Application of Complex Analysis on Solving Some Definite Integrals

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Abstract

This paper studies two types of definite integrals and uses Maple for verification. The closed forms of the two types of definite integrals can be obtained mainly using Cauchy integral theorem and Cauchy integral formula. In addition, some examples are used to demonstrate the calculations.

Keywords: Definite integrals, closed forms, Cauchy integral theorem, Cauchy integral formula, Maple.

1 Introduction

In calculus and engineering mathematics, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. This article considers the following two types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

\[ \int_{0}^{2\pi} \frac{A}{[(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta, \] (1)

and

\[ \int_{0}^{2\pi} \frac{B}{[(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta. \] (2)
Where \( A = e^r \cos \theta [-r^2 \sin (r \sin \theta - \theta) - (b+d)r \cos (r \sin \theta) + (a+c) r \sin (r \sin \theta) + (ad+bc) \cos (r \sin \theta + \theta) - (ac-bd) \sin (r \sin \theta + \theta)] \), \( B = e^r \cos \theta [r^2 \cos (r \sin \theta - \theta) - (b+d) r \sin (r \sin \theta) - (a+c) r \cos (r \sin \theta) + (ad+bc) \sin (r \sin \theta + \theta) + (ac-bd) \cos (r \sin \theta + \theta)] \), \( r, \theta, a, b, c, d \) are real numbers, \( 0 < a^2 + b^2 < c^2 + d^2 \), and \( |r| \neq \sqrt{a^2 + b^2}, \sqrt{c^2 + d^2} \). The closed forms of the two types of definite integrals can be obtained using Cauchy integral theorem and Cauchy integral formula; these are the major results of this paper (i.e., Theorems 1 and 2). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Yu [4-27], Yu and B.-H. Chen [28], and Yu and Sheu [29-30] used some methods, for example, complex power series, integration term by term theorem, differentiation with respect to a parameter, Parsevals theorem, and area mean value theorem to evaluate some types of integrals. In this paper, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

## 2 Preliminaries and Results

Some notations, formulas and theorems used in this paper are introduced below.

### 2.1 Notations

Let \( z = a + ib \) be a complex number, where \( i = \sqrt{-1} \), and \( a, b \) are real numbers. \( a \), the real part of \( z \), denoted as \( \text{Re}(z) \); \( b \), the imaginary part of \( z \), denoted as \( \text{Im}(z) \).

### 2.2 Formulas and theorems

#### 2.2.1 Euler’s formula:

\[ e^{ix} = \cos x + i \sin x, \text{ where } x \text{ is a real number.} \]

Two important methods in complex analysis are used in this study, which can be found in [31, p 109] and [31, p121] respectively.

#### 2.2.2 Cauchy integral theorem:

Let \( f \) be analytic on a simply connected region \( D \) and \( C \) be a simple closed curve in \( D \), then \( \int_C f(z)dz = 0. \)

#### 2.2.3 Cauchy integral formula:

If \( f \) is analytic on a region \( G \), \( C \) is a simple closed curve in \( G \), and \( a \) is inside \( C \), then \( f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}dz. \)

Before the major results can be derived, a lemma is needed.
Lemma 1 Assume that $z, \lambda, \mu$ are complex numbers, $0 < |\lambda| < |\mu|$, $r$ is a real number, $n$ is a positive integer, and $C$ is a circle centered at 0 with radius $|r|$. Then the contour integral

$$\int_C \frac{e^z}{(z - \lambda)(z - \mu)} \, dz = \begin{cases} 
0 & \text{if } 0 < |r| < |\lambda|, \\
\frac{2\pi i e^\lambda}{\lambda - \mu} & \text{if } |\lambda| < |r| < |\mu|, \\
\frac{2\pi i (e^\lambda - e^\mu)}{\lambda - \mu} & \text{if } |r| > |\mu|.
\end{cases}$$

Proof. Because

$$\frac{e^z}{(z - \lambda)(z - \mu)} = \frac{1}{\lambda - \mu} \left( \frac{e^z}{z - \lambda} - \frac{e^z}{z - \mu} \right).$$

