Serret Integral , 1844

Edgar Valdebenito

March 5 , 2010

Revisited 1/21/2016 1:00:03 PM

Abstract

In this note we show some formulas related to an integral of Serret , 1844.

Algunas Fórmulas Relacionadas con la Integral de Serret

Edgar Valdebenito

Marzo 5 , 2010

Resumen

Se muestran algunas fórmulas relacionadas con la integral de Serret (1844) :

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$$

1.Introducción.

Teorema. (Serret). $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$

Proof:

Considerando el cambio de variable: $x = \tan\phi$ se tiene,

$$\begin{aligned}\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx &= \int_0^{\pi/4} \ln(1+\tan\phi)\,d\phi = \int_0^{\pi/4} \ln\left(\sqrt{2}\cos\left(\frac{\pi}{4}-\phi\right)\frac{1}{\cos\phi}\right)d\phi \\ &= \frac{\pi}{8}\ln 2 + \int_0^{\pi/4} \ln\cos\left(\frac{\pi}{4}-\phi\right)d\phi - \int_0^{\pi/4} \ln\cos(\phi)\,d\phi\end{aligned} \tag{1}$$

Por simetría con respecto al punto $\phi = \frac{\pi}{8}$, se tiene,

$$\int_0^{\pi/4} \ln\cos\left(\frac{\pi}{4}-\phi\right)d\phi = \int_0^{\pi/4} \ln\cos(\phi)\,d\phi \tag{2}$$

Y el teorema queda probado.

En esta nota se muestran algunas fórmulas relacionadas con la integral de Serret.

2.Fórmulas.

$$\int_1^2 \frac{\ln x}{2-2x+x^2}\,dx = \frac{\pi}{8}\ln 2 \tag{3}$$

$$\int_0^{\ln 2} \frac{x\,e^x}{2-2e^x+e^{2x}}\,dx = \frac{\pi}{8}\ln 2 \tag{4}$$

$$\int_1^{\infty} \frac{1}{1+x^2}\ln\left(1+\frac{1}{x}\right)dx = \frac{\pi}{8}\ln 2 \tag{5}$$

$$\int_0^{\ln\left(\sqrt{2}+1\right)} \frac{\ln\left(1+\sinh x\right)}{\cosh x}\,dx = \frac{\pi}{8}\ln 2 \tag{6}$$

$$\int_0^{\infty} \frac{x-\ln\cosh x}{\cosh\left(2x\right)}\,dx = \frac{\pi}{8}\ln 2 \tag{7}$$

$$\int_{\ln\left(\sqrt{2}+1\right)}^{\infty} \frac{1}{\cosh x}\ln\left(1+\frac{1}{\sinh x}\right)dx = \frac{\pi}{8}\ln 2 \tag{8}$$

$$\begin{cases} \pi\ln 2 = 32\displaystyle\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{c_k\,3^{-2n+k-2}}{\left(2n-2k+1\right)\left(2n-k+2\right)} \\ c_{n+2} = 2\,c_{n+1}-5\,c_n\ ,c_0 = 1, c_1 = 2 \end{cases} \tag{9}$$

$$\frac{\pi}{8}\ln 2 = \frac{1}{1+a}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\sum_{s=0}^{m}\binom{n}{k}\binom{k}{m}\binom{m}{s}\left(\frac{a}{1+a}\right)^n\frac{\left(-1/a\right)^k}{\left(k-m+s+1\right)\left(k+m+s+2\right)}\quad ,a>1 \tag{10}$$

$$\frac{\pi}{8}\ln\left(\frac{2}{\left(1+a\right)^2}\right) = \frac{1}{1+a}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\sum_{s=0}^{k+1}\binom{n-k}{m}\binom{k+1}{s}\frac{\left(-1\right)^{m+s+1}\left(a/\left(1+a\right)\right)^{n-m-s+1}\left(1/\left(1+a\right)\right)^{m+s}}{\left(k+1\right)\left(2m+s+1\right)}\quad ,a>0 \tag{11}$$

$$\frac{\pi}{8}\ln 2 = \sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{\left(-1\right)^{k+n}\,n!}{2^{n+2}\left(k-n+1\right)\left(\frac{k-n+2}{2}\right)_{n+1}} \tag{12}$$

$$\frac{\pi}{8}\ln 2 = \int_a^1\frac{\ln\left(1+x\right)}{1+x^2}\,dx + \sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{2n+1}\left(\left(1+a^{2n+1}\right)\ln\left(1+a\right) - \sum_{k=0}^{2n}\frac{\left(-1\right)^k a^{k+1}}{k+1}\right)\quad ,0\le a\le 1 \tag{13}$$

