A Short Note on Quantum Entropies

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Published in physic.philica.com

Abstract

In quantum statistical mechanics, the extension of the classical Gibbs entropy is the von Neumann entropy, obtained from a quantum-mechanical system described by means of its density matrix. Here we shortly discuss this entropy and the use of generalized entropies instead of it.

Article body

A Short Note on Quantum Entropies

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Abstract: In quantum statistical mechanics, the extension of the classical Gibbs entropy is the von Neumann entropy, obtained from a quantum-mechanical system described by means of its density matrix. Here we shortly discuss this entropy and the use of generalized entropies instead of it.

Keywords: Entropy, Quantum entropy, von Neumann entropy, Non-additive entropy, Tsallis entropy, Kaniadakis entropy.

Entropy is involved in classical theory of information in the form of Shannon entropy. However, a generalized entropy, such as Tsallis entropy or other entropies, can be used to measure information too. For instance, mutual information is usually obtained from Shannon entropy, but it can be estimated also from Tsallis or Kaniadakis entropy [1,2]. In the case of systems obeying the quantum mechanics, information and changes of information are measured by von Neumann entropy as $S(\rho) = -\mathrm{Tr}\,(\rho \ln\rho)$, where the density matrix ρ is involved. If we express this matrix is a diagonal basis, given by eigenvectors $|i\rangle$ and eigenvalues η_i, the density matrix becomes $\rho = \Sigma_i \eta_i |i\rangle\langle i|$, and von Neumann entropy becomes $S = -\Sigma_i (\eta_i \ln\eta_i)$, where the summation runs across the range of eigenvalues.

Of course, the entropy measures of the classical information theory can be generalized to the quantum case to have mutual quantum information, joint entropies, conditional quantum entropies and so on. Then, given a bipartite quantum state described by the density matrix ρ_{AB}, the entropy of the joint system (A,B) is $S(\rho_{AB})$ and entropies of the subsystems are $S(\rho_A)$,

$S(\rho_B)$ respectively. By analogy with the classical conditional entropy, one defines the conditional quantum entropy as $S(\rho_{A|B}) = S(\rho_{AB}) - S(\rho_B)$, where we have an entropy containing the conditional density operator $\rho_{A|B}$ [3-6].

Let us concentrate on the mutual information obtained from the mutual entropy of a bipartite system. The definition of a quantum mutual entropy is modulated on the classical case too. In classical terms, given two subsystems A and B, and their probability distribution of two variables p(a,b), the two marginal distributions are given by: $p(a) = \Sigma_b\, p(a,b)$; $p(b) = \Sigma_a\, p(a,b)$.

The classical mutual information I(A;B) is defined by:

$$(1) \qquad I(A;B) = S(p(a)) + S(p(b)) - S(p(a,b)) = S(A) + S(B) - S(A,B),$$

In (1), S denotes the Shannon entropy. Note that it is giving information on the dependence or independence of subsets A and B. In the case of independent subsets the mutual information is zero. For a generalized entropy, the mutual information is given by a different equation according to its specific generalized additivity [1,2].

The mutual information can also be described as a relative entropy between p(a,b) and p(a)p(b) [6]. It follows from the property of relative entropy that $I(A;B) \geq 0$ and equality holds if and only if $p(a,b) = p(a)p(b)$, which is the condition of independency of the subsets A and B.

The quantum mechanical counterpart is obtained using von Neumann entropy. Then entropy S(A,B) becomes $S(\rho_{AB}) = -\mathrm{Tr}(\rho_{AB} \ln \rho_{AB})$. From the probability distribution p(a,b), the marginal distributions are obtained. Instead of a simple sum, here we have a partial trace. So one can assign to ρ a state on the subsystem A by $\rho_A = \mathrm{Tr}_B\,(\rho_{AB})$, where Tr_B is the partial trace with respect to system B. After, entropy $S(\rho_A)$ is calculated. $S(\rho_B)$ is defined in the same manner [6].

