} & 1 & 0 \\
5 & = & 0 & 1 & \bar{1} & 1. & \bar{1} & 1 & 0 \\
\hline
S_1 & = & & \bar{1} & & \bar{1}. & & \bar{1} & \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
C_1' & & & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
C_1'' & & 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
\hline
10 & = & 1 & \bar{1} & 0 & \bar{1}. & 0 & \bar{1} & 1
\end{array}
$$

The first step of the mirror-symmetrical summation is to summarize all the input ternary numerals with even indices (2,0,-2). The ternary numerals of all digits with odd indices (3,1,-1,-3) are delayed at the first step. The second step is the summation of all the carry-overs, which appear at the first step, with the input ternary numerals of the digits with odd indices. It is important to emphasize that the result of the summation

$$10=1\,\bar{1}\,0\,\bar{1}.\,0\,\bar{1}\,1 \tag{10.49}$$

is a positive number because the ternary representation (10.49) begins with the positive unit 1 and the result of the summation (10.49) is represented in mirror-symmetrical form.

An analysis of all the above examples of ternary mirror-symmetrical summation shows that both the final result of the summation and all intermediate results are mirror-symmetrical numbers, that is, the property of mirror symmetry is an invariant of mirror-symmetrical summation. This means that mirror-symmetrical summation (and subtraction) possesses the important mathematical property of "mirror symmetry" which allows one to check the ternary mirror-symmetrical summation.

10.7.5. *The "Doubling" Mirror-Symmetrical Summator*

Note that the summator in Fig. 10.7 consists of two mirror-symmetrical parts with respect to the 0th single-digit summator $4\Sigma_0$. The left-hand part of the summator in Fig. 10.7 acts as if "performing" the main numerical loading. However, its right-hand part is used for checking the output information according to the principle of "mirror symmetry." There is the possibility of decreasing the "structural redundancy" of the summator in Fig. 10.7 if we use the "doubling" interpretation of the ternary mirror-symmetrical summators as is shown in Fig. 10.8.

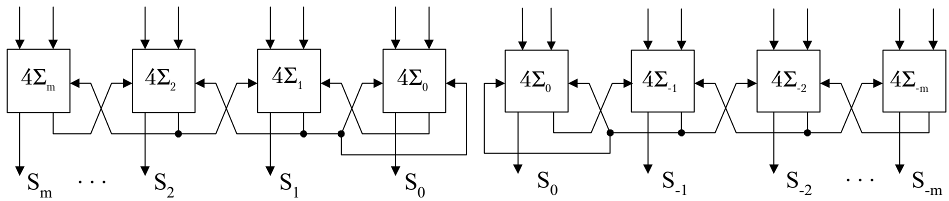


Figure 10.8. The "doubling" interpretation of the ternary mirror-symmetric summator

The ternary mirror-symmetrical summator in Fig. 10.8 consists of two parts. Each part has its own adder of the kind $4\Sigma_0$. Then, the carry-over output of the single-digit summator $4\Sigma_1$ in the left-hand part of the summator is connected with two inputs of the single-digit summator $4\Sigma_0$ and the carry-over output of the single-digit summator $4\Sigma_{-1}$ of the right-hand part of the summator is connected with two inputs of the "doubling" single-digit summator $4\Sigma_0$. This can provide the correct mirror-symmetrical summation in the left-hand and right-hand parts of the summator.

This means that we can use only one part of the summator in Fig. 10.8 for the ternary mirror-symmetrical computations. At the concluding stage of computations we can restore the final ternary mirror-symmetrical representation according to the principle of "mirror symmetry." This means also that the summator in Fig. 10.8 is fault-tolerant mirror-symmetrical summator. Theoretically the lost of the right-hand or left-hand parts of the summator in Fig. 10.8 does not influence on the correct functioning of the summator because all information from the lost part can be restored according to the property of "mirror symmetry."

We can point to a number of the important advantages of the mirror-symmetrical summation and subtraction from the "technical" point of view:

1. The mirror-symmetrical subtraction comes to the mirror-symmetrical summation by the use of the rule (10.47).

2. The mirror-symmetrical summation is performed in the “direct” code, that is, without the use of the notions of the “inverse” and “additional” codes.
3. The sign of the summable numbers is defined automatically because it coincides with the sign of the highest significant ternary numeral of the ternary mirror-symmetrical representation of the summation result.
4. The summation result is always represented in mirror-symmetrical form that allows one to check the process of ternary mirror-symmetrical summation.
5. There is a possibility to design the fault-tolerant mirror-symmetrical summator that allows one to design fault-tolerant arithmetical devices on the basis of ternary mirror-symmetrical arithmetic.

10.8. Mirror-Symmetrical Multiplication and Division

10.8.1. *Mirror-Symmetrical Multiplication*

The following trivial identity for the golden mean powers underlies the mirror-symmetrical multiplication:

$$\tau^{2n}\times\tau^{2n}=\tau^{2(n+m)}. \tag{10.50}$$

A rule for mirror-symmetrical multiplication of two single-digit ternary mirror-symmetrical numbers is given in Table 10.7.

The comparison of Table 10.7 and Table 10.2 shows that the rule of ternary mirror-symmetrical multiplication coincides with the rule of classical ternary-symmetrical multiplication.

The ternary mirror-symmetrical multiplication is carried out in “direct” code. The general algorithm for multiplication of two multi-digit mirror-symmetrical numbers results in the formation of the partial products in accordance with Table 10.7 and their summation in accordance with the rule for mirror-symmetrical summation.

Table 10.7. Mirror-symmetric multiplication $a_k \times b_k$

b_k / a_k	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	0	1

Example 10.7. Multiply the negative mirror-symmetrical number $-6=\bar{1}\,0\,1.\,0\,1$ by the positive mirror-symmetrical number $2=1\,\bar{1}.1$:

$$
\begin{array}{r}
\bar{1}\ 0\ 1.\ 0\ \bar{1} \\
1\ \bar{1}.\ 1\phantom{\ 0\ \bar{1}} \\
\hline
\bar{1}\ 0.\ 1\ 0\ \bar{1} \\
1\ 0\ \bar{1}.\ 0\ 1\phantom{\ \bar{1}} \\
\bar{1}\ 0\ 1\ 0.\ \bar{1}\phantom{\ 1\ \bar{1}} \\
\hline
\bar{1}\ 1\ 0\ \bar{1}.\ 0\ 1\ \bar{1}
\end{array}
$$

The multiplication result in Example 10.7 is formed through the sum of three partial products. The first partial product $\bar{1}0.10\bar{1}$ is the result of multiplication of the mirror-symmetrical multiplier $-6=\bar{1}01.01$ by the lowest positive unit 1 of the mirror-symmetrical multiplier $2=1\bar{1}.1$, the second partial product $10\bar{1}.01$ is the result of multiplication of the same number $-6=\bar{1}01.01$ by the middle negative unit $\bar{1}$ of number $2=1\bar{1}.1$, and, lastly, the third partial product $\bar{1}010.\bar{1}$ is the result of multiplication of the same number $-6=\bar{1}01.01$ by the highest positive unit 1 of number $2=1\bar{1}.1$.

Note that the product $-12=\bar{1}10\bar{1}.01\bar{1}$ is represented in mirror-symmetrical form! As its highest digit is a negative unit $\bar{1}$, it follows that the product is a negative mirror-symmetrical number.

10.8.2. *Mirror-Symmetrical Division*

The ternary mirror-symmetrical division is carried out in accordance with the division rule of the classical ternary-symmetrical number system (see Section 10.3). The general algorithm for the ternary mirror-symmetrical division amounts to the sequential subtraction of the shifted divisor that is then multiplied by the next ternary numeral of the quotient.

Example 10.8. Divide the ternary mirror-symmetrical number $24=110\bar{1}.011$ (the dividend) by the ternary mirror-symmetrical number $2=1\bar{1}.1$ (the divisor).

The *first step* of the division consists of shifting the divisor $2=1\bar{1}.1$ four digits to the left. As the outcome of this shift, we obtain the shifted divisor in the form $1\bar{1}10.000$. By comparing the dividend with the shifted divisor, that is, the numbers $110\bar{1}.011$ and $1\bar{1}10.000$, we conclude that the signs of the compared numbers are the same. In this case we have to write the positive unit 1 to the 3rd digit of the quotient. After that we have to subtract the shifted divisor from the dividend. The subtraction of two numbers amounts to the summation by means of ternary inversion of the subtrahend, that is, the shifted divisor $1\bar{1}10.000$. Hence, at the first step we add the two ternary numbers, namely the ternary-symmetrical dividend and the ternary inverse shifted divisor, as follows:

$$
\begin{array}{rl}
 & 1\ \ 1\ \ 0\ \bar{1}.\ 0\ 1\ 1 \\
 & \underline{\bar{1}\ \ 1\ \ \bar{1}\ 0.\ 0\ 0\ 0} \\
 & 0\ \ \bar{1}\ \ \bar{1}\ \bar{1}.\ 0\ 1\ 1 \rightarrow Q_1 = 100.0. \\
 & \underline{1 \leftrightarrow 1} \\
D_1 = & 1\ \ \bar{1}\ \ 0\ \bar{1}.\ 0\ 1\ 1
\end{array}
$$

Hence, the outcome of division in the first step results in the first intermediate quotient $Q_1 = 1\,0\,0.0$ and the first intermediate dividend $D_1 = 1\,\bar{1}\,0\,\bar{1}.\,0\,1\,1.$

The *second step* of the division is a repetition of the first step for the first intermediate dividend $D_1 = 1\,\bar{1}\,0\,\bar{1}.\,0\,1\,1$. Note that number D_1 contains a positive unit in the same highest digit as the initial dividend. This means that the divisor has to be shifted four digits to the left and then number D_1 has to be compared with the shifted divisor $1\,\bar{1}\,1\,0.\,0\,0\,0$. Since the signs of the compared numbers are the identical, we can form the following intermediate quotient of the kind $Q_2 = 1\,0\,0.0$ in the second step and after that carry out the summation of the number $D_1 = 1\,\bar{1}\,0\,\bar{1}.\,0\,1\,1$ with the new ternary inverse shifted divisor. Hence, the second step results in the following:

$$
\begin{array}{rl}
 & 1\ \bar{1}\ 0\ \bar{1}.\ 0\ 1\ 1 \\
 & \underline{\bar{1}\ 1\ \bar{1}\ 0.\ 0\ 0\ 0} \rightarrow Q_2 = 100.0. \\
D_2 = & 0\ 0\ \bar{1}\ \bar{1}.\ 0\ 1\ 1
\end{array}
$$

Hence, with the outcome of the division at the second step, we obtain the second intermediate quotient $Q_2 = 1\,0\,0.0$ and the second intermediate dividend $D_2 = 0\,0\,\bar{1}\,\bar{1}.\,0\,1\,1$. As the higher significant numeral of number $D_2 = 0\,0\,\bar{1}\,\bar{1}.\,0\,1\,1$ is a negative unit $\bar{1}$, it means that the number D_2 is negative.

The third step consists of the following. By comparing the negative number $D_2 = \bar{1}\,\bar{1}.\,0\,1\,1$ with the divisor $2 = 1\,\bar{1}.\,1$ (the positive number), we can form the third intermediate negative quotient $Q_3 = \bar{1}.\,0$. After that we should subtract the divisor $2 = 1\,\bar{1}.\,1$ multiplied by the negative unit $\bar{1}$, that is, number $-2 = \bar{1}\,1.\,\bar{1}$, from the number D_2. Taking into consideration the fact that the ternary subtraction is the summing of the ternary inverse number $-2 = \bar{1}\,1.\,\bar{1}$, meaning that in the case of a negative quotient the next step of the division amounts to the summation of the divisor with the preceding intermediate dividend D_2, that is,

$$
\begin{array}{rl}
 & \bar{1}\ \ \bar{1}.\ \ 0\ 1\ 1 \\
 & \underline{1\ \ \bar{1}.\ \ 1\ \ \ \ \ \ } \\
 & 0\ \ 1.\ \ 1\ 1\ 1 \rightarrow Q_3 = \bar{1}.0. \\
 & \underline{\bar{1} \leftrightarrow \bar{1}\ \ \ \ \ \ } \\
D_3 = & \bar{1}\ \ 1.\ \ 0\ 1\ 1
\end{array}
$$

The fourth step:

$$\begin{array}{rcccccc} & \bar{1} & 1. & 0 & 1 & 1 & \\ & 1 & \bar{1}. & 1 & & & \to Q_4 = \bar{1}.0. \\ \hline D_4 = & 0 & 0. & 1 & 1 & 1 & \end{array}$$

$Q_4 = 1.0$

The fifth step:

$$\begin{array}{rcccc} & 0. & 1 & 1 & 1 \\ & 0. & \bar{1} & 1 & \bar{1} \\ \hline & 0. & 0 & \bar{1} & 0 \to Q_5 = 0.01. \\ & & 1 & \leftrightarrow & 1 \\ \hline D_5 = & 0. & 1 & \bar{1} & 1 \end{array}$$

The sixth step:

$$\begin{array}{rcccc} & 0. & 1 & \bar{1} & 1 \\ & 0. & \bar{1} & 1 & \bar{1} \to Q_6 = 0.01. \\ \hline & 0. & 0 & 0 & 0 \end{array}$$

The division is over. The result of the division is formed by means of the summing of the intermediate quotients $Q_1+Q_2+Q_3+Q_4+Q_5+Q_6$ as follows:

$$\begin{array}{ccccccc} & 1 & 0 & 0. & 0 & & \\ & 1 & 0 & 0. & 0 & & \\ & & & \bar{1}. & 0 & & \\ & & & \bar{1}. & 0 & & \\ & & & 0. & 0 & 1 & \\ & & & 0. & 0 & 1 & \\ \hline & \bar{1} & 0 & 1. & 0 & \bar{1} & \\ & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\ 1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\ \hline 1 & \bar{1} & 0 & 1. & 0 & \bar{1} & 1 \end{array}$$

Note that the division result $12 = 1\bar{1}01.0\bar{1}1$ is represented in mirror-symmetrical form.

10.8.3. *The Main Arithmetical Advantages of Mirror-Symmetrical Multiplication and Division*

We can formulate all arithmetical advantages of mirror-symmetrical multiplication and division as follows:

1. Mirror-symmetrical multiplication and division is reduced to mirror-symmetrical summation.
2. Mirror-symmetrical multiplication and division are performed in the "direct" code, that is, without the use of the notions of the "inverse" and "additional" codes.
3. The sign of the results of mirror-symmetrical multiplication and division is automatically determined because it is equal in sign with the highest significant ternary numeral of the ternary mirror-symmetrical representation of the result of mirror-symmetrical multiplication and division.
4. The results of mirror-symmetrical multiplication and division are always represented in mirror-symmetrical form which allows one to check the process of ternary mirror-symmetrical multiplication and division.
5. We can design the fault-tolerant summator on the basis of the ternary mirror-symmetrical multi-digit summator (see Fig. 10.8) and the mirror-symmetrical multiplication and division amount to mirror-symmetrical summation, meaning that we can design a fault-tolerant arithmetical device on the basis of ternary mirror-symmetrical arithmetic.

10.9. Typical Devices of Ternary Mirror-Symmetrical Processors

10.9.1. *A Converter for Binary "Golden" Code to Ternary Mirror-Symmetrical Representation*

The simplicity of the rule for conversion of binary golden representations to ternary mirror-symmetrical representations is one of the important technical advantages of ternary mirror-symmetrical representation. The combinatorial circuit for this conversion is shown in Fig. 10.9.

The converter in Fig. 10.9 consists of identical logic circuits of the kind LC_i. The logic circuit LC_i transforms the binary 3-digit golden code combination $a_{i+1}\,a_i\,a_{i-1}$ represented in minimal form into the ternary numeral b_i according to Table 10.8.

Table 10.8.
Conversion of a binary golden representation to a ternary mirror-symmetrical representation

a_{i+1}	a_i	a_{i-1}	$\to$	b_i	=	b_i'	b_i''
0	0	0	$\to$	0	=	0	0
0	0	1	$\to$	1	=	0	1
0	1	0	$\to$	1	=	0	1
1	0	0	$\to$	$\bar{1}$	=	1	0
1	0	1	$\to$	0	=	0	0

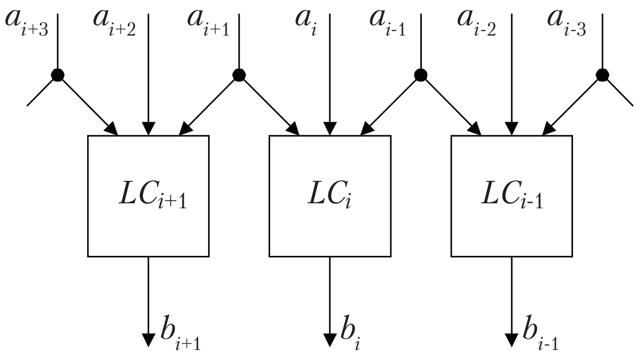


Figure 10.9. The combinative circuit for conversion of binary golden representations to ternary mirror-symmetric representations

If we use the binary numerals $b_i' b_i''$ for encoding ternary numerals, we obtain the following simple logical equalities describing the functioning of logic circuits of the kind LC_i:

$$b_i' = a_{i+1} \wedge \overline{a_i} \wedge \overline{a_{i-1}}$$

$$b_i'' = \left(\overline{a_{i-1}} \wedge \overline{a_i} \wedge a_{i-1}\right) \vee \left(\overline{a_{i-1}} \wedge a_i \overline{a_{i-1}}\right).$$

10.9.2. *Technical Realization of Mirror-Symmetrical Checking*

It is well-known that the ternary mirror-symmetrical representation of the integer

$$N=b_m\, b_{m-1} \ldots b_2\, b_1\, b_0.\, b_{-1}\, b_{-2} \ldots b_{-(m-1)}\, b_{-m} \tag{10.51}$$

consists of two parts, where b_i equals the ternary numerals $\{\overline{1}, 0, 1\}$.

In accordance with the property of "mirror symmetry," the left-hand part of the ternary mirror-symmetrical representation (10.51) is a mirror reflection of its right-hand part with respect to the 0-th digit b_0. This means that for the ternary representation (10.51), the following "checking equalities" are valid:

$$\begin{array}{l} b_m = b_{-m} \\ b_{m-1} = b_{-(m-1)} \\ \ldots \\ b_1 = b_{-1} \end{array}. \tag{10.52}$$

Let us now introduce the "double" digit b_0' with index 0. Then, we may write one more "checking equality":

$$b_0' = b_0. \tag{10.53}$$

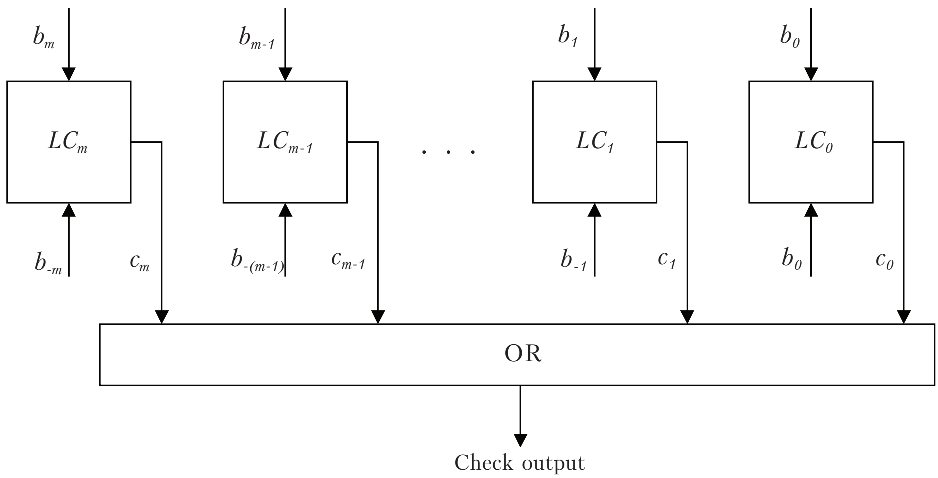


Figure 10.10. The combinatorial circuit for the checking of mirror-symmetrical representation

The logical circuit for the parallel check of the ternary mirror-symmetrical number (10.51) according to mirror-symmetrical properties (10.52) and (10.53) is shown in Fig. 10.10.

The combinative circuit for checking mirror-symmetrical representation (10.51) consists of $(m+1)$ identical logic circuits of the kind LC_i (i=0,1,2,3,...,m) and the logic element OR (Fig. 10.10). The output of element OR is the checking output for the combinatorial circuit in Fig. 10.10.

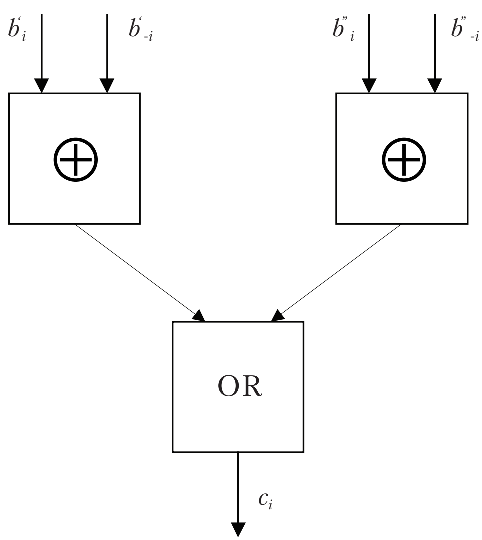


Figure 10.11. The binary realization of the logic circuit LC_i

If we use binary code in the representation of ternary numerals, that is,
$\bar{1}=10, 0=00, 1=01,$
we can represent the logic circuit LC_i as a Boolean logic circuit that amounts to two summators in modulo 2 and the logic element *OR* (Fig. 10.11).

10.9.3. *Technical Realization of Ternary Mirror-Symmetrical Arithmetic*

Comparing ternary-symmetrical arithmetic based upon the following property of ternary numbers:

$$3^n + 3^n = 3^{n+1} - 3^n = 1+1 = 1\,\bar{1} \qquad (10.54)$$

with ternary mirror-symmetrical arithmetic based upon the following golden mean property

$$\tau^{2n} + \tau^{2n} = \tau^{2(n+1)} - \tau^{2n} + \tau^{2(n-1)} = 1+1 = 1\,\bar{1}\,1, \qquad (10.55)$$

we see the similarity between (10.54) and (10.55) from a technical point of view. It is true that the rule for formation of an intermediate sum and carry-over at the summation of two ternary single-digit mirror-symmetrical numbers follows from (10.55). This rule amounts to the following. The identity (10.55) displays the rule for formation of intermediate sum $\bar{1}$ and carry-over 1 at the summation of 1+1. In accordance with (10.55) at the summation of 1+1, the intermediate sum equal to $\bar{1}$ and the carry-over equal to 1 appear. However, carry-over 1 is spreading to the left-hand and right-hand digits with respect to the initial digit. We can see that the rules of the formation for intermediate sum and carry-over at the addition of the ternary symmetrical and ternary mirror-symmetrical single-digit numbers coincide. The only difference is in the spreading of the carry-over. For the case (10.54) the carry-over is spreading to the left, that is, to the side of the higher digit, for the case (10.55) the carry-over is simultaneously spreading symmetrically with respect to the initial digit, that is, to the left-hand and right-hand adjacent digits.

A very important technical conclusion follows from this examination, namely that the logic circuits for the realization of the single-digit transformation of ternary-symmetrical arithmetic and ternary mirror-symmetrical arithmetic are identical.

10.9.4. *Ternary Mirror-Symmetrical Accumulator*

In Fig. 10.7 above we developed the ternary mirror-symmetrical multidigit summator. This summator is the basis for the mirror-symmetrical accumulator (Fig. 10.12), that is, the main device of the ternary mirror-symmetrical processor.

The accumulator has a traditional structure and consists of a ternary mirror-symmetrical multi-digit summator *AD*, an intermediate ternary register RG_1, which memorizes the sum S_1+A that appears at the *AD*-output and at the accumulating ternary register RG_2.

The ternary mirror-symmetrical accumulator in Fig. 10.12 is a universal device of a ternary mirror-symmetrical processor and underlies the mirror-symmetrical counter, summator, subtractor, multiplier and divisor. By using these additional devices, we can design the following computer devices:

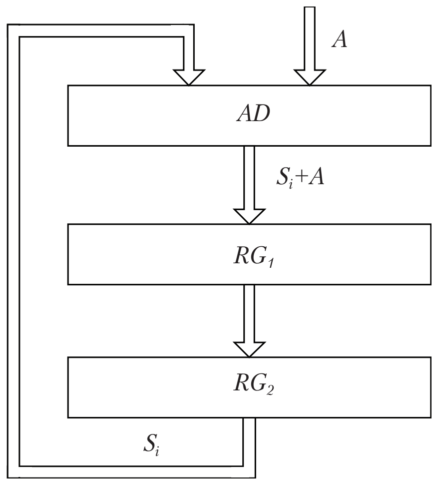


Figure 10.12. Mirror-symmetric accumulator

(a) If we send the positive (1) or negative ($\bar{1}$) units sequentially to input *A*, we turn the accumulator into a summing or subtracting mirror-symmetrical counter.
(b) If we place the ternary inverter before input *A*, we turn the accumulator into a mirror-symmetrical subtractor.
(c) If we place the device for the formation of partial products $Ab_i\tau^i$ before input *A*, we turn the accumulator into a multiplier.
(d) As the mirror-symmetrical division is tantamount to the shift of the divisor and its subtraction from the dividend, the accumulator can be used for the performance of ternary mirror-symmetrical division.

10.10. Matrix and Pipeline Mirror-Symmetrical Summators

10.10.1. *Matrix Mirror-Symmetrical Summator*

It is well known that digital signal processors put forward high demands on the speed of arithmetical devices. The different special structures (matrix, pipeline, etc.) are elaborated for this purpose. We can show that ternary mirror-symmetrical arithmetic gives rise to interesting possibilities for the design of fast arithmetic processors.

Now let us examine the matrix ternary multi-digit mirror-symmetrical summator in Fig. 10.13. Each cell of the summator in Fig. 10.13 is a full single-digit

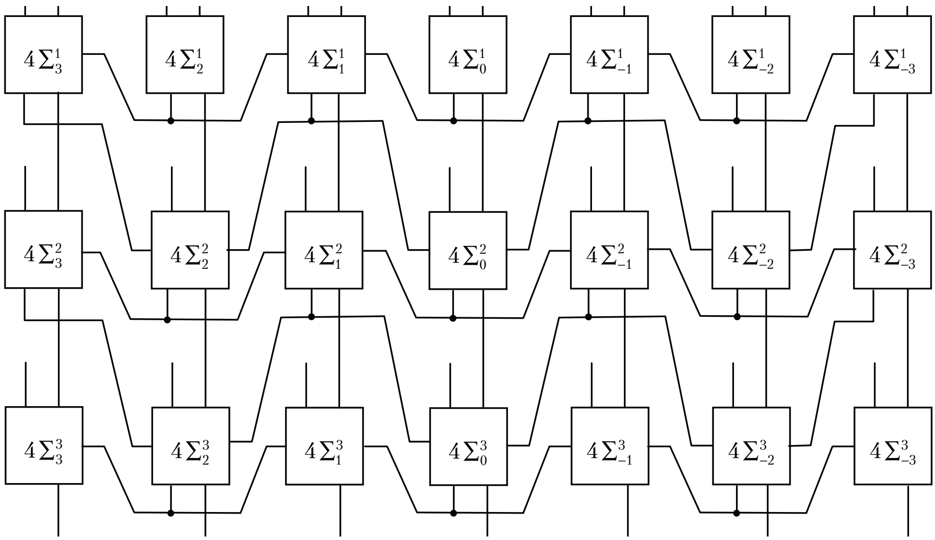


Figure 10.13. Matrix ternary mirror-symmetric summator

ternary-symmetrical summator that has 4 inputs and 2 outputs (see Fig. 10.6-b). The matrix summator in Fig. 10.13 consists of 21 single-digit full summators that are disposed in the form of a 7×3-matrix. Each ternary single-digit summator has the designation, where the number 4 means that the summator has 4 ternary inputs, the indexes *i* and *k* mean that the summator belongs to the *i*-th digit of the ternary mirror-symmetrical representation (10.26) and that the summator is placed in the *k*-th row of the matrix summator in Fig. 10.13.

Inputs of the single-digit summators $4\Sigma_3^1 - 4\Sigma_{-3}^1$ in the first row are the multi-digit input of the matrix ternary-symmetrical summator in Fig. 10.13. The output of the intermediate sum of each single-digit summator is connected with the corresponding input of the adjacent single-digit summator in the same column.

The outputs of the intermediate sum of single-digit summators

$$4\Sigma_3^1, 4\Sigma_2^1, 4\Sigma_1^1, 4\Sigma_0^1, 4\Sigma_{-1}^1, 4\Sigma_{-2}^1, 4\Sigma_{-3}^1$$

of the last row build up the multi-digit output of the matrix mirror-symmetrical summator.

The main peculiarity of the matrix mirror-symmetrical summator in Fig. 10.13 consists of the special design of the connections between carry-over outputs of the single-digit summators with inputs of the adjacent single-digit summators. The carry-over outputs of all the single-digit summators with the even lower-indices (2, 0, -2) are connected with the corresponding inputs of the adjacent single-digit summators placed in the same row. However, the carry-over outputs of all the single-digit summators with the odd lower-indices (3, 1, -1, -3) are connected with the corresponding inputs of adjacent single-digit summators placed in the lower row. Note that such organization of the carry-over connections allows for the elimination of the above "swing" phenomenon.

Let's examine the operation of the matrix mirror-symmetrical summator for the case of the summation of two equal ternary mirror-symmetrical numbers:

$A = 0\,1\,1\,1.\,1\,1\,0$ and $B = 0\,1\,1\,1.\,1\,1\,0$.

The addition is carried out in 2 stages. Each stage consists of using only one row of the single-digit summators and two steps.

10.10.1.1. *The first stage*

In accordance with Fig. 10.13 *the first step* of the first stage is the following: the single-digit summators in the first row with the even lower-indices ($4\Sigma_2^1, 4\Sigma_0^1, 4\Sigma_{-2}^1$) form the intermediate sums, which enter the inputs of the summators in the second row, and the carry-overs, which enter the corresponding inputs of the single-digit summators with odd lower-indices in the first row ($4\Sigma_3^1, 4\Sigma_1^1, 4\Sigma_{-1}^1, 4\Sigma_{-3}^1$). Such transformation of the code information can be represented in the following form:

$$\begin{array}{ccccccc}
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
\hline
 & \bar{1} & & \bar{1} & & \bar{1} & \\
 & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1
\end{array}.$$

Hence, the first step is the formation of intermediate sums and carry-overs at the outputs of single-digit summators with even lower-indices (2, 0, -2).

In *the second step* of the first stage, the single-digit summators with odd lower-indices (3, 1, -1, -3) begin to operate. In accordance with the entered carry-overs, they form intermediate sums and carry-overs, which enter the single-digit summators in the lower row, that is,

$$\begin{array}{ccccccc}
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
0 & 1 & 1 & 1. & 1 & 1 & 0 \\
\hline
 & \bar{1} & & \bar{1} & & \bar{1} & \\
 & \downarrow & 1 & \leftrightarrow & 1 & \downarrow & \\
1 & \leftrightarrow & 1 & & 1 & \leftrightarrow & 1 \\
\hline
1 & \bar{1} & 1 & \bar{1}. & 1 & \bar{1} & 1 \\
 & 1 & \leftrightarrow & 1 & \downarrow & & \\
 & & & 1 & \leftrightarrow & 1 &
\end{array}.$$

The first stage is finished. We see that the results of the first stage are composed of some intermediate sums and carry-overs, which enter the summators in the lower row.