$$\begin{cases} \dfrac{\pi}{8}\ln 2 = \displaystyle\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{c_k}{\left(n-k+1\right)\left(n+2\right)2^{n+2}} \\ c_{n+2} = 2\left(c_{n+1}-c_n\right), c_0 = 1, c_1 = 2 \end{cases} \tag{14}$$

$$\frac{\pi}{8}\ln 2 = -\int_0^{1/2} \frac{\ln(1-x)}{1-2x+2x^2}\,dx \tag{15}$$

$$\frac{\pi}{8}\ln 2 = -\int_{1/2}^{1} \frac{\ln x}{1-2x+2x^2}\,dx \tag{16}$$

$$\frac{\pi}{8}\ln 2 = \int_0^{1} \frac{\ln(2-x)}{2-2x+x^2}\,dx \tag{17}$$

$$\frac{\pi}{8}\ln 2 = \int_0^{\infty} \frac{e^{-x}\ln(1+e^{-x})}{1+e^{-2x}}\,dx \tag{18}$$

$$\left(\frac{\ln 2}{8}+\frac{\ln(b-a)}{4}\right)\pi = (b-a)\int_a^{b} \frac{\ln(b-2a+x)}{a^2+(b-a)^2-2ax+x^2}\,dx \quad ,a<b \tag{19}$$

$$\begin{cases} \dfrac{\pi}{8}\ln 2 = \displaystyle\sum_{n=0}^{\infty}\frac{c_n a^{n+2}}{n+2}+\int_a^1 \frac{\ln(1+x)}{1+x^2}\,dx \quad ,0\le a\le 1 \\ c_{n+2} = \dfrac{(-1)^n}{n+3}-c_n \quad ,c_0=1, c_1=-1/2 \end{cases} \tag{20}$$

$$\frac{\pi}{8}\ln 2 = \frac{2}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}\sum_{k=0}^{n}\binom{n}{k}(-2)^k\sum_{m=0}^{2k}\binom{2k}{m}(-1)^m\left(\frac{2^{m+1}\ln 2}{m+1}-\frac{2^{m+1}-1}{(m+1)^2}\right) \tag{21}$$

$$\frac{\pi}{4}\ln 2 = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\ln 2+\sum_{k=1}^{n-1}\frac{(-1)^k}{k}\right)-\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+3}\sum_{n=0}^{k}\frac{(-1)^n}{n+1} \tag{22}$$

$$\begin{cases} \dfrac{\pi}{8}\ln 2 = \displaystyle\sum_{n=0}^{\infty}(-1)^n I_n = \frac{1}{2}I_0+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(I_n-I_{n+1}\right) \\ I_n = \dfrac{1}{2n(2n+1)^2}+\dfrac{2n-1}{2n+1}I_{n-1} \quad ,I_0 = 2\ln 2-1 \end{cases} \tag{23}$$

$$\frac{\pi}{8}\ln 2 = \int_0^1\int_a^1 \frac{x}{(1+x^2)(1+yx)}\,dxdy+\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+2}}{n+2}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{n-2k+1} \quad ,0<a<1 \tag{24}$$

$$\frac{\pi}{8}\ln 2 = \int_a^1 \frac{\ln(1+x)}{1+x^2}\,dx+\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+2}}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}-\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+3}}{2n+3}\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\right) \quad ,0\le a\le 1 \tag{25}$$

$$\begin{cases} \frac{\pi}{8}\ln 2-\tan^{-1}\left(\frac{1-a}{1+a}\right)\ln\left(1+a\right)=a\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(-1\right)^{n}a^{2n-k+1}}{\left(k+1\right)\left(2n-k+2\right)}+ \\ \qquad +\frac{1-a}{1+a^{2}}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(-1\right)^{k}c_{n-k}}{\left(k+1\right)\left(n+2\right)}\left(\frac{1-a}{1+a}\right)^{k+1} \\ c_{n+2}=-\left(\frac{2a\left(1-a\right)}{1+a^{2}}c_{n+1}+\frac{\left(1-a\right)^{2}}{1+a^{2}}c_{n}\right)\quad ,c_{0}=1,c_{1}=-\frac{2a\left(1-a\right)}{1+a^{2}}\quad ,0\le a\le 1 \end{cases} \tag{26}$$