Case 1. If $0 < |r| < |\lambda|$, it follows from Cauchy integral theorem that

$$\int_C \frac{e^z}{(z - \lambda)(z - \mu)} \, dz = 0.$$

Case 2. If $|\lambda| < |r| < |\mu|$, we have

$$\int_C \frac{e^z}{(z - \lambda)(z - \mu)} \, dz = \frac{1}{\lambda - \mu} \int_C \frac{e^z}{z - \lambda} \, dz - \frac{1}{\lambda - \mu} \int_C \frac{e^z}{z - \mu} \, dz$$

(by Eq. (6))

$$= \frac{1}{\lambda - \mu} \int_C \frac{e^z}{z - \lambda} \, dz$$

(by Cauchy integral theorem)

$$= \frac{2\pi i e^\lambda}{\lambda - \mu}.$$ 

(by Cauchy integral formula)

Case 3. If $|r| > |\mu|$, then

$$\int_C \frac{e^z}{(z - \lambda)(z - \mu)} \, dz = \frac{1}{\lambda - \mu} \int_C \frac{e^z}{z - \lambda} \, dz - \frac{1}{\lambda - \mu} \int_C \frac{e^z}{z - \mu} \, dz$$

(by Cauchy integral formula)

$$= \frac{2\pi i (e^\lambda - e^\mu)}{\lambda - \mu}.$$ 

The closed form of the definite integral (1) can be obtained below.

Theorem 1 Suppose that $r, \theta, a, b, c, d$ are real numbers, $0 < a^2 + b^2 < c^2 + d^2$, $|r| \neq \sqrt{a^2 + b^2}, \sqrt{c^2 + d^2}$, and let $A = e^{r \cos \theta} [-r^2 \sin (r \sin \theta - \theta) - (b + d) r \cos (r \sin \theta) + (a + c) r \sin (r \sin \theta) + (a d + b c) \cos (r \sin \theta + \theta) - (a c - b d) \sin (r \sin \theta + \theta)]$, then the definite integral

$$\int_0^{2\pi} \frac{A}{[(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta$$

13
\[ \begin{array}{ll}
\text{Case 1.} & \text{If } |r| < \sqrt{a^2 + b^2}, \\
& \frac{2\pi e^n(b-d \cos(b-(a-c) \sin b))}{r[(a-c)^2+(b-d)^2]} i f |r| < \sqrt{a^2 + b^2}, \\
& \frac{2\pi [(b-d)(e^n \cos b - e^n \cos d) - (a-c)(e^n \sin b - e^n \sin d)]}{r[(a-c)^2+(b-d)^2]} i f |r| > \sqrt{c^2 + d^2}. 
\end{array} \tag{7} \tag{8} \tag{9} \]

**Proof.** Let \( z = re^{i\theta}, \lambda = a + ib, \) and \( \mu = c + id, \) then

\[
e^{z_{\lambda}}(z-\mu)
= [r e^{i\theta} - (a+ib)] [r e^{i\theta} - (c+id)]
= e^{r \cos \theta} e^{i r \sin \theta} \cdot \frac{e^{r \cos \theta} e^{i r \sin \theta}}{(r \cos \theta - a) + (r \sin \theta - b)] [r \cos \theta - c) + i (r \sin \theta - d)]}
= e^{r \cos \theta} e^{i r \sin \theta} [r e^{i \theta} - (a+ib)] [r e^{i \theta} - (c+id)]
= e^{r \cos \theta} \{r e^{i (r \sin \theta - 2 \theta)} - [(a+c) - i(b+d)] e^{i(r \sin \theta - \theta)} + [(ac-bd) - i(ad+bc)] e^{i(r \sin \theta)\}}
\]

Therefore, if \( r \neq 0, \) then

\[
\int_0^{2\pi} \frac{A}{[(r \cos \theta - a)^2 + (r \sin \theta - b)^2][r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta = \frac{1}{r} \text{Re} \left[ \int_C \frac{e^z}{(z-\lambda)(z-\mu)} \, dz \right]. \tag{11} \]