10.10.1.2. *The second stage*

The single-digit summators of the second row with even lower-indices ($4\Sigma_2^2, 4\Sigma_0^2, 4\Sigma_{-2}^2$) form intermediate sums, which enter the corresponding inputs of the summators in the lower row, and the carry-overs, which enter the corresponding inputs of summators in the same row with odd lower-indices ($4\Sigma_3^2, 4\Sigma_1^2, 4\Sigma_{-1}^2, 4\Sigma_{-3}^2$), that is,

$$\begin{array}{ccccccc}
1 & \bar{1} & 1 & \bar{1}. & 1 & \bar{1} & 1 \\
 & 1 & \leftrightarrow & 1 & \downarrow & & \\
 & & & 1 & \leftrightarrow & 1 & \\
\hline
1 & 0 & 1 & 1. & 1 & 0 & 1
\end{array}$$

As all carry-overs formed in this stage become equal to 0, it means that the summation is finished with the second stage (note: this is valid only for the given case). The sum obtained enters the inputs of the summators in the lower row $4\Sigma_3^3 - 4\Sigma_{-3}^3$ and then appears at the output of the matrix summator.

10.10.2. *Pipeline Mirror-Symmetrical Summator*

There are two directions for the extension of the functional possibilities of the matrix mirror-symmetrical summator in Fig. 10.13. If we set the ternary registers, which consist of the flip-flop-flaps (see Fig. 10.5-b) between the adjacent rows of the single-digit summators, then the above matrix summator turns into the *Pipeline Ternary Mirror-Symmetrical Summator*. In fact, the code information from the preceding rows of the single-digit summators is memorized in the corresponding registers. After that, summators of the preceding row become prepared for further processing. Then, summators of the lower row process the code information, which enters the summators of the lower row. However, simultaneously, summators of the top row start processing new input code information. "This means that since the given moment we start to obtain the sums of the numbers A_1+B_1, A_2+B_2, ..., A_n+B_n, which enter the input of the summator during the time period $2\Delta\tau$, where $\Delta\tau$ is the delay time of the single-digit summator."

10.10.3. *Pipeline Mirror-Symmetrical Multiplier*

Another possible extension of functional possibilities of the pipeline summator is as follows. We can see in Fig. 10.13 that all single-digit summators of the lower rows have "free" inputs. We can use these inputs as the new multi-digit inputs of the pipeline summator. By using these multi-digit inputs, we can turn the pipeline summator into the pipeline multiplier. In this case the mirror-symmetrical multiplication of two mirror-symmetrical numbers $A(1)\times B(1)$ is carried out in the following manner. The summators of the first row summarize the first two partial products $P_1^1 + P_2^1$. This code information enters the summators in the second row. If we send the 3rd partial product P_3^1 to the "free" inputs of the summators in the second row, we obtain the sum $P_1^1 + P_2^1 + P_3^1$ on the outputs of the summators in the second row. In this case the first row can start adding the first two partial products of the next pair of multiplied numbers $A(2)\times B(2)$. The "free" inputs of the summators in the 3rd row are used to receive the next partial product P_4^1 of the first pair of the multiplied numbers $A(1)\times B(1)$, etc. We can now see that the pipeline summator in Fig. 10.13 allows multiplying many mirror-symmetrical numbers in the pipeline regime. In this connection the multiplication speed is determined by the time $2\Delta\tau$, where $\Delta\tau$ is the delay time of the single-digit summator.

10.11. Ternary Mirror-Symmetrical Digit-to-Analog Converter

10.11.1. *The "Golden" Resistor Divider for Ternary Mirror-Symmetrical Representation*

In chapter 9 we examined different variants of the "golden" resistor dividers. All these dividers are based upon the electrical circuit in Fig. 9.1. The difference between the dividers consists of the choice of resistance values of different resistors of the circuit in Fig. 9.1. In this Section we will observe a new kind of "golden" divisor that is connected with ternary mirror-symmetrical representation (10.26). Choose the resistors of the divisor in Fig. 9.1 as follows:

$$R1 = R2;\ R3 = \tau R, \tag{10.56}$$

that is, all resistors which form the divisor in Fig. 9.1 have equal resistance R, except the end resistors with resistance $R3 = \tau R$, where $\tau = (1+\sqrt{5})/2$, which is of course the golden mean.

It is easy to calculate the equivalent resistances R_{e1} and R_{e2} of the resistive circuits to the left and to the right of the "connecting points" 0,1,2,3,4 and also the equivalent resistance R_{e3} of the divider in the "connecting points" 0,1,2,3,4:

$$R_{e1} = \frac{R\times\tau R}{R+\tau R} = \tau^{-1}R \tag{10.57}$$

$$R_{e2}=R+\tau^{-1}R=\tau R \tag{10.58}$$

$$R_{e3}=\frac{\tau R\times\tau^{-1}R}{\tau R+\tau^{-1}R}=\frac{1}{2}R. \tag{10.59}$$

Further it is easy to show that if we connect the electrical generator I to some "connecting point," we form at this point the electrical voltage U that is transmitted to the adjacent "connecting point" with voltage transmission coefficient τ^{-2}, that is, the electrical voltage in the adjacent points is equal to

$$U/\tau^2. \tag{10.60}$$

10.11.2. *Ternary Mirror-Symmetrical Digit-to-Analog Converter*

The golden resistor divider based upon the resistors (10.56) can be used for designing the ternary mirror-symmetrical digit-to-analog converter (DAC) represented in Fig. 10.14. This DAC consists of 5 digits (n in the general case). The middle point C corresponds to the 0th digit a_0 of the input "golden" mirror-symmetrical representation $a_2\, a_1\, a_0.\, a_{-1}\, a_{-2}$ ($a_m\, a_{m-1} \ldots a_0.\, a_{-1}\, a_{-2} \ldots, a_{-m}$ in general case of the number N). The ternary digits a_i $(i=0,\pm1,\pm2,\pm3,\ldots)$ control the special circuits I_0, which are connected with the corresponding connection points of the "golden" mirror-symmetrical divider. The circuit I_0 consists of the standard electrical generator I_0 and the 3-position electrical key that are controlled by the ternary digits a_i according to the following rule. If $a_i=1$, then the standard electrical current is switched on to the corresponding point of the "golden" mirror-symmetrical resistor divider in the "positive," that is,

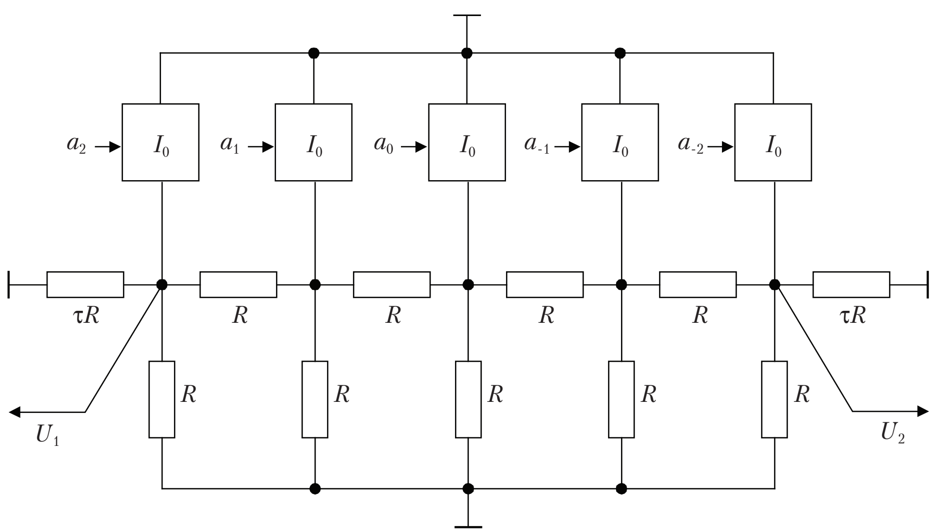


Figure 10.14. The golden mirror-symmetrical *DAC*

$+I_0$. If $a_i = -1$, then the standard electrical current is switched on to the corresponding point of the golden mirror-symmetrical resistor divider in the "negative," that is, $-I_0$. At last, if $a_i = 0$, then the standard electrical current I_0 is not switched on to the corresponding connection point.

The golden mirror-symmetrical DAC has two mirror-symmetrical outputs, U_1 and U_2. Taking into consideration the basic properties (10.57)-(10.59) of the golden mirror-symmetrical divider, we can prove that the mirror-symmetrical outputs U_1 and U_2 are connected with the golden mirror-symmetrical input code $a_m a_{m-1} ... a_1 a_o . a_{-1} a_{-2} ... a_{-m}$ as follows:

$$U_1 = U_2 = \frac{I_0}{2} \sum_{i=-m}^{m} a_i \tau^{2i}. \tag{10.61}$$

The fundamental checking property of the "golden" mirror-symmetrical *DAC* is the following:

$$U_1 = U_2. \tag{10.62}$$

A breach of the equality (10.62) is an indication of errors in the *DAC*. Thus, the golden mirror-symmetrical *DAC* in Fig. 10.14 is a self-checking *DAC* that allows for persistent checking of the *DAC* operation according to (10.62).

10.12. Conclusion

1. On the one hand, at the dawn of the computer era the problem of choosing the "optimal" number system for electronic computers was brilliantly solved by eminent American physicist and mathematician John von Neumann, who gave emphatic preference to the binary system in electronic computers. The famous "John von Neumann Principles" include three basic ideas, which underlie all modern electronic computers: the *Binary System, Binary (Boolean) Logic,* and the *Binary Memory Element ("Flip-Flop").*

2. On the other hand, the outstanding Soviet engineer Nikolay Brousentsov suggested another fundamental principle for computer design [180]. This principle was called the "Brousentsov Ternary Principle" [104] because it was based upon the "ternary" approach: the*Ternary-Symmetrical Numeral System, Ternary Logic,* and the *Ternary Memory Element ("Flip-Flap-Flop").* That idea was used in the first ternary "Setun" computer designed at Moscow University in 1958. The ternary-symmetrical numeral system has a number of essential advantages in comparison to the

binary system. The Russian scientist Dmitry Pospelov and American scientist Donald Knuth each expressed an opinion that one day the replacement of "flip-flop" by "flip-flap-flop" would occur.

3. In 2002, Alexey Stakhov developed the so-called *Ternary Mirror-Symmetrical Arithmetic* based upon *Ternary Mirror-Symmetrical Representation* [104]. This numeral system is an original synthesis of the classical ternary-symmetrical system and Bergman's system [86]. This is a positional numeral system using the ternary numerals {1, 0, -1} for number representation. However, its radix is the irrational number $\tau^2 = \left(3+\sqrt{5}\right)/2$. In this numeral system all integers are represented in *Mirror-Symmetrical Form*. This means that at the representation of integers, the 0th digit divides the number representation into two mirror-symmetrical parts. When increasing a number, its ternary mirror-symmetrical representation widens symmetrically both to the left and to the right with respect to the 0th digit. This unique mathematical property creates a very simple method of checking numbers in computers. It is demonstrated that the mirror-symmetrical property is invariant with respect to arithmetic operations, that is, the results of all arithmetic operations have mirror-symmetrical forms. This means that mirror-symmetrical arithmetic can be used for designing fault-tolerant processors and computers.

4. In conclusion the author would like to express his gratitude to the outstanding scientists of contemporary computer science, Dr. **Donald Knuth**, Emeritus Professor at Stanford University and author of the famous book *The Art of Computer Programming* [181], and Dr. **Nikolay Brousentsov** of Moscow University, the designer of the first ternary computer "Setun." They were the first scientists to congratulate this author on his publication of the article *"Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-Symmetrical Arithmetic"* [104]. High appreciation for ternary mirror-symmetrical arithmetic [104] by these outstanding computer specialists gives hope that it will become a source for new computer projects in the near future.

Chapter 11

A New Coding Theory Based on a Matrix Approach

11.1. A History of Coding Theory

11.1.1. *Types of Coding*

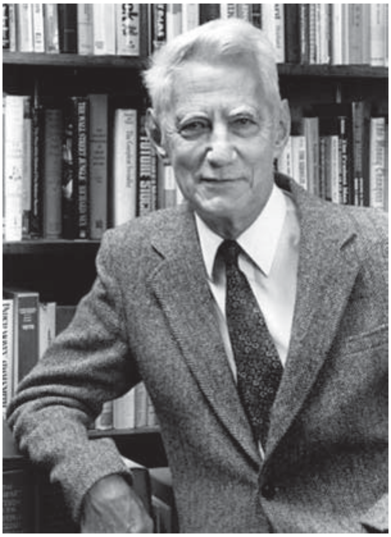
We can find definitions of the notions of "code," "coding," "encoding/decoding" in *Wikipedia, the Free Encyclopedia*. In communications, *Code* or *Coding* is a rule for converting a piece of information (for example, a letter, word or phrase) into another form or representation, not necessarily of the same type. In communications and information processing, *Encoding* is a process, by which information from a source is converted into symbols to be communicated or transmitted. *Decoding* is the reverse process that converts these code symbols back into information understandable to a receiver. The three basic purposes for using coding in information systems are: *Compression of Information, Detection and Correction of Errors in Information Channels* and *Ensuring Confidentiality*. Accordingly, there are three types of coding:

1. *Source Coding* (Data compression)
2. *Channel Coding* (Error detection and correction)
3. *Cryptographic Coding* (Ensuring confidentiality)

The first, *Source Coding*, compresses the data from a source in order to transmit it more efficiently. We see this practice every day on the Internet where the common "Zip" data compression is used to reduce the network load and to make files smaller. The second, *Channel Coding,* adds extra bits, commonly called *Redundancy Bits*, to data bits to protect the transmission of data from noise that is present on the transmission channel. The third, *Cryptographic Coding*, is used for ensuring the confidentiality of communications.

11.1.1.1. *Claude Elwood Shannon*

The source coding theory is a part of *Information Theory* created by American scientist Claude Shannon [151].



Claude Elwood Shannon
(1916-2001)

Claude Elwood Shannon, an American electrical engineer and mathematician, was called "the father of information theory." He was also the founder of a practical theory for designing digital circuits.

In 1932 Shannon entered the University of Michigan, where he took a course that acquainted him with the work of George Boole. Graduating in 1936 with two bachelor's degrees, electrical engineering and mathematics, he then began graduate study at the Massachusetts Institute of Technology, where he worked on Vannevar Bush's differential analyzer, an analog computer. While studying the complicated circuits of the differential analyzer, Shannon saw that Boole's concepts could be employed with great utility. An article drawn from his 1937 master's thesis, *A Symbolic Analysis of Relay and Switching Circuits*, was published in the 1938 issue of *Transactions of the American Institute of Electrical Engineers*. Here Shannon proved that Boolean algebra and binary arithmetic can be used for simplification of circuits on electromechanical relays then used in telephone routing switches. He also proved that it is possible to use relay circuits to solve Boolean algebra problems. By using this property of electrical switches, he proved that Boolean logic is the basic concept underlying all electronic digital computers. Shannon's work became the foundation for the practical design of digital circuits.

11.1.1.2. *Mathematical Theory of Communication*

In 1948 Shannon published the article *A Mathematical Theory of Communication* in two parts in the July and October issues of the *Bell System Technical Journal*. This work focused on the problem of how to best encode the information a sender wants to transmit. In this fundamental work he used probability theory in introducing *Information Entropy* as a measure of indeterminacy. Information entropy became known as the dominant concept of "information theory." The book, *The Mathematical Theory of Communication* co-authored with Warren Weaver, is accessible to non-specialists. Shannon's concepts were also popularized in John Robinson Pierce's *Symbols, Signals, and Noise*. Another notable paper Shannon published in 1949 is *Communication Theory of Secrecy Systems*, a major contribution to the development of the mathematical theory of cryptography. He is also credited with introducing *Sampling Theory*, connected with representing a continuous-time signal by a discrete set of samples.

Information Theory is a discipline in applied mathematics involving the quantification of data with the goal of enabling as much data as possible to be reliably stored on a medium and/or transmitted over a channel. The measure of data, known as *Information Entropy*, is usually expressed by the average number of bits needed for storage or communication. Applications of fundamental topics of information theory include ZIP files, lossless data compression, and DSL (channel coding). It is a field at the crossroads of mathematics, statistics, computer science, physics, and electrical engineering, whose impact was crucial to the invention of the CD, the feasibility of mobile phones, the development of the Internet, the study of linguistics and of human perception, amongst numerous other fields.

The main concepts of information theory can be grasped by examination of the most widespread means of human communication: language. Two important aspects of a good language are as follows: First of all, the most widespread words (e.g., “a,” “the,” “I”) should be shorter than less wide-spread words (e.g., “benefit,” “generation,” “mediocre”), so that sentences will not be too long. Such a tradeoff in word length is analogous to data compression and is the essential aspect of source coding. Source coding and channel coding are the fundamental concepts of information theory.

The decisive event, which established the discipline of information theory and brought it to immediate worldwide attention, was the publication of Claude E. Shannon’s (1946) classic article, *A Mathematical Theory of Communication,* in the *Bell System Technical Journal* in July and October of 1948. In this revolutionary and groundbreaking article (work which Shannon had substantially completed at Bell Labs by the end of 1944), Shannon for the first time introduced the quantitative model of communication as a statistical process underlying information theory.

11.1.2. *Error-Correction Codes*

Error-Correction Code or *ECC* is a code in which each data signal conforms to specific rules of construction so that deviations from this construction in the received signal can generally be automatically detected and corrected [177, 182]. It is used in computer data storage, for example in dynamic RAM, and in data transmission. Some examples include Hamming code, Reed-Solomon code, Reed-Muller code, Binary Golay code, convolution code, turbo code [182]. The simplest error-correction codes, for example Hamming codes, can correct single-bit errors (single error correction) and detect double-bit errors (double error detection). Other codes can detect or correct multi-bit errors. ECC memory provides greater data accuracy and extended system uptime by protecting against "soft errors" in computer memory.

11.1.2.1. *Codes Predating Hamming*

A number of simple error-detection codes were used prior to Hamming codes.

Parity. Parity adds a single bit that indicates whether the number of bits in the preceding data was even or odd. If a single bit is changed in transmission, the message will change parity and the error can be detected at this point. (Note that the bit that changed may have been the parity bit itself!) The most common convention is that a parity value of 1 indicates that there is an odd number of 1's in the data, and a parity value of 0 indicates that there is an even number of 1's in the data. In other words, the data and the parity bit together should contain an even number of 1's. Parity checking is not very robust, since if the number of bits changed is even, the check bit will be valid and the error will not be detected. Moreover, parity does not indicate which bit contained the error, even when it can detect error. Thus, the data must be discarded entirely, and re-transmitted from the source.

Two-out-of-five code. In the 1940s Bell used a slightly more sophisticated code known as the two-out-of-five code. This code ensured that every block of five bits (known as a *5-block*) had exactly two 1's. The computer could tell if there was an error if in its input there were not exactly two 1's in each block. Two-of-five was still only able to detect single bits; if one bit flipped to 1 and another to 0 in the same block, the two-of-five rule remained true and the error would go undiscovered.

Repetition. Another code in use at the time repeated every data bit several times in order to ensure that it got through. For instance, if the data bit to be sent was a 1, an n=3 *Repetition Code* would send "111." If the three bits received were not identical, an error occurred. If the channel were clean enough, most of the time only one bit would change in each triple. Therefore, 001, 010, and 100 each corresponded to a 0 bit, while 110, 101, and 011 corresponded to a 1 bit, as though the bits counted as "votes" towards what the original bit was. A code with this ability to reconstruct the original message in the presence of errors is known as an error-correction code. However, such codes cannot correctly repair all errors. In our example, if the channel flipped two bits and the receiver got "001," the system would detect the error, but conclude that the original bit was 0, which is incorrect. If we increase the number of times we duplicate each bit to four, we can detect all two-bit errors but cannot correct them (the votes "tie"); at five, we can correct all two-bit errors, but not all three-bit errors.

11.1.2.2. *Richard Wesley Hamming*

Richard Wesley Hamming was an American mathematician whose work had numerous implications for computer science and telecommunications.

Richard Wesley Hamming was a professor at the University of Louisville during World War II, and left to work on the Manhattan Project in 1945, programming one of the earliest electronic digital computers to calculate the solution to equations provided by the project's physicists. The objective of the program was to discover if the detonation of an atomic bomb would ignite the atmosphere. The result of the computation was that this would not occur, and so the United States used the bomb, first in a test in New Mexico, and then twice against Japan.

Richard Wesley Hamming (1915 – 1998)

Later he worked at the Bell Telephone Laboratories from 1946 to 1976, where he collaborated with Claude E. Shannon. In July 1976 he moved to the Naval Postgraduate School, where he worked as an Adjunct Professor until 1997, when he became Professor Emeritus.

11.1.2.3. *Hamming Code*

In telecommunication, a *Hamming Code* is a linear error-correction code named after its inventor, Richard Hamming. Hamming codes can detect and correct single-bit errors, and can detect (but not correct) double-bit errors. In contrast, the simple parity code cannot detect errors where two bits are transposed, nor correct the errors it finds. If more error-correction bits are included with a message, and if those bits can be arranged in such a way that different incorrect bits produce different error results, then bad bits could be identified. In a 7-bit message, there are seven possible single bit errors, so three error control bits could potentially specify not only that an error had occurred, but also which bit caused the error.

Hamming studied the existing coding schemes, including two-of-five, and generalized their concepts. To start with he developed a nomenclature to describe the system, including the number of data bits and error-correction bits in a block. For instance, parity includes a single bit for any data word, so assuming ASCII words with 7-bits, Hamming described this as an (8,7)-code, with eight bits in total, of which 7 are data. The repetition example would be a (3,1)-code, following the same logic.

Hamming also noticed the problems with flipping two or more bits, and described this as the "distance" (it is now called the *Hamming Distance*, after him). Parity has a distance of 2, as any two bit flips will be invisible. The (3,1) repetition has a distance of 3, as three bits need to be flipped in the same triple to obtain another code word with no visible errors. A (4,1) repetition (each bit is repeated four times) has a distance of 4, so flipping two bits in the same group will no longer go undiscovered.

11.1.2.4. *General Principles of Error Detection and Correction*

The main idea of the Hamming code and other error-correction codes (Reed-Solomon code, Reed-Muller code, Golay code, and so on) is set forth in [177, 182] and consists of the following. First consider the initial code combination that consists of n data bits. We add to the initial code combination m error-correction bits and build up the k-digit code combination of the error-correction code, or (k,n)-code, where $k=n+m$. The error-correction bits are formed from the data bits as the sums by module 2 of certain groups of data bits. It is clear that there are 2^n different k-digit binary combinations of error-correction code $a_1\ a_2\ a_3\ ...\ a_n$. These binary combinations are referred to as *Allowed Binary Combinations*. By using k digits we can form $2^k=2^{n+m}$ different binary combinations. Then we divide them into two non-crossing groups, the 2^n *Allowed Binary Combinations* and the 2^k-2^n *Prohibited Binary Combinations*. We can send to a channel one of the 2^n allowed binary combination. Under influence of noise in the channel, this binary combination can turn into one of the 2^k possible binary combinations. This means that there are $N=2^n2^k$ possible transitions. A principle of error detection is based on the fact that the allowed binary combination becomes the prohibited binary combination. The number of detectable transitions is $N_d=2^n(2^k-2^n)$. If we calculate the ratio N_d/N, we obtain the first numerical parameter of the error-correction code, called the *Coefficient of Potential Detection Ability*:

$$S_d=\frac{N_d}{N}=\frac{2^n(2^k-2^n)}{2^n2^k}=1-\frac{1}{2^m}, \tag{11.1}$$

where m is the number of error-correction bits.

A principle in error correction consists of the following. All the 2^k-2^n prohibited binary combinations are divided into the 2^n non-crossing sets M_1, M_2,

M_3, ... , M_n, where 2^n is the number of allowed binary combinations. Every allowed binary combination is assigned to one of the 2^n sets: $a_1 \to M_1$, $a_2 \to M_2$, $a_3 \to M_3$, ... The principle of error correction consists of the following. If we receive the prohibited code combination, which belongs to the set M_i, we assume that the allowed binary combination a_i has been transmitted. This means that we correct all erroneous binary combinations of the set M_i, if they are formed from the allowed binary combinations a_i. In the opposite case, a correction of the error is fulfilled incorrectly. It is clear that the number of the correctable erroneous transitions N_c is equal to the number of all prohibited combinations, that is, $N_c=2^k-2^n$.

The *Coefficient of Potential Correction Ability* is calculated as a ratio of all the correctable erroneous transitions N_c to all the detectable transitions, that is,

$$S_c = \frac{N_c}{N_d} = \frac{2^k - 2^n}{2^n(2^k - 2^n)} = \frac{1}{2^n}, \tag{11.2}$$

where n is a number of data bits in the code combination of error-correction code.

The coefficients (11.1) and (11.2) characterize the potential ability of the error-correction code to detect and correct errors. Besides these important coefficients, for the characterization of error-correction code abilities, we can use the concept of *Minimal Code Distance* or *Hamming Distance* $d_{\min}$. As is well known, a code distance between the binary combinations a_i and a_j characterizes the degree of difference between them and is equal to the number of distinct bits. For example, the code distance between binary combinations a_i=100101 and a_j=110110 is equal to d=3. If we compare in pairs all code combinations of the given error-correction code and calculate the code distances between them, we can find the minimal code distance or Hamming distance $d_{\min}$. The Hamming distance characterizes the ability of the error-correction code to guarantee detection and correction of certain errors. If we denote the numbers of the guaranteed detectable and correctable errors in the code combination by r and t, respectively, then there are the following simple correlations between r, t and the Hamming distance $d_{\min}$:

$$d_{\min} = r + 1 \tag{11.3}$$

$$d_{\min} = 2t + 1\,. \tag{11.4}$$

The correlation (11.3) means that the error-correction code with the Hamming distance (11.3) can detect (with guarantee) all errors of length r or less. The correlation (11.4) means that the error-correction code with the Hamming distance (11.4) can correct with guarantee all errors of length t or less.

A possibility to detect and correct is provided due to the redundancy of error-correction code. An *Absolute Redundancy* of an error-correction code is determined by the number m of error-correction bits. The *Relative Redundancy* of an error-correction code is determined by the ratio

$$R=\frac{m}{k}=\frac{k-n}{k}=1-\frac{n}{k}. \tag{11.5}$$

Note that this reasoning is valid for all error-correction codes, that is, the results (11.1)–(11.5) have a fundamental character for all error-correction (k,n)-codes.

Let us consider the application of the correlations (11.1)–(11.5) for characterization of the redundancy, detection and correction abilities of the known error-correction codes. The "parity" code has the following parameters: m=1 and d_{min}=2. This means that the coefficient of potential detecting ability (11.1) for the "parity" code is S_d=0.5 (or equal to 50%) and this code guarantees detection of all single errors in the code combination (r=1). The Hamming (15,11)-code is characterized by the following numerical parameters: k=15, n=11, m=4, $d_{\min}$=3. This means that the coefficient of potential detection ability (11.1) of this code is equal to S_d=0.9375 (93.75%), this code guarantees the detection of all single and double errors in the code combination (r=2) and the correction of all single errors (t=1) in the 15-digit code combination of the Hamming (15,11)-code. If we use the correlation (11.2) we can calculate that the potential correction ability of this code is S_c=0.0004882 (0.04882%). According to (11.5) the relative redundancy of the Hamming (15,11)-code is R=0.267 (26.7%).

11.1.2.5. *The Main Shortcomings of the Existing Error-Correction Codes*

Formula (11.1) shows that the coefficient of potential detection ability of the error-correction code increases very quickly and aims at 100% with the increase of m. And this fact confirms the high effectiveness of the error-correction codes to detect errors. However, the formula (11.2) shows that the coefficient of potential correction ability diminishes with the increase of the data bits n. For example, the Hamming (15,11)-code allows one to detect $2^{11}(2^{15}-2^{11})$=62,914,560 erroneous transitions; it can correct only $2^{15}-2^{11}$=30,720 of these erroneous transitions, that is, it can correct only 30,720/62,914,560=0.0004882 (0.04882%) erroneous transitions from the general number of transitions. There is a real question here about practical use of codes with such a low potential correction ability. The experts in the field of coding theory cannot answer this question. However, in all textbooks on coding theory, they describe the above example of the Hamming (15,11)-code as one of its achievements. As the formula (11.2) has a general character for all existing error-correction codes, we can conclude from this consideration that all existing error-correction codes have very low correction ability.

Another more fundamental shortcoming of all known error-correction codes is the fact that the very small information elements (i.e. bits and their combinations) are themselves objects of detection and correction. There is the question: is it possible to create a theory of error-correction codes in which the larger information elements, for example, numbers or even files, are the objects of detection and correction?

Author of this book developed a new coding theory based on the matrix approach in his book [44] and article [113]. One of the main purposes of this present chapter is to develop in detail a new coding theory that differs fundamentally from the classical algebraic coding theory [177, 182].

11.1.3. *Cryptographic Coding*

11.1.3.1. *What is Cryptography?*

The word *Cryptography* arises from the Greek *kryptys* ("hidden") and *grapho* ("write"). At present, cryptography is considered a branch of *Information Theory*. It is a central contributor to several fields: information security and related issues, particularly, authentication, and access control. One of cryptography's primary purposes is to hide the meaning of the transmitted messages. At present, cryptography also contributes to computer science. Cryptography is central to the techniques used in computer and network security for such things as access control and information confidentiality. Cryptography is also used in many applications in everyday life; the security of ATM cards, computer passwords, and electronic commerce - all depend on cryptography.

Also cryptography at present refers almost exclusively to *Encryption*, the process of converting ordinary information (*Plaintext*) into something unintelligible (*Ciphertext*). *Decryption* is the reverse procedure that allows transformation of the unintelligible ciphertext into plaintext. A *Cipher* is a pair of algorithms, the one performing encryption and the other decryption. The detailed operation of a cipher is controlled both by the algorithm and, in each instance, by a *Key*. This is a secret parameter (known only to the communicants) for the cipher algorithm. Keys are important since ciphers without variable keys are trivially breakable and therefore not useful. Historically, ciphers were often used directly for encryption or decryption without additional procedures.

Until the modern era, cryptography was connected solely with message confidentiality (i.e. encryption) - conversion of messages from a comprehensible form into an incomprehensible one, and back again at the other end, rendering it unreadable by interceptors or eavesdroppers without secret knowledge (namely, the key needed for decryption). In recent decades, the field has expanded beyond confidentiality concerns to include techniques for authentication of message integrity or sender/receiver identity, digital signatures, interactive proofs, and computation security.

11.1.3.2. *Symmetric-Key Cryptography*

There are two kinds of cryptography, *Symmetric-Key Cryptography* and *Public-Key Cryptography*. Symmetric-key cryptography refers to encryption methods, in which both the sender and receiver share the same key (or, less commonly, in which their keys are different, but related in an easily computable way). This was the only kind of encryption publicly known until publication of Diffie and Hellman's 1976 paper [183].

The modern study of symmetric-key ciphers relates mainly to the study of *Block Ciphers* and *Stream Ciphers* and their applications. A block cipher takes a block of plaintext and a key as an input, and a block of ciphertext of the same size as an output. The Data Encryption Standard (DES) is a block cipher, which has been designated as the cryptography standard by the U.S. government. Despite its deprecation as an official standard, DES remains quite popular; it is used across a wide range of applications, from ATM encryption to e-mail privacy and secure remote access.

In cryptography, a *Stream Cipher* is a symmetric cipher where plaintext bits are combined with a pseudorandom cipher bit stream (keystream), typically by an exclusive-or (xor) operation. In a stream cipher the plaintext digits are encrypted one at a time, and the transformation of successive digits varies during the encryption.

11.1.3.3. *Public-Key Cryptography*

A significant shortcoming of symmetric ciphers is the key management. Each distinct pair of communicating parties must, ideally, share a different key. The number of keys required increases by the square of the number of network members, which very quickly requires complex key management schemes to keep them all straight and secret. The difficulty of establishing a secret key between two communicating parties, when a secure channel does not already exist between them, also presents a chicken-and-egg problem, which is a considerable practical obstacle for cryptography users in the real world.

In a groundbreaking 1976 paper [183], Whitfield Diffie and Martin Hellman proposed the concept of *Public-key* (also, more generally, called *Asymmetric-key*) cryptography in which two different but mathematically related keys are used - a *Public Key* and a *Private Key*. A public key system is so constructed that calculation of the private key from the public key is computationally impractical, even though they are necessarily related. Instead, both keys are generated secretly, as an interrelated pair.

In public-key cryptosystems [183-186] the public key may be freely distributed, while its paired private key must remain secret. The public key is typically used for encryption, while the private or secret key is used for decryption. Diffie and Hellman showed that public-key cryptography was possible by presenting the Diffie-Hellman key exchange protocol. In 1978, Ronald Rivest, Adi Shamir, and Len Adleman invented RSA, another public-key system.