$$\frac{\pi}{8}\ln 2=\int_{1/2}^{1}\frac{\ln\left(1+x\right)}{1+x^{2}}dx+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\frac{\left(-1\right)^{k+m}\left(\left(n-k-m+2\right)^{-2m-k-2}-\left(n-k-m+3\right)^{-2m-k-2}\right)}{\left(k+1\right)\left(k+2m+2\right)} \tag{27}$$

$$\begin{cases} \pi\ln 2=2\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}c_{n}}{n+2} \\ c_{n+2}=\frac{\left(-1\right)^{n}}{\left(n+3\right)2^{n+2}}-c_{n+1}-\frac{1}{2}c_{n}\quad ,c_{0}=1,c_{1}=-5/4 \end{cases} \tag{28}$$

$$\begin{cases} \frac{\pi}{8}\ln 2=\sum_{n=0}^{\infty}\frac{c_{2n}}{\left(2n+1\right)2^{2n}} \\ c_{n}=\frac{8}{15n}\left(\left(1-n\right)c_{n-3}-\left(\frac{5}{2}n-1\right)c_{n-2}-\left(\frac{11}{4}n-\frac{5}{4}\right)c_{n-1}\right),n\ge 3 \\ c_{0}=\frac{4}{5}\ln\frac{3}{2},c_{1}=\frac{8}{15}-\frac{16}{25}\ln\frac{3}{2},c_{2}=-\frac{136}{225}-\frac{16}{125}\ln\frac{3}{2} \end{cases} \tag{29}$$

$$\begin{cases} \pi\ln\left(\frac{\sqrt[4]{3}}{\sqrt[8]{8}}\right)=\frac{4}{45}\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{\left(-1\right)^{k}c_{k}}{\left(2n-k+2\right)\left(2n+3\right)10^{k}3^{2n-k}} \\ c_{n+2}=-4c_{n+1}-20c_{n}\quad ,c_{0}=1,c_{1}=-4 \end{cases} \tag{30}$$

$$\begin{aligned} \frac{\pi}{8}\ln 2=a\int_{0}^{1}\frac{\ln\left(1+ax\right)}{1+a^{2}x^{2}}dx+\left(b-a\right)\int_{0}^{1}\frac{\ln\left(1+a+\left(b-a\right)x\right)}{1+a^{2}+2a\left(b-a\right)x+\left(b-a\right)^{2}x^{2}}dx+ \\ +\left(1-b\right)\int_{0}^{1}\frac{\ln\left(1+b+\left(1-b\right)x\right)}{1+b^{2}+2b\left(1-b\right)x+\left(1-b\right)^{2}x^{2}}dx\quad ,0\le a\le b\le 1 \end{aligned} \tag{31}$$

$$\frac{\pi}{8}\ln 2=-1+\ln 2+\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(2n+3\right)\left(2n+5\right)}\sum_{k=0}^{n}\frac{\left(-1\right)^{k}}{k+1}+\sum_{n=0}^{\infty}\frac{a^{n+2}}{n+2}\sum_{k=0}^{[n/2]}\frac{\left(-1\right)^{k}}{n-2k+1}-\int_{a}^{1}\frac{\ln\left(1-x\right)}{1+x^{2}}dx\ ,0<a<1 \tag{32}$$

$$\frac{\pi}{8}\ln 2=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}=\frac{1}{4}+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+2}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}-\frac{1}{n+3}\sum_{k=0}^{[(n+1)/2]}\frac{(-1)^k}{2k+1}\right) \tag{33}$$

$$\frac{\pi}{8}\ln 2=\sum_{n=0}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n+3}\right)\sum_{k=0}^{n}\frac{(-1)^k}{2k+1} \tag{34}$$

$$\frac{\pi}{8}\ln 2=\frac{i}{2}\left(\int_1^{1+i}\frac{1}{x}\ln\left(1-\left(\frac{1+i}{2}\right)x\right)dx-\int_1^{1-i}\frac{1}{x}\ln\left(1-\left(\frac{1-i}{2}\right)x\right)dx\right) \tag{35}$$

$$\frac{\pi}{8}\ln 2=\tan^{-1}(a)\ln(1+a)+\int_a^1\frac{\tan^{-1}x}{1+x}dx-\int_0^a\frac{\ln(1+x)}{1+x^2}dx \quad ,0\le a\le 1 \tag{36}$$

$$G=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a^{2n-1}}{(2n-1)^2}-\frac{1}{2i}\left(\int_{a(1+i)}^{1+i}\frac{1}{x}\ln\left(1-\left(\frac{1+i}{2}\right)x\right)dx-\int_{a(1-i)}^{1-i}\frac{1}{x}\ln\left(1-\left(\frac{1-i}{2}\right)x\right)dx\right) \ ,0\le a\le 1 \tag{37}$$

