Quasi-interpolation method for numerical solution of Volterra integral equations

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Abstract

In this article, a numerical method based on quasi-interpolation method is used for the numerical solution of the linear Volterra integral equations of the second kind. Also, we approximate the solution of Volterra integral equations by Nystrom's method. Some examples are given and the errors are obtained for the sake of comparison.

Keywords : Volterra integral equations, Nystrom method, Quasi-interpolation.

1 Introduction

This paper a quasi-interpolation procedure is presented for the numerical solution of the linear Volterra integral equation of the second kind:

$$u(s) = f(s) + \lambda \int_{-1}^{s} k(s,t)u(t)dt, \quad -1 \leq s,t \leq 1, \tag{1}$$

where, the function $f(s)$ is known, $k(s,t)$ is the kernel function which is known, continuous and λ is a constant, the aim is to find the unknown function $y(s)$ which is solution of equation equation 1. For the numerical solution of these equations several numerical approaches have been proposed. Many different authors present numerical solution for equation 1 by using other method. In [1], Maleknejad and Aghazadeh applied a Taylor-series expansion method to Volterra integral equations. Rashidinia and Zarebnia [2], introduced a numerical method for linear Volterra integral equations

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of the second kind, also Maleknejad and Shahrezaee [3], studied some numerical methods for linear integral equations of the second kind. In this study we use quasi-interpolation method for solving Volterra integral equation 1.
The outline of the paper is as follows. First, in Section 2 we review some of the main properties of quasi function and quasi-interpolation that are necessary for the formulation of the discrete system. In Section 3, we illustrate how the quasi-interpolation and Nystrom's methods may be used to replace equation 1 by an explicit system of linear algebraic equations. In Section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical schemes by considering some numerical examples.

2 Quasi-interpolation and Approximation of integral operators

We consider the following linear Volterra integral equations of the second kind:

$$u(s) = f(s) + \lambda \int_{-1}^{s} k(s,t)u(t)\,dt. \tag{2}$$

If we let $z(t) = \frac{s+1}{2}t + \frac{s-1}{2}$, $t \in [-1,1]$, then the integral equation in above can be written as:

$$u(s) = f(s) + \vartheta \int_{-1}^{1} k(s,z(t))u(z(t))dt, \tag{3}$$

where $\vartheta = \frac{s+1}{2}\lambda$. The equation 3 is a Fredholm integral equation. We can solve it by quasi-interpolation method to obtain unknown $u(s)$.

Now, we consider the integral operator

$$T : C([-1,1]) \to C([-1,1]), \quad (Tu)(s) := \mu \int_{-1}^{1} k(s,z(t))u(z(t))\,dt\,, \tag{4}$$

where $\mu = \frac{s+1}{2}$ and $k : [-1,1]^2 \to \mathbf{R}$ is continuous kernel.
We approximate $Tu(s)$ with the trapezoidal rule in point mh. Let $N \in \mathbf{N}$ and $h = \frac{1}{N}$. We obtain

$$\begin{aligned} Tu(s) &= \mu \int_{-1}^{1} k(s,z(t))u(z(t))\,dt \\ &\approx \mu \sum_{m=-N}^{N} \frac{h}{2} k(s,z(mh))u(z(mh)))(2-\delta_{|m|N})\,, \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

We let the operator $T_N : C([-1,1]) \to C([-1,1])$ as follow:

$$T_N u(s) := \mu \sum_{m=-N}^{N} \frac{h}{2} k(s, z(mh)) u(z(mh))(2 - \delta_{|m|N}). \tag{5}$$

Thus we can write

$$Tu(s) \approx T_N u(s).$$

definition 1. The error function is defined by

$$erf(a,b) := \frac{2}{\sqrt{\pi}} \int_a^b e^{-t^2}\, dt,$$

where $a, b \in \mathbf{R}$ and $a \le b$.
The quasi-interpolation for $d > 0$ is defined as follows:

$$u_{d,N} z(t) := \mu \sum_{m=-N}^{N} u(z(mh)) \frac{e^{-\frac{(z(t)-z(mh))^2}{dh^2}}}{\sqrt{\pi d}},$$
$$u_{d,N} : [-1,1] \to \mathbf{R}.$$

By replacing u with the quasi-interpolation we obtain $Tu(s)$ as follow:

$$\begin{aligned} Tu(s) &= \mu \int_{-1}^{1} k(s, z(t)) u(z(t))\, dt \\ &\approx \mu \int_{-1}^{1} k(s, z(t)) u_{d,N}(z(t))\, dt \\ &= \mu^2 \sum_{m=-N}^{N} u(z(mh)) \int_{-1}^{1} k(s, z(t)) \frac{e^{-\frac{(z(t)-z(mh))^2}{dh^2}}}{\sqrt{\pi d}}\, dt. \end{aligned}$$