Public-key algorithms are most often based on the computational complexity of "hard" problems, often from number theory. The "hardness" of RSA is related to the *Integer Factorization Problem*, while Diffie-Hellman and DSA are related to the *Discrete Logarithm Problem*. More recently, *Elliptic Curve Cryptography* was developed. Elliptic curve cryptography is based on theoretical problems involving elliptical curves. Because of the complexity of the underlying problems, most public-key algorithms involve operations such as *Modular Multiplication* and *Exponentiation*, which are much more computationally expensive than the techniques used in most block ciphers, especially with typical key sizes.

11.2. Non-singular Matrices

11.2.1. *A Definition of Non-singular Matrices*

It is known that a square matrix A is called *non-singular*, if its determinant is not equal to zero, that is

$$\det A \neq 0 \cdot \tag{11.6}$$

In linear algebra [144], the non-singular square $(n \times n)$-matrix is called *invertible* because every non-singular matrix A has *inverse matrix* A^{-1}, which is connected with the matrix A with the following correlation:

$$AA^{-1} = I_n, \tag{11.7}$$

where I_n is identity $(n \times n)$-matrix.

11.2.2. *Non-singular (2×2)-matrices*

Let us consider a square non-singular (2×2)-matrix

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{11.8}$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are some real numbers. It is clear that the determinant of the non-singular matrix A is equal:

$$\det A = a_{11}a_{22} - a_{12}a_{21} \neq 0\,. \tag{11.9}$$

Inversion of this matrix can be done easily as follows [190]:

$$A^{-1}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}=\frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \tag{11.10}$$

As is well known, determinant of a square matrix equals to the product of its *eigenvalues* [189]. Note that the *eigenvalues* of the matrix (11.8) can be obtained as follows [189]. Let us consider a square (2×2)-matrix constructed from the matrix (11.8):

$$A-\lambda I=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}, \tag{11.11}$$

where I is an identity (2×2)-matrix and λ is continuous variable.

Determinant of the matrix (11.11) is called *characteristic polynomial* of the matrix A:

$$\begin{aligned}\det(A-\lambda I)&=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}\\ &=\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})\end{aligned} \tag{11.12}$$

Notice that the characteristic polynomial (11.12) can be written in terms of the *trace* $\mathrm{tr}(A)=a_{11}+a_{22}$ and the *determinant* $\det(A)=a_{11}a_{22}-a_{12}a_{21}$ of the matrix A as follows:

$$\det(A-\lambda I)=\lambda^2-\mathrm{tr}(A)\lambda+\det(A)\,. \tag{11.13}$$

It follows from the polynomials (11.12) and (11.13) that a *characteristic equation* of the matrix A

$$\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})=\lambda^2-\mathrm{tr}(A)\lambda+\det(A)=0\,. \tag{11.14}$$

Two roots of the equation (11.14)

$$\begin{aligned}\lambda_{1,2}&=\frac{a_{11}+a_{22}}{2}\pm\sqrt{\frac{(a_{11}+a_{22})^2}{4}+a_{12}a_{21}-a_{11}+a_{22}}\\ &=\frac{1}{2}\left[\mathrm{tr}(A)-4\det(A)\right]\end{aligned} \tag{11.15}$$

is called *eigenvalues* of the matrix A.

Of course, we can study more general class of nonsingular ($n \times n$)-matrices of the kind

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{11.16}$$

with

$$\det A \neq 0 \cdot \tag{11.17}$$

Note that all *Fibonacci matrices* (6.2), (6.4), (6.28), (6.30), (6.70), (6.72), (6.95), (6.97) and *"golden" matrices* (6.128), (6.129), (6.148), (6.149), studied in Chapter 6, are particular cases of the nonsingular matrices of the kind (11.8) and (11.16). We would remind that the determinants of all *Fibonacci matrices* (6.2), (6.4), (6.28), (6.30), (6.70), (6.72), (6.95), (6.97) and *"golden" matrices* (6.128), (6.129), (6.148), (6.149) are equal only to (+1) or (-1), which corresponds to the main properties of nonsingular matrices given by (11.9) and (11.17).

11.3. Fibonacci Encoding/Decoding Method Based Upon Matrix Multiplication

11.3.1. *The Fibonacci Encoding/Decoding Method Based on the $Q_{p,m}$-Matrices*

Consider the following encoding/decoding method based upon the application of the Fibonacci matrices. We can use the direct Fibonacci matrices of the kind (6.2), (6.4), (6.28), (6.30), (6.70), (6.72), (6.95), (6.97) as encoding matrices. Now, let us represent the initial message M in the form of the square 2×2-matrix for the cases (6.2), (6.4), (6.28), (6.30) or in the form of the square $(p+1)\times(p+1)$-matrix for the cases (6.70), (6.72), (6.95), (6.97). The only condition that the matrix M should be non-singular matrix with the determinant det $M\neq 0$. As is known [190], each nonsingular matrix A has its *invertible* matrix A^{-1}. All Fibonacci matrices of the kind (6.2), (6.4), (6.28), (6.30), (6.70), (6.72), (6.95), (6.97) have inverse matrices.

The Fibonacci encoding amounts to the multiplication of the initial matrix M by the encoding matrix of the kind (6.2), (6.4), (6.28), (6.30), (6.70), (6.72), (6.95), (6.97). A code matrix E appears as the result of this multiplication. The Fibonacci decoding amounts to the multiplication of the code matrix E by the decoding Fibonacci matrices inverse to (6.2), (6.4), (6.28), (6.30), (6.70), (6.72), (6.95), (6.97). For the case of the Q-matrix (6.4), the Fibonacci inverse matrices are given by (6.66), (6.67). For the case of the G_m-matrix (6.72) the inverse matrices are given by (6.93) and (6.94).

For the case of the Fibonacci $Q_{p,m}$-matrices (6.97), the Fibonacci encoding/decoding method is given by Table 11.1.

Table 11.1. Fibonacci encoding/decoding method based on the $Q_{p,m}$-matrices

Encoding	Decoding
$M \times Q_{p,m}^{n} = E$	$E \times Q_{p,m}^{-n} = M$

Note that the Fibonacci encoding/decoding method given by Table 11.1 for the cases $p>0$ and $m>0$ gives infinite variants of the Fibonacci encoding/decoding methods because every Fibonacci $Q_{p,m}$-matrix $Q_{p,m}^{n}$ and its inverse matrix $Q_{p,m}^{-n}$ "generate" their own Fibonacci encoding/decoding method.

11.3.2. *Fibonacci Encoding/Decoding Method Based on the G_m and Q-Matrices*

We can use the above Fibonacci G_m-matrices to design a new variant of the matrix encoding/decoding method. As mentioned above, for this case the initial information is represented in the form of the non-singular data (2×2)-matrix M. The direct matrix G_m^n given by (6.72) is used as the encoding matrix and the inverse matrices (6.93) or (6.94) are used as the decoding matrices. For this case the Fibonacci encoding/decoding method is given in Table 11.2.

Table 11.2. Fibonacci encoding/decoding method based on the G_m-matrices

Encoding	Decoding
$M \times G_m^n = E$	$E \times G_m^{-n} = M$

Note that for the case m=1 the Fibonacci G_m-matrix (6.72) becomes the classical Fibonacci Q-matrix (6.4). Also the Fibonacci $Q_{p,m}$-matrix (6.97) for the case m=1 and p=1 becomes the classical Fibonacci Q-matrix (6.4). This means that for these cases the Fibonacci encoding/decoding methods given by Tables 11.1 and 11.2 coincide and they become the simplest Fibonacci encoding/decoding method given by Table 11.3.

Table 11.3. Fibonacci encoding/decoding method based on the Q-matrix

Encoding	Decoding
$M \times Q^n = E$	$E \times Q^{-n} = M$

Here the encoding matrix Q^n is given by (6.4), that is, $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, and the decoding matrix Q^{-n} is given by (6.66) or

(6.67), that is, $Q^{-2k}=\begin{pmatrix} F_{2k-1} & -F_{2k} \\ -F_{2k} & F_{2k+1} \end{pmatrix}$ or $Q^{-2k-1}=\begin{pmatrix} -F_{2k} & F_{2k+1} \\ F_{2k+1} & -F_{2k+2} \end{pmatrix}$, respectively.

11.3.3. *An Example of the Fibonacci Encoding/Decoding Method Based on the Q-Matrix*

First of all, we study the simplest Fibonacci encoding/decoding method given in Table 11.3. Represent the initial message M in the form of the square (2×2)-matrix:

$$M=\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}. \tag{11.18}$$

We would remind that the matrix (11.18) should be *non-singular* or *invertible* matrix [190]. This means that its determinant

$$\det M = m_1m_4 - m_2m_3 \neq 0. \tag{11.19}$$

If the initial message M is represented in the form of the binary n-digit code

$$M = a_1a_2a_3...a_n\,, \tag{11.20}$$

we should divide the binary code (11.20) into the four parts $m_1m_2m_3m_4$ so that the condition (11.19) is fulfilled. Below we suppose that the matrix (11.18) is always non-singular.

Suppose that we have chosen the Fibonacci matrix Q^5 as the encoding matrix, that is:

$$Q^5=\begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}. \tag{11.21}$$

According to Table 6.1 its inverse matrix has the following form:

$$Q^{-5}=\begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix}. \tag{11.22}$$

Then the Fibonacci encoding of the matrix (11.18) consists in its multiplication by the direct encoding matrix (11.21), that is:

$$M\times Q^5=\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\times\begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}=\begin{pmatrix} 8m_1+5m_2 & 5m_1+3m_2 \\ 8m_3+5m_4 & 5m_3+3m_4 \end{pmatrix}=\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}=E, \tag{11.23}$$

where

$$\begin{aligned} e_1 &= 8m_1+5m_2; \quad e_2=5m_1+3m_2 \\ e_3 &= 8m_3+5m_4; \quad e_4=5m_3+3m_4. \end{aligned} \tag{11.24}$$

Then, the code message $E=e_1, e_2, e_3, e_4$ is sent to the channel.

The decoding of the code message E given by (11.23) is carried out in the following manner. The code matrix E (11.23) is multiplied by the inverse matrix (11.22):

$$\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix} = \begin{bmatrix} (-3)e_1 + 5e_2 & 5e_1 + (-8)e_2 \\ (-3)e_3 + 5e_4 & 5e_3 + (-8)e_4 \end{bmatrix} = \begin{pmatrix} e_1' & e_2' \\ e_3' & e_4' \end{pmatrix}. \tag{11.25}$$

By calculating the entries of the matrix (11.25) and by taking into account the formulas (11.24), we obtain:

$$e_1' = (-3)e_1 + 5e_2 = (-3)\times(8m_1 + 5m_2) + 5\times(5m_1 + 3m_2) = m_1.$$

By analogy, we can calculate:

$$e_2' = m_2;\; e_3' = m_3;\; e_4' = m_4,$$

that is,

$$\begin{pmatrix} e_1' & e_2' \\ e_3' & e_4' \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = M.$$

11.4. The Main Checking Relations of the Fibonacci Encoding/Decoding Method

11.4.1. *Determinants of the Code Matrix*

We mentioned above that the code matrix E is the outcome of the Fibonacci encoding for the matrix methods given by Tables 11.1, 11.2 and 11.3. We can write the code matrices for Tables 11.1, 11.2 and 11.3 respectively as follows:

$$E = M \times Q_{p,m}^n \tag{11.26}$$

$$E = M \times G_m^n \tag{11.27}$$

$$E = M \times Q^n. \tag{11.28}$$

By using a general matrix property for the determinants [158]

$$\det(A \times B) = \det A \times \det B,$$

we can write the following expressions for the determinants of the matrices (11.26) - (11.28):

$$\det E = \det[M \times Q_{p,m}^n] = \det M \times \det Q_{p,m}^n \tag{11.29}$$

$$\det E = \det[M \times G_m^n] = \det M \times \det G_m^n \tag{11.30}$$

$$\det E = \det[M \times Q^n] = \det M \times \det Q^n. \tag{11.31}$$

By using the identities (6.5), (6.76) and (6.119), we can rewrite the formulas (11.29) - (11.31) as follows:

$$\det E = \det M \times (-1)^{pn} \left(\text{for the } Q_{p,m}\text{ - matrix}\right) \tag{11.32}$$

$$\det E = \det M \times (-1)^{n} \left(\text{for the } G_{m}\text{ - matrix}\right) \tag{11.33}$$

$$\det E = \det M \times (-1)^{n} \left(\text{for the } Q\text{ - matrix}\right). \tag{11.34}$$

Our examination resulted in some quite unexpected properties of the Fibonacci encoding/decoding methods given by Tables 11.1, 11.2 and 11.3. It turns out that very strict mathematical relations (11.32)-(11.34) connect the data matrix M and the code matrix E for the Fibonacci encoding/decoding methods given by Tables 11.1-11.3. It is clear that the identities (11.32) - (11.34) can be used as the main "checking relations" of the Fibonacci coding/decoding methods given by Tables 11.1, 11.2 and 11.3. According to (11.32) - (11.34) the determinant of the code matrix E is determined entirely by the determinant of the data matrix M.

11.4.2. *Connections between the Elements of the Code Matrix*

Besides the identities (11.32)-(11.34), other mathematical correlations between the elements of the code matrix E exist. Now, let us consider the simplest Fibonacci coding/decoding method for the case p=1 (Table 11.3). For this case we can represent the code matrix E as follows:

$$E = M \times Q^n = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \times \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}, \tag{11.35}$$

where the elements e_1, e_2, e_3, e_4 are equal to

$$e_1 = F_{n+1}m_1 + F_n m_2 \tag{11.36}$$

$$e_2 = F_n m_1 + F_{n-1} m_2 \tag{11.37}$$

$$e_3 = F_{n+1}m_3 + F_n m_4 \tag{11.38}$$

$$e_4 = F_n m_3 + F_{n-1} m_4 . \tag{11.39}$$

After the Fibonacci decoding, the data matrix M can be represented as follows:

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = E \times Q^{-n} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times Q^{-n}. \tag{11.40}$$

For the case n=2k+1 we can use the inverse matrix (6.67), that is, the matrix $Q^{-n} = \begin{pmatrix} -F_{n-1} & F_n \\ F_n & -F_{n+1} \end{pmatrix}, n = 2k+1$ as the decoding matrix and then the formula (11.40) takes the following form:

$$M=\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}=\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}\times\begin{pmatrix} -F_{n-1} & F_n \\ F_n & -F_{n+1} \end{pmatrix}. \tag{11.41}$$

It follows from (11.41) that the elements of the matrix (11.41) can be calculated according to the following formulas:

$$m_1=-F_{n-1}e_1+F_ne_2 \tag{11.42}$$

$$m_2=F_ne_1-F_{n+1}e_2 \tag{11.43}$$

$$m_3=-F_{n-1}e_3+F_ne_4 \tag{11.44}$$

$$m_4=F_ne_3-F_{n+1}e_4 \tag{11.45}$$

As the elements $m_1m_2m_3m_4$ of the data matrix (11.18) are always non-negative integers, we can write the equalities (11.42)-(11.45) as the following non-equalities:

$$-F_{n-1}e_1+F_ne_2\ge 0 \tag{11.46}$$

$$F_ne_1-F_{n+1}e_2\ge 0 \tag{11.47}$$

$$-F_{n-1}e_3+F_ne_4\ge 0 \tag{11.48}$$

$$F_ne_3-F_{n+1}e_4\ge 0\,. \tag{11.49}$$

We can rewrite the non-equalities (11.46) and (11.47) as follows:

$$\frac{F_{n+1}}{F_n}e_2\le e_1\le\frac{F_n}{F_{n-1}}e_2. \tag{11.50}$$

By analogy, we can rewrite the non-equalities (11.48) and (11.49) as follows:

$$\frac{F_{n+1}}{F_n}e_4\le e_3\le\frac{F_n}{F_{n-1}}e_4. \tag{11.51}$$

As the ratio of the adjacent Fibonacci numbers aims for the golden mean, the following approximate equalities, which connect the elements of the code matrix (11.35), come from (11.50) and (11.51):

$$e_1\approx\tau e_2 \tag{11.52}$$

$$e_3\approx\tau e_4\,, \tag{11.53}$$

where $\tau=\left(1+\sqrt{5}\right)/2$ is the golden mean.

By analogy, for the case $n=2k$ we can use the inverse Fibonacci matrix (6.66) as a decoding matrix and then we can write the approximate equalities that are similar to (11.52) and (11.53) and connect the pairs of the adjacent elements e_1 and e_3, e_2 and e_4 of the code matrix (11.35).

Thus, we have found the additional mathematical relations (11.52) and (11.53) that connect the elements of the code matrix (11.35). Note that the code matrix (11.35) corresponds to the simplest case p=1. It is clear that we can find similar approximate mathematical correlations that connect the code matrix E for the Fibonacci encoding/decoding methods given by Tables 11.1 and 11.2.

We can estimate the absolute value of the maximal relative "approximation error" for the equalities (11.52) and (11.53). For this purpose we should estimate the absolute value of the difference

$$\Delta = \left|\frac{F_n}{F_{n-1}} - \frac{F_{n+1}}{F_n}\right| = \left|\frac{F_n^2 - F_{n+1}F_{n-1}}{F_{n-1}F_n}\right|. \tag{11.54}$$

For the simplification of the formula (11.54) we can use the Cassini formula (6.6), that is,

$$\det Q^n = F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

and the Binet formula (2.68) for the classical Fibonacci numbers, that is,

$$F_n = \begin{cases} \dfrac{\tau^n + \tau^{-n}}{\sqrt{5}} \text{ for } n = 2k+1 \\ \dfrac{\tau^n - \tau^{-n}}{\sqrt{5}} \text{ for } n = 2k \end{cases}.$$

According to (6.6) the absolute value of the expression (6.6) is equal:

$$\left|F_n^2 - F_{n+1}F_{n-1}\right| = 1. \tag{11.55}$$

According to the Binet formula (2.68) we can write the following approximate formulas for the calculation of the Fibonacci numbers F_{n-1} and F_n:

$$F_{n-1} \approx \frac{\tau^{n-1}}{\sqrt{5}} \text{ and } F_n \approx \frac{\tau^n}{\sqrt{5}}. \tag{11.56}$$

Then, using (11.55) and (11.56), after simple transformations of (11.54) we can write the formula for the maximal relative "approximation error" Δ:

$$\Delta = \frac{5}{\tau^{2n-1}} = \frac{5}{(1.618)^{2n-1}}. \tag{11.57}$$

The values of the "approximation error" Δ for different n are given by Table 11.4. We can see from (11.57) and Table 11.4 that the maximal relative "approximation error" Δ given by (11.57) is decreasing quickly when n increases. For example, for the case n=7 the relative "approximation error" Δ=0.0096 (about 1%). This means that for a "large enough" n ($n \geq 7$) the "approximate" equalities (11.52) and (11.53) approach "precise" equalities.

Table 11.4. Relative "approximation error" Δ

n	1	2	4	6	8	10
Δ	3.09023	1.18041	0.17223	0.02513	0.00367	0.00054

11.5. Error Detection and Correction

11.5.1. *A Notion of "Errors"*

The notion of "errors" for the above Fibonacci encoding/decoding method differs from the similar notion in the classical theory of error-correction codes [177, 182]. As is known, a "single" error is a transition of a bit into an opposite state ($1\to0$ or $0\to1$). In our coding theory a "single" error is the transition of any element of the code matrix E into an "erroneous" state. If, for example, the "true" element e_1 is the 2-digit decimal number 17, then the "single" error in the element e_1 is the transition of the number 17 into one of the decimal numbers 0-16 or 18-99. It is clear that there are four $\left(C_4^1=4\right)$ variants of the "single" errors in the code matrix (11.35):

$$\begin{pmatrix} x & e_2 \\ e_3 & e_4 \end{pmatrix}; \begin{pmatrix} e_1 & y \\ e_3 & e_4 \end{pmatrix}; \begin{pmatrix} e_1 & e_2 \\ z & e_4 \end{pmatrix}; \begin{pmatrix} e_1 & e_2 \\ e_3 & v \end{pmatrix}, \tag{11.58}$$

where x, y, z, v are the "erroneous elements" of the code matrix E.

Note that if the data message M is represented in the form of the binary code, then the notion of a "single error" in the code matrix element, represented by the binary combinations of length k, corresponds to the notion of the "group error" used in the classical error-correction code theory [177, 182].

Now, let us consider the case of "double" errors. There are six $\left(C_4^2=6\right)$ variants of the "double" errors:

$$\begin{pmatrix} x & y \\ e_3 & e_4 \end{pmatrix}; \begin{pmatrix} x & e_2 \\ z & e_4 \end{pmatrix}; \begin{pmatrix} x & e_2 \\ e_3 & v \end{pmatrix}; \begin{pmatrix} x & y \\ z & e_4 \end{pmatrix}; \begin{pmatrix} e_1 & y \\ z & e_4 \end{pmatrix}; \begin{pmatrix} e_1 & y \\ e_3 & v \end{pmatrix}; \begin{pmatrix} e_1 & e_2 \\ z & v \end{pmatrix}. \tag{11.59}$$

It is clear that there are four $\left(C_4^1=4\right)$ variants of the "triple" errors:

$$\begin{pmatrix} x & y \\ z & e_4 \end{pmatrix}; \begin{pmatrix} e_1 & y \\ e_3 & v \end{pmatrix}; \begin{pmatrix} x & e_2 \\ z & v \end{pmatrix}; \begin{pmatrix} e_1 & y \\ z & v \end{pmatrix} \tag{11.60}$$

and one variant of the "fourfold" error:

$$\begin{pmatrix} x & y \\ z & v \end{pmatrix}. \tag{11.61}$$

It is clear that in total we have 15 possible "errors" in the code matrix (11.35).

11.5.2. *Detection of "Errors"*

Thus, following the above reasoning we may regard the Fibonacci encoding/decoding method given by Table 11.3 (also in general by Tables 11.1 and 11.2) as a transformation of the data matrix M into the code matrix E by means of the multiplication of the data matrix M by the encoding matrix Q^n (6.4) (in the general case by $Q^n_{p,m}$ or G^n_m). This transformation results in the appearance of the strict "checking relations" (11.32), (11.33), (11.34), (11.52) and (11.53) that connect the elements of the code matrix E. These "checking relations" allow for detection and correction of "errors" in code matrix E.

Let us examine the application of "checking relations" (11.34), (11.52) and (11.53) to the detection of "errors" in the code matrix (11.35). For verification of the "checking relation" (11.34) we must calculate the determinant of the data matrix (11.18) according to the formula (11.19) and then send the det M to the "channel." Remember that the matrix M is non-singular and its determinant (11.19) is not equal to zero. As shown below, the element det M is the main "checking element" for the Fibonacci encoding/decoding method.

The "recipient" receives the elements e_1, e_2, e_3, e_4 of the code matrix (11.35) together with the "checking element" det M and then calculates the determinant of the code matrix (11.35) according to the formula:

$$\det E = e_1 \times e_4 - e_2 \times e_3. \tag{11.62}$$

After the calculation of det E by the formula (11.62), the "recipient" verifies the "checking relation" (11.34) by means of comparison of the det E (11.62) with det M received from the channel. If the "checking relation" (11.34) is valid, it means that the transmission of the code matrix E via the "channel" was carried out correctly and we can decode the code matrix E according to (11.41). If the "checking relation" (11.34) is disturbed, this means that some "errors" appear in the elements e_1, e_2, e_3, e_4 of the code matrix (11.35) or the "checking element" det M in the process of their transmission via the channel. In this case, we can use the additional "checking relations" (11.52) and (11.53) for verification of "errors" in the code matrix E. If the "checking relations" (11.52) and (11.53) are fulfilled with sufficient exactness, the "recipient" can decide that the "errors" are in the "checking element" det M, however, the code matrix E is correct. In this case, the code matrix E can be decoded according to (11.41).

However, we should give specific attention to the protection of the "checking element" det M from noise, in particular, to the detection of errors in the det M, because the det M plays a role of the "main checking element" for the Fibonacci method of encoding/decoding. If the "checking relation" (11.34) is disturbed, it is important to know the cause of this, whether it is the det M or the elements e_1, e_2, e_3, e_4 of the code matrix E. This is why, it is desirable to use classical error-correction codes for the additional protection of the det M. We mentioned above that the classical error-correction codes [177, 182] have a high degree of detection ability. For example, the "parity" code with only one error correction bit allows the detection of 50% of possible errors. If we are convinced in the correctness of det *M*, we can proceed to the correction of errors. If det *M* is not correct, we should again repeat the transmission of det *M*.

11.5.3. *Correction of "Single" Errors*

Let us examine the case when det *M* is correct and the code matrix $E=\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ has possible errors. In order to verify errors in the code matrix *E*, the "recipient" needs to calculate det *M* by the formula (11.62) and then to compare it with det *M* received from the channel according to the "checking relation" (11.34). If our verification shows a presence of errors in the code matrix *E*, then we should try to correct those errors.

For the correction of errors we form different hypotheses about possible errors in the code matrix *E* and then check these hypotheses. The first step is to check the hypothesis about the possible "single" errors of the kind (11.58). Note that in the general case we do not know what element of matrix *E* is "erroneous." In the case of "single" errors, we have to verify the four error situations given by (11.58). For checking the "erroneous situations" (11.58), we can write the following algebraic equations based on the "checking relation" (11.34):

$$xe_4 - e_2e_3 = (-1)^n \det M \ (e_1 \text{ is erroneous}) \tag{11.63}$$

$$e_1e_4 - ye_3 = (-1)^n \det M \ (e_2 \text{ is erroneous}) \tag{11.64}$$

$$e_1e_4 - e_2z = (-1)^n \det M \ (e_3 \text{ is erroneous}) \tag{11.65}$$

$$e_1v - e_2e_3 = (-1)^n \det M \ (e_4 \text{ is erroneous}). \tag{11.66}$$

It follows from (11.63)-(11.66) the four different formulas for calculation of the possible "single" errors:

$$x = \frac{(-1)^n \det M + e_2e_3}{e_4} \tag{11.67}$$

$$y = \frac{-(-1)^n \det M + e_1e_4}{e_3} \tag{11.68}$$

$$z = \frac{-(-1)^n \det M + e_1e_4}{e_2} \tag{11.69}$$

$$v = \frac{(-1)^n \det M + e_2e_3}{e_1}. \tag{11.70}$$

Formulas (11.67)-(11.70) give four possible variants for the calculation of "single" errors. However, we know that the elements e_1, e_2, e_3, e_4 of the code matrix (11.35) have to be integers! And we must choose the correct solutions to the equations (11.63)-(11.66) solely from amongst the cases of integer solutions x, y, z, v of the kind (11.67)-(11.70). If we have several integer solutions (11.67)-(11.70), we must choose that solution, which satisfies the additional "checking relations" (11.52) and (11.53). If the calculations by formulas (11.67)-(11.70) do not result in integer solutions, we have to conclude that our hypothesis about "single" errors in the code matrix E is incorrect and therefore the matrix E has "double" or "triple" errors.

11.5.4. *A Numerical Example of "Single" Error Correction*

Let us examine a numerical example of the Fibonacci coding/decoding method based on the application of the classical Fibonacci Q-matrix. Suppose that the sequence of "bytes" (8 bits) is the initial message transmitted via the channel.

Suppose we need to transmit via the channel the binary code combinations of a length of 4 bytes (32 bits). Then for Fibonacci encoding/decoding we divide the initial code combination into four parts of equal length by one byte representing the initial code combination in the form of a data (2×2) matrix of the kind (11.18). In this case every 1-byte element of the matrix (11.18) is equal to the 8-bit binary number that takes its values in the range $[0\div(2^8\text{-}1)]$. Suppose, the data matrix (11.18) has the following numerical form:

$$M = \begin{pmatrix} 200 & 26 \\ 166 & 150 \end{pmatrix}. \tag{11.71}$$

Note that the maximal value of every element of the matrix (11.71) is equal to 255.

We mentioned above that we can use different Q^n-matrices (n=1,2,3, ...). We use the Fibonacci matrices (11.21) and (11.22) as encoding and decoding matrices, respectively.

The Fibonacci encoding then consists of the following stages:

1. Multiply the initial matrix (11.71) by the encoding matrix (11.21):

$$E = M \times Q^5 = \begin{pmatrix} 200 & 26 \\ 166 & 150 \end{pmatrix} \times \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1730 & 1078 \\ 2078 & 1280 \end{pmatrix}. \tag{11.72}$$

2. Calculate the determinant of the data matrix (11.71):

$$\det M = (200\times150) - (166\times26) = 25684. \tag{11.73}$$

3. Send the four elements of the code matrix (11.72) to the channel together with the "checking element" det M given by (11.73).

Now, let us consider three variants of the message including a possible "single" error that can be received from the channel:

(a) The elements e_1, e_2, e_3, e_4 of the matrix (11.72) and the "checking element" det M (11.73) do not have "errors"

(b) The det M is received with the "error" and has, for example, the value det M=20325; however, all elements e_1, e_2, e_3, e_4 of the matrix (11.72) are received without "errors"

(c) The det M is received without "error," the element $e_3 = 2078$ of the matrix (11.72) is received with "error" and has the "erroneous" value $e_3 = 3705$.

Let us consider how the "recipient" acts for each situation.

11.5.4.1. *Situation (a)*

1. Calculate the $\det E = 1730 \times 1280 - 1078 \times 2078 = -25684$.

2. Compare the $\det E = -25684$ with the $\det M = 25684$ according to the "checking relation" (11.34). Remember that for this case the power of the encoding matrix (11.21) is equal to n=5. This means that the "checking relation" (11.34) for the case n=5 has the following form:

$$\det E = -\det M. \tag{11.74}$$

In fact, the "checking relation" (11.74) is valid for this case because $25684 = -(-25684)$. This means that the code matrix (11.72) is correct for situation (a) and we can decode the code matrix (11.72) by means of its multiplication by the decoding matrix (11.22):

$$\begin{pmatrix} 1730 & 1078 \\ 2078 & 1280 \end{pmatrix} \times \begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} 200 & 26 \\ 166 & 150 \end{pmatrix} = M. \tag{11.75}$$

11.5.4.2. *Situation (b)*

For situation (b), we have the "error" in the det M because the "erroneous" value of det M=20325, which differs from its true value as given by (11.73). For situation (b), the action of the "recipient" is as follows:

1. Calculate the det E=–25684.

2. By comparing the det E=–25684 with the det $M = 20325$, we can see that the "checking relation" (11.74) is disturbed because –25884≠–20325. This means that we have an "error" in the elements of code matrix E or in the "checking element" det M. We mentioned above that we should provide additional measures for the protection of the det M by using the classical error-correction codes. If the det M has an "error," it means that

we can apply the additional "checking relations" (11.52) and (11.53) to the code matrix E.

3. Check code matrix E according to the additional "checking relations" (11.52) and (11.53). With this purpose we calculate the ratios of adjacent row elements of the code matrix (11.72):

$$1730:1078=1.6; 2078:1280=1.62. \tag{11.76}$$

It follows from (11.76) that these ratios are near to the golden mean 1.618. This means that the "checking relations" (11.52) and (11.53) are valid for this case and we may conclude that we have an "error" only in the "checking element" det M but the elements e_1, e_2, e_3, e_4 of the code matrix E do not have "errors." Hence, we can decode the message e_1, e_2, e_3, e_4 according to (11.75). However, if the "checking relations" (11.52) and (11.53) are not valid, we should repeat transmission of det M until the "checking element" det M is transmitted without "error."

11.5.4.3. *Situation (c)*

Remember that for this hypothetical situation the element e_3 has been received with an "error" and has the "erroneous" value $e_3=3705$. However, assume that we do not know this. Then the code matrix E, which is received from the channel, has the following "erroneous" form:

$$E=\begin{pmatrix}1730 & 1078\\ 3705 & 1280\end{pmatrix}. \tag{11.77}$$

For situation (c) the action of the "recipient" is as follows:

1. Calculate $\det E=1730\times1280-1078\times3705=-1779590$.
2. By comparing det E =–1779590 with det M =25684, received from the channel, we can see that the checking relation (11.74) is disturbed. This means that we have "errors" in the elements of the code matrix E.
3. Check code matrix E according to the additional checking relations (11.52) and (11.53) by means of the calculation of the ratios of adjacent row elements of the matrix (11.76):

$$1730:1078=1.6; 3705:1280=2.89. \tag{11.78}$$

We can see from (11.78) that the checking relation (11.53) is disturbed for this case and therefore the possible "single" error is in the elements of the code matrix E. It is most probable that we have "single" errors in the elements e_3=3705 or e_4=1280, that is, one of the elements e_3 or e_4 is distorted. It is also possible that both the elements e_3 and e_4 are distorted (a case of "double" errors).