$$\frac{\pi}{8}\ln 2=\int_1^{\infty}\frac{\ln\left(2-x^{-1}\right)}{1-2x+2x^2}dx \tag{38}$$

$$\frac{\pi}{8}\ln 2=\int_1^{\infty}\frac{1}{1+x^2}\ln\left(1+\frac{x-1}{x+1}\right)dx=\int_1^{\infty}\frac{1}{1+x^2}\ln\left(\frac{2x}{1+x}\right)dx \tag{39}$$

$$\frac{\pi}{8}\ln 2=\int_0^{1}\frac{1}{1+x^2}\ln\left(1+\frac{1-x}{1+x}\right)dx=\int_0^{1}\frac{1}{1+x^2}\ln\left(\frac{2}{1+x}\right)dx \tag{40}$$

$$\frac{\pi}{4}\ln 2=\int_{1/2}^{1}\frac{1}{\sqrt{x(1-x)}}\ln\left(1+\sqrt{\frac{1-x}{x}}\right)dx \tag{41}$$

$$\frac{\pi}{8}\ln 2=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\left(\frac{1}{4n-3k+2}-\frac{1}{4n-3k+4}\right) \tag{42}$$

3.La Función Dilogaritmo.

Se define la función dilogarítmo $Li_2(z)$ por:

$$Li_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2} \quad ,z\in\mathbb{C},|z|\le 1 \tag{43}$$

Una relación importante es:

$$\frac{1}{2i}\left(Li_2(i)-Li_2(-i)\right)=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}=G \tag{44}$$

El número G se denomina constante de Catalan.

La integral de Serret se puede escribir como:

$$\frac{\pi}{8}\ln 2=\int_0^1\frac{\ln(1+x)}{1+x^2}dx=\underbrace{\frac{1}{2}\int_0^1\frac{\ln(1+x)}{1+ix}dx}_{I_1}+\underbrace{\frac{1}{2}\int_0^1\frac{\ln(1+x)}{1-ix}dx}_{I_2} \tag{45}$$

En I_1 con el cambio de variable $u=1+ix$ y usando (43), se tiene:

$$I_1=\frac{1}{2i}\left(\ln(1+i)\right)^2+\frac{1}{2i}Li_2\left(\frac{1+i}{2}\right)-\frac{1}{2i}Li_2(i) \tag{46}$$

En I_2 con el cambio de variable $u=1-ix$ y usando (43), se tiene:

$$I_2=-\frac{1}{2i}\left(\ln(1-i)\right)^2-\frac{1}{2i}Li_2\left(\frac{1-i}{2}\right)+\frac{1}{2i}Li_2(-i) \tag{47}$$

De (44),(45),(46),(47) , se tiene:

$$\frac{\pi}{8}\ln 2=G-\frac{1}{2i}\left(Li_2\left(\frac{1+i}{2}\right)-Li_2\left(\frac{1-i}{2}\right)\right) \tag{48}$$

poniendo

$$c_n=\frac{1}{2i}\left((1+i)^n-(1-i)^n\right)\quad ,n\in\mathbb{N} \tag{49}$$

Se tiene

$$c_{n+2}=2\left(c_{n+1}-c_n\right)\quad ,c_1=1,c_2=2,c_{4n}=0 \tag{50}$$

La fórmula (48) se transforma en:

$$G-\frac{\pi}{8}\ln 2=\sum_{n=1}^{\infty}\frac{c_n}{n^2 2^n} \tag{51}$$

Utilizando (50), $c_{4n}=0$, se tiene

$$G-\frac{\pi}{8}\ln 2=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^{2n}}\left(\frac{2}{(4n-3)^2}+\frac{2}{(4n-2)^2}+\frac{1}{(4n-1)^2}\right) \tag{52}$$

Referencias

[1] Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions. Nueva York: Dover , 1965.

[2] Boros, G., Moll, V.: Irresistible Integrals.Cambridge University Press,2004.

[3] Gradshteyn, I.S., and Ryzhik, I.M.: Table of Integrals,Series and Products. 5th ed.,ed. Alan Jeffrey. Academic Press, 1994.

[4] Spiegel,M.R.: Mathematical Handbook, McGraw-Hill Book Company , New York , 1968.

[5] Valdebenito, E.:, Pi Handbook , manuscript , unpublished , 1989 , (20000 formulas).