**Case 1.** If \( |r| < \sqrt{a^2 + b^2}, \) then Eq.(7) holds when \( r = 0, \) and hence we may assume that \( r \neq 0. \) Using Eqs. (3) and (11) yields

\[
\int_0^{2\pi} \frac{A}{[(r \cos \theta - a)^2 + (r \sin \theta - b)^2][r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta = 0.
\]

**Case 2.** If \( \sqrt{a^2 + b^2} < |r| < \sqrt{c^2 + d^2}, \) then

\[
\int_0^{2\pi} \frac{A}{[(r \cos \theta - a)^2 + (r \sin \theta - b)^2][r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta
= \frac{1}{r} \text{Re} \left[ \frac{2\pi e^n}{a+ib - (c+id)} \right] \tag{by Eqs. (4) and (11)}
= \frac{2\pi e^n}{r[(a-c)^2+(b-d)^2]} \text{Re} \left\{ [(b-d) + i(a-c)]e^{ib} \right\}
= \frac{2\pi e^n[(b-d) \cos b - (a-c) \sin b]}{r[(a-c)^2+(b-d)^2]}.
\]

14
Case 1. If $|r| > \sqrt{c^2 + d^2}$, then
\[
\int_0^{2\pi} \frac{B}{ [(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta
\]
\[= \frac{1}{r} \text{Re} \left[ \frac{2\pi i (e^{a + ib} - e^{c + id})}{(a + ib) - (c + id)} \right] \] (by Eqs. (5) and (11))
\[= \frac{2\pi}{r[(a-c)^2+(b-d)^2]} \text{Re} \left\{ [(b - d) + i(a - c)](e^{a + ib} - e^{c + id}) \right\}
\[= \frac{2\pi [(b - d)(e^a \cos b - e^c \cos d) - (a - c)(e^a \sin b - e^c \sin d)]}{r[(a-c)^2+(b-d)^2]} \] .

Next, we determine the closed form of the definite integral (2).

**Theorem 2** If the assumptions are the same as Theorem 1, and let $B = e^{r \cos \theta} \left[ r^2 \cos(r \sin \theta - \theta) - (b + d)r \sin(r \sin \theta) - (a + c)r \cos(r \sin \theta) + (ad + bc) \sin(r \sin \theta + \theta) + (ac - bd) \cos(r \sin \theta + \theta) \right]$, then the definite integral
\[
\int_0^{2\pi} \frac{B}{ [(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta
\]
\[= \begin{cases} 
2\pi e^a [(b - d) \sin b + (a - c) \cos b] & \text{if } |r| < \sqrt{a^2 + b^2}, \\
\frac{2\pi e^a [(b - d) \sin b + (a - c) \cos b]}{r[(a-c)^2+(b-d)^2]} & \text{if } \sqrt{a^2 + b^2} < |r| < \sqrt{c^2 + d^2}, \\
\frac{2\pi [(b - d)(e^a \cos b - e^c \cos d) - (a - c)(e^a \sin b - e^c \sin d)]}{r[(a-c)^2+(b-d)^2]} & \text{if } |r| > \sqrt{c^2 + d^2},
\end{cases}
\]

**Proof.** If $r \neq 0$, then using Eq. (10) yields
\[\int_0^{2\pi} \frac{B}{ [(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta = \frac{1}{r} \text{Im} \left[ \int_{C'} \frac{e^z}{(z - \lambda)(z - \mu)} \, dz \right]. \] (15)

Case 1. If $|r| < \sqrt{a^2 + b^2}$. Also, we may assume that $r \neq 0$. Using Eqs. (3) and (15) yields
\[\int_0^{2\pi} \frac{B}{ [(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta = 0.
\]

Case 2. If $\sqrt{a^2 + b^2} < |r| < \sqrt{c^2 + d^2}$, then
\[\int_0^{2\pi} \frac{B}{ [(r \cos \theta - a)^2 + (r \sin \theta - b)^2][(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} \, d\theta \]
\[= \frac{1}{r} \text{Im} \left[ \frac{2\pi i (e^{a + ib})}{(a + ib) - (c + id)} \right] \] (by Eqs. (4) and (15))
\[= \frac{2\pi e^a}{r[(a-c)^2+(b-d)^2]} \Im\{[(b-d) + i(a-c)](\cos b + i \sin b)\}
\]
\[= \frac{2\pi e^a[(b-d) \sin b + (a-c) \cos b]}{r[(a-c)^2+(b-d)^2]}.
\]