4. In this case the "recipient" can use the formulas (11.67)-(11.70) for the calculation of the erroneous elements. Remember that the "recipient" should verify different hypotheses about the possible "single" errors given by (11.58). Verification of these hypotheses is carried out by calculation of the possible values of elements e_1, e_2, e_3, e_4 according to the formulas (11.67)-(11.70).

Let's calculate the formulas (11.67)-(11.70) taking into account the "real" values of the elements e_1=1730, e_2=1078, e_3=3705, e_4=1280 and det M=25684, received from the channel:

$$x = \frac{-25684+1078\times 3705}{1280} = \frac{3968306}{1280} = 3100.239 \tag{11.79}$$

$$y = \frac{25684+1730\times 1280}{3705} = \frac{2240084}{3705} = 604.61106 \tag{11.80}$$

$$z = \frac{25684+1730\times 1280}{1078} = \frac{2240084}{1078} = 2078 \tag{11.81}$$

$$v = \frac{-25684+1078\times 3705}{1730} = \frac{3968306}{1730} = 2293.8184. \tag{11.82}$$

Let us analyze the results of the calculations given by (11.79) - (11.82). Only for the case (11.81) we have obtained a positive integer number; all the remaining solutions are fractions. This means that our hypothesis about a possible "single" error is true only in the case of (11.81), and therefore, we have a "single error" in element e_3. By substituting the calculated value z=2078 for the "erroneous" element e_3=3705 in the matrix (11.77) we can correct the "single" error in the element e_3.

After correction of the error, our code matrix E can be represented in the following form:

$$E = \begin{pmatrix} 1730 & 1078 \\ 2078 & 1280 \end{pmatrix}. \tag{11.83}$$

We can check the accuracy of our correction by using the checking relation (11.74). Indeed, the determinant of the matrix (11.83) is equal to

$$\det E = 1730\times 1280 - 1708\times 2078 = -25684. \tag{11.84}$$

By comparing det E=–25684 with det M=25684, received from the channel, we conclude that the checking relation (11.74) is true. This means that restoration of the "erroneous" element e_3 was done correctly.

11.5.5. *Correction of "Double" Errors*

Now let us consider the situations (11.59) when two matrix elements have been received with "errors." We examine two characteristic cases of "double" errors in the code matrix E:

(a) The erroneous elements are diagonal elements of the matrix (11.72).

(b) The erroneous elements are in the same row or in the same column of the code matrix (11.72).

Suppose that the code matrix E (11.72) is "destroyed" by the "diagonal" manner

$$E=\begin{pmatrix} x & 1078 \\ 2078 & v \end{pmatrix}, \tag{11.85}$$

by the "column" manner

$$E=\begin{pmatrix} x & 1078 \\ z & 1280 \end{pmatrix}, \tag{11.86}$$

or by the "row" manner

$$E=\begin{pmatrix} x & y \\ 2078 & 1280 \end{pmatrix}. \tag{11.87}$$

Then using the checking relation (11.74) for the case n=5, we can write the following equations for the determinants of the matrices (11.85) - (11.87):

$$xv-2078\times1078=-25684 \tag{11.88}$$

$$1280x-1078z=-25684 \tag{11.89}$$

$$1280x-2078y=-25684. \tag{11.90}$$

Note that, if our hypothesis about the "double" errors is valid, the Diophantine equations (11.88) - (11.90) have at least one integer solution.

Let us analyze the Diophantine equation (11.88). We can represent equation (11.88) in the form:

$$xv=2214400. \tag{11.91}$$

Taking into account the additional checking relations (11.52) and (11.53), we will search the integer solutions to the Diophantine equation (11.91) among integers that are near to the following numbers:

$$x=1.618\times1078=1744.204 \tag{11.92}$$

and

$$v=0.618\times2079=1284.294. \tag{11.93}$$

It is easy to calculate the following pair of integer solutions to (11.91), the integers 1730 and 1280, which are near to the numbers (11.92) and (11.93) and satisfy equation (11.91):

$$1730\times1280=2214400.$$

This means that our hypothesis about "double" errors of the kind (11.85) is true and we can correct the "double" error in the code matrix (11.85), if we take the calculated values of the elements x and v as follows:

$$x=1730;\, v=1280. \tag{11.94}$$

We can verify the validity of our correction because our restored code matrix has the form of the code matrix (11.83) that satisfies the checking relation (11.74).

Now let us examine the case of the "double" error given by (11.86). For this case, we can write the following Diophantine equation:

$$1280x-2078z=-25684. \tag{11.95}$$

Note that according to the additional checking relations (11.52) and (11.53) we have to search the integer solutions to the Diophantine equation (11.95) among the integers that are near to the following numbers:

$$x=1.618\times1078=1744.204 \tag{11.96}$$

and

$$z=1.618\times1280=2071.04. \tag{11.97}$$

By solving the Diophantine equation (11.95), we can find the following pair of integers

$$x=1730;\, z=2078, \tag{11.98}$$

which are near to the numbers (11.96) and (11.97), respectively, and are integer solutions to the Diophantine equation (11.95):

$$1280\times1730-1078\times2078=-25684.$$

For the case (11.87) the Diophantine equation has the following form:

$$1280x-2078y=-25684. \tag{11.99}$$

It is easy to find the following integer solutions to equation (11.99):

$$x=1730;\, y=1078. \tag{11.100}$$

Note that the solution (11.100) satisfies the approximate equality (11.52) because the ratio

$$x:y=1730:1078=1.6048$$

is close to the golden mean 1.618. Similar to the preceding cases, the integer solution (11.100) satisfies the main "checking relation" (11.74).

Thus, by using numerical examples, we have proven that we can correct all "double" errors in the code matrix (11.83), if our hypothesis about possible "double" errors is valid. By analogy, we can prove that we can correct all possible "double" errors for all situations given by (11.59).

11.5.6. *Correction of "Triple" Errors*

There are four cases of "triple" errors given by (11.60). Let us consider the "triple" errors in the matrix (11.83) of the following kind:

$$\begin{pmatrix} x & y \\ z & 1280 \end{pmatrix}. \tag{11.101}$$

By using the checking relations (11.74), (11.52) and (11.53), we can write the following equalities that connect the elements of the matrix (11.101):

$$x\times 1280 - y\times z = -\det M \tag{11.102}$$

$$x \approx \tau \times y \tag{11.103}$$

$$z \approx \tau \times e_4, \tag{11.104}$$

where τ=1.618 is the golden mean.

By solving the Diophantine equation (11.102) and taking into account the additional checking relations (11.103) and (11.104), we can find the correct values of x=1730, y=1078 and z=2078 that satisfy the checking relation (11.74). This means that the Fibonacci encoding/decoding method allows for correction of all possible "triple" errors of the kind (11.60).

11.6. Redundancy, Correction Ability, and the Advantages of the Fibonacci Encoding/Decoding Method

11.6.1. *Redundancy of the Fibonacci Encoding/Decoding Method*

For the estimation of code redundancy we can use the concept of *absolute* and *relative redundancy* [177]. Absolute redundancy is calculated as the difference:

$$r=s-l, \tag{11.105}$$

where s is the number of bits in the code message and l is the number of bits in the initial (or data) message. Then the relative redundancy can be calculated by the formula:

$$R=(s-l)/s. \tag{11.106}$$

Suppose that l=4t. As for the Fibonacci encoding, the initial message is represented in the form of data matrix (11.18), which means that each element of the data matrix (11.18) consists of t bits.

Now let us consider a general case where the encoding matrix Q^n has the form (6.4), that is, $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$. In this case the code matrix takes the form (11.35).

The code message, entering the channel, consists of the five elements, the "checking element" det M and the four elements e_1, e_2, e_3, e_4 given by (11.36)-(11.39). The "checking element" det M is the main source of redundancy of the code message entering the channel.

We need to calculate the number of bits necessary for the representation of the det M. With this purpose in mind, let us calculate the maximal value of the det M given by (11.19).

It is clear that the determinant (11.19) can reach its maximal value in that case when the product $m_1 \times m_4$ reaches its maximal value and the product $m_2 \times m_3$ reaches its minimal value. If we neglect the minimal product $m_2 \times m_3$ in comparison to the maximal product of $m_1 \times m_4$ and take the maximal value of m_1 and m_4 equal to 2^t, we can then estimate the maximal value of the det M as follows:

$$\left(\det M\right)_{\max} \approx 2^t \times 2^t = 2^{2t}. \qquad (11.107)$$

It follows from (11.107) that we need $2t$ bits for the binary representation of the det M.

The code matrix (11.35) is redundant with respect to the data matrix (11.18). By comparing the code matrix elements e_1, e_2, e_3, e_4, which are given by (11.36)-(11.39), with the data matrix elements m_1, m_2, m_3, m_4, we may conclude that for the case $n \geq 1$, the numerical values of the code matrix elements e_1, e_2, e_3, e_4 are more than the numerical values of the data matrix elements m_1, m_2, m_3, m_4. This means that for the binary representation of the code matrix elements e_1, e_2, e_3, e_4 we need no less than l=4t bits.

In order to obtain the lowest estimate of the redundancy of the Fibonacci coding/decoding method, we can use the following reasoning. Above we found that we needed $2t$ bits for the representation of the "checking element" det M. On the other hand, we need no less than $4t$ bits for the representation of the code matrix (11.35). It follows from this consideration that for the representation of the code message, entering the channel, we need no less than $6t$ bits. The ratio of the number of the "checking bits" ($2t$) to the general number of bits ($6t$) is the lowest estimation of the relative redundancy of the Fibonacci encoding/decoding method:

R_{FC}=2t/6t=1/3=0.333 (33.3%). (11.108)

It is clear that the real relative redundancy of this method exceeds the estimation of 33.3% because the lowest estimation (11.108) does not include the redundancy that is caused by the code matrix (11.35). It is clear that for the diminution of redundancy we should choose the coding matrix with the minimal value of n (for example n=1). However, for this case (according to Table 11.4) the equalities (11.52) and (11.53) become very approximate and we cannot use these checking relations for error detection and correction. If n increases, then according to Table 11.4 the equalities (11.52) and (11.53) become more and more "precise" and the "correction ability" of the Fibonacci method will increase. Thus, for each individual case we should decide what value of n is optimal for our application from the point of view of necessary redundancy and the required correction ability of the method.

11.6.2. *Correction Ability of the Fibonacci Encoding/Decoding Method*

Now, let us estimate the correction ability of the Fibonacci encoding/decoding method for two cases of n: (1) n=1 and (2) n>>1.

For the case n=1 the Fibonacci encoding/decoding method has a minimal relative redundancy given by (11.108). However, we can use only one checking relation (11.34) for this case. By using the checking relation (11.34), we can only correct the "single" errors given by (11.58). It is clear that there are 15 different errors in the code matrix $E=\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ including "single," "double," "triple" and "fourfold" errors, and we can only correct four of them (the "single" errors). Then we can estimate for this case the correction ability of the Fibonacci encoding/decoding method as follows:

$$S_{cor}=\frac{4}{15}=0.2667=26.67\%. \qquad (11.109)$$

If we compare this estimation with the potential correction ability of the Hamming (15,11)-code

S_c=0.0004882 (0.04882%), (11.110)

we can see that the correction ability of the Fibonacci encoding/decoding method given by (11.109) exceeds the potential correction ability of the Hamming code given by (11.110) by more than 500 times. Remember too that the relative redundancy of the Hamming (15,11)-code and the Fibonacci encoding/decoding method are equal to 0.267 (26.7%) and 0.333 (33.3%), respectively, that is, their relative redundancies are comparable.

We should also recall that the potential correction ability of the classical error-correction codes given by (11.2) do decrease exponentially with the increase of the data bits n. For example, for n=20, the coefficient of the potential correction ability of the error-correction code is equal to $S_c=1/2^{20}=0.0000009$ (0.00009%). For this case, the advantage of the Fibonacci encoding/decoding method in comparison to the classical error-correction code is still more impressive (by about 300,000 times).

However, for the case n>>1, we can use the additional checking relations given by (11.52) and (11.53). Above we have proved that for this case we can correct with guarantee the "single," "double," and "triple" errors in the code matrix $E=\begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$. It is clear that for this case the correction ability of the Fibonacci encoding/decoding method is defined by the ratio:

$$S_{cor}=\frac{14}{15}=0{,}9333\,(93{,}33\%). \tag{11.111}$$

It is clear that for the case n>>1, the correction ability of the Fibonacci encoding/decoding method given by (11.111) exceeds the potential correction ability of the Hamming (15,11)-code (11.110) by about 2,000 times. However, for the case n=20, the advantage of the Fibonacci encoding/decoding method in comparison to the classical error-correction code is more impressive (by more than 1,000,000 times).

11.6.3. *Advantages of the Fibonacci Encoding/Decoding Method*

Comparing the classical error-correction codes [177, 182] with the Fibonacci error-correction codes based on the matrix approach, we may note the following. A theory of error-correction codes developed by prominent American researcher Richard Wesley Hamming and others [177, 182] is one of the most important theoretical achievements in computer science. However, the existing error-correction codes have a number of fundamental shortcomings:

1. **The very low potential correction ability of all existing error-correction codes is the first shortcoming.** For example, the potential correcting ability of the Hamming (15,11)-code, which allows one to correct all "single" errors in the 15-digit code combination, is equal to 0.04882%. This means that the Hamming code can correct only about 0.05% of all possible errors that can appear in the code combination.
2. **It is well known that very small information elements, bits and their combinations, are the objects of detection and correction. This is next essential shortcoming of all existing error-correction codes.** The exist-

ing error-correction codes are at times named *Algebraic Codes* as they are based on modern algebra. Unfortunately, modern mathematical theory, such as matrix theory, is insufficiently used in modern coding theory.

The Fibonacci error-correction codes based on the matrix approach possess a number of essential peculiarities and advantages in comparison to classical error-correction codes:

1. The use of the matrix theory for designing new error-correction codes is the first peculiarity of the Fibonacci encoding/decoding method.

2. The large information units, in particular, matrix elements, are objects of detection and correction of errors in the Fibonacci encoding/decoding method. This fact is the first advantage of the Fibonacci encoding/decoding method in comparison to the classical error-correction codes [177, 182]. Note that there are no theoretical restrictions for the value of the numbers that can be matrix elements.

3. However, the most important advantage of the Fibonacci encoding/decoding method is the very high correction ability in comparison to the classical error-correction codes. As is demonstrated above, the correction ability of the Fibonacci error-correction codes exceeds the correction ability of the classical error-correction codes by more than 1,000,000 times. What is the basis for such high correction ability? Probably the main cause is a fundamentally new approach to coding theory. For the detection and correction of errors we use not only the "checking relations" (11.32)-(11.34), (11.52) and (11.53) but also the property of matrix elements to be integer numbers. This allows for effective application of the theory of Diophantine equations for error correction. In general, the combination of matrix theory with the Diophantine equation theory results in the creation of a new class of error-correction codes that exceed by more than 1,000,000 times the classical error-correction codes in their correction ability.

11.7. Matrix Cryptography

The idea of a *Matrix Approach* for the creation of a new theory of error-correcting codes [44, 113] can be used in the development of new cryptographic algorithms. First, in this section we discuss an application of *Matrix Cryptography* for *Digital Signals,* which allows an increase in the speed of encryption/decryption and its application as a cryptographic method for protection of communication systems operating in real time. Second, we discuss one modification of *Matrix Cryptography* called *"Golden" Cryptography,* which allows for the checking of informational processes in cryptographic systems.

11.7.1. *The Concept of Hybrid Cryptosystems*

In this study we are talking about cryptographic protection of *Digital Signals*. The term digital signal refers to discrete-time signals that have a discrete number of levels. Digital signals are digital representations of discrete-time signals, which are often derived from analog signals. In the *Digital Revolution,* the use of digital signals has increased significantly. Many modern media devices, especially the ones that connect with computers, use digital signals to represent signals that were traditionally represented as continuous-time signals: examples include measurement systems, mobile phones, music and video players, personal video recorders, and digital cameras.

Let us represent a digital signal X in the form of a sequence of samples, that is,

$$X = \{x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, ..., x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, ...\}. \tag{11.112}$$

It is clear that for many cases there is a problem with the cryptographic protection of the digital signal (11.112). First of all, it is important to protect mobile phones from forbidden eavesdropping. It is also necessary to protect music or video information from forbidden access. And it is very important to protect many measurement systems from forbidden access, and so on. Such problems exist for video recorders and digital cameras.

It is well-known that a majority of continuous-time signals are signals representing information in real time. Of course, cryptographic algorithms used for cryptographic protection of digital signals must be sufficiently fast-acting algorithms.

Let us consider from this point of view *Public-key Algorithms* [183] used widely in modern cryptographic praxis. Many recognized specialists are critically evaluating the advantages of public-key cryptography and are paying close attention to the shortcomings of public-key cryptography. For example, Richard A. Molin writes [187]: *"Public-key methods are extremely slow compared with symmetric-key methods. In latter discussions we will see how both the public-key and symmetry-key cryptosystems come to be used, in concert, to provide the best of all worlds combining the efficiency of the symmetric-key ciphers with the increased security of public-key ciphers, called hybrid systems."* As noted in [188] *"we have symmetric key algorithms that are very fast and strong, but are really bad at key management. We have asymmetric key algorithms that are really good at key management, but are terribly slow. Real-world systems are usually hybrids, using each* *technology, symmetric and asymmetric, where it is strong. Normally, these hybrid systems will use asymmetric key cryptography to do the key management, and use symmetric key cryptography to do bulk encryption/decryption."*

The concept of a *Hybrid Cryptosystem* is a new direction in cryptography [187]. The main goal is to combine the high security of a public-key cryptosystem with the high speed of a symmetric-key cryptosystem. A hybrid cryptosystem can be constructed using two separate cryptosystems:

- a public-key cryptosystem for the **transmission of cryptographic keys**,
- a symmetric-key cryptosystem for **data transmission**.

Such approach increases interest in the development of hybrid cryptography based upon new cryptographic algorithms, particularly, *Matrix Cryptography*.

11.7.2. *General Principle of Matrix Cryptography*

Let us consider a non-singular ($n\times n$)-matrix E and its inverse matrix E^{-1}, which are connected by the identity:

$$E\times E^{-1}=I_n, \tag{11.113}$$

where I_n is *identity* ($n\times n$)-*matrix*.

Note that the matrices E and E^{-1} play the role of *encryption* and *decryption matrices* for matrix cryptography.

Now let us consider a *data* square matrix X (plaintext) with the same size as the matrix E. Then we can write the product of the matrices E and X as follows:

$$Y=E\times X\cdot \tag{11.114}$$

The procedure (11.114) is called *Matrix Encryption*. As a result of the matrix encryption we get a *Code Matrix* Y (ciphertext).

If we now multiply the code matrix Y by a *Decryption Matrix* E^{-1}, we get:

$$E^{-1}\times Y=E^{-1}\times\left(E\times X\right)=\left(E^{-1}\times E\right)\times X=I_n\times X=X\,. \tag{11.115}$$

The procedure (11.115) is called *Matrix Decryption*.

The identities (11.114) and (11.115) give rise to a general principle, which can be used in coding theory and cryptography. This principle, for the first time, was formulated in the book [44].

Consider the sequence $\{x_i\}_{i=1}^{N}$, representing data for a one-dimensional digital signal. The elements of this sequence can be rearranged in the form of ($m\times m$)-matrices. This results in a sequence of square matrices:

$$\begin{pmatrix} x_1 & \cdots & x_m \\ \vdots & \ddots & \vdots \\ x_{m^2-m+1} & \cdots & x_{m^2}\end{pmatrix},\begin{pmatrix} x_{m^2+1} & \cdots & x_{m^2+m} \\ \vdots & \ddots & \vdots \\ x_{2m^2-m+1} & \cdots & x_{2m^2}\end{pmatrix},\cdots\,. \tag{11.116}$$

The elements of this new sequence are defined by:

$$X_k = \begin{pmatrix} x_{(k-1)m^2+1} & \cdots & x_{(k-1)m^2+m} \\ \vdots & \ddots & \vdots \\ x_{km^2-m+1} & \cdots & x_{km^2} \end{pmatrix}. \tag{11.117}$$

The sequence of square matrices $\{X_k\}_{k=1}^{K}$ contains the same data as sequence $\{x_i\}_{i=1}^{N}$. To encrypt the sequence $\{x_i\}_{i=1}^{N}$, it can be reformulated to build a sequence of square matrices.

Let us consider the formation of encryption and decryption matrices E and E^{-1}. For the simplest case, the encryption matrix E can be chosen randomly from some set of encryption matrices. The decryption matrix E^{-1} is computed from the encryption matrix E according to a special algorithm for calculation of *Inverse Matrices* [190]. For example, we can use the formula (11.10) for computation of the invertible (2×2) matrix.

In the generalized case, we can use different encryption and decryption matrices E_k and E_k^{-1} for the encryption/decryption of every element of the sequence (11.116). Such a method increases the cryptographic power of matrix cryptography.

Suppose that $\{E_k\}_{k=1}^{K}$ is a sequence of encrypted non-singular square matrices of the same size as the data matrix (11.117). Multiplication of E_k and X_k yields:

$$Y_k = E_k X_k \tag{11.118}$$

and thus the sequence of $\{Y_k\}_{k=1}^{K}$ is defined. In inverting the operations done in Eq. (11.115) to create matrix sequence $\{X_k\}_{k=1}^{K}$, the matrix sequence $\{Y_k\}_{k=1}^{K}$ is transformed into scalar sequence $\{y_i\}_{i=1}^{N}$. This new sequence is an encrypted form of the original sequence $\{x_i\}_{i=1}^{N}$. The cryptographic key, which is used here, is the sequence of non-singular matrices $\{E_k\}_{k=1}^{K}$. Some methods for defining this encryption sequence of non-singular matrices are discussed below.

11.7.3. *Matrix Cryptography of Digital Sound Signals*

In this section we use the scientific results obtained by a follower of the author, the Iranian researcher Mostapha Kalami Heris (Ferdowsi University of Mashhad).

One possible way to define non-singular matrices in the encryption sequence $\{E_k\}_{k=1}^{K}$ is to define these matrices as real powers of a non-singular matrix, named a *Kernel Matrix*.

Assume that B is a kernel non-singular matrix. Let us prove that any real power $p \neq 0$ of the kernel non-singular matrix B with $\det B \neq 0$, that is, the matrix B^p is also non-singular. Indeed, we can represent the determinant of the matrix B^p as follows:

$$\det\left(B^p\right) = \left(\det B\right)^p. \tag{11.119}$$

It follows from (11.119) that for cases $\det B \neq 0$ and $p > 0$

$$\det\left(B^p\right) \neq 0, \tag{11.120}$$

that is, the matrix B^p is also non-singular.

So the matrices B^p and B^{-p} can be used to create a sequence of encryption and decryption non-singular matrices. Assume that the encryption sequence has the form of $\{E_k\}_{k=1}^{K}$. Suppose $\{p_k\}_{k=1}^{K}$ is a sequence of real numbers. Then elements of the encryption sequence can be defined by:

$$E_k = B^{p_k}, \tag{11.121}$$

which is the definition of an encryption sequence with the real powers of a non-singular kernel matrix B.

A Digital Sound Signal is an example of a one-dimensional data signal. Note that if the signal is recorded in a multi-channel mode, any channel of this sound is one-dimensional. In this subsection, a single channel of sound is studied and the cryptographic algorithm is applied to a single channel. Digital sound signals are saved and stored in many formats in the computers. The amplitudes of samples are saved in the file as data, and later the sound will be played back by audio hardware. When all amplitudes of sound are multiplied or divided by a constant, the volume of sound is the only thing which is changing. If the relative value of amplitudes remains unchanged, the sound is unchanged. Since the value of amplitudes has to lie in the range of [-1,1], all sounds in the algorithm are normalized to have the largest absolute value of amplitudes equal to 1.

Suppose $x[n]$ is a digital signal represented as the sequence form $\{x_i\}_{i=1}^{N}$. To apply the matrix cryptography to this signal, the following non-singular matrix is used, for example, as kernel matrix:

$$B = \begin{pmatrix} 0.2 & -1 \\ 1 & -0.5 \end{pmatrix}. \tag{11.122}$$

The powers of this matrix are computed where powers are real numbers, in the range of [-5, 5].

A block diagram of the encryption/decryption process is shown in Fig. 11.1. The input signal $x[n]$ is passed through an encryption process which yields the *Encrypted Signal* $y[n]$. Also, the latter signal is passed through the decryption process yielding the *Recovered Signal* $\hat{x}[n]$.

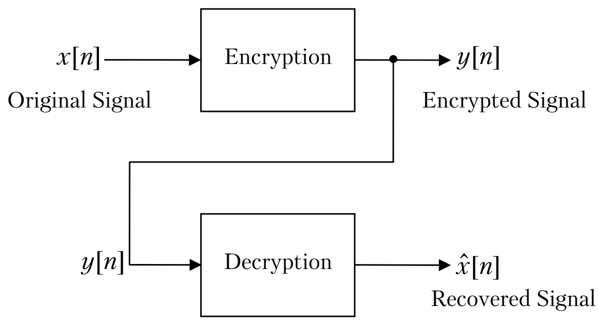


Figure 11.1. Block diagram of encryption and decryption processes

According to the size of B, the elements of sequence $\{x_i\}_{i=1}^{N}$ corresponding to the samples of sound signal, must be reformed to build a sequence of (2×2) square matrices, like to $\{X_k\}_{k=1}^{K}$. N and K are appropriate numbers and in this particular problem, $N=4K$. The elements of the power sequence $\{p_k\}_{k=1}^{K}$ are randomly generated with a uniform distribution over the range [5,5]. The sequence of encryption matrices $\{E_k\}_{k=1}^{K}$ is computed by the rule $E_k = B^{p_k}$. The encrypted matrix sequence $\{Y_k\}_{k=1}^{K}$ is an element-by-element multiplication of the encryption sequence $\{E_k\}_{k=1}^{K}$ and data sequence $\{X_k\}_{k=1}^{K}$. Flattening the elements of matrix sequence $\{Y_k\}_{k=1}^{K}$ yields the numerical sequence of the encrypted data $\{\hat{y}_i\}_{i=1}^{N}$. This sequence may be assumed to be a representation of an encrypted sound signal having the same duration as the original sound signal. As mentioned earlier, the amplitude of digital sounds are bounded in the range [-1, 1], and the elements of $\{\hat{y}_i\}_{i=1}^{N}$ may not lie in this range. Hence, the sequence $\{\hat{y}_i\}_{i=1}^{N}$ must be normalized to obtain the sequence $\{y_i\}_{i=1}^{N}$. The elements of this sequence are defined by:

$$y_i = \frac{\hat{y}_i}{\max\limits_{1\le j\le N} \hat{y}_j}. \tag{11.123}$$

The resulting sequence is an original sound signal in an encrypted form. Due to the non-singularity of all encryption matrices, the original data can be recovered completely from the encrypted data. The decryption algorithm is just like the encryption algorithm. The only difference is the sequence of powers. In the decryption phase, the negated form of the power sequence of the encryption phase must be used.

As is well-known, the binary representation used in digital computers cannot represent all real numbers accurately and round-off errors are unavoidable. The numbers appearing in this encryption/decryption algorithm are real numbers in general, and most of them cannot be represented using a finite number of bits. Because of the already mentioned round-off error, the recovered signal and original signal are not necessarily equal and there exists an encryption/decryption error in the algorithm. The existence of this error is one of the limitations in the implementation of matrix cryptographic algorithms in digital computers, and restricts the application of this algorithm to fields which do not require exact recovery of encrypted data.

11.7.4. *Matrix Cryptography of Digital Images*

In this subsection we discuss the application of *matrix cryptography* to digital images. This application was developed by Mostapha Kalami Heris. Images are 2-dimensional signals and the digital image can be represented as a 2-dimensional sequence of color data. One useful color coding approach used in digital computers is *RGB* coding. This coding is based upon the fact that every color can be represented as a linear combination of three base light beams: *Red, Green* and *Blue*. Digital images are saved in digital computers as a 2-dimensional array of color data. Each element of this array is called a *pixel*. Every pixel in a digital image has a color composed of three elements: red, green and blue. Each element of a color in the 24-bit color coding standard, has 8 bits of data. So the elements of a 24-bit *RGB* color are represented as positive integers ranging from 0 to 255. It is standard practice to scale this range into the range of real numbers between 0 and 1. The decomposition of color of every pixel in a digital image into its corresponding red, green and blue elements yields three numerical 2-dimensional arrays. Each array is in the form of $x[i, j]$ where $i=1,2,...,M$ and $j=1,2,...,N$. M and N are the vertical and horizontal sizes of the picture, respectively. It is assumed that values of elements of this array are normalized to be in the range [0, 1].

To apply matrix encryption to a sample image, the method of the previous subsection is used to create an encryption sequence. Note that all sequences in this subsection are assumed to be 2-dimensional and have the appropriate size and number of elements. The example of *kernel matrix*, which is used in this subsection, is given by (11.124).

This matrix is non-singular and has non-negative eigenvalues. So all of its real powers are real matrices which can be used as encryption matrices. To use this matrix as a kernel, the data in the original image must be reformed into

(8×8)-matrices. Simply put, the original image is divided into (8×8)-image blocks and the corresponding data for each block is used to create the source matrix sequence. The sequence of powers is generated as a random variable, uniformly distributed in the range [-10, 10]. Such a cryptography algorithm, based upon the non-singular kernel matrix of the kind of (11.124), can be used for a cryptographic protection of images.

$$B=\begin{pmatrix} 0.2 & -1 & 0 & 0.1 & 0.3 & 0 & 0 & 0.2 \\ 1 & -0.5 & -0.1 & 0 & 0 & -0.7 & -0.2 & 0 \\ 0 & -0.2 & 0.3 & -1 & 0 & -0.1 & 0.1 & 0 \\ 0.2 & 0 & 1 & 0.6 & 0.1 & 0 & 0 & 0.5 \\ 0.3 & 0 & 0 & 0.1 & 0.1 & -1 & 0 & 0.1 \\ 0 & -0.8 & -0.1 & 0 & 1 & -0.5 & -0.1 & 0 \\ 0 & -0.2 & 0.2 & 0 & 0 & -0.2 & 0.2 & -1 \\ 0.2 & 0 & 0 & 0.5 & 0.2 & 0 & 1 & 0.4 \end{pmatrix}. \tag{11.124}$$

11.7.5. *The Problem of Checking Information in Cryptosystems*

Studying applications of cryptosystems in communication systems, one sometimes loses sight of an important problem, one which exists in practically all communication systems. We are talking about the problem of protecting the information processing from *malfunctions* and *failures* in the encoder and decoder, as well as, from *noises* in the communication channel itself. This problem is especially important for special communication systems, for example, for cosmic or military communication systems.

It is well-known that existing cryptographic methods and algorithms [183-187] were created for "ideal conditions" when one assumes that the encoder, the communication channel, and the decoder operate "ideally," that is, the coder carries out an "ideal" transformation of plaintext into ciphertext, the communication channel "ideally" transmits a ciphertext from the encoder to the decoder, and the latter performs an "ideal" transformation of ciphertext into plaintext. It is clear that the slightest breach of the "ideal" transformation or transmission is a catastrophe for the cryptosystem because the true message cannot be delivered to the "recipient."

Consider for a moment the transformation of *plaintext* into *ciphertext*. Before sending the ciphertext to the communication channel we must be convinced that the ciphertext is consistent with the plaintext. We can be convinced of this by means of an inverse transformation of the ciphertext into the plaintext. In symmetric-key cryptosystems we can perform an inverse transformation because the "sender" knows the cryptographic key. However, we cannot make this inverse transformation in a public-key cryptosystem because the secret key is not known to the "sender." It follows from this reasoning that **public-key cryptosystems are the most vulnerable for errors that may appear in the encoder.** This means that the basic advantage of the public-key cryptosystem becomes a shortcoming when we are talking about designing reliable cryptosystems. Despite the extraordinary importance of protecting cryptosystems from "noises," "malfunctions," and "failures," this problem is not adequately highlighted in modern cryptographic literature [183-187]. Sending a ciphertext through a communication channel in public-key cryptosystems, we must rely on the good will of God!