We approximate the Volterra integral operator by replacing $k(s, z(t))$ with $k(s, z(mh))$ as:

$$\begin{aligned} Tu(s) &\approx \mu^2 \sum_{m=-N}^{N} u(z(mh)) \int_{-1}^{1} k(s, z(t)) \frac{e^{-\frac{(z(t)-z(m))^2}{dh^2}}}{\sqrt{\pi d}}\, dt \\ &\approx \mu^2 \sum_{m=-N}^{N} k(s, z(mh)) u(z(mh)) \int_{-1}^{1} \frac{e^{-\frac{(z(t)-z(m))^2}{dh^2}}}{\sqrt{\pi d}}\, dt \\ &= \mu \sum_{m=-N}^{N} \frac{h}{2} k(s, z(mh)) u(z(mh)) erf\left(\frac{\mu(m-N)}{\sqrt{d}}, \frac{\mu(m+N)}{\sqrt{d}} \right). \end{aligned}$$

We define the operator

$$T_{d,N} : C([-1,1]) \to C([-1,1])$$

by

$$(T_{d,N}u)(s) := \mu \sum_{m=-N}^{N} \frac{h}{2} k(s, z(mh))u(z(mh)) erf\left(\frac{\mu(m-N)}{\sqrt{d}}, \frac{\mu(m+N)}{\sqrt{d}}\right), \tag{6}$$

we have

$$(Tu)(s) \approx (T_{d,N}u)(s).$$

Lemma 1. It holds:

$$\lim_{d\to 0} erf\left(\frac{\mu(m-N)}{\sqrt{d}}, \frac{\mu(m+N)}{\sqrt{d}}\right) = 2 - \delta_{|m|N}.$$

proof. We have

$$\lim_{d\to 0} erf(\frac{\mu(m-N)}{\sqrt{d}}, \frac{\mu(m+N)}{\sqrt{d}}) = \lim_{d\to 0} \frac{2}{\sqrt{\pi}} \int_{\frac{\mu(m-N)}{\sqrt{d}}}^{\frac{\mu(m+N)}{\sqrt{d}}} e^{-t^2}\, dt,$$

and

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2}\, dt = 1,$$

then we obtain

$$\lim_{d\to 0} erf(\frac{\mu(m-N)}{\sqrt{d}}, \frac{\mu(m+N)}{\sqrt{d}}) = \begin{cases} 1, & m = -N, \\ 2, & |m| \neq N, \\ 1. & m = N, \end{cases}$$

therefore

$$\lim_{d\to 0} erf\left(\frac{\mu(m-N)}{\sqrt{d}}, \frac{\mu(m+N)}{\sqrt{d}}\right) = 2 - \delta_{|m|N}.$$

Lemma 2. It holds:

$$lim_{d\to 0} T_{d,N} = T_N.$$

proof. [4].

3 Application to Volterra integral equations

Consider the linear Volterra integral equation

$$X - TX = b, \tag{7}$$

where $b \in C[-1,1]$ and T is defined by equation 4. For $||k||_\infty < \frac{1}{2}$, the equation 7 has a unique solution $u \in C([-1,1])$.
Because if u_1 and u_2 be solutions for equation 7, then we have

$$u_1(s) - \mu \int_{-1}^{1} k(s, z(t))u_1(z(t))\, dt = b(s), \tag{8}$$

and

$$u_2(s) - \mu \int_{-1}^{1} k(s, z(t))u_2(z(t))\, dt = b(s), \tag{9}$$

from equation 8 and equation 9 we can write

$$F(s) = \mu \int_{-1}^{1} k(s, z(t))R(z(t))\, dt,$$

where

$$F(s) = u_1(s) - u_2(s), \qquad R(t) = u_1(z(t)) - u_2(z(t)).$$

Therefore

$$\begin{aligned}\int_{-1}^{1} |F(s)|^2\, ds &\leq \int_{-1}^{1} |R(t)|^2\, dt \int_{-1}^{1}\int_{-1}^{1} \mu^2 |k(s, z(t))|^2\, dt\, ds \\ &< \frac{\mu^3}{12}\int_{-1}^{1} |R(t)|^2 dt,\end{aligned}$$

The above equation is in contradiction with our assumption. Hence we conclude that $u_1(t) = u_2(t)$.

