

# Derivation Of The Schwarzschild Radius Without General Relativity

*In this paper I present an alternative method of deriving the Schwarzschild radius of a black hole. The method uses three of the Planck units formulas: the Planck mass, the Planck momentum and the Planck length. Therefore it is shown that General Relativity (GR) is not necessary to derive the formula for the radius of a black hole. Thus, compared to GR, the simplification achieved by this derivation is enormous.*

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## 1. Introduction

*and I stumbled on the key: the Planck units, a primitive but marvelous quantum gravity theory*

In 1783 John Mitchell derived the correct formula for the Schwarzschild radius using a wrong formulation [1] based on classical or Newtonian physics. He applied the principle of conservation of energy to a photon emitted from a star of mass  $M$

$$\frac{1}{2}mv^2 = \frac{GMm}{r} \quad (1.1)$$

Where  $m$  denotes the equivalent mass of a photon (Mitchell supposed, at least implicitly, that the equivalent inertial mass of the photon was identical to its equivalent gravitational mass. Thus, from an implicit point of view, Einstein was not the first to introduce the equivalence principle). In equation (1.1) Mitchell replaced the escape velocity  $v$  of a photon with the speed of light,  $c$ . Thus, he was able to write

$$\frac{1}{2}c^2 = \frac{GM}{r} \quad (1.2)$$

Then he solved this equation for  $r$  to get the following correct result for the Schwarzschild radius of a black hole

$$r = \frac{2GM}{c^2} \quad (1.3)$$

However, we must emphasize that, despite the correctness of the above result, Mitchell's formulation was wrong and it should not be considered the proof of the formula of the Schwarzschild radius for black holes. If we want to obtain a correct derivation of this formula we should use either GR, quantum gravity or the Planck units (an extremely primitive form of quantum gravity). In this paper I have chosen the method based on the Planck units.

In 2014 I published an article entitled *The Special Quantum Gravitational Theory of Black Holes (Wormholes and Cosmic Time Machines)* [2] where I derived (a) a general formula for the temperature of a black hole and (b) the formula for the black hole entropy. In that formulation I used the universal uncertainty principle (UUP) [3] as the cornerstone of the theory along with the Schwarzschild radius formula discovered by Carl Schwarzschild in 1915. The issue was that I wanted to derive the thermodynamic properties of black holes without using any results from Einstein's General Theory of Relativity; and although, at that time, I could have included the derivation presented here, I did not do so because I wanted to write a relatively short paper. Now I feel it is time to introduce the method I left out at that time so that my previous formulation on black holes does not rest on Einstein's GR. The mathematics used in this paper are simple enough so that the derivation may be easily understood by the non-expert. Appendix I includes the nomenclature used throughout this paper.

## 2. Derivation of the Schwarzschild Radius

The idea is to find, if possible, a relation based on (but not identical) the de Broglie law from the Planck units so that the formula will have the following form

$$R = K \frac{h}{P_p} \quad (2.1)$$

Where

- $h$  = Planck's constant
- $R$  = radius of a black hole
- $K$  = dimensionless constant to be determined
- $P_p$  = Planck momentum

From here we shall derive a special formula for the radius of a black hole (a black hole of minimum size). Then, we shall generalize the formula so that it coincides with the famous black hole radius formula found by Carl Schwarzschild.

I shall start the derivation with the formula for the Planck mass,  $M_p$

$$\text{Planck mass} \quad M_p \equiv \sqrt{\frac{hc}{2\pi G}} \quad (2.2)$$

and the formula for the Planck momentum,  $P_p$

Planck momentum  $P_P \equiv \sqrt{\frac{hc^3}{2\pi G}}$  (2.3)

Then we multiply these two equations as follows

$$M_P P_P = \sqrt{\frac{hc}{2\pi G} \frac{hc^3}{2\pi G}}$$
 (2.4)

Which gives

$$M_P P_P = \frac{hc^2}{2\pi G}$$
 (2.5)

Now we invert each side. This yields

$$\frac{1}{M_P P_P} = \frac{2\pi G}{hc^2}$$
 (2.6)

Which may also be expressed as follows

$$\frac{h}{2\pi P_P} = \frac{GM_P}{c^2}$$
 (2.7)

Now we multiply and divide the second side of this equation by the mass,  $M$ , of a star

$$\frac{h}{\pi P_P} = \frac{2GM}{c^2} \times \frac{M_P}{M}$$
 (2.8)

