Integrals , Mathematical Constants

Edgar Valdebenito

June - 2006

Abstract

A collection of integral formulas involving constants classic shows

Introduction

Recall the definition of the constants: π , e , γ , G , $\ln 2$:

(1) $$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.141592 \ldots$$

(2) $$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281 \ldots$$

(3) $$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = 0.577215 \ldots$$

(4) $$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965 \ldots$$

(5) $$\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 0.693147 \ldots$$

In this note we show integral formulas involving constants : π , e , γ , G , $\ln 2$.

Formulas

(6) $$e = 2 + \int_1^{\infty} \left(1 + \frac{1}{x}\right)^x \left(\ln\left(1 + \frac{1}{x}\right) - \frac{1}{1+x}\right) dx$$

(7) $$e = \left(1 + \frac{1}{m}\right)^m + \int_m^{\infty} \left(1 + \frac{1}{x}\right)^x \left(\ln\left(1 + \frac{1}{x}\right) - \frac{1}{1+x}\right) dx \quad , m \in \mathbb{N}$$

(8) $$e = \left(1 + \frac{1}{m}\right)^m + \int_0^{1/m} (1 + x)^{1/x} \left(\frac{\ln(1+x)}{x^2} - \frac{1}{x(1+x)}\right) dx \quad , m \in \mathbb{N}$$

(9) $e = \left(1+\frac{1}{m}\right)^m + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{n+1} \int_m^{\infty} \left(1+\frac{1}{x}\right)^x x^{-n-1}\, dx \quad , m \in \mathbb{N} - \{1\}$

(10) $e = \left(1+\frac{1}{m}\right)^m + \int_0^{1/m} \int_0^1 \frac{(1+x)^{1/x}}{x(1+x\,y)} dy\, dx - \int_0^{1/m} \frac{(1+x)^{1/x}}{x} dx \quad , m \in \mathbb{N}$

(11) $e = \left(1+\frac{1}{m}\right)^m + \int_m^{\infty} \int_1^{\infty} \left(1+\frac{1}{x}\right)^x \frac{1}{y(1+x\,y)^2} dy\, dx \quad , m \in \mathbb{N}$

(12) $\pi = \cosh 1 \int_{-1}^{1} \frac{\cos x}{1+x^2} dx + \sinh 1 \int_{-1}^{1} \frac{x \sin x}{1+x^2} dx + \cos 1 \int_{-1}^{1} \frac{e^x}{1+x^2} dx - \sin 1 \int_{-1}^{1} \frac{x e^x}{1+x^2} dx$

(13) $\pi = 2\cosh 1 \int_0^1 \frac{\cos x}{1+x^2} dx + 2\sinh 1 \int_0^1 \frac{x \sin x}{1+x^2} dx + 2\cos 1 \int_0^1 \frac{\cosh x}{1+x^2} dx - 2\sin 1 \int_0^1 \frac{x \sinh x}{1+x^2} dx$

(14) $\sqrt{\pi} = 2\sqrt{a}\, u\, e^{-au^2} + 2\sqrt{a} \int_u^{\infty} e^{-ax^2} dx + 2\int_v^1 \sqrt{-\ln x}\, dx \quad , a > 0 , u \geq 0 , v = e^{-au^2}$

(15) $\pi \ln 2 = 2\int_u^{\pi/2} (-\ln(\sin x))\, dx + 2\int_v^{\infty} \sin^{-1}(e^{-x})\, dx \quad , 0 \leq u \leq {}^{\pi}/_{2} , v = -\ln(\sin u)$

(16) $\pi^2 = 4\,u\,v + 4\int_u^{\infty} \ln\left(\frac{e^x+1}{e^x-1}\right) dx + 4\int_v^{\infty} \ln\left(\frac{e^x+1}{e^x-1}\right) dx \quad , u \geq 0, v = \ln\left(\frac{e^u+1}{e^u-1}\right)$

(17) $\pi = 2\,u\,v + 2\int_u^{\infty} \frac{1}{\cosh x} dx + 2\int_v^1 \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) dx \quad , u \geq 0, v = \frac{1}{\cosh u}$

(18) $\pi^3 = 16\,u\,v + 16\int_u^{\pi/4} (\ln \tan x)^2\, dx + 16\int_v^{\infty} \tan^{-1}\left(e^{\sqrt{x}}\right) dx \quad , 0 \leq u \leq {}^{\pi}/_{4} , v = (\ln \tan u)^2$