3.1 Approximation with quasi-interpolation

We estimate solution of the linear integral equation 7 using quasi-interpolation. Substituting T with T_N from equation 6, we have

$$X_{d,N} - T_{d,N}X_{d,N} = b, \tag{10}$$

similarly we obtain the linear system in the point jh as follows:

$$\sum_{m=-N}^{N}[\delta_{jm}-\mu\frac{h}{2}k(jh,z(mh))erf\left(\frac{\mu(m-N)}{\sqrt{d}},\frac{\mu(m+N)}{\sqrt{d}}\right)]$$
$$u(z(mh))=b(jh) \quad ,\mu=\frac{s+1}{2}. \tag{11}$$

3.2 Approximation with Nystrom's method

In this section we obtain approximate of integral equations 1 by Nystrom's method [5].
Let u be the solution of equation 7 and $s\in[-1,1]$, we 'll have

$$u(s)-(TGu)(s)=b(s).$$

If we employ the trapezoidal rule for the quadrature procedure and we approximate equation

$$X_N-T_NGX_N=b, \tag{12}$$

with T_N is defined in equation 5. We can obtain the nonlinear system of equation in the points jh.

$$u(jh)-\sum_{m=-N}^{N}\frac{h}{2}k(jh,mh)Gu_m(2-\delta_{|m|N})=b(jh), \tag{13}$$

that $j=\{-N,...,N\}$ and the values $u_m=X_N(mh)$ are then approximation for $u(mh)$.

Lemma 3. It hold

$$\lim_{N\to\infty}||(I-T)^{-1}(T_N-T)T_N||=0,$$

and for all $N\in\mathbf{N}$ with

$$||(I-T)^{-1}(T_N-T)T_N||<1,$$

it holds. Furthermore, the approximate equation 12 has a unique solution and for the sequence (X_N) of approximate solutions it hold:

$$\lim_{N\to\infty}||X_N-u||_\infty=0,$$

and the linear system (13) has a unique solution.

proof. [4].

Lemma 4. There exist $N_0 \in \mathbf{N}$ and $d_0 > 0$, such that for all $N \geq N_0$ and $d \leq d_0$, the both Linear system equation 12 and equation 13 have unique solution and $d \to 0$ the unique solution of the system 13 converges to the unique of the system 11.
proof. [4].

Lemma 5. Let N_0, d_0, U and U_d be as Lemma 4. For all $N \geq N_0$ and $d \leq d_0$ the approximate equation 12 has a unique solution.
proof. [4].

4 Numerical examples

In this section, we use the above proposed method in Examples with detailed explanations. we compare the results of numerical solution this method with the solution of the Nystrom's method.

Example 1 For the following Linear Volterra integral equation:

$$u(s) = e^{-s^2} + \frac{s}{2}(\frac{1}{e} - e^{-s^2}) + \int_{-1}^{s} stu(t)\,dt,$$

with exact solution $u(s) = e^{-s^2}$, table 1 shows for $N = 5$ the solution u of the Linear system 11 and the solution u_d of the Linear system 13 with $d = 0.1$ and $d = 0.01$.

Example 2 Consider the volterra integral equation

$$u(s) = \cos 1 + s\cos 1 - \sin 1 + \int_{-1}^{\infty} (t-s)u(t)\,dt,$$

with exact solution $u(s) = \sin s$. The table 2 illustrate the numerical results for $N = 3$ the solution u of the Linear system 11 and the solution u_d of the Linear system 13 with $d = 0.1$ and $d = 0.01$.

Example 3 For the following Linear Volterra integral equation:

$$u(s) = \frac{-1}{8} - \frac{s}{7} + s^6 - \frac{s^8}{56} + \int_{-1}^{s} (s-t)u(t)\,dt,$$

Table 1: Comparison of Nystrrom's method and Quasi-interpolation method with $N = 5$, $d = 0.01$ and $d = 0.001$.

U	$U_{0.01}$	$U_{0.001}$
0.3678794411714423	0.3678794411714423	0.3678794411714423
0.5700539682324104	0.5688183603449020	0.5700539097004954
0.7413087547119917	0.7412397588663719	0.7413087537159444
0.8806438016219256	0.8806201063495188	0.8806438005489951
0.9735879528999010	0.9735711917234211	0.9735879521333585
1.000000000000000	1.000000000000000	1.000000000000000
0.9463833924242510	0.9464118222321171	0.9463833937278310
0.8172136443885194	0.8172838167979040	0.8172136476096000
0.6395761006056206	0.6397029724920464	0.6395761064350057
0.4588103521657759	0.4590105244866359	0.4588103613711525
0.3276911166590817	0.3279828344536390	0.3276911300850199

Table 2: Comparison of Nystrrom's method and Quasi-interpolation method withe $N = 5$, $d = 0.01$ and $d = 0.001$.

