The distribution of prime numbers in an interval

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May 16, 2015

Abstract

The Goldbach theorem and the twin prime theorem are homologous. the paper from the prime origin, derived the equations of the twin prime theorem and the Goldbach theorem, and new prime number theorem.

Keywords the Goldbach theorem; the twin prime theorem; prime number theorem

Mathematics Subject Classifications (2010) 11A41;11R04

Notation

p : a prime number.

p : an odd prime.

$\pi'(\mathrm{p}^2)$: the number of primes in the open interval $(\mathrm{p},\ \mathrm{p}^2)$.

$T'(\mathrm{p}^2)$: the number of twin prime pairs $(p, p+2)$ in the open interval $(\mathrm{p},\ \mathrm{p}^2)$.

$G(x)$: the number of prime p in the open interval $(\mathrm{p},\ \mathrm{p}^2)$. p is the largest prime number less than $\sqrt{x}$, and $x-p$ is prime number, x is a large even integer.

$\pi(\mathrm{p})$: denotes not more than p of prime numbers.

$p|x$: p divides x.

$\sim$: denotes equialence relation. $f(x) \sim g(x)$, namely: $Lim \dfrac{f(x)}{g(x)} = 1$, when x tends to infinity.

$\mathcal{O}$: mean big O notation describes the limiting behavior of a function when the argument tends towards a particular value or infinity, usually in terms of simpler functions.

$Li(x)$: express the logarithmic integral function or integral logarithm $Li(x)$ is a special function such as $Li(x) = \int_2^x \frac{dt}{\ln t}$.

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1. Prime number theorem [1] [2]

Let $\pi'(\mathrm{p}^2)$ is the number of primes in the open interval $(\mathrm{p},\ \mathrm{p}^2)$, p is an odd prime,

$$\pi'(\mathrm{p}^2) > \frac{1}{2}\left(\ \mathrm{p}^2 \cdot \prod_{2\le p\le \mathrm{p}}\left(1-\frac{1}{p}\right) - \pi(\mathrm{p}) - 1\ \right) \qquad (1)$$

where p is a prime number.

$\pi(\mathrm{p})$ is not more than p of prime numbers.

Lemma 1

Let $\pi'(\mathrm{p}^2)$ is the number of primes in the open interval $(\mathrm{p},\ \mathrm{p}^2)$, p is an odd prime,

Let $\pi'_k(\mathrm{p}^2)$ is the number of odd p_k between $k\,\mathrm{p}^2$ to $(k+1)\,\mathrm{p}^2$, $k \ge 1$

and $(p_k,\ p) = 1, 3 \le p \le \mathrm{p}$, p is a prime number, p is an odd prime,

Let $g(\mathrm{p}^2) = \pi'_k(\mathrm{p}^2) - \pi'(\mathrm{p}^2)$

$$\left|g(\mathrm{p}^2)\right| \le \pi'(\mathrm{p}^2) + \pi(\mathrm{p}) + 1 \qquad (2)$$

where $\pi(\mathrm{p})$ is not more than p of prime numbers.

The proof of lemma 1

Reduction to absurdity.

The proof of prime number theorem

Proof

By lemma 1 and Chinese remainder theorem, it can be derived

$$1 + \pi'(\mathrm{p}^2) + (\pi'(\mathrm{p}^2) + g(\mathrm{p}^2))\cdot\left(\mathrm{p}\cdot\prod_{2\le p\le \mathrm{p}} p \cdot \frac{1}{\mathrm{p}^2} - 1\right) = \mathrm{p}\cdot\prod_{2\le p\le \mathrm{p}}(p-1) \qquad (3)$$

Hence proving

$$\pi'(\mathrm{p}^2) > \frac{1}{2}\left(\ \mathrm{p}^2 \cdot \prod_{2\le p\le \mathrm{p}}\left(1-\frac{1}{p}\right) - \pi(\mathrm{p}) - 1\ \right)$$

where p is a prime number.

$\pi(\mathrm{p})$ is not more than p of prime numbers.