11.7.6. *"Golden" Cryptography*

Let us consider the application of so-called *"golden" Q- and G_m-matrices* introduced in Chapter 6 for the creation of a new cryptographic method called *"golden" cryptography* [114, 118]. Recall (see Chapter 6) that the *"golden" Q-matrices* are the following matrices:

$$Q^{2x}=\begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix} Q^{2x+1}=\begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix}. \quad (11.125)$$

The first peculiarity of the *"golden" Q-matrices* (11.125) is the fact that the *symmetric hyperbolic Fibonacci sine* and *cosine* given by (5.16) and (5.17), that is, the functions $sF(x)=\left(\tau^{2x}-\tau^{-2x}\right)\big/\sqrt{5}$ and $cF(x)=\left(\tau^{2x+1}+\tau^{-(2x+1)}\right)\big/\sqrt{5}$, where $\tau=\left(1+\sqrt{5}\right)/2$ is the *golden mean*, are in fact elements of the *"golden" Q-matrices.* This means that the "golden" Q-matrices (11.125) are functions of the continuous variable x. The second peculiarity of the "golden" matrices (11.125) is the fact that their determinants are equal to 1 and -1:

$$\det Q^{2x}=1 \text{ and } \det Q^{2x+1}=-1. \quad (11.126)$$

It follows from (11.126) that the *"golden" Q-matrices* (11.125) are *nonsingular* or *invertible* matrices. Their inverse matrices (see Chapter 6) are the following:

$$Q^{-2x}=\begin{pmatrix} cFs(2x-1) & -sFs(2x) \\ -sFs(2x) & cFs(2x+1) \end{pmatrix};$$
$$Q^{-2x-1}=\begin{pmatrix} -sFs(2x) & cFs(2x+1) \\ cFs(2x+1) & -sFs(2x+2) \end{pmatrix}. \quad (11.127)$$

The *"golden" G_m-matrices* (see Chapter 6) are the following matrices:

$$G_m^{2x} = \begin{pmatrix} cF_m(2x+1) & sF_m(2x) \\ sF_m(2x) & cF_m(2x-1) \end{pmatrix};$$
$$G_m^{-2x+1} = \begin{pmatrix} sF_m(2x+2) & cF_m(2x+1) \\ cF_m(2x+1) & sF_m(2x) \end{pmatrix}. \tag{11.128}$$

The first peculiarity of the *"golden" G_m-matrices* (11.128) is the fact that *hyperbolic Fibonacci m-sine* and *m-cosine* given by (5.103) and (5.104), that is, the functions $sF_m(x) = \frac{\Phi_m^x - \Phi_m^{-x}}{\sqrt{4+m^2}}$ and $cF_m(x) = \frac{\Phi_m^x + \Phi_m^{-x}}{\sqrt{4+m^2}}$, where $\Phi_m = \left(m + \sqrt{4+m^2}\right)/2$ is the *metallic mean*, are elements of the *"golden" G_m-matrices*. This means that the "golden" G_m-matrices (11.128) are functions of the continuous variables x and m>0. The second peculiarity of the "golden" G_m-matrices (11.128) is the fact that their determinants are equal to 1 and -1:

$$\det G_m^{2x} = 1 \text{ and } \det G_m^{2x+1} = -1. \tag{11.129}$$

It follows from (11.129) that the *"golden" G_m-matrices* (11.128) are *non-singular* or *invertible* matrices. Their inverse matrices (see Chapter 6) have the following form:

$$G_m^{-2x} = \begin{pmatrix} cF_m(2x-1) & -sF_m(2x) \\ -sF_m(2x) & cF_m(2x+1) \end{pmatrix};$$
$$G_m^{-2x-1} = \begin{pmatrix} -sF_m(2x) & cF_m(2x+1) \\ cF_m(2x+1) & -sF_m(2x+2) \end{pmatrix}. \tag{11.130}$$

The idea of "golden" cryptography is similar to that of the Fibonacci encoding/decoding method described in Section 11.6. Let us here represent *plaintext* in the form of a non-singular square (2×2)-matrix M:

$$M = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}. \tag{11.131}$$

Note that there are $4! = 4 \times 3 \times 2 \times 1 = 24$ variants (permutations) to form the matrix (11.131) from the four elements a_1, a_2, a_3, a_4. Let us designate the i-th permutation by P_i (i=1, 2, ..., 24). The first step of cryptographic protection of the *plaintext* a_1, a_2, a_3, a_4 is a choice of the permutation P_i. Then we choose the direct "golden" matrices (11.125) or (11.128) as *encryption matrices* and their inverse matrices (11.127) or (11.130) as *decryption matrices*.

Let us now consider the following encryption/decryption algorithms based on matrix multiplication (see Table 11.5 and Table 11.6).

Table 11.5. Encryption/decryption algorithm based on the "golden" Q-matrices

Encryption	Decryption
$M \times Q^{2x} = E_1(x)$	$E_1(x) \times Q^{-2x} = M$
$M \times Q^{2x+1} = E_2(x)$	$E_2(x) \times Q^{-2x-1} = M$

Table 11.6. Encryption/decryption algorithm based on the "golden" G_m-matrices

Encryption	Decryption
$M \times G_m^{2x} = E_3(x,m)$	$E_3(x,m) \times G_m^{-2x} = M$
$M \times G_m^{2x+1} = E_4(x,m)$	$E_4(x,m) \times G_m^{-2x-1} = M$

Here M is the *plaintext* (11.130) that is formed according to the permutation P_i; $E_1(x), E_2(x)$, $E_3(x,m), E_4(x,m)$ are *code matrices* or *ciphertexts*; Q^{2x}, Q^{2x+1}, G_m^{2x}, G_m^{2x+1} are *encryption matrices*; Q^{-2x}, Q^{-2x-1}, G_m^{-2x}, G_m^{-2x-1} are *decryption matrices*. For the encryption/decryption method given in Table 11.5 we can use the variable x as a *cryptographic key*. For the encryption/decryption method given in Table 11.6 we can use the variables x and $m>0$ as components of a *cryptographic key*. This means that in dependence on the value of the keys x and m there are an infinite number of transformations of a *plaintext* M into a *ciphertext* $E(x)$ or $E(x,m)$.

In general the cryptographic key K consists of three components: permutation P_i and the variables x and m, that is,

$K = \{P, x, m\}$.

11.7.7. *Checking Information in the "Golden" Cryptosystem*

11.7.7.1. *The Main Checking Relations for the "Golden" Cryptography*

In order to protect information in cryptosystems from "noise," "malfunction," and "failure," we really should provide for information "checking" at all stages of the transformations, that is, we should provide for checking

(1) a transformation of plaintext into ciphertext in the encoder

(2) a transmission of the ciphertext in communication channels

(3) a transformation of ciphertext into plaintext in the decoder.

For the "golden" cryptography we can use the following identities, which connect determinants of *plaintext* and *ciphertext*. Let us calculate determinants of the code matrices $E_1(x), E_2(x), E_3(x,m), E_4(x,m)$:

$$\det E_1(x) = \det M \times \det Q^{2x}; \det E_2(x) = \det M \times \det Q^{2x+1};$$
$$\det E_3(x,m) = \det M \times \det G_m^{2x}; \det E_4(x,m) = \det M \times \det G_m^{2x+1}. \quad (11.132)$$

Taking into account (11.126) and (11.129), we can rewrite the formulas (11.132) as follows:

$$\det E_1(x) = \det M; \det E_2(x) = -\det M;$$
$$\det E_3(x,m) = \det M; \det E_4(x,m) = -\det M. \tag{11.133}$$

The identities (11.133) play a major role in the "main checking relations" for "golden" cryptography.

11.7.7.2. *Checking Encoder*

For "checking" an encoder we can use the fundamental identities (11.133). For this purpose we first compute the determinant of the *plaintext* (11.131):

$$\det M = a_1a_4 - a_2a_3 . \tag{11.134}$$

Then we compute the determinants det E of the code matrices $E_1(x)$, $E_2(x)$, $E_3(x,m)$, $E_4(x,m)$ and then compare the determinants det E and det M according to (11.133). If the identity (11.133) is valid, it means that the "golden" encryption is valid and we can send the code matrix E via the channel. In the contrary case, one must begin anew the "golden" encryption.

11.7.7.3. *Checking Channel and Decoder*

For "checking" the channel we will use the determinant det M taken by module k, that is, $[\det M]_{\mathrm{mod}\,k}$. Then we send $[\det M]_{\mathrm{mod}\,k}$ to the channel after the *ciphertext E*. After receiving the code matrix E from the channel we should compute the determinant det E and perform the module k operation, that is, calculate $[\det E]_{\mathrm{mod}\,k}$. By comparing $[\det E]_{\mathrm{mod}\,k}$ with $[\det M]_{\mathrm{mod}\,k}$ received from the channel, we can check for the correctness of the transmission of the *ciphertext E* via the channel. If $[\det E]_{\mathrm{mod}\,k} = [\det M]_{\mathrm{mod}\,k}$, then the *ciphertext E* is correct and the decoder can transform the *ciphertext E* into the *plaintext M*. After the transformation $E \to M$, we should calculate det M according to (11.134) and then compare it with det E. If one of the identities (11.133) is valid, it means that the transformation $E \to M$ is correct. Thus, all information transformations in the cryptosystem can be checked and the reliability of the cryptosystem is increased.

11.7.8. *Project for a Cryptographic Mobile Phone*

A *mobile phone* (also known as a *wireless phone* or *cellular phone*) is an electronic device used for mobile voice or data communication over a network of specialized base stations known as *cell sites*. In addition to the standard voice function of a mobile phone, *telephone*, current mobile phones may support many additional services, and accessories, such as *SMS for text messaging*, *email*, *packet switching* for access to the Internet, *gaming*, *camera with video recorder*, *MMS for sending and receiving photos and video* and so on.

We can suggest a project for the *Mobile Phone* based on a *hybrid matrix cryptosystem*. In addition to all traditional electronic components, a new *Cryptographic Mobile Phone* should contain a hybrid cryptosystem, consisting of a public-key cryptosystem and symmetric-key cryptosystem based on matrix cryptography for sound signals and images. This approach provides cryptographic protection of all important information, in particular, voice and video information by using a hybrid cryptosystem based on matrix cryptography, and also protects all text information (SMS, email and so on) by using a public-key cryptosystem. It is clear that the *Cryptographic Mobile Phone* protects human rights and the freedom of the individual more effectively than our existing cryptographic systems.

11.8. Conclusion

1. The *theory of information and error-correction codes* developed by American researchers Claude Shannon, Richard Wesley Hamming and others is one of the most important theoretical achievements in computer science. However, existing error-correction codes have a number of fundamental shortcomings. The first one is the very low potential correction ability of error-correction codes. For example, the potential correction ability of the Hamming (15,11)-code, which allows for correction of all single errors in the 15-digit code combination, is equal to 0.04882%. This means that the Hamming code can correct only about 0.05% of all errors that can appear in the code combination. The next shortcoming is the fact that the very small information elements, bits and their combinations, are the objects of detection and correction. The existing error-correction codes are sometimes named *Algebraic Codes* because they are based on modern algebra. Unfortunately, important modern theories like matrix theory are insufficiently employed in modern coding theory.

2. The Fibonacci Q-matrix was developed in the work of American mathematician Verner Hoggatt [16], and is considered to be one of the foremost achievements in contemporary *Fibonacci number theory*. Recently the author, Alexey Stakhov, generalized the concept of the Fibonacci Q-matrix and introduced the notion of generalized Fibonacci matrices [103, 113] based upon the Fibonacci p-numbers [20], Fibonacci m-numbers [118] and Fibonacci (p,n)-numbers [154].

3. A new theory of error-correction codes based upon the generalized Fibonacci matrices [113, 118] has the following advantages compared with the algebraic error-correction code theory [177, 182]: (1) the Fibonacci coding/decoding method amounts to matrix multiplication, that is, to the well-known algebraic operation that is carried out very efficiently in modern computers; (2) the main practical feature of the Fibonacci encoding/decoding method is the fact that large information units, in particular, matrix elements, are objects of error detection and correction; (3) the simplest Fibonacci coding/decoding method (where p=1) can guarantee the restoration of all "erroneous" (2×2)-code matrices having "single," "double" or "triple" errors; (4) the potential correction ability of this method for the simplest case p=1 is equal to 26.67% (for n=1) or 93.33% (for n>>1) which exceeds essentially (in thousands and millions times) the potential correction ability of all well-known algebraic error-correction codes (0.04882% for the Hamming code). This means that the new coding theory based on the matrix approach is of great practical importance for modern computer science.

4. There are two directions in cryptography, *Symmetric Cryptosystems with a Secret Key* and *Asymmetric Cryptosystems with Public and Secret Keys*. Since publication of an article by W. Diffie and M. E. Hellman titled *New Directions in Cryptography* [183], the topic of *Public-Key Cryptosystems* [183-186] has attracted a great deal of attention and become a basis for the development of modern cryptosystems. Unfortunately, Richard A. Molin's opinion [187] that "public-key methods are extremely slow compared with symmetric-key methods" makes their application for informational systems operating in real time (for example, telephone systems) much more difficult. Concepts of the *Hybrid Cryptosystem* [187, 188] and *Matrix Cryptography* developed in this present book can lead to the design of a *Cryptographic Mobile Phone*. This approach provides cryptographic protection to all voice and video information. The *Cryptographic Mobile Phone* in turn protects human rights and the freedom of the individual to communicate under secure conditions more effectively than existing cryptographic systems. And such a *Cryptographic Mobile Phone* could become a very important step in the creation of a *Harmonic Society* based on the *Principles of Harmony and the Golden Section* already developed in ancient science.

Epilogue

Dirac's Principle of Mathematical Beauty and the Mathematics of Harmony:
Clarifying the Origins and Development of Mathematics

E.1. Introduction

In the Epilogue, we try to give a review of the basic results obtained in the present book from the point of view of *Dirac's Principle of Mathematical Beauty*. For convenience of the readers, we have duplicated in the Epilogue some of the most important formulas of the *Mathematics of Harmony*. Also we have included a number of the newest mathematical results, which were obtained on the concluding stage of preparing the camera-ready manuscript. First of all, we mention the scientific results obtained by the author together with the outstanding Russian mathematician Samuil Aranson – *Fibonacci-Lorenz transformations*, which have direct relation to special theory of relativity, and the solution to *Hilbert's Fourth Problem*, one of the unsolved Hilbert's problems [191].

E.1.1. *Dirac's Principle of Mathematical Beauty*

Recently the author studied the contents of a public lecture: *The complexity of finite sequences of zeros and units, and the geometry of finite functional spaces* [192] by eminent Russian mathematician and academician Vladimir Arnold, presented before the Moscow Mathematical Society on May 13, 2006. Let us consider some of its general ideas. Arnold says:

1. *In my opinion, mathematics is simply a part of physics, that is, it is an experimental science, which discovers for mankind the most important and simple laws of nature.*
2. *We must begin with a beautiful mathematical theory. Dirac states: **"If this theory is really beautiful, then it necessarily will appear as a fine model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathemati-***

cal theory and to interpret them as predictions of new laws of physics." *Thus, according to Dirac, all new physics, including relativistic and quantum, develop in this way.*

At Moscow University there is a tradition that the distinguished visiting-scientists are requested to write on a blackboard a self-chosen inscription. When Dirac visited Moscow in 1956, he wrote *"A physical law must possess mathematical beauty."* This inscription is the famous *Principle of Mathematical Beauty* that Dirac developed during his scientific life. No other modern physicist has been preoccupied with the concept of beauty more than Dirac.

Thus, *Dirac's Principle of Mathematical Beauty* is the primary criterion for a mathematical theory to be considered as a model of physical phenomena. Of course, there is an element of subjectivity in the definition of the "beauty" of mathematics, but the majority of mathematicians agrees that "beauty" in mathematical objects and theories nevertheless exist. Let's examine some which have a direct relation to the theme of this book.

E.1.2. *Platonic Solids*

We can find the beautiful mathematical objects in Euclid's *Elements*. As is well known, the Book XIII of Euclid's *Elements* was devoted to a geometric theory of 5 regular polyhedrons called *Platonic Solids* (Fig. 3.4). And really these remarkable geometrical figures got very wide applications in theoretical natural sciences, in particular, in crystallography (Shechtman's quasi-crystals), chemistry (fullerenes), biology and they are brilliant confirmation of *Dirac's Principle of Mathematical Beauty*.

E.1.3. *Binomial Coefficients, the Binomial Formula, and Pascal's Triangle*

For the given non-negative integers n and k, there is the following beautiful formula that sets the *binomial coefficients*:

$$C_n^k = \frac{n!}{k!(n-k)!}, \tag{E.1}$$

where $n! = 1\times2\times3\times\ldots\times n$ is a *factorial* of n.

One of the most beautiful mathematical formulas, the *binomial formula,* is based upon the binomial coefficients:

$$(a+b)^n = a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}b^2 + \ldots + C_n^k a^{n-k}b^k + \ldots + C_n^{n-1} ab^{n-1} + b^n. \tag{E.2}$$

There is a very simple method for calculation of the binomial coefficients based on their following graceful properties called *Pascal's rule*:

$C_{n+1}^{k} = C_{n}^{n-1} + C_{n}^{k}$. (E.3)

Using the recursive relation (E.3) and taking into consideration that $C_n^0 = C_n^n = 1$ and $C_n^k = C_n^{n-k}$, we can construct the following beautiful table of binomial coefficients called *Pascal's triangle* (see Table E.1).

Table E.1. Pascal's triangle

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1
\end{array}
$$

Here we attribute "beautiful" to all the mathematical objects above. They are widely used in both mathematics and physics.

E.1.4. *Fibonacci and Lucas Numbers, the Golden Mean and Binet Formulas*

Let us consider the simplest recurrence relation:

$F_n = F_{n-1} + F_{n-2}$, (E.4)

where $n=0,\pm1,\pm2,\pm3,\ldots$. This recurrence relation was introduced for the first time by the famous Italian mathematician Leonardo of Pisa (nicknamed *Fibonacci*). For the seeds

$F_0=0$ and $F_1=1$ (E.5)

the recurrence relation (E.4) generates a numerical sequence called the *Fibonacci numbers* (see Table E.2).

Table E.2. Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	–1	2	–3	5	–8	13	–21	34	–55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	–1	3	–4	7	–11	18	–29	47	–76	123

In the 19th century the French mathematician Francois Edouard Anatole Lucas (1842-1891) introduced the so-called *Lucas numbers* (see Table E.2) given by the recurrence relation

$$L_n=L_{n-1}+L_{n-2} \tag{E.6}$$

with the seeds

$$L_0=2 \text{ and } L_1=1. \tag{E.7}$$

It follows from Table E.2 that the Fibonacci and Lucas numbers build up two infinite numerical sequences, each possessing graceful mathematical properties. As can be seen from Table E.2, for the odd indices $n=2k+1$ the elements F_n and F_{-n} of the Fibonacci sequence coincide, that is, $F_{2k+1}=F_{-2k-1}$, and for the even indices $n=2k$ they are opposite in sign, that is, $F_{2k}=-F_{-2k}$. For the Lucas numbers L_n all is vice versa, that is, $L_{2k}=L_{-2k}; L_{2k+1}=-L_{-2k-1}$.

In the 17th century the famous astronomer Giovanni Domenico Cassini (1625-1712) deduced the following beautiful formula, which connects three adjacent Fibonacci numbers in the Fibonacci sequence:

$$F_n^2-F_{n-1}F_{n+1}=(-1)^{n+1}. \tag{E.8}$$

This wonderful formula evokes a reverent thrill, if one recognizes that it is valid for any value of n (n can be any integer within the limits of $-\infty$ to $+\infty$). The alternation of +1 and -1 in the expression (E.8) within the succession of all Fibonacci numbers results in the experience of genuine aesthetic enjoyment of its rhythm and beauty.

If we take the ratio of two adjacent Fibonacci numbers F_n/F_{n-1} and direct this ratio towards infinity, we arrive at the following unexpected result:

$$\lim_{n\to\infty}\frac{F_n}{F_{n-1}}=\tau=\frac{1+\sqrt{5}}{2}, \tag{E.9}$$

where τ is the famous irrational number, which is the positive root of the algebraic equation:

$$x^2=x+1. \tag{E.10}$$

The number τ has many beautiful names – *golden section, golden number, golden mean, golden proportion, and the divine proportion*. See Olsen, p. 2 [54].

Note that formula (E.9) is sometimes called *Kepler's formula* after Johannes Kepler (1571-1630) who deduced it for the first time.

In the 19th century French mathematician Jacques Philippe Marie Binet (1786-1856) deduced the two magnificent *Binet formulas:*

$$F_n=\frac{\tau^n-(-1)^n\tau^{-n}}{\sqrt{5}}. \tag{E.11}$$

$$L_n=\tau^n+(-1)^n\tau^{-n}. \tag{E.12}$$

The *golden section* or *division of a line segment in extreme and mean ratio* descended to us from Euclid's *Elements*. Over the many centuries the *golden mean* has been the subject of enthusiastic worship by outstanding scientists and thinkers including Pythagoras, Plato, Leonardo da Vinci, Luca Pacioli, Johannes Kepler and several others. In this connection, we should recall Kepler's saying concerning the *golden section*:

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first, we may compare to a measure of gold; the second we may name a precious stone."

Alexey Losev, the Russian philosopher and researcher into the aesthetics of Ancient Greece and the Renaissance, expressed his delight in the golden section and Plato's cosmology in the following words:

"From Plato's point of view, and generally from the point of view of all antique cosmology, the universe is a certain proportional whole that is subordinated to the law of harmonious division, the Golden Section... This system of cosmic proportions is sometimes considered by literary critics as a curious result of unrestrained and preposterous fantasy. Total anti-scientific weakness resounds in the explanations of those who declare this. However, we can understand this historical and aesthetic phenomenon only in conjunction with an integral comprehension of history, that is, by employing a dialectical and materialistic approach to culture and by searching for the answer in the peculiarities of ancient social existence."

We can ask the question: in what way is the "golden mean" reflected in contemporary mathematics? Unfortunately, the answer forced upon us is - only in the most impoverished manner. In mathematics, Pythagoras and Plato's ideas are considered to be a "curious result of unrestrained and preposterous fantasy." Therefore, the majority of mathematicians consider study of the golden section as a mere pastime, which is unworthy of the serious mathematician. Unfortunately, we can also find neglect of the golden section in contemporary theoretical physics. In 2006 "BINOM" publishing house (Moscow) published the interesting scientific book *Metaphysics: Century XXI* [57]. In the Preface to the book, its compiler and editor Professor Vladimirov (Moscow University) wrote:

"The third part of this book is devoted to a discussion of numerous examples of the manifestation of the 'golden section' in art, biology and our surrounding reality. However, paradoxically, the 'golden proportion' is not reflected in contemporary theoretical physics. In order to be convinced of this fact, it is enough to merely browse 10 volumes of Theoretical Physics by Landau and Lifshitz. The time has come to fill this gap in physics, all the more given that the "golden proportion" is closely connected with metaphysics and 'trinitarity' [the 'triune' nature of things]."

During several decades, the author has developed a new mathematical direction called *The Mathematics of Harmony* [20, 21, 24, 51, 55, 84, 87-119]. For the first time, the name of *The Mathematics of Harmony* was introduced by the author in 1996 in the lecture, *The Golden Section and Modern Harmony Mathematics* [100], presented at the session of the 7th International conference *Fibonacci Numbers and Their Applications* (Austria, Graz, July 1996).

E.2. The "Strategic Mistakes" in the Development of Mathematics

E.2.1. *Mathematics: The Loss of Certainty*

The book *Mathematics: The Loss of Certainty* [6] by Morris Kline (1908-1992) is devoted to the analysis of the crisis of 20th century mathematics. Kline wrote:

"The history of mathematics is crowned with glorious achievements but also a record of calamities. The loss of truth is certainly a tragedy of the first magnitude, for truths are man's dearest possessions and a loss of even one is cause for grief. The realization that the splendid showcase of human reasoning exhibits a by no means perfect structure but one marred by shortcomings and vulnerable to the discovery of disastrous contradictions at any time is another blow to the stature of mathematics. But these are not the only grounds for distress. Grave misgivings and cause for dissension among mathematicians stem from the direction which research of the past one hundred years has taken. Most mathematicians have withdrawn from the world to concentrate on problems generated within mathematics. They have abandoned science. This change in direction is often described as the turn to pure as opposed to applied mathematics."

Further we read:

"Science had been the life blood and sustenance of mathematics. Mathematicians were willing partners with physicists, astronomers, chemists, and engineers in the scientific enterprise. In fact, during the 17th and 18th centuries and most of the 19th, the distinction between mathematics and theoretical science was rarely noted. And many of the leading mathematicians did far greater work in astronomy, mechanics, hydrodynamics, electricity, magnetism, and elasticity than they did in mathematics proper. Mathematics was simultaneously the queen and the handmaiden of the sciences."

Kline notes that our great predecessors were not interested in the problems of "pure mathematics," which were put forward in the forefront of the 20th century mathematics. In this connection, Kline writes:

"However, pure mathematics totally unrelated to science was not the main concern. It was a hobby, a diversion from the far more vital and intriguing problems posed by the sciences. Though Fermat was the founder of the theory of numbers, he devoted most of his efforts to the creation of analytic geometry, to problems of the calculus, and to optics .... He tried to interest Pascal and Huygens in the theory of numbers but failed. Very few men of the 17th century took any interest in that subject."

Felix Klein (1849 –1925), who was the recognized head of the mathematical world at the boundary of the 19th and 20th centuries, considered it necessary to protest against striving for abstract, "pure" mathematics:

"We cannot help feeling that in the rapid developments of modern thought, our science is in danger of becoming more and more isolated. The intimate mutual relation between mathematics and theoretical natural science which, to the lasting benefit of both sides, existed ever since the rise of modern analysis, threatens to be disrupted."

Richard Courant (1888-1972), who headed the Institute of Mathematical Sciences of New York University, also treated disapprovingly the passion for "pure" mathematics. He wrote in 1939:

"A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from the definition and postulates that must be consistent but otherwise may be created by the free will of mathematicians. If this description were accurate, mathematics could not attract any intelligent person. It would be a game with definitions, rules, and syllogisms without motivation or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only guided by intrinsic necessity, can the free mind achieve results of scientific value."

At present, mathematicians turned their attention to the solution of old mathematical problems formulated by the great mathematicians of the past. *Fermat's Last Theorem* is one of them. This theorem can be formulated very simply. Let us prove that for $n>2$ any integers x, y, z do not satisfy the correlation $x^n + y^n = z^n$. The theorem was formulated by Fermat in 1637 in the margins of Diophantus of Alexandria's book *Arithmetica* along with a postscript that the witty proof he found was too long to be placed there. Over the years many outstanding mathematicians (including Euler, Dirichlet, Legandre and others) tried to solve this problem. The proof of *Fermat's Last Theorem* was completed in 1993 by Andrew Wiles, a British mathematician working in the United States at Princeton University. The proof required 130 pages in the *Annals of Mathematics*.

Johann Carl Friedrich Gauss (1777 –1855) was a recognized specialist in number theory, confirmed by the publication of his book *Arithmetical Researchers* (1801). In this connection, it is curious to find Gauss' opinion about *Fermat's Last Theorem*. Gauss explained in one of his letters why he did not study Fermat's problem. From his point of view, "Fermat's hypothesis is an isolated theorem, connected with nothing, and therefore this theorem holds no interest" [6]. We should not forget that Gauss treated with great interest all 19th century mathematical problems and discoveries. In particular, Gauss was the first mathematician who supported Lobachevski's researchers on non-Euclidean geometry. Without a doubt, Gauss' opinion about *Fermat's Last Theorem* somewhat diminishes Wiles' proof of this theorem. In this connection, we can ask the following questions:

1. What significance does Fermat's Last Theorem hold for the development of modern science?
2. Can we compare the solution of Fermat's problem with the discovery of Non-Euclidean geometry in the first half of the 19th century and other mathematical discoveries?
3. Is Fermat's Last Theorem an "aimless play of intellect" and its proof merely a demonstration of the imaginative power of human intellect - and nothing more?

Thus, following Felix Klein, Richard Courant and other famous mathematicians, Morris Kline asserted that **the main reason for the contemporary crisis in mathematics was the severance of the relationship between mathematics and theoretical natural sciences that is the greatest "strategic mistake" of 20th century mathematics.**

E.2.2. *The Neglect of the "Beginnings"*

Eminent Russian mathematician Andrey Kolmogorov (1903 - 1987) wrote a preface to the Russian translation of Lebegue's book *About the Measurement of Magnitudes* [3]. He stated that "there is a tendency among mathematicians to be ashamed of the origin of mathematics. In comparison with the crystal clarity of the theory of its development it seems unsavory and an unpleasant pastime to rummage through the origins of its basic notions and assumptions.

All building up of school algebra and all mathematical analysis might be constructed on the notion of real number without any mention of the measurement of specific magnitudes (lengths, areas, time intervals, and so on). Therefore, one and the same tendency shows itself at different stages of education and with different degrees of inclination to introduce numbers possibly sooner, and furthermore to speak only about numbers and relations between them. Lebegue protests against this tendency!"

In this statement, Kolmogorov recognized a peculiarity of mathematicians - the diffident attitude towards the "origins" of mathematics. However, long before Kolmogorov, Nikolay Lobachevski (1792–1856) also recognized this tendency:

"Algebra and Geometry have one and the same fate. Their very slow successes followed after the fast ones at the beginning. They left science in a state very far from perfect. It probably happened, because mathematicians turned all their attention towards the advanced aspects of analytics, and have neglected the origins of mathematics by being unwilling to dig in the field already harvested by them and now left behind."

However, just as Lobachevski demonstrated by his research that the "origins" of mathematical sciences, in particular, Euclid's *Elements* are an inexhaustible source of new mathematical ideas and discoveries. *Geometric Researches on Parallel Lines* (1840) by Lobachevski opens with the following words:

"I have found some disadvantages in geometry, reasons why this science did not until now step beyond the bounds of Euclid's Elements. We are talking here about the first notions surrounding geometric magnitudes, measurement methods, and finally, the important gap in the theory of parallel lines ...."

Thankfully, Lobachevski, unlike other mathematicians did not neglect concern with "origins." His thorough analysis of the *Fifth Euclidean Postulate* ("the important gap in the theory of parallel lines") led him to the creation of Non-Euclidean geometry – the most important mathematical discovery of the 19th century.

E.2.3. *The Neglect of the Golden Section*

Pythagoreans advanced for the first time the brilliant idea about the harmonic structure of the Universe, including not only nature and people, but also everything in the entire cosmos. According to the Pythagoreans, "harmony is an inner connection of things without which the cosmos cannot exist." At last, according to Pythagoras, harmony had numerical expression, that is, it is connected with the concept of number. Aristotle

(384 BC – 322 BC) noticed in his *Metaphysics* just this peculiarity of the Pythagorean doctrine:

"The so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things ... since, then, all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole cosmos to be a harmony and a number."

The Pythagoreans recognized that the shape of the Universe should be harmonious and all its "elements" connected with harmonious figures. Pythagoras taught that the Earth arose from cube, Fire from pyramid (tetrahedron), Air from octahedron, Water from icosahedron, the sphere of the Cosmos (the ether) from dodecahedron.

The famous Pythagorean doctrine of the "harmony of spheres" is of course connected with the harmony concept. Pythagoras and his followers held that the movement of heavenly bodies around the central world fire creates a wonderful music, which is perceived not by ear, but by intellect. The doctrine about the "harmony of the spheres," the unity of the microcosm and macrocosm, and the doctrine about proportions - unified together provide the basis of the Pythagorean doctrine.

The main conclusion, following from Pythagorean doctrine, is that harmony is objective; it exists independently from our consciousness and is expressed in the harmonious structure of the Universe from the macrocosm down to the microcosm. However, if harmony is in fact objective, it should become a central subject of mathematical research.