U	$U_{0.01}$	$U_{0.001}$
-0.8414709848078965	-0.8414709848078965	-0.8414709848078965
-0.7261192234397584	-0.7263051621262983	-0.7261192324194392
-0.5961848616826001	-0.5962043920885245	-0.5961848615851126
-0.4516678995364214	-0.4516631119388150	-0.4516678993170746
-0.2925683370012225	-0.2925594149411874	-0.2925683366112725
-0.1188861740770033	-0.1188722329099838	-0.1188861734677065
0.06937858923623619	0.06939866451680860	0.06937859011362358
0.2722259529384959	0.2722532776259417	0.2722259541327177
0.4896559170297760	0.4896916064174603	0.4896559185895758
0.7216684815100763	0.7217136508913642	0.7216684834841979
0.9682636463793969	0.9683194110476536	0.9682636488165841

with exact solution $u(s) = s^6$, table 3 shows for $N = 5$ the solution u of the Linear system 11 and the solution u_d of the Linear system 13 with $d = 0.01$ and $d = 0.001$.

Table 3: Comparison of Nystrrom's method and Quasi-interpolation method with $N = 5$, $d = 0.01$ and $d = 0.001$.

U	$U_{0.01}$	$U_{0.001}$
1.00000000000000000	1.00000000000000000	1.00000000000000000
0.2509327736631130	0.2508541735719969	0.2509327698002908
0.01706631750959488	0.01705517886440254	0.01706631742701674
-0.04128192846055437	-0.04128620819379357	-0.04128192864635520
-0.05638076424733475	-0.05638819747905596	-0.05638076457764734
-0.06252522985074627	-0.06253684408416107	-0.06252523036685968
-0.06354380527078891	-0.06356052976687878	-0.06354380601399223
-0.05560801050746269	-0.05563077440491833	-0.05560801151904498
-0.00442280556076759	-0.00445253799826068	-0.00442280688201793
0.2222806095692964	0.2222429794530942	0.2222806078970889
0.9641847948827292	0.9641383379491463	0.9641847928182755

Example 4 Consider the Volterra integral equation

$$u(s) = 2 - e^{s+1} + \int_{-1}^{s} (s-t)u(t)\,dt,$$

with exact solution $u(s) = 1$. table 4 shows for $N = 5$ the errors Linear systems 11 and 13 for $d = 0.001$ and $d = 0.0001$.

Table 4: Absolute errors on the Nystrrom's method and Quasi-interpolation method with $N = 5$, $d = 0.001$ and $d = 0.0001$..

$\lvert U_{ex} - U\rvert$	$\lvert U_{ex} - U_{0.001}\rvert$	$\lvert U_{ex} - U_{0.0001}\rvert$
0.000000000000001	0.000000000000001	0.000000000000001
0.001171859524375	0.001172014286205	0.001171859524375
0.002485494180033	0.002485460451403	0.002485494180033
0.003901087444930	0.003901035066331	0.003901087444930
0.005353441435689	0.005353369299922	0.005353441435689
0.006741728125986	0.006741635265749	0.006741728125986
0.007915725659340	0.007915611334512	0.007915725659340
0.008657436287128	0.008657300099894	0.008657436287128
0.008656655192545	0.008656497237827	0.008656655192545
0.007478638960151	0.007478460020654	0.007478638960151
0.004521482951976	0.004521284749420	0.004521482951976

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Example 5 Consider the Volterra integral equation

$$u(s) = 1 - \sin(1+s) + \int_{-1}^{s} \cos(s-t)u(t)\,dt,$$

with exact solution $u(s) = 1$. table 5 shows for $N = 5$ the errors Linear systems 11 and 13 for $d = 0.001$ and $d = 0.0001$.

Table 5: Absolute errors on the Nystrrom's method and Quasi-interpolation method with $N = 5$, $d = 0.001$ and $d = 0.0001$..

$\lvert U_{ex} - U\rvert$	$\lvert U_{ex} - U_{0.001}\rvert$	$\lvert U_{ex} - U_{0.0001}\rvert$
0.000000000000001	0.000000000000001	0.000000000000001
0.001335070102586	0.001335230400499	0.001335070102586
0.002682218170329	0.002682233126727	0.002682218170329
0.004049035966528	0.004049057372153	0.004049035966528
0.005434556536294	0.005434583209857	0.005434556536294
0.006825980097397	0.006826010550540	0.006825980097397
0.008196526907088	0.008196559413918	0.008196526907088
0.009504665781228	0.009504698459494	0.009504665781228
0.010694868238897	0.010694899139327	0.010694868238897
0.011699926860483	0.011699954060431	0.011699926860483
0.012444759359551	0.012444781056883	0.012444759359551

5 Conclusion

In this paper, the quasi-interpolation is used to solve the linear Volterra integral equations. The approximation of the linear integral equation, gained with this method, lead to the same numerical results as Nystrom's method with the trapezoidal rule. Also approximate of quasi-interpolation method and Nystroms method are convergence to exact solution.

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