(We assume that this star will be sufficiently massive to collapse into a black hole when it runs out of fuel). But, what length does the quantity  $h/\pi P_P$  represent? In order to answer this question we replace the Planck momentum on the first side of equation (2.8) by the second side of equation (2.3). This gives

$$\frac{h}{\pi P_P} = \frac{h}{\sqrt{\frac{hc^3}{2\pi G}}}$$
 (2.9)

Which, after some work, leads to

$$\frac{h}{\pi P_P} = 2\sqrt{\frac{hG}{2\pi c^3}}$$
 (2.10)

Where the square root of the second side is the definition of the Planck length,  $L_P$

Planck length  $L_P \equiv \sqrt{\frac{hG}{2\pi c^3}}$  (2.11)

From equations (2.10) and (2.11) we write

$$\frac{h}{\pi P_P} = 2 L_P \quad (2.12)$$

From equations (2.7) and (2.12) we may write the following equation

$$2 L_P = \frac{2 G M_P}{c^2} \quad (2.13)$$

Or

$$L_P = \frac{G M_P}{c^2} \quad (2.14)$$

But from previous studies carried out in Appendix 1 of reference [4] we know the mass of a black hole of minimum size (which, by the way, coincides with the initial mass of the universe). In other words, we know that (a) the minimum radius of a black hole is equal to the Planck length and (b) that the minimum mass of a black hole is equal to the Planck mass divided by 2, this is

$$M_{bhmin} = \frac{M_P}{2} \quad (2.15)$$

Then we multiply and divide the second side of equation (2.14) by 2. This yields

$$L_P = \frac{2 G M_P}{c^2} \frac{1}{2} \quad (2.16)$$

But according to equation (2.15) we may write

Minimum radius of a black hole  $L_P = \frac{2 G M_{bhmin}}{c^2}$  (2.17)

Therefore the quantity on the second side of this equation must be a radius (because  $L_P$  is a radius) and not a diameter. Therefore, it follows that the more general expression  $2 G M / c^2$  must also be a radius and not a diameter. With these considerations in mind we may generalize the last equation by replacing  $M_{bhmin}$  by  $M$  (the mass of a star) and the radius  $L_P$  by  $R$  (the radius of a collapsed star or black hole). With these two changes equation (2.17) becomes the general equation for the radius of a black hole (because equation 2.17 is a special equation for the radius of a black hole). Thus we have

Radius of a black hole  $R \equiv \frac{2 G M}{c^2}$  (2.18)

Therefore  $R$  must be the Schwarzschild radius,  $R_s$ , of the black hole and  $M$  must be its mass ( $M$  was, originally, the mass of a star that collapsed into a black hole). Thus, we finally write the formula for the Schwarzschild radius of a black hole as it is generally written in the literature

$$R_s \equiv \frac{2GM}{c^2} \quad (2.19)$$

### 3. Determination of the Constant $K$

Equation (2.8) may be rewritten as

$$\frac{h}{\pi P_p} \times \frac{M}{M_p} = \frac{2GM}{c^2} \quad (3.1)$$

Thus the dimensionless constant  $K$  may be defined as

$$K \equiv \frac{M}{\pi M_p} \quad (3.2)$$

Therefore the formula we want it is (see equation 2.1)

$$K \frac{h}{P_p} = R_s = \frac{2GM}{c^2} \quad (3.3)$$

### 4. Conclusions

This paper shows the power of the methodology based on the Planck units: we do not need General Relativity to obtain the formula for the Schwarzschild radius of a black hole. Thus, “quantum gravity” is somehow “embedded” in these units. Due to its simplicity, the method presented here is suitable to be taught in senior courses of secondary schools. Before I finish this article I would like to say that there is at least another method of deriving the Schwarzschild radius without GR. But this is a subject for another article.

### Appendix 1 Nomenclature

- $h$  = Planck's constant
- $G$  = Newton's gravitational constant
- $c$  = speed of light in vacuum
- $K$  = a constant to be determined
- $P_p$  = Planck momentum
- $m$  = equivalent mass of a photon

$M_p$  = Planck mass

$L_p$  = Planck length

$M_{bhmin}$  = minimum mass of a black hole

$M$  = mass of a black hole

$r$  = radius of a black hole (black hole radius or Schwarzschild radius)

$R$  = radius of a black hole (black hole radius or Schwarzschild radius)

$R_s$  = Schwarzschild radius

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## REFERENCES

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