The Pythagorean doctrine of numerical harmony in the Universe influenced the development of all subsequent doctrines about nature and the essence of aesthetics. This brilliant doctrine was reflected and developed in the works of great thinkers, in particular, in Plato's cosmology. In his works, **Plato** (428/427 BC – 348/347 BC) developed Pythagorean doctrine and especially emphasized the cosmic significance of harmony. He was firmly convinced that harmony can be expressed by numerical proportions. This Pythagorean influence was traced especially in his *Timaeus,* where Plato, after Pythagoras, developed a doctrine about proportions and analyzed the role of the regular polyhedra (Platonic Solids), which, in his opinion, underlie the Universe itself.

The *golden section,* which was called in that period the "division in extreme and mean ratio," played a special role in ancient science, including Plato's cosmology. Above we presented Kepler's and Losev's statements about the role of the golden section in geometry and Greek culture. Kepler's assertion raises the significance of the golden section up to the level of the *Pythagorean Theorem* - one of the most famous theorems of geometry. As a result of the unilateral approach to mathematical education each school-child knows the Pythagorean Theorem, but has a rather vague concept of the golden section - the second "treasure of geometry." The majority of school textbooks on geometry go back in their origin to Euclid's *Elements*. But then we may ask the question: why in the majority of them is there no real significant mention of the golden section, described for the first time in Euclid's *Elements*? The impression created is that "the materialistic pedagogy" have thrown out the golden section from mathematical education onto the dump heap of "doubtful scientific concepts" together with astrology and other so-called esoteric sciences (where the golden section is widely emphasized). We consider this sad fact to be one of the "strategic mistakes" of modern mathematical education.

Many mathematicians interpret the above Kepler's comparison of the *golden section* with Pythagorean Theorem as a great overstatement regarding the golden section. However, we should not forget that Kepler was not only a brilliant astronomer, but also a great physicist and great mathematician (in contrast to the mathematicians who criticize Kepler). In his first book *Mysterium Cosmographicum* (*The Cosmographic Mystery*), Kepler created an original model of the Solar System based on the *Platonic Solids*. He was one of the first scientists, who started to study the "Harmony of the Universe" in his book *Harmonices Mundi* (*Harmony of the World*). In *Harmony*, he attempted to explain the proportions of the natural world – particularly the astronomical and astrological aspects –in terms of music. The *Musica Universalis* or *Music of the Spheres*, studied by Ptolemy and many others before Kepler, was his main idea. From there, he extended his harmonic analysis to music, meteorology and astrology; harmony resulted from the tones made by the souls of heavenly bodies – and in the case of astrology, the interaction between those tones and human souls. In the final portion of the work (Book V), Kepler dealt with planetary motions, especially relationships between orbital velocity and orbital distance from the Sun. Similar relationships had been used by other astronomers, but Kepler – with Tycho's data and his own astronomical theories – treated them much more precisely and attached new physical significance to them.

Thus, the neglect of the "golden section" and its associated "idea of harmony" is one more "strategic mistake" in not only mathematics and mathematical education, but also theoretical physics. This mistake resulted in a number of other "strategic mistakes" in the development of mathematics and mathematical education.

E.2.4. *The One-sided Interpretation of Euclid's Elements*

Euclid's *Elements* is the primary work of Greek mathematics. It is devoted to the axiomatic construction of geometry, and led to the "axiomatic approach" widely used in mathematics. This view of the *Elements* is widespread in contemporary mathematics. In his *Elements* Euclid collected and logically analyzed all achievements of the previous period in the field of geometry. At the same time, he presented the basis of number theory. For the first time, Euclid proved the infinity of *prime numbers* and constructed a full theory of divisibility. At last, in Books II, VI and X, we find the description of a so-called geometrical algebra that allowed Euclid to not only solve quadratic equations, but also perform complex transformations on quadratic irrationals.

Euclid's *Elements* fundamentally influenced mathematical education. Without exaggeration it is reasonable to suggest, that the contents of mathematical education in modern schools is on the whole based upon the mathematical knowledge presented in Euclid's *Elements*.

However, there is another point of view on Euclid's *Elements* suggested by Proclus Diadochus (412-485), the best commentator on Euclid's *Elements*. The final book of Euclid's *Elements*, Book XIII, is devoted to a description of the theory of the five regular polyhedra that played a predominate role in Plato's cosmology. They are well known in modern science under the name *Platonic Solids*. Proclus did pay special attention to this fact. As is generally the case, the most important data are presented in the final part of a scientific book. Based on this fact, Proclus asserts that **Euclid created his *Elements* primarily not to present an axiomatic approach to geometry, but in order to give a systematic theory of the construction of the 5 Platonic Solids, in passing throwing light on some of the most important achievements of Greek mathematics.** Thus, "Proclus' hypothesis" allows one to suppose that it was well-known in ancient science that the "Pythagorean Doctrine about the Numerical Harmony of the Cosmos" and "Plato's Cosmology," based on the regular polyhedra, were embodied in Euclid's *Elements*, the greatest Greek work of mathematics. From this point of view, **we can interpret Euclid's *Elements* as the first attempt to create a "Mathematical Theory of Harmony" which was the primary idea in Greek science.**

This hypothesis is confirmed by the geometric theorems in Euclid's *Elements. The problem of division in extreme and mean ratio* described in Theorem II.11 is one of them. This division named later the golden section was used by Euclid for the geometric construction of the isosceles triangle with the angles 72°, 72° & 36° (the "golden" isosceles triangle) and then of the regular pentagon and dodecahedron. We ascertain with great regret that "Proclus' hypothesis" was not really recognized by modern mathematicians who continue to consider the axiomatic statement of geometry as the main achievement of Euclid's *Elements*. However, as Euclid's *Elements* are the beginnings of school mathematical education, we should ask the question: why do the golden section and Platonic Solids occupy such a modest place in modern mathematical education?

The narrow one-sided interpretation of Euclid's *Elements* is one more "strategic mistake" in the development of mathematics and mathematical education. This "strategic mistake" resulted in a distorted picture of the history of mathematics.

E.2.5. *The One-sided Approach to the Origin of Mathematics*

The traditional approach to the origin of mathematics consists of the following [1]. Historically, two practical problems stimulated the development of mathematics on its earlier stages of development. We are referring to the *count problem* and *measurement problem*. The *count problem* resulted in the creation of the first methods of number representation and the first rules for the fulfillment of arithmetical operations (including the Babylonian sexagecimal number system, Egyptian decimal arithmetic). The formation of the concept of *natural number* was the main result of this long period in the mathematics history. On the other hand, the "measurement problem" underlies the creation of geometry ("Measurement of the Earth"). The discovery of *incommensurable line segments* is considered to be the major mathematical discovery in this field. This discovery resulted in the introduction of *irrational numbers***,** the next fundamental notion of mathematics following natural numbers.

The concepts of *natural number* and *irrational number* are the major fundamental mathematical concepts, without which it is impossible to imagine the existence of mathematics. These concepts underlie "Classical Mathematics."

Neglect of the *harmony problem* and *golden section* by mathematicians has an unfortunate influence on the development of mathematics and mathematical education. As a result, we have a one-sided view of the origin of mathematics which is one more "strategic mistake" in the development of mathematics and mathematical education.

E.2.6. *The Greatest Mathematical Mystification of the 19th Century*

The "strategic mistake" influenced considerably on the development of mathematics and mathematical education, was made in the 19th century. We are talking about *Cantor's Theory of Infinite Sets*. Recall that George Cantor (1845 –1918) was a German mathematician, born in Russia. He is best known as the creator of *set theory*, which has become a fundamental theory in mathematics. Unfortunately, Cantor's set theory was perceived by the 19th century mathematicians without proper critical analysis.

The end of the 19th century was a culmination point in recognizing of *Cantor's set theory*. The official proclamation of the *set theory* as the mathematics fouindation was held in 1897: this statement was contained in *Hadamard's speech* on the First International Congress of Mathematicians in Zurich (1897). In his lecture the Great mathematician Jacques *Hadamard* (1865-1963) did emphasize that the main attractive reason of *Cantor's set theory* consists of the fact that for the first time in mathematics history the classification of the sets was made on the base of a new concept of "cardinality" and the amazing mathematical outcomes inspired mathematicians for new and surprising discoveries.

However, very soon the "mathematical paradise" based on *Cantor's set theory* was destroyed. Finding paradoxes in *Cantor's set theory* resulted in the crisis in mathematics foundations, what cooled enthusiasm of mathematicians to *Cantor's set theory*. The Russian mathematician Alexander Zenkin finished a critical analysis of *Cantor's set theory* and a concept of *actual infinity,* which is the main philosophical idea of *Cantor's set theory*.

After the thorough analysis of *Cantor's continuum theorem*, in which Alexander Zenkin gave the "logic" substantiation for legitimacy of the use of the *actual infinity* in mathematics, he did the following unusual conclusion [168]:

1. Cantor's proof of this theorem is not mathematical proof in Hilbert's sense and in the sense of classical mathematics.
2. Cantor's conclusion about non-denumerability of continuum is a "jump" through a potentially infinite stage, that is, Cantor's reasoning contains the fatal logic error of "unproved basis" (a jump to the "wishful conclusion").
3. Cantor's theorem, actually, proves, strictly mathematically, the potential, that is, not finished character of the infinity of the set of all "real numbers," that is, Cantor proves strictly mathematically the fundamental principle of classical logic and mathematics: *"Infinitum Actu Non Datur"* (Aristotle).

However, despite so sharp critical attitude to Cantor's theory of infinite sets, the theoretic-set ideas had appeared rather "hardy" and were applied in modern mathematical education. In a number of countries, in particular, in Russia, the revision of the school mathematical education on the base of *theoretic-set approach* was made. As is well known, the theoretic-set approach assumes certain mathematical culture. A majority of pupils and many mathematics teachers do not possess and cannot possess this culture. What as a result had happened? In the opinion of the known Russian mathematician, academician Lev Pontrjagin (1908-1988) [193], this brought "to artificial complication of the learning material and unreasonable overload of pupils, to the introduction of formalism in mathematical training and isolation of mathematical education from life, from practice. Many major concepts of school mathematics (such as concepts of function, equation, vector, etc.) became difficult for mastering by pupils... The theoretic-set approach is a language of scientific researches convenient only for mathematicians-professionals. The valid tendency of the mathematics development is in its movement to specific problems, to practice. Therefore, modern school mathematics textbooks are a step back in interpretation of this science, they are unfounded essentially because they emasculate an essence of mathematical method."

Thus, Cantor's theory of infinite sets based on the concept of "actual infinity" contains "fatal logic error" and cannot be considered as mathematics base. Its acceptance as mathematics foundation, without proper critical analysis, is one more "strategic mistake" in the mathematics development; Cantor's theory is one of the major reasons of the contemporary crisis in mathematics foundations. A use of theoretic-set approach in school mathematical education has led to artificial complication of the learning material, unreasonable overload of pupils and to the isolation of mathematical education from life, from practice.

E.2.7. *The Underestimation of Binet Formulas*

In the 19th century a theory of the "golden section" and Fibonacci numbers was supplemented by one important result. This was with the so-called *Binet formulas* for Fibonacci and Lucas numbers given by (E.11) and (E.12).

The analysis of the Binet formulas (E.11) and (E.12) gives one the opportunity to sense the beauty of mathematics and once again be convinced of the power of the human intellect! Actually, we know that the Fibonacci and Lucas numbers are always integers. But any power of the golden mean is an irrational number. As it follows from the Binet formulas, the integer numbers F_n and L_n can be represented as the difference or sum of irrational numbers, namely the powers of the golden mean! We know it is not easy to explain to pupils the concept of irrationals. For learning mathematics, the *Binet formulas* (E.10) and (E.11), which connect Fibonacci and Lucas numbers with the *golden mean* τ, are very important because they demonstrate visually a connection between integers and irrational numbers.

Unfortunately, in classical mathematics and mathematical education the *Binet formulas* did not get the proper kind of recognition as did, for example, "Euler formulas" and other famous mathematical formulas. Apparently, this attitude towards the Binet formulas is connected with the golden mean, which always provoked an "allergic reaction" in many mathematicians. Therefore, the Binet formulas are not generally found in mathematics textbooks.

However, the main "strategic mistake" in the underestimation of the *Binet formulas* is the fact that mathematicians could not see in the *Binet formulas* a prototype for a new class of hyperbolic functions – the hyperbolic Fibonacci and Lucas functions. Such functions were discovered roughly 100 years later by Ukrainian researchers Bodnar [52], Stakhov, Tkachenko, and Rozin [98, 106, 116, 118, 119]. If the hyperbolic functions on Fibonacci and Lucas had been discovered in the 19th century, hyperbolic geometry and its applications to theoretical physics would have received a new impulse in their development.

E.2.8. *The Underestimation of Felix Klein's Idea Concerning the Regular Icosahedron*

The name Felix Klein is well known in mathematics. In the 19th century Felix Klein tried to unite all branches of mathematics on the base of the regular icosahedron dual to the dodecahedron [58].

Klein interprets the regular icosahedron based on the "golden section" as a geometric object, connected with 5 mathematical theories: *Geometry, Galois Theory, Group Theory, Invariant Theory,* and *Differential Equations.* Klein's main idea is extremely simple: "Each unique geometric object is connected one way or another with the properties of the regular icosahedron." **Unfortunately, this remarkable idea was not developed in contemporary mathematics, which is one more "strategic mistake" in the development of mathematics.**

E.2.9. *The Underestimation of Bergman's Number System*

One "strange" tradition exists in mathematics. It is usually the case that mathematicians underestimate the mathematical achievements of their contemporaries. The epochal mathematical discoveries, as a rule, in the beginning go unrecognized by mathematicians. Sometimes they are subjected to sharp criticism and even to gibes. Only after approximately 50 years, as a rule, after the death of the authors of the outstanding mathematical discoveries, the new mathematical theories are recognized and take their place of worth in mathematics. The dramatic destinies of Lobachevski, Abel, and Galois are very well-known.

In 1957, the American mathematician George Bergman published the article *A number system with an irrational base* [86]. In this article Bergman developed a very unusual extension of the notion of the positional number system. He suggested that one use the golden mean $\tau = 1+\sqrt{5}/2$ as the basis of a special positional number system. If we use the sequences τ^i $(i = 0, \pm1, \pm2, \pm3, ...)$ as "digit weights" of the "binary" number system, we get the "binary" number system with irrational base τ:

$$A = \sum_{i=-\infty}^{+\infty} a_i \tau^i , \qquad \text{(E.13)}$$

where A is a real number, a_i are binary numerals 0 or 1, $i = 0, \pm 1, \pm 2, \pm 3, ...$, τ^i is the weight of the i-th digit, τ is the base of the number system (E.13).

Unfortunately, Bergman's article [86] was not noticed by mathematicians of that period. Only the journalists were surprised by the fact that George Bergman made his mathematical discovery at the age of 12! In this connection, TIME Magazine published an article about mathematical talent in America. In 50 years, according to "mathematical tradition" the time had come to evaluate the role of Bergman's system for the development of contemporary mathematics.

The "strategic" importance of Bergman's system is the fact that **it overturns our ideas about positional number systems, moreover, our ideas about correlations between rational and irrational numbers.**

As is well known, historically natural numbers were first introduced, after them rational numbers as ratios of natural numbers, and later – after the discovery of the "incommensurable line segments" - irrational numbers, which cannot be expressed as ratios of natural numbers. By using the traditional positional number systems (binary, ternary, decimal and so on), we can represent any natural, real or irrational number by using number systems with a base of (2, 3, 10 and so on). The base in Bergman's system [86] is the golden mean. By using Bergman's system (E.13), we can represent all natural, real and irrational numbers. As Bergman's system (E.13) is fundamentally a new positional number system, its study is very important for school mathematical education because it expands our ideas about the positional principle of number representation.

The "strategic mistake" of 20th century mathematicians is that they took no notice of Bergman's mathematical discovery, which can be considered as the major mathematical discovery in the field of number systems (following the Babylonian discovery of the positional principle of number representation and also decimal and binary systems).

E.3. Three "Key" Problems of Mathematics and a New Approach to the Mathematics Origins

The main purpose of the "Harmony Mathematics" is to overcome the "strategic mistakes," which arose in mathematics in the process of its development.

We can see that three "key" problems – *the "count problem," the "measurement problem,"* and *the "harmony problem"* - underlie the origin of mathematics (see Fig. I.1). The first two "key" problems resulted in the creation of two fundamental notions of mathematics – *natural number* and *irrational number* that underlie *classical mathematics*. The *harmony problem* connected with the *division in extreme and mean ratio* (Theorem II.11 of Euclid's *Elements*) resulted in the origin of *Harmony Mathematics* – a new interdisciplinary direction of contemporary science, which is related to contemporary mathematics, theoretical physics, and computer science. This approach leads to a conclusion, which is startling for many mathematicians. It proves to be, in parallel with classical mathematics, one more mathematical direction – the *Harmony Mathematics* – already developing in ancient science. Similarly to the *Classical Mathematics*, the *Harmony Mathematics* has its origin in Euclid's *Elements*. However, the *Classical Mathematics* focuses its attention on the "axiomatic approach," while the *Harmony Mathematics* is based on the golden section (Theorem II.11) and Platonic Solids described in Book XIII of Euclid's *Elements*. Thus, Euclid's *Elements* is the source of two independent directions in the development of mathematics – the *Classical Mathematics* and the *Harmony Mathematics*.

For many centuries, the main focus of mathematicians was directed towards the creation of the "Classical Mathematics," which became the *Czarina of Natural Sciences*. However, the forces of many prominent mathematicians - since Pythagoras, Plato and Euclid, Pacioli, Kepler up to Lucas, Binet, Vorobyov, Hoggatt and so forth - were directed towards the development of the basic concepts and applications of Harmony Mathematics. Unfortunately, these important mathematical directions developed separately from one other. The time has come to unite "Classical Mathematics" and "Harmony Mathematics." This unusual union can lead to new scientific discoveries in mathematics and the natural sciences. Some of the latest discoveries in the natural sciences, in particular, Shechtman's quasi-crystals based on Plato's icosahedron and fullerenes (Nobel Prize of 1996) based on the Archimedean truncated icosahedron do demand this union. All mathematical theories should be united for one unique purpose: to discover and explain Nature's Laws.

A new approach to the mathematics origins (see Fig. I.1) is very important for school mathematical education. This approach introduces in a very natural manner the idea of harmony and the golden section into school mathematical education. This provides pupils access to ancient science and to its main achievement – the harmony idea – and to tell them about the most important architectural and sculptural works of ancient art based upon the golden section (including pyramid of Khufu (Cheops), Nefertiti, Parthenon, Doryphorus, Venus).

E.4. The Generalized Fibonacci Numbers and the Generalized Golden Proportions

E.4.1. *The Generalized Fibonacci P-numbers, the Generalized P-proportions, the Generalized Binet Formulas and the Generalized Lucas P-numbers*

Pascal's triangle is recognized as one of the most beautiful objects of mathematics. And we can expect further beautiful mathematical objects stemming from Pascal's triangle. In the recent decades, many mathematicians found a connection between Pascal's triangle and Fibonacci numbers independent of each other. The generalized *Fibonacci p-numbers,* which can be obtained from Pascal's triangle as its "diagonal sums" [20] are the most important of them. For a given integer p=0, 1, 2, 3, ... , they are given by the recursive relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1);\ F_p(0)=0,\ F_p(1)= F_p(2)=...= F_p(p)=1. \quad \text{(E.14)}$$

It is easy to see for the case p=1 that the above recursive formula is reduced to the recursive formula for classical Fibonacci numbers:

$$F_1(n) = F_1(n-1) + F_1(n-2);\ F_1(0)=0,\ F_1(1)=1. \quad \text{(E.15)}$$

It follows from (E.14) that the Fibonacci p-numbers express more complicated "harmonies" than the classical Fibonacci numbers given by (E.15). Note that the recursive formula (E.14) generates an infinite number of different recursive numerical sequences because every p generates its own recursive sequences, in particular, the binary numbers 1, 2, 4, 8, 16, ... for the case p=0 and the classical Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... for the case p=1.

It is important to note that the recursive relation (E.14) expresses some deep mathematical properties of Pascal's triangle (the "diagonal sums" of Pascal's triangle). The Fibonacci p-numbers are represented by the binomial coefficients as follows [20]:

$$F_p(n+1)=C_n^0+C_{n-p}^1+C_{n-2p}^3+C_{n-4p}^4+...+C_{n-kp}^k+... , \quad (E.16)$$

where the binomial coefficient $C_{n-kp}^k=0$ for the case $k>n-kp$.

Note that for the case p=0 the formula (E.16) is reduced to the well-known formula of combinatorial analysis:

$$2^n=C_n^0+C_n^1+...+C_n^n. \quad (E.17)$$

It is easy to prove [4] that in the limit $(n\to\infty)$ the ratio of the adjacent Fibonacci p-numbers $F_p(n)/F_p(n\text{-}1)$ aims for some numerical constant, that is,

$$\lim_{n\to\infty}\frac{F_p(n)}{F_p(n-1)}=\tau_p, \quad (E.18)$$

where τ_p is the positive root of the following algebraic equation:

$$x^{p+1}=x^p+1, \quad (E.19)$$

which for p=1 is reduced to the "golden" algebraic equation (E.10) given by the classical golden mean (E.9).

Note that the result (E.16) is a generalization of Kepler's formula (E.9) for classical Fibonacci numbers (p=1).

The positive roots of Eq. (E.17) were named the *golden p-proportions* [20]. It is easy to prove [20] that the powers of the golden p-proportions are connected between themselves by the following identity:

$$\tau_p^n=\tau_p^{n-1}+\tau_p^{n-p-1}=\tau_p\times\tau_p^{n-1}, \quad (E.20)$$

that is, each power of the golden p-proportion is connected with the preceding powers by the "additive" relation $\tau_p^n=\tau_p^{n-1}+\tau_p^{n-p-1}$ and by the "multiplicative" relation $\tau_p^n=\tau_p\times\tau_p^{n-1}$ (similar to the classical golden mean).

It is proved in [111] that the Fibonacci p-numbers can be represented in the following analytical form:

$$F_p(n)=k_1(x_1)^n+k_2(x_2)^n+...+k_{p+1}(x_{p+1})^n, \quad (E.21)$$

where n=0,±1,±2,±3,..., $x_1, x_2, ..., x_{p+1}$ are the roots of Eq. (E.19), and $k_1, k_2, ..., k_{p+1}$ are constant coefficients that depend on the initial elements of the Fibonacci p-series, and are solutions to the following system of algebraic equations:

$$F_p(0)=k_1+k_2+...+k_{p+1}=0$$
$$F_p(1)=k_1x_1+k_2x_2+...+k_{p+1}=1$$
$$F_p(2)=k_1(x_1)^2+k_2(x_2)^2+...+k_{p+1}(x_{p+1})^2=1$$
...
$$F_p(p)=k_1(x_1)^p+k_2(x_2)^p+...+k_{p+1}(x_{p+1})^p=1.$$

Note that for the case p=1, the formula (E.21) is reduced to the Binet formula (E.11) for the classical Fibonacci numbers.

In [111] the generalizid Lucas p-numbers are introduced. They are repreented in the following analytical form:

$$L_p(n)=(x_1)^n+(x_2)^n+\ldots+(x_{p+1})^n, \tag{E.22}$$

where $n=0,\pm1,\pm2,\pm3,\ldots$, $x_1, x_2, \ldots, x_{p+1}$ are the roots of Eq. (E.19).

Note that for the case p=1, the formula (E.22) is reduced to the Binet formula (E.12) for the classical Lucas numbers.

Directly from (E.22) we can deduce the following recurrence relation

$$L_p(n)= L_p(n-1)+L_p(n-p-1), \tag{E.23}$$

which at the seeds

$$L_p(0)=p+1 \text{ and } L_p(1)=L_p(2)=\ldots=L_p(p)=1 \tag{E.24}$$

produces a new class of numerical sequences – *Lucas p-numbers*. They are a generalization of the classical Lucas numbers for the case p=1.

Thus, a study of Pascal's triangle produces the following beautiful mathematical results:

1. The generalized Fibonacci p-numbers are expressed through binomial coefficients by the graceful formula (E.16).
2. A new class of mathematical constants τ_p (p=0, 1, 2, 3, ...), express some important mathematical properties of Pascal's triangle and possess unique mathematical properties (E.20).
3. A new class of algebraic equations (E.19), which are a wide generalization of the classical "golden" equation (E.10).
4. A generalization of Binet formulas for Fibonacci and Lucas p-numbers.

Discussing applications of Fibonacci p-numbers and golden p-proportions to contemporary theoretical natural sciences, we find two important applications:

1. **Asymmetric division of biological cells**. The authors of [171] proved that the generalized Fibonacci p-numbers can model the growth of biological cells. They conclude that "binary cell division is regularly asymmetric in most species. Growth by asymmetric binary division may be represented by the generalized Fibonacci equation .... Our models, for the first time at the single cell level, provide a rational basis for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology, founded on the occurrence of the regular asymmetry of binary division."
2. **Structural harmony of systems**. Studying the process of system self-organization in different aspects of nature, Belarusian philosopher Eduard Soroko formulated in [25] the "Law of Structural Harmony of Sys-

tems" based on the golden p-proportions: "The generalized golden proportions are invariants that allow natural systems in the process of their self-organization to find a harmonious structure, a stationary regime for their existence, and structural and functional stability."

E.4.2. *The generalized Fibonacci λ-numbers, "Metallic Means" by Vera Spinadel, Gazale Formulas and General Theory of Hyperbolic Functions*

Another generalization of Fibonacci numbers was introduced recently by Vera W. Spinadel [42], Midchat Gazale [45], Jay Kappraff [47] and other scientists. We are talking about the generalized Fibonacci λ-numbers that for a given positive real number $\lambda>0$ are given by the recurrence relation:

$$F_\lambda(n)=\lambda F_\lambda(n-1)+F_\lambda(n-2); F_\lambda(0)=0, F_\lambda(1)=1. \tag{E.25}$$

Notice that here we use λ as in the article [191] instead m as in Section 4.12.

First of all, we note that the recurrence relation (E.25) is reduced to the recurrence relation (E.4) for the case $\lambda=1$. For other values of λ, the recurrence relation (E.25) generates an infinite number of new recurrence numerical sequences.

The following characteristic algebraic equation follows from (E.25):

$$x^2-\lambda x-1=0, \tag{E.26}$$

which for the case $\lambda=1$ is reduced to (E.10). A positive root of Eq. (E.26) generates an infinite number of new "harmonic" proportions – "Metallic Means" by Vera Spinadel [42], which are expressed by the following general formula:

$$\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}. \tag{E.27}$$

Note that for the case $\lambda=1$ the formula (E.27) gives the classical golden mean $\Phi_1=\frac{1+\sqrt{5}}{2}$. The metallic means possess the following unique mathematical properties:

$$\Phi_\lambda=\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\ldots}}}} \qquad \Phi_\lambda=\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\ldots}}}, \tag{E.28}$$

which are generalizations of similar properties for the classical golden mean $\Phi_1=\tau(\lambda=1)$:

$$\tau=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}; \qquad \tau=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}. \tag{E.29}$$

Note that the expressions (E.27), (E.28) and (E.29), without doubt, satisfy Dirac's "Principle of Mathematical Beauty" and emphasize a fundamental characteristic of both the classical golden mean and the metallic means.

Recently, by studying the recurrence relation (E.25), the Egyptian mathematician Midchat Gazale [45] deduced the following remarkable formula given by Fibonacci λ-numbers:

$$F_\lambda(n)=\frac{\Phi_\lambda^n-(-1)^n\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}, \tag{E.30}$$

where λ>0 is a given positive real number, Φ_λ is the metallic mean given by (E.27), $n=0,\pm1,\pm2,\pm3,\dots$. The author of the article [118] named the formula (E.30) in [118] *formula Gazale for the Fibonacci λ-numbers* after Midchat Gazale. The similar Gazale formula for the Lucas λ-numbers is deduced in [118]:

$$L_\lambda(n)=\Phi_\lambda^n+(-1)^n\Phi_\lambda^{-n}. \tag{E.31}$$

First of all, we note that "Gazale formulas" (E.30) and (E.31) are a wide generalization of Binet formulas (E.11) and (E.12) for the classical Fibonacci and Lucas numbers $(\lambda=1)$.

The most important result is that the Gazale formulas (E.30) and (E.31) result in a general theory of hyperbolic functions [118].

Hyperbolic Fibonacci λ-sine

$$sF_\lambda(x)=\frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}=\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x-\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{E.32}$$

Hyperbolic Fibonacci λ-cosine

$$cF_\lambda(x)=\frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}=\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x+\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{E.33}$$

Hyperbolic Lucas λ-sine

$$sL_\lambda(x)=\Phi_\lambda^x-\Phi_\lambda^{-x}=\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x-\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x} \tag{E.34}$$

Hyperbolic Lucas λ-cosine

$$cL_\lambda(x)=\Phi_\lambda^x+\Phi_\lambda^{-x}=\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x+\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}. \tag{E.35}$$

Note that the hyperbolic Fibonacci and Lucas λ-functions coincide with the Fibonacci and Lucas λ-numbers for the discrete values of the variable $x=n=0,\pm1,\pm2,\pm3,\dots$, that is,

$$F_\lambda(n)=\begin{cases} sF_\lambda(n), n=2k \\ cF_\lambda(n), n=2k+1 \end{cases}$$

$$L_\lambda(n)=\begin{cases} cL_\lambda(n), n=2k \\ sL_\lambda(n), n=2k+1 \end{cases}. \tag{E.36}$$

The formulas (E.32)-(E.35) provide an infinite number of hyperbolic models of nature because every real number λ originates its own class of hyperbolic functions of the kind (E.32)-(E.35). As is proved in [118], these functions have, on the one hand, the "hyperbolic" properties similar to the properties of classical hyperbolic functions, and on the other hand, "recursive" properties similar to the properties of the Fibonacci and Lucas λ-numbers (E.30) and (E.31). In particular, the classical hyperbolic functions are a partial case of the hyperbolic Lucas λ-functions (E.34) and (E.35). For the case $\lambda_e = e-1/e \approx 2.35040238...$, the classical hyperbolic functions are connected with hyperbolic Lucas λ-functions by the following simple relations:

$$sh(x)=\frac{sL_\lambda(x)}{2} \text{ and } ch(x)=\frac{cL_\lambda(x)}{2}. \tag{E.37}$$

Note that for the case λ=1, the hyperbolic Fibonacci and Lucas λ-functions (E.32)-(E.35) coincide with the *symmetric hyperbolic Fibonacci and Lucas functions* introduced by Alexey Stakhov and Boris Rozin in the article [106]:

Symmetrical hyperbolic Fibonacci sine and cosine

$$sFs(x)=\frac{\tau^x-\tau^{-x}}{\sqrt{5}};\ cFs(x)=\frac{\tau^x+\tau^{-x}}{\sqrt{5}} \tag{E.38}$$

Symmetrical hyperbolic Fibonacci sine and cosine

$$sLs(x)=\tau^x-\tau^{-x};\ cLs(x)=\tau^x+\tau^{-x} \tag{E.39}$$

where $\tau=\frac{1+\sqrt{5}}{2}$.

In the book [37], the Ukrainian researcher Oleg Bodnar used Stakhov and Rozin's symmetric hyperbolic Fibonacci and Lucas functions (E.38) and (E.39) for the creation of a graceful geometric theory of phyllotaxis. This means that **the symmetrical hyperbolic Fibonacci and Lucas functions (E.38) and (E.39) and their generalization – the hyperbolic Fibonacci and Lucas λ-functions (E.32)-(E.35) – can be ascribed to the fundamental mathematical results of modern science because they "reflect phenomena of Nature," in particular, phyllotaxis phenomena** [37]. These functions set a general theory of hyperbolic functions that is of fundamental importance for contemporary mathematics and theoretical physics.