An appropriate definition of quantum mutual information should be $I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$. S(ρ) is additive for independent systems. Given two density matrices ρ_A , ρ_B describing independent systems, we have:

$$(2) \qquad S(\rho_{AB}) = S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B), \text{ and } I(\rho_{AB}) = 0.$$

Moreover, $I(\rho_{AB}) \leq 2 \min[S(\rho_A), S(\rho_B)]$ for a corollary of Araki-Leib theorem [5]. This implies that quantum systems can be supercorrelated.

Here some examples [7]. Let us consider independent particles so that: $\rho_A = \rho_B = ½(\,|0\rangle\langle 0| + |1\rangle\langle 1|\,)$, we have that $\rho_{AB} = \rho_A \otimes \rho_B$. $S(A) = S(B) = -½\log_2 ½ - ½\log_2 ½ = 1$, $S(A,B)=2$, $I(A;B)=0$ and $S(A|B)=1$. Let us consider fully correlated particles: $\rho_{AB} = ½\,(\,|00\rangle\langle 00| + |11\rangle\langle 11|\,)$. The density matrix for A is given by: $\rho_A = \mathrm{Tr}_B\,(\rho_{AB}) = ½\,(\,|0\rangle\langle 0| + |1\rangle\langle 1|\,)$. $S(A) = 1 = S(A,B)$ and $S(A;B)=1$ and $S(A|B)=0$. And we can have also entangled particles: $|\psi_{AB}\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle)$. The density matrix is given by: $\rho_{AB} = |\,\psi_{AB}\rangle\langle\psi_{AB}\,|$. We can calculate the density matrix of A: $\rho_A = \mathrm{Tr}_B(\rho_{AB}) = ½\,(\,|0\rangle\langle 0| + |1\rangle\langle 1|\,)$. Therefore $S(A)=1$, $S(A,B)=0$, $S(A;B)=2$ and $S(A|B)=-1$ [7].

After these examples with von Neumann entropy, let us consider the case of the generalized Tsallis entropy [8]. In this paper we find the quantum Tsallis entropy. It is given by:

$$(3) \qquad S_q(\rho) = (\mathrm{Tr}\,\rho^q - 1)/(1-q).$$

Let us remember that Tsallis entropy is non-additive, that is, for independent subsystems the joint entropy is different from the sum of the entropies. As a consequence, in the quantum case, for product states [8]:

$$(4) \qquad S_q(\rho_{AB}) = S_q(\rho_A \otimes \rho_B) = S_q(\rho_A) + S_q(\rho_B) + (1-q)\, S_q(\rho_A) S_q(\rho_B).$$

And, also in the case of commutating operators, a correlation is induced by this non additivity [7]. Such a correlation disappers when $q \rightarrow 1$. Quantum Tsallis entropy had been also discussed in [9].

What can we obtain, if we use the Kaniadakis entropy? It is an entropy having the following generalized additivity for independent systems $S_\kappa(A,B) = S_\kappa(A)Y_\kappa(B)+S_\kappa(B)Y_\kappa(A)$, where $S_\kappa = (\Sigma_i p_i^{1-\kappa} - \Sigma_i p_i^{1+\kappa})/(2\kappa)$; $Y_\kappa = (\Sigma_i p_i^{1-\kappa} + \Sigma_i p_i^{1+\kappa})/2$ (see [10] for a discussion of the generalized additivity of Tsallis and Kaniadakis entropies, and the references therein). The summations run across i-states of the system with probabilities p_i. In [11], the quantum Kaniadakis entropy is given in the framework of a generalized additivity involving only entropies S, not function Y, because the authors used a formula which is valid in the case of equiprobable states. In a more general case then, let us stress that we have to add a quantum Y function:

$$(5) \qquad S_\kappa(\rho)=(\mathrm{Tr}\, \rho^{1+\kappa} - \mathrm{Tr}\, \rho^{1-\kappa})/(2\kappa) \ ; \ Y_\kappa(\rho)=(\mathrm{Tr}\, \rho^{1+\kappa} + \mathrm{Tr}\, \rho^{1-\kappa})/2.$$

And therefore, besides the quantum entropy, we have also to consider another quantity, which is the quantum version of function Y, fundamental for any discussion of conditional and mutual information as obtained in the Kaniadakis formalism [2].

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