Infinity Product for Constant $e = 2.718281 \ldots$

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Abstract

In this note we show an infinite product for the constant e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718281 \ldots$$

Resumen

En esta nota mostramos un producto infinito para la constante e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718281 \ldots$$

Keywords: constant e , infinity product , recurrence.

1. Introducción

Sea $f(n)$ la sucesión definida por:

$$f(n) = n!\sum_{k=0}^{n}\frac{1}{k!} \; , n = 0,1,23, \ldots$$

La sucesión $f(n)$ satisface la recurrencia:

$$f(n+1)=(n+1)f(n)+1 \ , f(0)=1$$

Algunos valores de $f(n)$ son:

$$\{f(n): n \in \mathbb{N} \cup \{0\}\} = \{1,2,5,16,65,326,1957, \dots\}$$

La sucesión $f(n)$, aparece en un producto infinito para la constante e

2. Producto Infinito

Sea $m \in \mathbb{N} = \{1,2,3,4, \dots\}$, se tiene:

$$e = \prod_{n=1}^{\infty} \frac{f(mn)\big(m(n-1)\big)!}{f\big(m(n-1)\big)(mn)!} = \left(\frac{f(m)}{m!}\right)\left(\frac{f(2m)m!}{f(m)(2m)!}\right)\left(\frac{f(3m)(2m)!}{f(2m)(3m)!}\right) \dots$$

El producto infinito se puede escribir como:

$$e = \prod_{n=1}^{\infty} \frac{f(mn)}{f\big(m(n-1)\big)(m(n-1)+1)_m}$$

donde $(x)_m = x(x+1)(x+2)\dots(x+m-1)$.

Otra forma equivalente es:

$$e = \prod_{n=1}^{\infty} \frac{f(mn)}{f\big(m(n-1)\big)} \binom{mn}{m(n-1)}^{-1} \frac{1}{m!}$$

donde $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

3. Ejemplos

Ejemplo 1: $m = 1$

$$e = \frac{2}{1!}\left(\frac{5 \cdot 1!}{2 \cdot 2!}\right)\left(\frac{16 \cdot 2!}{5 \cdot 3!}\right) \dots = \frac{2}{1_1}\left(\frac{5}{2 \cdot 2_1}\right)\left(\frac{16}{5 \cdot 3_1}\right) \dots$$

Ejemplo 2 : $m = 2$

$$e = \frac{5}{2!}\left(\frac{65 \cdot 2!}{5 \cdot 4!}\right)\left(\frac{1957 \cdot 4!}{65 \cdot 6!}\right) \dots = \frac{5}{1_2}\left(\frac{65}{5 \cdot 3_2}\right)\left(\frac{1957}{65 \cdot 5_2}\right) \dots$$

Ejemplo 3 : $m = 3$

$$e = \frac{16}{3!}\left(\frac{1957 \cdot 3!}{16 \cdot 6!}\right)\left(\frac{986410 \cdot 6!}{1957 \cdot 9!}\right) \dots = \frac{16}{1_3}\left(\frac{1957}{16 \cdot 4_3}\right)\left(\frac{986410}{1957 \cdot 7_3}\right) \dots$$

Referencias

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