**Case 3.** If \(|r| > \sqrt{c^2 + d^2}\), we have
\[
\int_{0}^{2\pi} e^{r \cos \theta} \frac{(r \cos \theta - a)^2 + (r \sin \theta - b)^2}{[(r \cos \theta - c)^2 + (r \sin \theta - d)^2]} d\theta = 0.
\]
This follows from the integral (15) (by Eqs. (5) and (15)).

\[
= \frac{2\pi}{r[(a-c)^2+(b-d)^2]} \Im\{[(b-d) + i(a-c)](e^{a+ib} - e^{c+id})\}
\]
\[= \frac{2\pi}{r[(a-c)^2+(b-d)^2]} \Im\{[(b-d)(e^a \sin b - e^c \sin d) + (a-c)(e^a \cos b - e^c \cos d)]\}
\]
\[= \frac{2\pi}{r[(a-c)^2+(b-d)^2]} \Im\{[(b-d)(e^a \sin b - e^c \sin d) + (a-c)(e^a \cos b - e^c \cos d)]\}.
\]

**3 Examples**

In the following, for the two types of definite integrals in this study, we provide some examples and use Theorems 1 and 2 to obtain their closed forms. On the other hand, Maple is used to calculate the approximations of these definite integrals and their solutions for verifying our answers.

**3.1 Example**

In Theorem 1, let \(r = 3\), \(a = 2\), \(b = -4\), \(c = 6\), and \(d = -3\), then using Eq. (7) yields
\[
\int_{0}^{2\pi} e^{3 \cos \theta} \frac{-9 \sin (3 \sin \theta - \theta) + 21 \cos (3 \sin \theta) + 24 \sin (3 \sin \theta) - 30 \cos (3 \sin \theta)}{[(3 \cos \theta)^2 + (3 \sin \theta + 4)^2][(3 \cos \theta - 6)^2 + (3 \sin \theta + 3)^2]} d\theta = 0.
\]

Next, we use Maple to verify the correctness of Eq. (16).

\[
> \text{evalf}((\text{int}(\text{exp}(3 \text{cos(theta)})^((-9 \sin(3 \text{sin(theta)}-\text{theta})+21 \cos(3 \text{sin(theta)})+24 \sin(3 \text{sin(theta)})-30 \cos(3 \text{sin(theta)}))/((3 \cos(\theta-6)^2+(3 \sin(\theta+4)^2][(3 \cos(\theta-6)^2+(3 \sin(\theta+3)^2)))) \theta=0..2\text{Pi},18));}
\]

\[
0.
\]

If \(r = -5\), \(a = -3\), \(b = 2\), \(c = -7\), and \(d = 4\) in Theorem 1, then by Eq. (8) we obtain
\[
\int_{0}^{2\pi} e^{-5 \cos \theta} \frac{-25 \sin(-5 \sin \theta) + 30 \cos(-5 \sin \theta) + 50 \sin(-5 \sin \theta) - 26 \cos(-5 \sin \theta) - 13 \sin(-5 \sin \theta)}{[(-5 \cos \theta + 3)^2 + (-5 \sin \theta - 2)^2][(5 \cos \theta + 7)^2 + (-5 \sin \theta - 4)^2]} d\theta
\]
\[= \frac{\pi(\cos 2 + 2 \sin 2)}{25e^3}.
\]
We also use Maple to verify the correctness of Eq. (17).
\[
\text{evalf(int(exp(-5*cos(theta))*(-25*sin(-5*sin(theta)-theta)+30*cos(-5*sin(theta))+50*sin(-5*sin(theta)-26*cos(-5*sin(theta)+theta)-13*sin(-5*sin(theta)+theta))/((-5*cos(theta)+3)^2+(-5*sin(theta)-2)^2)*((-5*cos(theta)+7)^2+(-5*sin(theta)-4)^2)),theta=0..2*Pi),18);}
\]
0.00877431438238486168