2. The twin prime theorem

Let $T'(\mathrm{p}^2)$ is the number of twin prime pairs $(p, p+2)$ in the open interval $(\mathrm{p},\ \mathrm{p}^2)$, p is an odd prime,

$$T'(\mathrm{p}^2) > \frac{1}{2}\left(\mathrm{p}^2 \cdot 2C \cdot \prod_{2 \le p \le \mathrm{p}} \left(1 - \frac{1}{p}\right)^2 - \pi(\mathrm{p}) - 1\right) \tag{4}$$

where ($\mathrm{p} < p,\ p+2 < \mathrm{p}^2$), p is a prime number.

$\pi(\mathrm{p})$ is not more than p of prime numbers.

Among which

$$C = \prod_{3 \le p \le \mathrm{p}} \left(1 - \frac{1}{(p-1)^2}\right) \tag{5}$$

Lemma 2

Let $T'(\mathrm{p}^2)$ is the number of twin prime pairs $(p, p+2)$ in the open interval $(\mathrm{p},\ \mathrm{p}^2)$, $\mathrm{p} < p,\ p+2 < \mathrm{p}^2$, p is a prime number, p is an odd prime.

Let $T_k'(\mathrm{p}^2)$ is the number of odd P_k between $k\,\mathrm{p}^2$ to $(k+1)\,\mathrm{p}^2$, $k \ge 1$,

and $(P_k, p) = 1$, $(P_k - 2, p) = 1$, $3 \le p \le \mathrm{p}$, p is a prime number,

p is an odd prime.

Let $f(\mathrm{p}^2) = T_k'(\mathrm{p}^2) - T'(\mathrm{p}^2)$

$$\left| f(\mathrm{p}^2) \right| \le T'(\mathrm{p}^2) + \pi(\mathrm{p}) + 1 \tag{6}$$

where $\pi(\mathrm{p})$ is not more than p of prime numbers.

The proof of lemma 2

Reduction to absurdity.

The proof of the twin prime theorem

Proof

By lemma 2 and Chinese remainder theorem, it can be derived

$$1 + T'(\mathrm{p}^2) + \left(T'(\mathrm{p}^2) + f(\mathrm{p}^2)\right) \cdot \left(\mathrm{p} \cdot \prod_{2 \le p \le \mathrm{p}} p \cdot \frac{1}{\mathrm{p}^2} - 1\right) = \mathrm{p} \cdot \prod_{3 \le p \le \mathrm{p}} (p-2) \tag{7}$$

Hence proving

$$T'(\mathrm{p}^2) > \frac{1}{2}\left(\frac{\mathrm{p}^2}{2} \cdot \prod_{3 \le p \le \mathrm{p}} \left(1 - \frac{2}{p}\right) - \pi(\mathrm{p}) - 1\right) \tag{8}$$

or $$T'(\mathrm{p}^2) > \frac{1}{2}\left(\mathrm{p}^2 \cdot 2C \cdot \prod_{2 \le p \le \mathrm{p}} \left(1 - \frac{1}{p}\right)^2 - \pi(\mathrm{p}) - 1\right)$$

where ($\mathrm{p} < p$, $p+2 < \mathrm{p}^2$), p is a prime number.

$\pi(\mathrm{p})$ is not more than p of prime numbers.

Among which

$$C = \prod_{3 \le p \le \mathrm{p}} \left(1 - \frac{1}{(p-1)^2}\right)$$

3. The Goldbach theorem

Let $G(x)$ is the number of prime p in the open interval $(\mathrm{p},\ \mathrm{p}^2)$, p is the largest prime number less than $\sqrt{x}$, and $x-p$ is prime number, x is a large even integer.

$$G(x) > \frac{1}{2}\left(\mathrm{p}^2 \cdot 2C \cdot \prod_{2 \le p \le \mathrm{p}} \left(1 - \frac{1}{p}\right)^2 \cdot \prod_{\substack{p \mid x \\ 3 \le p \le \mathrm{p}}} \frac{(p-1)}{(p-2)} - \pi(\mathrm{p}) - 1\right) \tag{9}$$

where ($\mathrm{p} < p < \mathrm{p}^2$), p is a prime number.

$\pi(\mathrm{p})$ is not more than p of prime numbers.