(19) $\pi^3 = -16\,u\,v + 16\int_0^{u} (\ln \tan x)^2\, dx + 16\int_0^{v} \tan^{-1}\left(e^{\sqrt{x}}\right) dx \quad , 0 \leq u \leq {}^{\pi}/_{4} , v = (\ln \tan u)^2$

(20) $\pi \frac{(2m)!}{2^{2m+1}(m!)^2} = u\,v + \int_u^{\pi/2} (\sin x)^{2m}\, dx - \int_0^{v} \sin^{-1}\left(\sqrt[2m]{x}\right) dx \quad , m \in \mathbb{N}, 0 \leq u \leq {}^{\pi}/_{2} , v = (\sin u)^{2m}$

(21) $G = u\,v - \int_u^{\pi/4} \ln \tan x\, dx + \int_v^{\infty} \tan^{-1}(e^{-x})\, dx \quad , 0 \leq u \leq {}^{\pi}/_{4} , v = -\ln \tan u$

(22) $G = -u\,v - \int_0^u \ln\tan x\,dx + \int_0^v \tan^{-1}(e^{-x})\,dx \quad ,0 \le u \le {}^{\pi}/_{4}, v = -\ln\tan u$

(23) $\pi\ln 2 + 2G = 4\,u\,v - 4\int_u^{\pi/4} \ln\sin x\,dx + 4\int_v^{\infty} \sin^{-1}(e^{-x})\,dx \quad ,0 \le u \le {}^{\pi}/_{4}, v = -\ln\sin u$

(24) $\pi^2 = 12\,u\,v + 12\int_u^{\infty} \ln(1+e^{-x})\,dx - 12\int_v^{\ln 2} \ln(e^x - 1)\,dx \quad ,u \ge 0, v = \ln(1+e^{-u})$

(25) $\pi^2 = -12\,u\,v - 12\int_0^{v} \ln(e^x - 1)\,dx + 12\int_0^{u} \ln(1+e^{-x})\,dx \quad ,u \ge 0, v = \ln(1+e^{-u})$

(26) $\pi(a-b) = u\,v + \int_u^{\infty} \ln\left(\frac{a^2+x^2}{b^2+x^2}\right)dx + \int_v^{2\ln(a/b)} \sqrt{\frac{a^2-b^2e^x}{e^x-1}}\,dx \quad ,u \ge 0, v = \ln\left(\frac{a^2+u^2}{b^2+u^2}\right)$

(27) $\pi\frac{(-1)^n}{4} + \sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1} = u\,v + \int_u^{\pi/4}(\tan x)^{2n}\,dx - \int_0^{v} \tan^{-1}\left(\sqrt[2n]{x}\right)dx \quad ,n \in \mathbb{N}, 0 \le u \le {}^{\pi}/_{4}, v = (\tan u)^{2n}$

(28) $\pi = 4\int_0^{\infty} \frac{1}{(1+x^2)^2}\,dx = 4\int_0^{1} \sqrt{\frac{1}{\sqrt{x}} - 1}\,dx$

(29) $\pi = 4\,u\,v + 4\int_u^{\infty} \frac{1}{(1+x^2)^2}\,dx + 4\int_v^{1} \sqrt{\frac{1}{\sqrt{x}} - 1}\,dx \quad ,u \ge 0, v = (1+u^2)^{-2}$

(30) $\pi = 4\,u\,v + 4\int_u^{\pi/2}(1+(\tan x)^m)^{-1}\,dx + 4\int_v^{1} \tan^{-1}\left(\sqrt[m]{\frac{1}{x} - 1}\right)dx \quad ,0 \le u \le {}^{\pi}/_{2}, v = (1+(\tan u)^m)^{-1}$

(31) $\pi\ln 2 = 8\int_0^{\pi/4} \ln(1+\tan x)\,dx = 8\int_0^{\ln 2} \tan^{-1}(e^x - 1)\,dx$

(32) $\pi\ln 2 = 8\,u\,v - 8\int_0^{v} \tan^{-1}(e^x - 1)\,dx + 8\int_u^{\pi/4} \ln(1+\tan x)\,dx \quad ,0 \le u \le {}^{\pi}/_{4}, v = \ln(1+\tan u)$