We propose that hyperbolic Fibonacci and Lucas λ-functions, corresponding to the different values of λ, can model different physical phenomena. For example, when λ=2 the recursive relation (E.25) is reduced to

$$F_2(n) = 2F_2(n-1) + F_2(n-2);\ F_2(0)=0,\ F_2(1)=1, \tag{E.40}$$

which gives the so-called *Pell numbers*: 0, 1, 2, 5, 12, 29, ... In this connection, the formulas for the golden λ-proportion and hyperbolic Fibonacci and Lucas λ-numbers take for the case λ=2 the following forms, respectively:

$$\Phi_2 = 1+\sqrt{2} \tag{E.41}$$

$$sF_2(x) = \frac{\Phi_2^x - \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x}\right] \tag{E.42}$$

$$cF_2(x) = \frac{\Phi_2^x + \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}\right] \tag{E.43}$$

$$sL_2(x) = \Phi_2^x - \Phi_2^{-x} = \left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x} \tag{E.44}$$

$$cL_2(x) = \Phi_2^x + \Phi_2^{-x} = \left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}. \tag{E.45}$$

It is appropriate to give the following comparative table (Table E3), which

Table E3. Connection of the Golden Mean with Metallic Means

The Golden Mean (λ = 1)	**The Metallic Means (λ > 0)**
$\tau = \frac{1+\sqrt{5}}{2}$	$\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$
$\tau = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ldots}}}}$	$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{\ldots}}}}$
$\tau = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$	$\Phi_\lambda = \lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\cfrac{1}{\lambda+\ldots}}}$
$\tau^n = \tau^{n-1}+\tau^{n-2} = \tau\times\tau^{n-1}$	$\Phi_\lambda^n = \lambda\Phi_\lambda^{n-1}+\Phi_\lambda^{n-2} = \Phi_\lambda\times\Phi_\lambda^{n-1}$
$F_n = \frac{\tau^n-(-1)^n\tau^{-n}}{\sqrt{5}}$	$F_\lambda(n) = \frac{\Phi_\lambda^n-(-1)^n\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}$
$L_n = \tau^n+(-1)^n\tau^{-n}$	$L_\lambda(n) = \Phi_\lambda^n+(-1)^n\Phi_\lambda^{-n}$
$sFs(x) = \frac{\tau^x-\tau^{-x}}{\sqrt{5}};\ cFs(x) = \frac{\tau^x+\tau^{-x}}{\sqrt{5}}$	$sF_\lambda(x) = \frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}};\ cF_\lambda(x) = \frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$
$sLs(x) = \tau^x-\tau^{-x};\ cLs(x) = \tau^x+\tau^{-x}$	$sL_\lambda(x) = \Phi_\lambda^x-\Phi_\lambda^{-x};\ cL_\lambda(x) = \Phi_\lambda^x+\Phi_\lambda^{-x}$

gives a relationship between the golden mean and metallic means as new mathematical constants of Nature.

Table E4. The main formulas for the "golden" Fibonacci goniometry

Formulas for the classical hyperbolic functions	Formulas for the hyperbolic Fibonacci λ – functions »
$sh(x)=\frac{e^x-e^{-x}}{2};\ ch(x)=\frac{e^x+e^{-x}}{2}$	$sF_\lambda(x)=\frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}};\ cF_\lambda(x)=\frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}$
$sh(x+2)=2sh(1)ch(x+1)+sh(x)$ $ch(x+2)=2sh(1)sh(x+1)+ch(x)$	$sF_\lambda(x+2)=\lambda cF_\lambda(x+1)+sF_\lambda(x)$ $cF_\lambda(x+2)=\lambda sF_\lambda(x+1)+cF_\lambda(x)$
$sh^2(x)-ch(x+1)ch(x-1)=-ch^2(1)$ $ch^2(x)-sh(x+1)sh(x-1)=ch^2(1)$	$[sF_\lambda(x)]^2-cF_\lambda(x+1)cF_\lambda(x-1)=-1$ $[cF_\lambda(x)]^2-sF_\lambda(x+1)sF_\lambda(x-1)=1$
$ch^2(x)-sh^2(x)=1$	$[cF_\lambda(x)]^2-[sF_\lambda(x)]^2=\frac{4}{4+\lambda^2}$
$sh(x+y)=sh(x)ch(x)+ch(x)sh(x)$ $sh(x-y)=sh(x)ch(x)-ch(x)sh(x)$	$\frac{2}{\sqrt{4+\lambda^2}}sF_\lambda(x+y)$ $=sF_\lambda(x)cF_\lambda(x)+cF_\lambda(x)sF_\lambda(x)$ $\frac{2}{\sqrt{4+\lambda^2}}sF_\lambda(x-y)$ $=sF_\lambda(x)cF_\lambda(x)-cF_\lambda(x)sF_\lambda(x)$
$ch(x+y)=ch(x)ch(x)+sh(x)sh(x)$ $ch(x-y)=ch(x)ch(x)-sh(x)sh(x)$	$\frac{2}{\sqrt{4+\lambda^2}}cF_\lambda(x+y)$ $=cF_\lambda(x)cF_\lambda(x)+sF_\lambda(x)sF_\lambda(x)$ $\frac{2}{\sqrt{4+\lambda^2}}cF_\lambda(x-y)$ $=cF_\lambda(x)cF_\lambda(x)-sF_\lambda(x)sF_\lambda(x)$
$ch(2x)=2sh(x)ch(x)$	$\frac{1}{\sqrt{4+\lambda^2}}cF_\lambda(2x)=sF_\lambda(x)cF_\lambda(x)$
$[ch(x)\pm sh(x)]^n=ch(nx)\pm sh(nx)$	$[cF_\lambda(x)\pm sF_\lambda(x)]^n$ $=\left(\frac{2}{\sqrt{4+\lambda^2}}\right)^{n-1}[cF_\lambda(nx)\pm sF_\lambda(nx)]$

A beauty of these formulas is charming. This gives a right to suppose that Dirac's "Principle of Mathematical Beauty" is fully applicable to the metallic means and hyperbolic Fibonacci and Lucas m-functions. And this, in its turn, gives hope that these mathematical results can become a base of theoretical natural sciences.

Another table (See Table E4) gives the basic formulas for the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$ in comparison with corresponding formulas for the classical hyperbolic functions $sh(x)$ and $ch(x)$.

Remark. For the hyperbolic Lucas λ-functions $sL_\lambda(x)$ and $cL_\lambda(x)$ the corresponding formulas can be gotten by multiplication of the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$ by constant factor $\sqrt{4+\lambda^2}$ according to the correlations (41).

The table for the hyperbolic Fibonacci λ-functions $sF_\lambda(x)$ and $cF_\lambda(x)$, with regard to the above remark for the hyperbolic Lucas λ-functions $sL_\lambda(x)$ and $cL_\lambda(x)$, makes up a base of the *"Golden" Fibonacci goniometry* [192].

This table is very convincing confirmation of the fact that we are talking about a new class of hyperbolic functions, which keep all well-known properties of the classical hyperbolic functions $sh(x)$ and $ch(x)$, but, in addition, they posses additional ("recursive") properties, which unite them with remarkable numerical sequences – *Fibonacci* and *Lucas* λ-*numbers* $F_\lambda(n)$ and $L_\lambda(n)$.

E.5. A New Geometric Definition of Number

E.5.1. *Euclidean and Newtonian Definition of Real Number*

The first definition of a number was made in Greek mathematics. We are talking about the "Euclidean definition of natural number":

$$N = \underbrace{1+1+\ldots+1}_{N}. \tag{E.46}$$

In spite of the utmost simplicity of the *Euclidean definition* (E.46), it played a decisive role in mathematics, in particular, in number theory. This definition underlies many important mathematical concepts, for example, the concept of *prime* and *composite* numbers, and also *divisibility* that is one of the major concepts of number theory. Over the centuries, mathematicians developed and defined more exactly the concept of a number. In the 17th century, that is, in the period of the creation of new science, in particular, new mathematics, a number of methods for the study of "continuous" processes were developed and the concept of a real number again moves into the foreground.

Most clearly, a new definition of this concept was given by Isaac Newton, one of the founders of mathematical analysis, in his *Arithmetica Universalis* (1707):

"We understand a number not as a set of units, but as the abstract ratio of one magnitude to another magnitude of the same kind taken for the unit."

This formulation gives us a general definition of numbers, rational and irrational. For example, the binary system

$$A=\sum_{i=-\infty}^{+\infty} a_i 2^i \tag{E.47}$$

is an example of *Newton's definition,* when we choose the number of 2 for the unit and represent a number as the sum of the powers of number 2.

E.5.2. *Number Systems with Irrational Radices as a New Definition of Real Number*

Let us consider the set of the powers of the golden p-proportions:

$$S=\{\tau_p^i\},\ p=0,1,2,3,...;\ i=0,\pm1,\pm2,\pm3,.... \tag{E.48}$$

Using (E.48), we can construct the following method of positional representation of real number A:

$$A=\sum_{i=-\infty}^{+\infty} a_i \tau_p^i\ , \tag{E.49}$$

where a_i is the binary numeral of the i-th digit; τ_p^i is the weight of the i-th digit; τ_p is the radix of the numeral system (E.47), $p=0,1,2,3,...;$ $i=0,\pm1,\pm2,\pm3,...$. The positional representation (E.49) is called *code of the golden p-proportion* [24].

Note that for the case p=0 the sum (E.49) is reduced to the classical binary representation (E.47). For the case p=1, the sum (E.49) is reduced to Bergman's system (E.13). For the case $p\to\infty$, the sum (E.49) strives for the expression similar to (E.46).

In the author's article [105], a new approach to geometric definition of real numbers based on (E.49) was developed. A new theory of real numbers based on the definition (E.49) contains a number of unexpected results concerning number theory. Let us study these results as applied to Bergman's system (E.13). We shall represent a natural number N in Bergman's system (E.13) as follows:

$$N=\sum_{i=-\infty}^{+\infty} a_i \tau^i\ . \tag{E.50}$$

The following theorems are proved in [105]:

1. Every natural number N can be represented in the form (E.50) as a finite sum of the golden powers τ^i $(i = 0, \pm1, \pm2, \pm3, ...)$. Note that this theorem is not a trivial property of natural numbers.

2. ***Z*-property of natural numbers**. If we substitute in (E.50) the Fibonacci number F_i for the power of the golden mean τ^i $(i = 0, \pm1, \pm2, \pm3, ...)$, then the sum that appears as a result of such a substitution is equal to 0 independent of the initial natural number N, that is,

$$\sum_{i=-\infty}^{+\infty} a_i F_i = 0. \tag{E.51}$$

3. ***D*-property of natural numbers**. If we substitute in (E.50) the Lucas number L_i for the power of the golden mean τ^i $(i = 0, \pm1, \pm2, \pm3, ...)$, then the sum that appears as a result of such a substitution is equal to the double sum (50) independent of the initial natural number N, that is,

$$\sum_{i=-\infty}^{+\infty} a_i L_i = 2N. \tag{E.52}$$

4. ***F*-code of natural number *N***. If we substitute in (E.50) the Fibonacci number F_{i+1} for the power of the golden mean τ^i $(i = 0, \pm1, \pm2, \pm3, ...)$, then the sum that appears as a result of such a substitution is a new positional representation of the same natural number N called the *F-code of natural number N,* that is,

$$N = \sum_{i=-\infty}^{+\infty} a_i F_{i+1} \; (i = 0, \pm1, \pm2, \pm3, ...). \tag{E.53}$$

5. ***L*-code of natural number *N***. If we substitute in (E.50) the Lucas number L_{i+1} for the power of the golden mean τ^i $(i = 0, \pm1, \pm2, \pm3, ...)$, then the sum that appear as a result of such substitution is a new positional representation of the same natural number N called *L-code of natural number N,* that is,

$$N = \sum_{i=-\infty}^{+\infty} a_i L_{i+1} \; (i = 0, \pm1, \pm2, \pm3, ...). \tag{E.54}$$

Note that similar properties are proved for the code of the golden p-proportion given by (E.49).

Thus, we have discovered new properties of natural numbers (***Z*-property, *D*-property, *F*- and *L*-codes**) that confirm the fruitfulness of such an approach to number theory [105]. These results are of great importance for computer science and could become a source for new computer projects.

As the study of the positional binary and decimal systems are an important part of mathematical education, the number systems with irrational radices given by (E.13) and (E.49) are of general interest for mathematical education.

E.6. Fibonacci and "Golden" Matrices

E.6.1. *Fibonacci Matrices*

For the first time, a theory of the *Fibonacci Q-matrix* was developed in the book [16] written by the eminent American mathematician Verner Hoggatt – founder of the *Fibonacci Association* and *The Fibonacci Quarterly*.

The article [158] devoted to the memory of Verner E. Hoggatt contained a history and extensive bibliography of the Q-matrix and emphasized Hoggatt's contribution to its development. Although the name of the Q-matrix was introduced before Verner E. Hoggatt, he was the first mathematician who appreciated the mathematical beauty of the Q-matrix and introduced it into Fibonacci number theory. Thanks to Hoggatt's work, the idea of the Q-matrix "caught on like wildfire among Fibonacci enthusiasts. Numerous papers appeared in 'The Fibonacci Quarterly' authored by Hoggatt and/or his students and other collaborators where the Q-matrix method became the central tool in the analysis of Fibonacci properties" [158].

The Q-matrix

$$Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \tag{E.55}$$

is a generating matrix for Fibonacci numbers and the following wonderful properties:

$$Q^n=\begin{pmatrix}F_{n+1} & F_n\\ F_n & F_{n+1}\end{pmatrix} \tag{E.56}$$

$$\det Q^n = F_{n+1}F_{n+1}-F_n^2=(-1)^n. \tag{E.57}$$

Note that there is a direct relation between the Cassini formula (E.8) and the formula (E.57) given for the determinant of the matrix (E.56).

In article [103], the author introduced a generating matrix for Fibonacci p-numbers called Q_p*-matrix* (p=0, 1, 2, 3, ...):

$$Q_p=\begin{pmatrix}1 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & 0 & 0 & \cdots & 0 & 0\end{pmatrix}. \tag{E.58}$$

The following properties of Fibonacci p-numbers are proved in [103]:

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix} \tag{E.59}$$

$$\det Q_p^n = (-1)^{pn}, \tag{E.60}$$

where p=0, 1, 2, 3, ... ; and n=0, ±1, ±2, ±3, ...

The generating matrix G_λ for the Fibonacci λ-numbers $F_\lambda(n)$

$$G_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \tag{E.61}$$

was introduced in [118]. The following properties of the G_λ-matrix (E.61) are proved in [118]:

$$G_\lambda^n = \begin{pmatrix} F_\lambda(n+1) & F_\lambda(n) \\ F_\lambda(n) & F_\lambda(n-1) \end{pmatrix} \tag{E.62}$$

$$\det G_\lambda^n = (-1)^n. \tag{E.63}$$

The general property of the Fibonacci Q-, Q_p-, and G_λ -matrices consists of the following. The determinants of the Fibonacci Q-, Q_p-, and G_λ -matrices and all their powers are equal to +1 or -1. This unique property emphasizes mathematical beauty in the Fibonacci matrices and unites them into a special class of matrices, which are of fundamental interest for matrix theory.

E.6.2. *The "Golden" Matrices*

Integer numbers – the classical Fibonacci numbers, the Fibonacci p- and m-numbers - are elements of Fibonacci matrices (E.56), (E.59) and (E.62). In [114] a special class of the square matrices called **"golden" matrices** was introduced. Their peculiarity is the fact that the hyperbolic Fibonacci functions (E.38) or the hyperbolic Fibonacci λ-functions (E.32) and (E.33) are elements of these matrices. Let us consider the simplest of them [114]:

$$Q^{2x} = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix}; \quad Q^{2x+1} = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix}. \tag{E.64}$$

If we calculate the determinants of the matrices (E.64), we obtain the following unusual identities:

$$\det Q^{2x} = cFs(2x+1)\times cF(2x-1)-\left[sFs(2x)\right]^2 = 1$$
$$\det Q^{2x+1} = sFs(2x+2)\times sF(2x)-\left[cFs(2x+1)\right]^2 = -1. \tag{E.65}$$

The “golden” matrices based on hyperbolic Fibonacci λ-functions (E.32) and (E.33) take the following form [118]:

$$G_\lambda^{2x} = \begin{pmatrix} cF_\lambda(2x+1) & sF_\lambda(2x) \\ sF_\lambda(2x) & cF_\lambda(2x-1) \end{pmatrix};\ G_\lambda^{2x+1} = \begin{pmatrix} sF_\lambda(2x+2) & cF_\lambda(2x+1) \\ cF_\lambda(2x+1) & sF_\lambda(2x) \end{pmatrix}. \tag{E.66}$$

It is proved [118] that the “golden” G_λ -matrices (E.67) possess the following unusual properties:

$$\det G_\lambda^{2x} = 1;\quad \det G_\lambda^{2x+1} = -1\ . \tag{E.67}$$

The mathematical beauty of “golden” matrices (E.64) and (E.66) are confirmed by their unique mathematical properties (E.65) and (E.67).

E.7. Applications in Computer Science: the “Golden” Information Technology

E.7.1. *Fibonacci Codes, Fibonacci Arithmetic and Fibonacci Computers*

The concept of *Fibonacci computers* suggested by the author in a speech *Algorithmic Measurement Theory and Foundations of Computer Arithmetic* given to the joint meeting of Computer and Cybernetics Societies of Austria (Vienna, March 1976) and described in the book [20] is one of the more important ideas of modern computer science. The essence of the concept amounts to the following: modern computers are based on a binary system (E.57), which represents all numbers as sums of the binary numbers with binary coefficients, 0 and 1. However, the binary system (E.47) is non-redundant and does not allow for detection of errors, which could appear in the computer during the process of its exploitation. In order to eliminate this shortcoming, the author suggested [20] the use of *Fibonacci p-codes*

$$N = a_n F_p(n) + a_{n-1}F_p(n-1) + \ldots + a_i F_p(i) + \ldots + a_1 F_p(1), \tag{E.68}$$

where N is a natural number, $a_i \in \{0, 1\}$ is a binary numeral of the i-th digit of the code (E.68); n is the digit number of the code (E.68); $F_p(i)$ is the i-th digit weight calculated in accordance with the recurrence relation (E.14).

Thus, Fibonacci p-codes (E.68) represent all numbers as the sums of Fibonacci p-numbers with binary coefficients, 0 and 1. In contrast to the classical binary number system, the Fibonacci p-codes (E.68) are redundant positional methods of number representation. This redundancy can be used for checking different transformations of numerical information in the computer, including arithmetical operations. A Fibonacci computer project was developed by the author in the former Soviet Union from 1976 right up to the disintegration of the Soviet Union in 1991. 65 foreign patents in the U.S., Japan, England, France, Germany, Canada and other countries are official juridical documents, which confirm Soviet priority in Fibonacci computers.

E.7.2. *Ternary Mirror-Symmetrical Arithmetic*

Computers can be constructed by using different number systems. The ternary computer "Setun" designed in Moscow University in 1958 was the first computer based not on a binary system but on a ternary system [180]. The ternary mirror-symmetrical number system [104] is an original synthesis of the classical ternary system [180] and Bergman's system (E.13) [86]. It represents integers as the sum of golden mean squares with ternary coefficients {-1, 0, 1}. Each ternary representation consists of two parts that are disposed symmetrically with respect to the 0th digit. However, one part is mirror-symmetrical to another part. At the increase of a number, its ternary mirror-symmetrical representation is expanding symmetrically to the left and to the right with respect to 0-th digit. This unique mathematical property produces a very simple method for checking numerical information in computers. It is proved [104] that the mirror-symmetric property is invariant with respect to arithmetical operations, that is, the results of all arithmetical operations have mirror-symmetrical form. This means that the mirror-symmetrical arithmetic can be used for designing self-controlling and fault-tolerant processors and computers.

Stakhov's article *Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-Symmetrical Arithmetic* [104] published in *The Computer Journal* (England) got a high approval from two outstanding computer specialists - Donald Knuth, Professor-Emeritus of Stanford University and the author of the famous book *The Art of Computer Programming*, and Nikolay Brousentsov, Professor at Moscow University, a principal designer of the first ternary computer "Setun." And this fact gives hope that the ternary mirror-symmetrical arithmetic [104] can become a source of new computer projects in the near future.

E.7.3. *A New Theory of Error-Correcting Codes Based upon Fibonacci Matrices*

The error-correcting codes [177, 182] are used widely in modern computer and communication systems for the protection of information from noise.

The main idea of error-correcting codes consists in the following [177, 182]. Let us consider the initial code combination that consists of n data bits. We add to the initial code combination m error-correction bits and build up the k-digit code combination of the error-correcting code, or (k,n)-code, where $k = n+m$. The error-correction bits are formed from the data bits as the sums by module 2 of certain groups of the data bits. There are two important coefficients, which characterize an ability of error-correcting codes to detect and correct errors [177].

The potential detection ability

$$S_d = 1 - \frac{1}{2^m} \quad . \tag{E.69}$$

The potential correction ability

$$S_c = \frac{1}{2^n} \, , \tag{E.70}$$

where m is the number of error-correction bits, n is the number of data bits.

The formula (E.70) shows that the coefficient of potential correcting ability diminishes potentially to 0 as the number n of data bits increases. For example, the Hamming (15,11)-code allows one to detect $2^{11}\times(2^{15} - 2^{11}) =$ 62,914,560 erroneous transitions; at that rate it can only correct $2^{15} - 2^{11} =$ 30,720 erroneous transitions, that is, it can correct only 30,720/62,914,560 = 0.0004882 (0.04882%) erroneous transitions. If we take n=20, then according to (E.70) the potential correcting ability of the error-correcting (k,n)-code diminishes to 0.00009%. Thus, the potential correcting ability of the classical error-correcting codes [177, 182] is very low. This conclusion is of fundamental importance! One more fundamental shortcoming of all known error-correcting codes is the fact that the very small information elements, bits and their combinations are objects of detection and correction.

The new theory of error-correcting codes [44, 113] that is based on Fibonacci matrices has the following advantages in comparison to the existing theory of algebraic error-correcting codes [177,182]:

1. The Fibonacci coding/decoding method is reduced to matrix multiplication, that is, to the well-known algebraic operation that is carried out so well in modern computers.
2. The main practical peculiarity of the Fibonacci encoding/decoding method is the fact that large information units, in particular, matrix elements, are objects of detection and correction of errors.
3. The simplest Fibonacci coding/decoding method (p=1) can guarantee the restoration of all “erroneous” (2×2)-code matrices having “single,” “double” and “triple” errors.

4. The potential correcting ability of the method for the simplest case $p=1$ is between 26.67% and 93.33% which exceeds the potential correcting ability of all known algebraic error-correcting codes by 1,000,000 or more times. This means that a new coding theory based upon the matrix approach is of great practical importance for modern computer science.

E.7.4. *Matrix and "Golden" Cryptography*

Matrix and "golden" cryptography developed by the author together with Iranian researcher Mostapha Kalami Heris is a new cryptographic method, which provides cryptographic protection of informational systems operating in real scale of time. A new *Cryptographic Mobil Phone* should contain hybrid cryptosystem, which consists of public-key cryptosystem and symmetric-key cryptosystem based on matrix cryptography for sound signals and images. Such an approach provides an cryptographic protection of all important information, in particular, voice and video information by using hybrid cryptosystem based on matrix cryptography and also all text information (SMS, email and so on) by using public-key cryptosystem. It is clear that *Cryptographic Mobil Phone* allows protecting human rights and freedom of individual more effectively than existing cryptographic systems.

E.8. Fundamental Discoveries of Modern Science Based Upon the Golden Section and "Platonic Solids"

E.8.1. *Shechtman's Quasi-Crystals*

It is necessary to note that right up to the last quarter of the 20th century the use of the golden mean in theoretical science, in particular, in theoretical physics, was very rare. In order to be convinced of this, it is enough to browse 10 volumes of *Theoretical Physics* by Landau and Lifshitz. We cannot find any mention about the golden mean and Platonic solids. The situation in theoretical science changed following the discovery of *Quasi-crystals* by the Israel researcher *Dan Shechtman* in 1982 [148].

One type of quasi-crystal was based upon the regular icosahedron described in Euclid's *Elements*! Quasi-crystals are of revolutionary importance for modern theoretical science. First of all, this discovery is the moment of a great triumph for the "icosahedron-dodecahedron doctrine," which proceeds throughout all the history of the natural sciences and is a source of deep and useful scientific ideas. Secondly, the quasi-crystals shattered the conventional idea that there was an insuperable watershed between the mineral world where the "pentagonal" symmetry was prohibited, and the living world, where the "pentagonal" symmetry is one of most widespread. Note that Dan Shechtman published his first article about the quasi-crystals in 1984, that is, exactly 100 years after the publication of Felix Klein's *Lectures on the Icosahedron* ... (1884) [58]. This means that this discovery is a worthy gift to the centennial anniversary of Klein's book [58], in which Klein predicted the outstanding role of the icosahedron in the future development of science.

E.8.2. *Fullerenes (Nobel Prize for Chemistry of 1996)*

The discovery of *fullerenes* is one of the more outstanding scientific discoveries of modern science. This discovery was made in 1985 by Robert F. Curl, Harold W. Kroto and Richard E. Smalley. The title "fullerenes" refers to the carbon molecules of the type C_{60}, C_{70}, C_{76}, C_{84}, in which all atoms are on a spherical or spheroid surface. In these molecules the atoms of carbon are located at the vertexes of regular hexagons and pentagons that cover the surface of a sphere or spheroid. The molecule C_{60} plays a special role amongst fullerenes. This molecule is based upon the Archimedean truncated icosahedron. The molecule C_{60} is characterized by the greatest symmetry and as a consequence is of the greatest stability. In 1996 Robert F. Curl, Harold W. Kroto and Richard E. Smalley won the Nobel Prize in chemistry for this discovery.

E.8.3. *El-Naschie's E-infinity Theory*

Prominent theoretical physicist and engineering scientist Mohammed S. El Naschie is a world leader in the field of golden mean applications to theoretical physics, in particular, quantum physics [60] – [72]. El Naschie's discovery of the golden mean in the famous physical two-slit experiment — which underlies quantum physics — became the source of many important discoveries in this area, in particular, of *E*-infinity theory. It is also necessary to note that the important contribution of Slavic researchers in this area. The book [53] written by Belarusian physicist Vasyl Pertrunenko is devoted to applications of the golden mean in quantum physics and astronomy.

E.8.4. *Bodnar's Geometry*

According to the law of phyllotaxis the numbers on the left-hand and right-hand spirals on the surface of phyllotaxis objects are always adjacent Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Their ratios 1/1, 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, ... are called a *symmetry order* of phyllotaxis objects. The phyllotaxis phenomena excited the best minds of humanity during the centuries since Johannes Kepler. The "puzzle of phyllotaxis" consists of the fact that a majority of bio-forms change their phyllotaxis orders during their growth. It is known, for example, that sunflower disks that are located on different levels of the same stalk have different phyllotaxis orders; moreover, the greater the age of the disk, the higher its phyllotaxis order. This means that during the growth of the phyllotaxis object, a natural modification (increase) in symmetry happens and this modification of symmetry obeys the law:

$$\frac{2}{1}\to\frac{3}{2}\to\frac{5}{3}\to\frac{8}{5}\to\frac{13}{8}\to\frac{21}{13}\to\dots\ . \tag{E.71}$$

The law (E.71) is called *Dynamic Symmetry*.

Recently Ukrainian researcher Oleg Bodnar developed a very interesting geometric theory of phyllotaxis [37]. He proved that phyllotaxis geometry is a special kind of non-Euclidean geometry based upon the "golden" hyperbolic functions similar to hyperbolic Fibonacci and Lucas functions (E.38) and (E.39). Such approach allows one to explain geometrically how the "Fibonacci spirals" appear on the surface of phyllotaxis objects (for example, pine cones, ananas, and cacti) in the process of their growth and thus dynamic symmetry (E.71) appears. Bodnar's geometry is of essential importance because it concerns fundamentals of the theoretical natural sciences, in particular, this discovery gives a strict geometrical explanation of the phyllotaxis law and dynamic symmetry based upon Fibonacci numbers.

E.8.5. *Petoukhov's "Golden" Genomatrices*

The idea of the genetic code is amazingly simple. The record of the genetic information in ribonucleic acids (RNA) of any living organism, uses the "alphabet" that consists of four "letters" or the nitrogenous bases: *Adenine* (*A*), *Cytosine* (*C*), *Guanine* (*G*), *Uracil* (*U*) (in DNA instead of the *Uracil* it uses the related *Thymine* (*T*)). Petoukhov's article [59] is devoted to the description of an important scientific discovery—the *golden genomatrices*, which affirm the deep mathematical connection between the golden mean and the genetic code.

E.8.6. *Fibonacci-Lorenz Transformations and "Golden" Interpretation of the Universe Evolution*

As is known, *Lorentz's transformations* used in special relativity theory (SRT) are the transformations of the coordinates of the events (x, y, z, t) at the transition from one inertial coordinate system (ICS) K to another ICS K', which is moving relatively to ICS K with a constant velocity V.

The transformations were named in honor of Dutch physicist Hendrik Antoon Lorentz (1853-1928), who introduced them in order to eliminate the contradictions between *Maxwell's electrodynamics* and *Newton's mechanics*. *Lorentz's transformations* were first published in 1904, but at that time their form was not perfect. The French mathematician Jules Henri Poincaré (1854-1912) brought them to modern form.

In 1908, that is, three years after the promulgation of SRT, the German mathematician Hermann Minkowski (1864-1909) gave the original geometrical interpretation of *Lorentz's transformations*. In *Minkowski's space*, a geometrical link between two ICS K and K' are established with the help of *hyperbolic rotation*, a motion similar to a normal turn of the Cartesian system in *Euclidean space*. However, the coordinates of x' and t' in the ICS K' are connected with the coordinates of x and t of the ICS K by using classical hyperbolic functions.

Thus, *Lorentz's transformations* in *Minkowski's geometry* are nothing as the relations of *hyperbolic trigonometry* expressed in physics terms. This means that *Minkowski's geometry* is hyperbolic interpretation of SRT and therefore it is a revolutionary breakthrough in geometric representations of physics, a way out on a qualitatively new level of relations between physics and geometry.

Alexey Stakhov and Samuil Aranson put forward in [191] the following hypotheses concerning the SRT :

1. The first hypothesis concerns the *light velocity in vacuum*. As is well known, the main dispute concerning the SRT, basically, is about the *principle of the constancy of the light velocity in vacuum*. In recent years a lot of scientists in the field of cosmology put forward a hypothesis, which puts doubt the permanence of the light velocity in vacuum - a fundamental physical constant, on which the basic laws of modern physics are based[194]. Thus, **the first hypothesis is that the light velocity in vacuum was changed in the process of the Universe evolution.**

2. Another fundamental idea involves with the factor of the *Universe self-organization* in the process of its evolution [195, 196]. According to modern view [196], a few stages of self-organization and degradation can be identified in process of the Universe development: *initial vacuum, the emergence of su-*

perstrings, the birth of particles, the separation of matter and radiation, the birth of the Sun, stars, and galaxies, the emergence of civilization, the death of Sun, the death of the Universe. The main idea of the article [196] is to unite the fact of the *light velocity change* during the Universe evolution with the factor of its *self-organization*, that is, to introduce a dependence of the light velocity in vacuum from some *self-organization parameter* ψ, which does not have dimension and is changing within: $(-\infty < \psi < +\infty)$. The light velocity in vacuum c is depending on the "self-organization" parameter $\psi(-\infty < \psi < +\infty)$ and this dependence has the following form:

$$c = c(\psi) = \bar{c}(\psi)c_0 . \tag{E.72}$$

As follows from (E.72), the *light velocity in vacuum* is a product of the two parameters: c_0 and $\bar{c}(\psi)$. The parameter c_0=*const*, having dimension [$m.sec^{-1}$], is called *normalizing factor*. It is assumed in [191] that constant parameter c_0 is equal to *Einstein's light velocity in vacuum* $\left(2.988\times10^{-8}\ m.\sec^{-1}\right)$ divided by the *golden mean* $\tau = \left(1+\sqrt{5}\right)/2 \approx 1.61803$. The dimensionless parameter $\bar{c}(\psi)$ is called *non-singular normalized Fibonacci velocity of light in vacuum*.