Since

$$C = \prod_{3 \le p \le \mathrm{p}} \left(1 - \frac{1}{(p-1)^2}\right) \tag{10}$$

When $x = 2^n$,

$$G(x) > \frac{1}{2}\left(\mathrm{p}^2 \cdot 2C \cdot \prod_{2 \le p \le \mathrm{p}} \left(1 - \frac{1}{p}\right)^2 - \pi(\mathrm{p}) - 1\right) \tag{11}$$

Lemma 3

Let $G(x)$ is the number of prime p in the open interval $(\mathrm{p},\ \mathrm{p}^2)$, p is the largest prime number less than $\sqrt{x}$, and $x-p$ is prime number, x is a large even integer.

$$\mathrm{p}\cdot\prod_{2\le p\le \mathrm{p}} p = \mathrm{M} \qquad\qquad \mathrm{p}\cdot\prod_{2\le p\le \mathrm{p}} p\cdot\frac{1}{\mathrm{p}^2} = \mathrm{m}$$

Let $G_k(x)$ is the number of odd P_k between $k\,\mathrm{p}^2$ to $(k+1)\,\mathrm{p}^2$, $1\le k\le \mathrm{m}$, and $(P_k,\ p)=1$, $(\mathrm{M}+x-P_k\ ,\ p)=1$, $3\le p\le \mathrm{p}$, p is a prime number, p is the largest prime number less than $\sqrt{x}$, x is a large even integer.

Let $g(x)=G_k(x)-G(x)$

$$\left|\,g(x)\right|\le G(x)+\pi(\mathrm{p})+1 \qquad (12)$$

where $\pi(\mathrm{p})$ is not more than p of prime numbers.

The proof of lemma 3

Reduction to absurdity.

The proof of the Goldbach theorem

Proof

By lemma 3 and Chinese remainder theorem, it can be derived

$$1+G(x)+(G(x)+g(x))\cdot\Big(\mathrm{p}\cdot\prod_{2\le p\le \mathrm{p}} p\cdot\frac{1}{\mathrm{p}^2}-1\Big)=\mathrm{p}\cdot\prod_{3\le p\le \mathrm{p}}(p-2)\cdot\prod_{p|x\ \ 3\le p\le \mathrm{p}}\frac{(p-1)}{(p-2)} \qquad (13)$$

where $(x-1\ ,\ p)=1$, $2\le p\le \mathrm{p}$, p is a prime number

or

$$G(x)+(G(x)+g(x))\cdot\Big(\mathrm{p}\cdot\prod_{2\le p\le \mathrm{p}} p\cdot\frac{1}{\mathrm{p}^2}-1\Big)=\mathrm{p}\cdot\prod_{3\le p\le \mathrm{p}}(p-2)\cdot\prod_{p|x\ \ 3\le p\le \mathrm{p}}\frac{(p-1)}{(p-2)} \qquad (14)$$

Hence proving

$$G(x)>\frac{1}{2}\Big(\frac{\mathrm{p}^2}{2}\cdot\prod_{3\le p\le \mathrm{p}}\Big(1-\frac{2}{p}\Big)\cdot\prod_{p|x\ \ 3\le p\le \mathrm{p}}\frac{(p-1)}{(p-2)}-\pi(\mathrm{p})-1\Big) \qquad (15)$$

or

$$G(x)>\frac{1}{2}\Big(\mathrm{p}^2\cdot 2C\cdot\prod_{2\le p\le \mathrm{p}}\Big(1-\frac{1}{p}\Big)^2\cdot\prod_{p|x\ \ 3\le p\le \mathrm{p}}\frac{(p-1)}{(p-2)}-\pi(\mathrm{p})-1\Big)$$

where ($\mathrm{p} < p < \mathrm{p}^2$), p is a prime number.

$\pi(\mathrm{p})$ is not more than p of prime numbers.

Since
$$C = \prod_{3 \le p \le \mathrm{p}} (1 - \frac{1}{(p-1)^2})$$

When $x = 2^n$,

$$G(x) > \frac{1}{2} (\mathrm{p}^2 \cdot 2C \cdot \prod_{2 \le p \le \mathrm{p}} (1 - \frac{1}{p})^2 - \pi(\mathrm{p}) - 1)$$

References

[1] J.B. Rosser and L.scloenfeld. approximate formulas for some functions of prime numbers . Illinois J. Math. Volume 6,Issue 1 (1962),64-94

[2] G.H.Hardy and E.M.Wright. An Introduction To The Theory of Numbers,section 22.8 and 22.19. the Oxford University Press,4ed,1959