\[
\text{evalf(Pi*(cos(2)+2*sin(2))/(25*exp(3)),18);}
\]
0.00877431438238486168

On the other hand, let \(r = 9, a = 2, b = -3, c = 4,\) and \(d = -5\) in Theorem 1, it follows from Eq. (9) that
\[
\int_0^{2\pi} e^{9\cos \theta} \left[ -81 \sin(9 \sin \theta - \theta) + 72 \cos(9 \sin \theta + 7) - 22 \cos(9 \sin \theta + \theta) + 7 \sin(9 \sin \theta + \theta) \right] d\theta
\]
\[
= \frac{\pi e^2 (9 \sin 3 - 9 \sin 5 - e^4 (5 \sin 5))}{18}. \tag{18}
\]
\[
\text{evalf(int(exp(9*cos(theta))*(-81*sin(9*sin(theta)-theta)+72*cos(9*sin(theta))+54*sin(9*sin(theta))-22*cos(9*sin(theta)+theta)+7*sin(9*sin(theta)+theta))/(((9*cos(theta)-2)^2+(9*sin(theta)+3)^2)*((9*cos(theta)-4)^2+(9*sin(theta)+5)^2)),theta=0..2*Pi),18);}
\]
-13.2995442906699572

\[
\text{evalf(Pi*(exp(2)*(cos(3)-sin(3))-exp(4)*(cos(5)-sin(5)))/18,18);}
\]
-13.2995442906699576

3.2 Example

If \(r = 2, a = -3, b = 2, c = -5,\) and \(d = 1\) in Theorem 2, it follows from Eq. (12) that
\[
\int_0^{2\pi} e^{2\cos \theta} \left[ 4 \cos(2 \sin \theta - \theta) + 16 \cos(2 \sin \theta) - 13 \sin(2 \sin \theta + \theta) + 13 \cos(2 \sin \theta + \theta) \right] d\theta = 0. \tag{19}
\]
Using Maple to verify the correctness of Eq. (19) as follows:
\[
\text{evalf(int(exp(2*cos(theta))*(4*cos(2*sin(theta)-theta)+16*cos(2*sin(theta))-13*sin(2*sin(theta)+theta)+13*cos(2*sin(theta)+theta))/((2*cos(theta)+3)^2+(2*sin(theta)-2)^2)*((2*cos(theta)+5)^2+(2*sin(theta)-1)^2)),theta=0..2*Pi),18);}
\]
0.

Let \(r = 6, a = 4, b = -3, c = 8,\) and \(d = -2\) in Theorem 2, then by Eq. (13) we have
\[
\int_0^{2\pi} e^{6\cos \theta} \left[ 36 \cos(6 \sin \theta - \theta) + 30 \sin(6 \sin \theta) - 72 \cos(6 \sin \theta) - 32 \sin(6 \sin \theta + \theta) + 26 \cos(6 \sin \theta + \theta) \right] d\theta
\]
\[ \pi e^{4(\sin 3 - 4 \cos 3)} \frac{1}{51}. \]  

In addition, if \( r = 11, a = -1, b = 5, c = -6, \) and \( d = 7 \) in Theorem 2, it follows from Eq. (14) that

\[ \int_{0}^{2\pi} e^{11 \cos \theta} \left[ 121 \cos(11 \sin \theta + \theta) - 132 \sin(11 \sin \theta) + 77 \cos(11 \sin \theta) - 37 \sin(11 \sin \theta + \theta) - 29 \cos(11 \sin \theta + \theta) \right] d\theta \]

\[ = 2\pi \left[ \exp(-1) \cdot (5 \cos 5 - 2 \sin 5) - \exp(-6) \cdot (5 \cos 7 - 2 \sin 7) \right] \]

\[ = \frac{2\pi \cdot \exp(-1) \cdot (5 \cos 5 - 2 \sin 5) - \exp(-6) \cdot (5 \cos 7 - 2 \sin 7)}{319}. \]  

4 Conclusion

In this study, we mainly use Cauchy integral theorem and Cauchy integral formula to solve some definite integrals. In fact, the applications of the two methods are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers.

References