3. The *"golden" Fibonacci goniometry* is used for the introduction of the *Fibonacci-Lorentz transformations,* which are a generalization of the classical *Lorentz transformations*. We are talking about the matrix

$$\Omega(\psi) = \begin{pmatrix} cFs(\psi-1) & sFs(\psi-2) \\ sFs(\psi) & cFs(\psi-1) \end{pmatrix}, \tag{E.73}$$

whose elements are *symmetric hyperbolic functions sFs, cFs,* introduced by Alexey Stakhov and Boris Rozin in [106]. The matrix $\Omega(\psi)$ of the kind (E.73) is called *non-singular two-dimensional Fibonacci-Lorentz matrix* and the transformations

$$\begin{pmatrix} \xi \\ x_1 \end{pmatrix} = \begin{pmatrix} cFs(\psi-1) & sFs(\psi-2) \\ sFs(\psi) & cFs(\psi-1) \end{pmatrix} \begin{pmatrix} \xi' \\ x_1' \end{pmatrix}$$

are called *non-singular two-dimensional Fibonacci-Lorentz transformations*.

The above approach to the SRT led to the new ("golden") cosmological interpretation of the Universe evolution before, in the moment, and after the bifurcation, called Big Bang. Based on this approach, Alexey Stakhov and Samuil Aranson have obtained in [191] *new cosmological model of the Universe evolution* (*Stakhov-Aranson's model*), beginning with the Big Bang (T=0) both to positive direction of increasing time T (material Universe) and with a turn of the time arrow to inverse direction, that is, to increasing the time T to the negative direction (anti-material Universe).

According to [191], at the process of the evolution of the *material Universe* (increasing the time T into positive direction) the two "bifurcations" appeared. The first "bifurcation" corresponds to the *Big Bang*. The second "bifurcation" corresponds to the *origin of light* and transition of the Universe from *Dark Ages* to *Shining Period*, when the first stars lighted up the Universe appeared. The light velocity immediately after the second "bifurcation" was very high, much higher than its modern value. As the Universe evolution, the light velocity begun to drop and with the further increase in the time T in the positive direction the light velocity slowly (as if "freezing") seeks to the limit value 300 000 [km.sec^{-1}].

As the *anti-material Universe,* Stakhov and Aranson's model consists of the fact that immediately after the first "bifurcation" – the *Big Bang* (the time T is negative and close to zero), the light velocity is also close to zero. Then, as the *anti-material Universe* evolution (increasing the time T in a negative direction), the light velocity is slowly seeking to the limit value equal to $300000/\Phi^2\left[\text{km.sec}^{-1}\right]$, where Φ is the *golden mean.*

Thus, the "golden" model of the Universe evolution based on *Fibonacci-Lorentz transformations* (*Stakhov-Aranson's model*) [191] differs essentially from the classical model of the Universe evolution based on classical *Lorentz transformations* in the following:

1. In *Stakhov-Aranson's model* [191] the light velocity in vacuum was changed in process of the Universe evolution.
2. *Stakhov-Aranson's model* [191] singles out the two directions of the Universe evolution after the *Big Bang* – the evolution of the *material Universe* (the time T increases in positive direction) and the evolution of *anti-material Universe* (the time T increases in negative direction). In the process of the *material Universe* evolution after the *Big Bang* (the first "bifurcation") the second "bifurcation" appears. This "bifurcation" corresponds to the origin of light and transition of the Universe from *Dark Ages* to *Shining Period.*
3. The light velocity immediately after the second "bifurcation" was very high, however, as the Universe evolution, the light velocity slowly seeks to the limit value 300 000 [km.sec^{-1}]. As the *anti-material Universe* evolution (increasing the time T in a negative direction), the light velocity is slowly seeking to the limit value equal to $300000/\Phi^2\left[\text{km.sec}^{-1}\right]$, where Φ is the *golden mean.*

E.8.7. *Hilbert's Fourth Problem*

In the lecture *Mathematical Problems* presented at the Second International Congress of Mathematicians (Paris, 1900), David Hilbert (1862-1943) had formulated his famous 23 mathematical problems. These problems determined considerably the development of mathematics of 20th century. This lecture is a unique phenomenon in the mathematics history and in mathematical literature. The Russian translation of Hilbert's lecture and its comments are given in the works [197-199]. In particular, *Hilbert's Fourth Problem* is formulated in [197] as follows:

"Whether it is possible from the other fruitful point of view to construct geometries, which with the same right can be considered the nearest geometries to the traditional Euclidean geometry."

Note that Hilbert considered that *Lobachevski's geometry* and *Riemannian geometry* are nearest to the Euclidean geometry.

In mathematical literature *Hilbert's Fourth Problem* is sometimes considered as formulated very vague what makes difficult its final solution. The famous German mathematician Georg Hamel (1877-1952) [200] was the first one who tried to solve Hilbert's Fourth Problem.

In [198] American geometer Herbert Busemann analyzed the whole range of issues related to *Hilbert's Fourth Problem* and also concluded that the question related to this issue, unnecessarily broad. Note also the book [199] by Alexei Pogorelov (1919-2002) is devoted to a partial solution to *Hilbert's Fourth Problem*. The book identifies all, up to isomorphism, implementations of the axioms of classical geometries (Euclid, Lobachevski and elliptical), if we delete the axiom of congruence and refill these systems with the axiom of "triangle inequality."

In spite of critical attitude of mathematicians to *Hilbert's Fourth Problem*, we should emphasize great importance of this problem for mathematics, particularly for geometry. Without doubts, Hilbert's intuition led him to the conclusion that *Lobachevski's geometry* and *Riemannian geometry* do not exhaust all possible variants of non-Euclidean geometries. *Hilbert's Fourth Problem* directs attention of researchers at finding new *non-Euclidean geometries*, which are the nearest geometries to the traditional *Euclidean geometry*.

The most important mathematical result presented in [191] is a new approach to *Hilbert's Fourth Problem* based on the *hyperbolic Fibonacci λ-functions* (32) and (33). The main mathematical result of this study is a creation of infinite set of the *isometric λ-models of Lobachevski's plane* that is directly relevant to *Hilbert's Fourth Problem*.

As is known [201], the classical model of *Lobachevski's plane* in *pseudo-spherical coordinates* $(u,v), 0<u<+\infty, -\infty<v<+\infty$ with the Gaussian curvature K=–1 (Beltrami's interpretation of hyperbolic geometry on pseudosphere) has the following form:

$$(ds)^2 = (du)^2 + sh^2(u)(dv)^2, \tag{E.74}$$

where *ds* is an element of length and *sh*(*u*) is the hyperbolic sine.

Based on the hyperbolic Fibonacci λ-functions (E.32) and (E.33), Alexey Stakhov and Samuil Aranson deduced in [191] the *metric λ-forms of Lobachevski's plane* given by the following formula:

$$(ds)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}\left[sF_\lambda(u)\right]^2(dv)^2, \tag{E.75}$$

where $\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$ is the *metallic mean* and $sF_\lambda(u)$ is *hyperbolic Fibonacci λ-sine* (E.32).

Let us study partial cases of the *metric λ-forms of Lobachevski's plane* corresponding to the different values of λ:

1. **The *golden metric form* of Lobachevski's plane**. For the case $\lambda=1$ we have $\Phi_1 = \frac{1+\sqrt{5}}{2} \approx 1.61803$ – the *golden mean*, and hence the form (E.75) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_1)(du)^2 + \frac{5}{4}\left[sFs(u)\right]^2(dv)^2 \tag{E.76}$$

where $\ln^2(\Phi_1) = \ln^2\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.231565$ and $sFs(u) = \frac{\Phi_1^u - \Phi_1^{-u}}{\sqrt{5}}$ is symmetric *hyperbolic Fibonacci sine* (E.38).

2. **The *silver metric form* of Lobachevski's plane**. For the case $\lambda=2$ we have $\Phi_2 = 1+\sqrt{2} \approx 2.1421$ - the *silver mean,* and hence the form (E.74) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_2)(du)^2 + 2\left[sF_2(u)\right]^2(dv)^2, \tag{E.77}$$

where $\ln^2(\Phi_2) \approx 0.776819$ and $sF_2(u) = \frac{\Phi_2^u - \Phi_2^{-u}}{2\sqrt{2}}$.

3. **The *bronze metric form* of Lobachevski's plane**. For the case $\lambda=3$ we have $\Phi_3 = \frac{3+\sqrt{13}}{2} \approx 3.30278$ - the *bronze mean,* and hence the form (E.75) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_3)(du)^2 + \frac{13}{4}\left[sF_3(u)\right]^2(dv)^2 \tag{E.78}$$

where $\ln^2(\Phi_3) \approx 1.42746$ and $sF_3(u) = \frac{\Phi_3^u - \Phi_3^{-u}}{\sqrt{13}}$.

4. **The *cooper metric form* of Lobachevski's plane**. For the case $\lambda = 4$ we have $\Phi_4 = 2 + \sqrt{5} \approx 4.23607$ - the *cooper mean,* and hence the form (E.75) is reduced to the following:

$$(ds)^2 = \ln^2(\Phi_4)(du)^2 + 5\left[sF_4(u)\right]^2(dv)^2, \tag{E.79}$$

where $\ln^2(\Phi_4) \approx 2.08408$ and $sF_4(u) = \dfrac{\Phi_4^u - \Phi_4^{-u}}{2\sqrt{5}}$.

5. **The *classical metric form* of Lobachevski's plane**. For the case $\lambda = \lambda_e = 2sh(1) \approx 2.350402$ we have $\Phi_{\lambda_e} = e \approx 2.7182$ - *Napier number,* and hence the form (E.75) is reduced to the classical *metric forms of Lobachevski's plane* given by (E.74).

Thus, the formula (E.75) sets an infinite number of *metric forms of Lobachevski's plane*. The formula (E.74) given the *classical metric form of Lobachevski's plane* is a partial case of the formula (E.75). This means that there are infinite number of *Lobachevski's "golden" geometries,* which *"can be considered the nearest geometries to the traditional Euclidean geometry"* (David Hilbert). Thus, the formula (E.75) can be considered as a solution to *Hilbert's Fourth Problem.*

E.9. Conclusion

The following conclusions come from this study:

E.9.1. *The First Conclusion*

The first conclusion touches on a question of the origins of mathematics and its development. This conclusion can be much unexpected for many mathematicians. **We affirm that since the Greek period, the two mathematical doctrines – the *Classical Mathematics* and the *Harmony Mathematics* – begun to develop in parallel and independent of one another.** They both originated from one and the same source – Euclid's *Elements*, the greatest mathematical work of the Greek mathematics. Geometric axioms, the beginnings of algebra, theory of numbers, theory of irrationals and other achievements of the Greek mathematics were borrowed from Euclid's *Elements* by the *Classical Mathematics*. On the other hand, a *problem of division in extreme and mean ratio* (Theorem II.11) called later the *golden section* and a geometric theory of *regular polyhedrons* (Book XIII), expressed the Harmony of the Cosmos in Plato's Cosmology, were borrowed from Euclid's *Elements* by the *Mathematics of Harmony*. **We affirm that Euclid's *Elements* were the first attempt to reflect in mathematics the major scientific idea of the Greek science, the idea of Harmony. According to Proclus, the creation of geometric theory of *Platonic Solids* (Book XIII of Euclid's *Elements*) was the main purpose of Euclid's *Elements*.**

E.9.2. *The Second Conclusion*

The second conclusion touches on the development of *number theory*. **We affirm that the new constructive definitions of real numbers based on *Bergman's system* (E.13) and *the codes of the golden p-proportion* (E.49) overturn our ideas about rational and irrational numbers** [105]. A special class of irrational numbers – the *golden mean* and *golden p-proportions* - becomes a base of new number theory because all rest real numbers can be reduced to them by using the definitions (E.13) and (E.49). New properties of natural numbers (*Z-property* (E.51), *F-code* (E.53) and *L-code* (E.54)), following from this approach, confirm a fruitfulness of this approach to number theory.

E.9.3. *The Third Conclusion*

The third conclusion touches on the development of *hyperbolic geometry*. We affirm that a new class of hyperbolic functions – *the hyperbolic Fibonacci and Lucas λ-functions* (E.32)-(E.35) [118] – can become inexhaustible source for the development of *hyperbolic geometry*. **We affirm that the formulas (E.32)-(E.35) give an infinite number of hyperbolic functions similar to the classical hyperbolic functions, which underlie *Lobachevski's geometry*.** This affirmation can be referred to one of the main mathematical results of the *Mathematics of Harmony*. A solution to *Hilbert's Fourth Problem* [191] confirms a fruitfulness of this approach to hyperbolic geometry.

E.9.4. *The Fourth Conclusion*

The forth conclusion touches on the applications of the *Mathematics of Harmony* in *theoretical natural sciences*. We affirm that the *Mathematics of Harmony* is inexhaustible source of the development of *theoretical natural sciences*. This confirms by the newest scientific discoveries based on the *golden mean* and *Platonic Solids* (*quasi-crystals, fullerenes, golden genomatrices, E-*

infinity theory and so on). **The *Mathematics of Harmony* offers for *theoretical natural sciences* a tremendous amount of new recurrence relations, new mathematical constants, and new hyperbolic functions, which can be used in theoretical natural sciences for the creation of new mathematical models of natural phenomena and processes.** A new approach to the relativity theory and Universe evolution based on the *hyperbolic Fibonacci and Lucas functions* is a confirmation of fruitfulness of this approach.

E.9.5. *The Fifth Conclusion*

The fifth conclusion touches on the "golden" model of the Universe evolution based on *Fibonacci-Lorentz transformations* (*Stakhov-Aranson's model*) [191]. *Stakhov-Aranson's model* [191] singles out the two directions of the Universe evolution after the *Big Bang* – the evolution of the *material Universe* (the time T increases in positive direction) and the evolution of *anti-material Universe* (the time T increases in negative direction). In the process of the *material Universe* evolution after the *Big Bang* (the first "bifurcation") the second "bifurcation" appears. This "bifurcation" corresponds to the origin of light and transition of the Universe from *Dark Ages* to *Shining Period*. The light velocity immediately after the second "bifurcation" was very high, however, as the Universe evolution, the light velocity slowly seeks to the limit value 300 000 [km.sec^{-1}]. As the *anti-material Universe* evolution (increasing the time T in a negative direction), the light velocity is slowly seeking to the limit value equal to $300000/\Phi^2\left[\text{km.sec}^{-1}\right]$, where Φ is the *golden mean*.

E.9.6. *The Sixth Conclusion*

The sixth conclusion touches on the applications of the *Mathematics of Harmony* in computer science. **We affirm that the *Mathematics of Harmony* is a source for the development of new information technology – *the "Golden" Information Technology* based on the *Fibonacci codes* (E.68), *Bergman's system* (E.13), *codes of the golden p-proportions* (E.49), *"golden" ternary mirror-symmetrical representation* [104] and following from them new computer arithmetic's: *Fibonacci arithmetic, "golden" arithmetic, and ternary mirror-symmetrical arithmetic,* which can become a source of new computer projects.** Also this conclusion is confirmed by the *new theory of error-correcting codes based on Fibonacci matrices* [113] and the *matrix and "golden" cryptography* [114, 118].

E.9.7. *The Seventh Conclusion*

The seventh conclusion touches on the applications of the *Mathematics of Harmony* in modern *mathematical education*. **We affirm that the *Mathematics of Harmony* should become a base for the reform of modern mathematical education on the base of the *ancient idea of Harmony* and *golden section*.** Such an approach can increase an interest of pupils to studying mathematics because this approach brings together mathematics and natural sciences. A study of mathematics turns into fascinating search of new mathematical regularities of Nature.

E.9.8. *The Eighth Conclusion*

The eighth conclusion touches on the general role of the *Mathematics of Harmony* in the progress of contemporary mathematics. **We affirm that the *Mathematics of Harmony* can overcome a contemporary crisis in the development of the 20th century mathematics what resulted in the severance of the relationship between mathematics and theoretical natural sciences** [6]. The *Mathematics of Harmony* is a true "Mathematics of Nature" incarnated in many wonderful structures and phenomena of the Universe (*pine cones, pineapples, cacti, heads of sunflowers, quasi-crystals, fullerenes, genetic code, Universe evolution* and so on) and it can give birth to new scientific discoveries.

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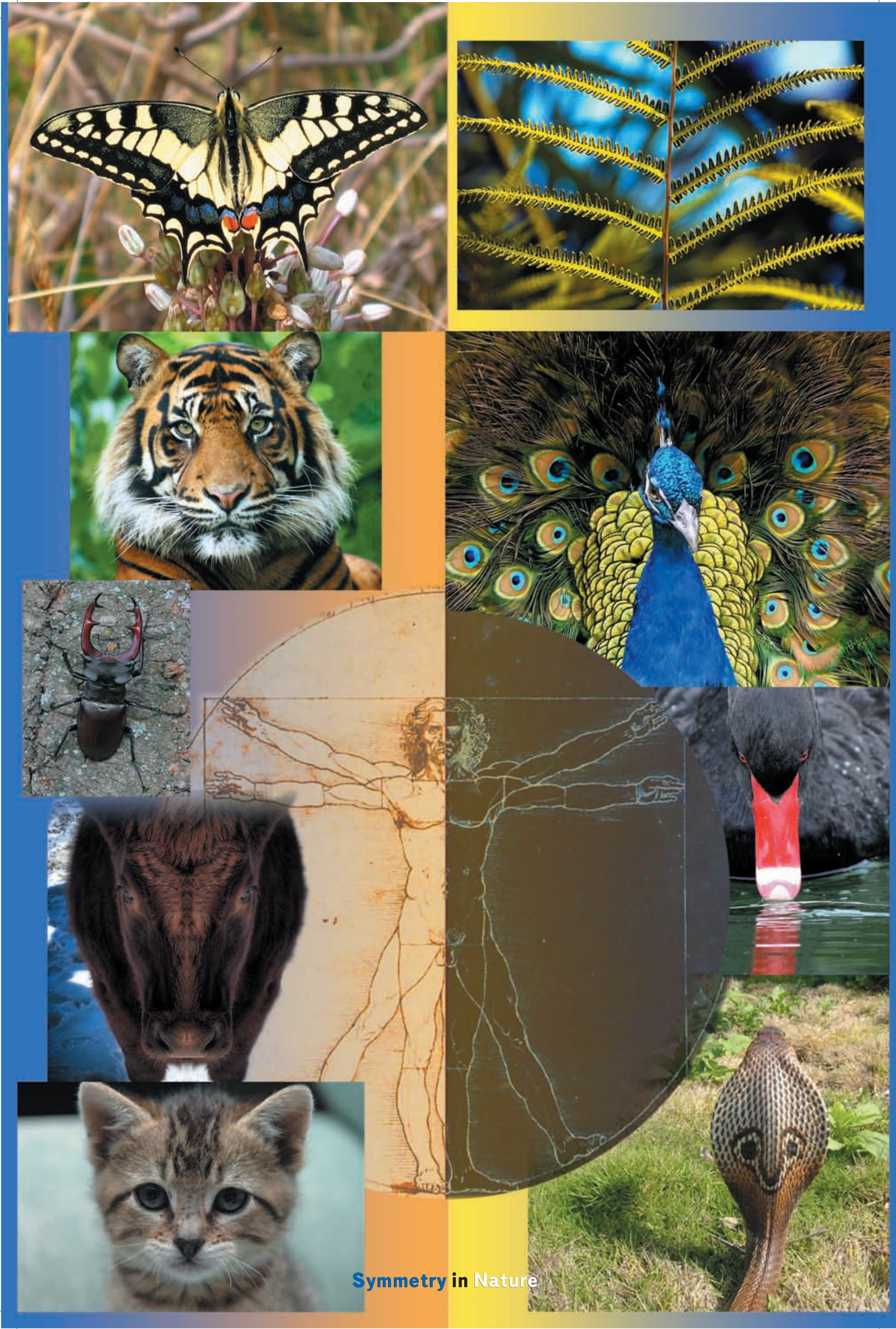
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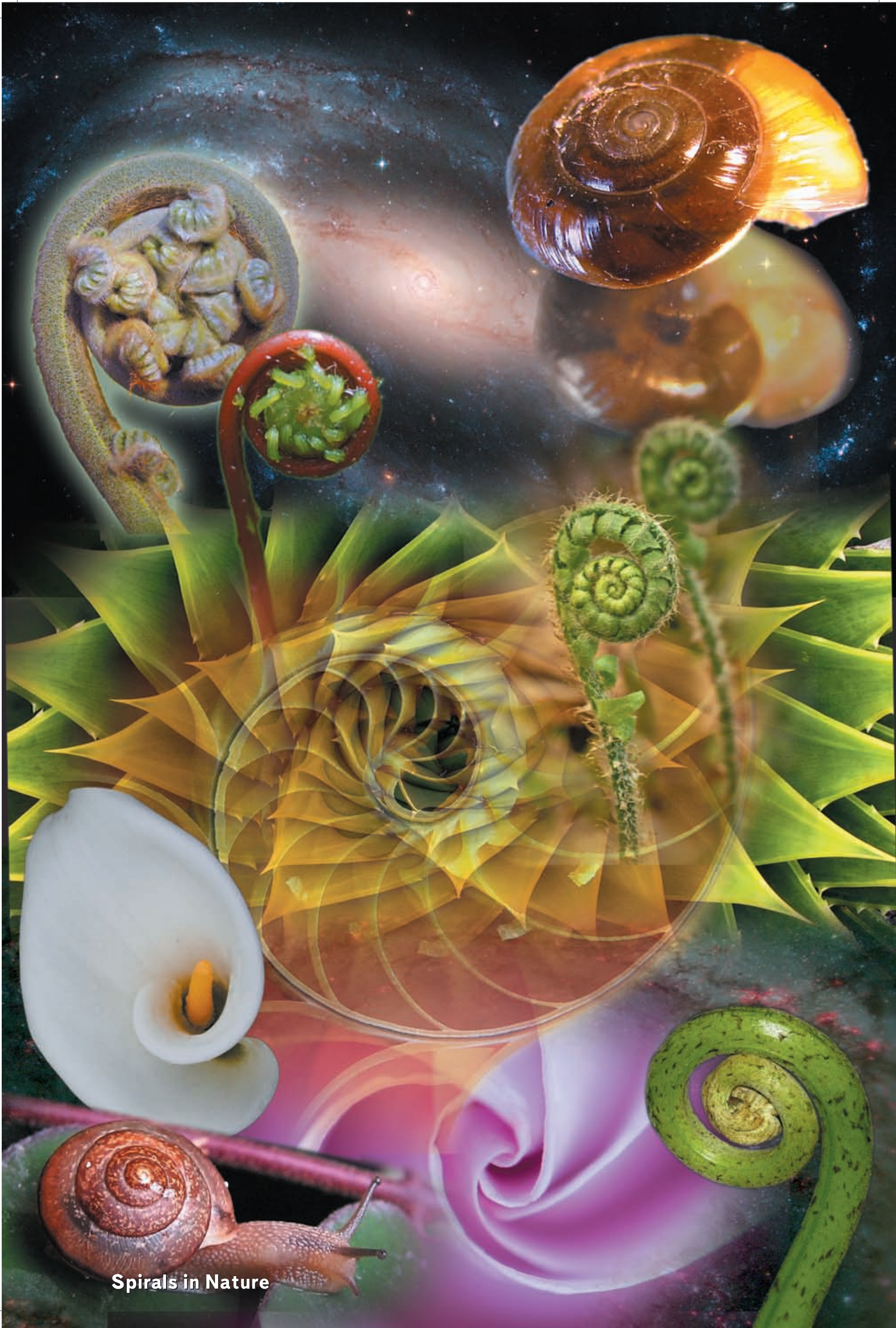
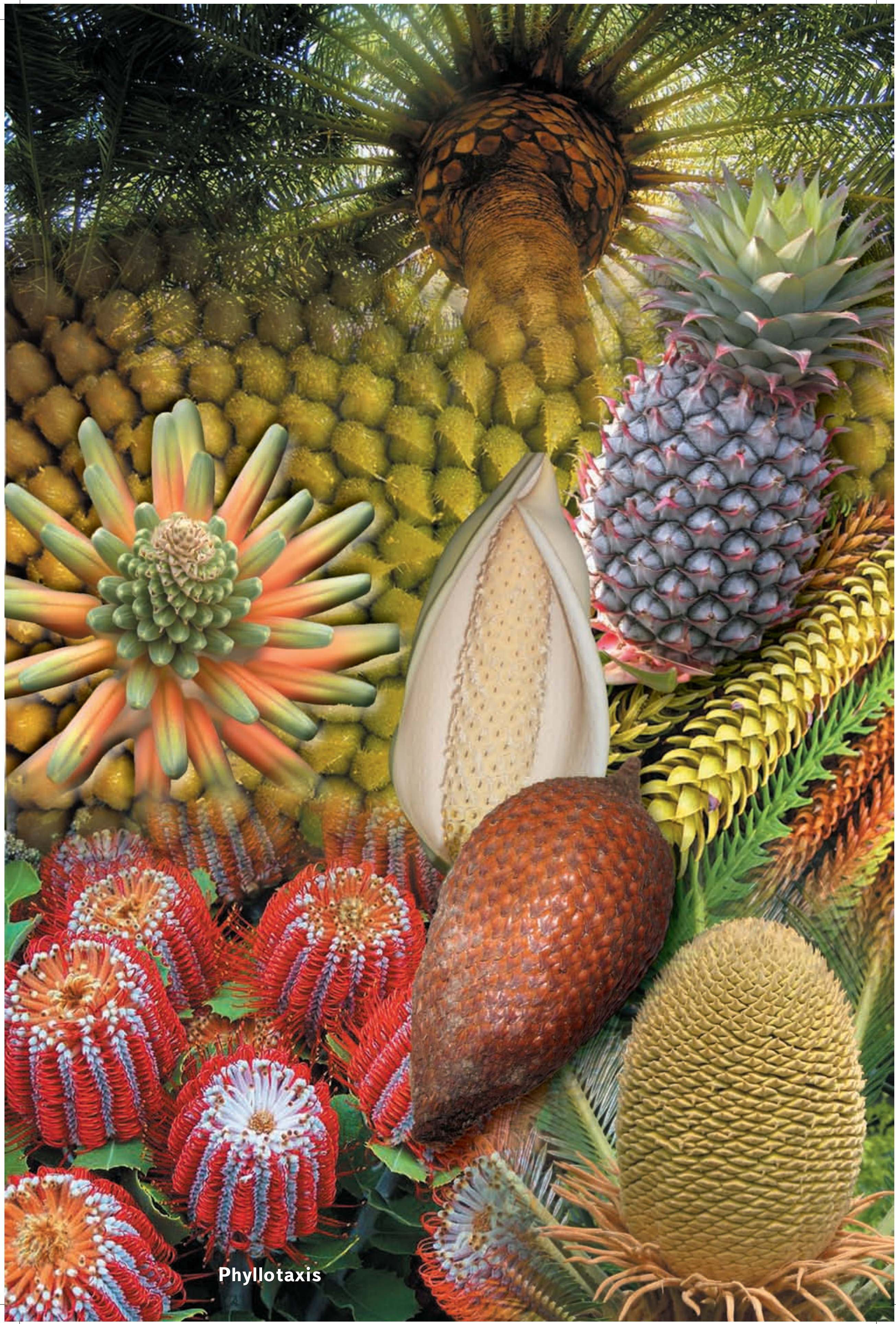
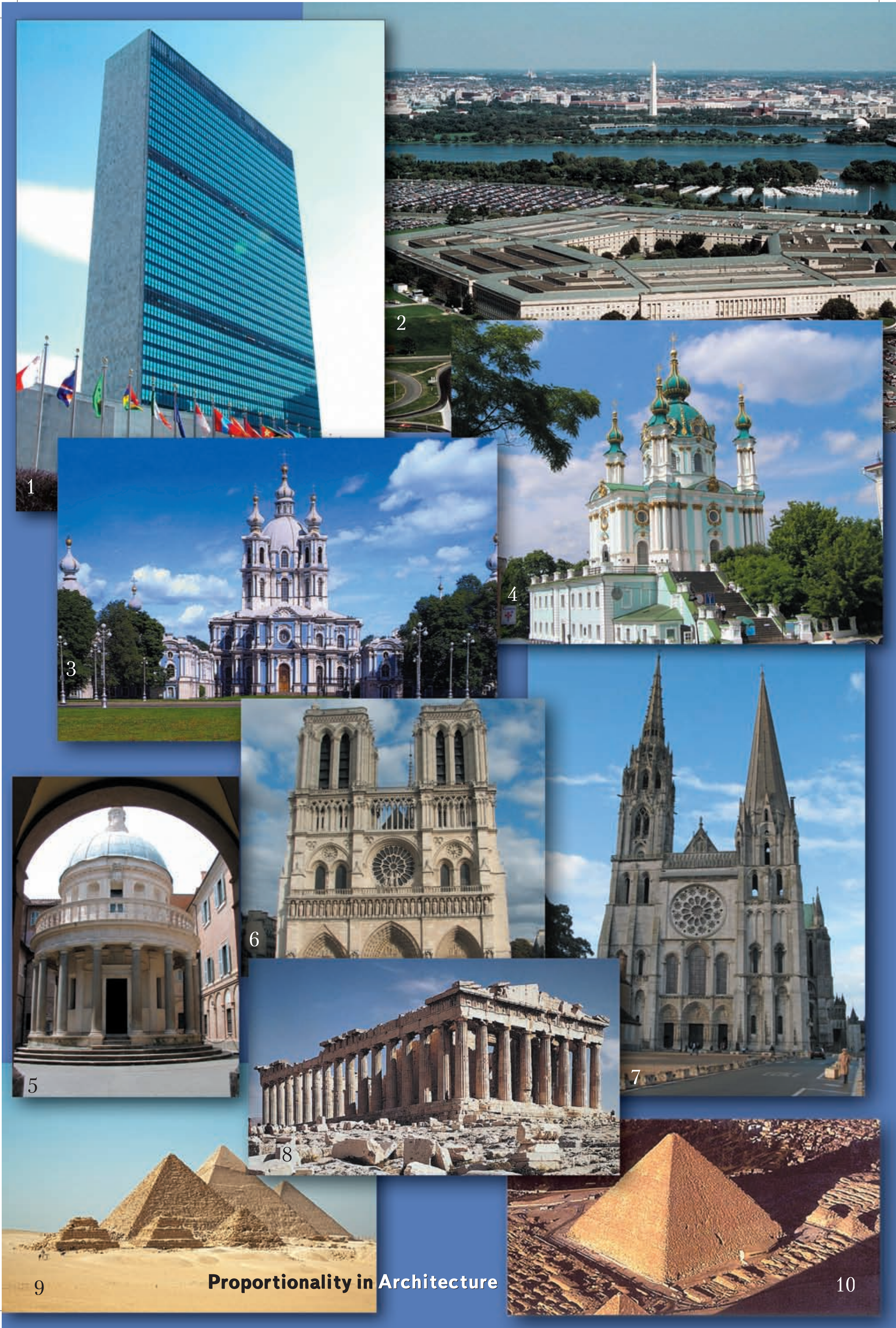
Pentagonal Symmetry



Spirals in Nature


Phyllotaxis



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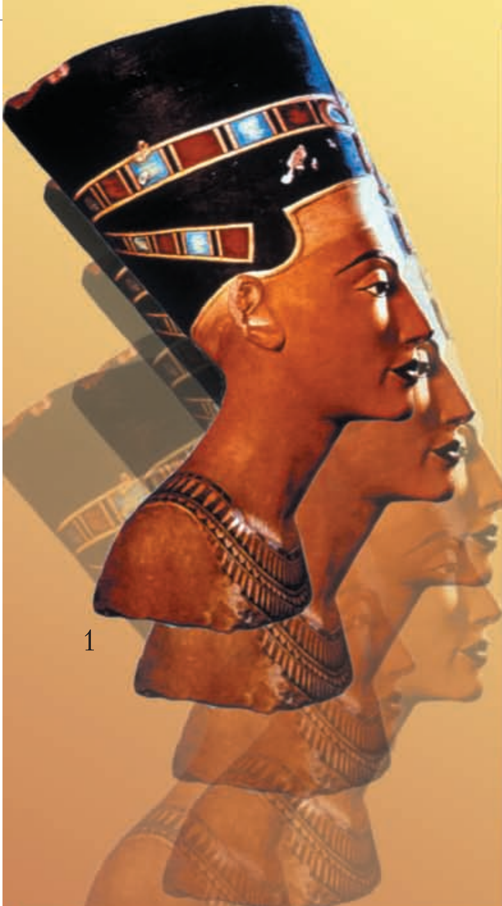

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