(33) $\pi + 2\ln 2 - 4 = 2\,u\,v + 2\int_u^{1} \ln(1+x^2)\,dx - 2\int_0^{v} \sqrt{e^x - 1}\,dx \quad ,0 \le u \le 1, v = \ln(1+u^2)$

(34) $\pi - 4\tan^{-1}\left(\frac{1}{n}\right) + 2 = 4\,u\,v + 4\int_u^n \frac{1}{1+x^2}\,dx + 4\int_v^{1/2}\sqrt{\frac{1}{x}-1}\,dx \quad ,n \in \mathbb{N} - \{1\}, 1 \le u \le n, v = (1+u^2)^{-1}$

(35) $\gamma = u\,v + (1-z)w - \int_{-\infty}^{v} e^{-e^{-x}}\,dx - \int_u^z \ln\left(\ln\frac{1}{x}\right)dx + \int_w^{\infty}\left(1-e^{-e^{-x}}\right)dx \quad ,0 \le u \le e^{-1} \le z \le 1, v = -\ln\left(\ln\frac{1}{u}\right), w = -\ln\left(\ln\frac{1}{z}\right)$

(36) $\pi = -3\,u\,v + 3\int_0^u \frac{1}{1+x^6}\,dx + 3\int_0^v \sqrt[6]{\frac{1}{x}-1}\,dx \quad ,u \ge 0, v = (1+u^6)^{-1}$

(37) $\pi = 3\,u\,v + 3\int_u^{\infty} \frac{1}{1+x^6}\,dx + 3\int_v^1 \sqrt[6]{\frac{1}{x}-1}\,dx \quad ,u \ge 0, v = (1+u^6)^{-1}$

(38) $\pi = \sqrt{2}\,u\,v - \sqrt{2}\int_0^v \tan^{-1}(x^2)\,dx + \sqrt{2}\int_u^{\pi/2}\sqrt{\tan x}\,dx \quad ,0 \le u < \pi/2, v = \sqrt{\tan u}$

(39) $\pi = \sqrt{2}\left(\frac{\pi}{2} - u\right)v + \sqrt{2}\int_0^u \sqrt{\tan x}\,dx + \sqrt{2}\int_v^{\infty}\tan^{-1}(x^{-2})\,dx \quad ,0 \le u \le \pi/2, v = \sqrt{\tan u}$

(40) $\pi\sqrt{2} = 4\,u\,v + 4\int_u^{\infty}\frac{1}{1+x^4}\,dx + 4\int_v^1 \sqrt[4]{\frac{1}{x}-1}\,dx \quad ,u \ge 0, v = (1+u^4)^{-1}$

(41) $\pi\sqrt{2} = -4\,u\,v + 4\int_0^u \frac{1}{1+x^4}\,dx + 4\int_0^v \sqrt[4]{\frac{1}{x}-1}\,dx \quad ,u \ge 0, v = (1+u^4)^{-1}$

(42) $\ln 2 = 2\,u\,v + 2\int_u^{\pi/4}\tan x\,dx - 2\int_0^v \tan^{-1}(x)\,dx \quad ,0 \le u \le \pi/4, v = \tan u$

(43) $\pi - 2\ln 2 = 4\,u\,v - 4\int_0^u \tan x\,dx + 4\int_v^1 \tan^{-1}(x)\,dx \quad ,0 \le u \le \pi/4, v = \tan u$

(44) $\pi\ln\left(1+\sqrt{2}\right) - 2G = 2\,u\,v + 2\int_v^{\ln(1+\sqrt{2})}\sin^{-1}(\sinh x)\,dx - 2\int_0^u \sinh^{-1}(\sin x)\,dx \quad ,0 \le u \le \pi/2, v = \sinh^{-1}(\sin u)$

(45) $G = u\,v + \int_u^{\pi/2}\sinh^{-1}(\sin x)\,dx - \int_0^v \sin^{-1}(\sinh x)\,dx \quad ,0 \le u \le \pi/2, v = \sinh^{-1}(\sin u)$

References

1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York:Dover , 1965.
2. I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals,Series, and Products (A.Jeffrey), Academic Press,New York, London, and Toronto, 1980.
3. M.R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York,1968.
4. E. Valdebenito, Pi Handbook, manuscript,unpublished,1989, (20000 formulas).