The Lorentz Force as Pure Geodesic Motion in Four-Dimensional Spacetime

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Abstract: We develop a linear metric element ds in ordinary four-dimensional spacetime which, when held stationary under worldline variations, leads to the gravitational equations of geodesic motion extended to include the Lorentz force law for both abelian and non-abelian gauge fields. We see that in the presence of an electromagnetic vector potential A^μ, all that is needed to obtain this result is to follow the well-known gauge theory prescription of replacing the kinetic momentum p^μ with a canonical momentum $\pi^\mu=p^\mu+eA^\mu$ in the mass / momentum relationship $m^2=p_\sigma p^\sigma$, and then to apply variational calculus to obtain the motion of charged particles in this potential. We also show how by this same prescription, Maxwell's classical source-free field equations become embedded within canonical extensions of the Riemann tensor and the gravitational field equation.

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Contents

1. Introduction

In §9 of his landmark 1916 paper [1], Albert Einstein first derived the geodesic equation of motion $d^2x^\mu / ds^2 = -\Gamma^\mu{}_{\alpha\beta}\left(dx^\alpha / ds\right)\left(dx^\beta / ds\right)$ for a particle in a gravitational field based on the variation $0 = \delta\int_A^B ds$ of the linear metric element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ between any two spacetime events A and B at which the worldlines of different observers meet so that their clocks and measuring rods and scales can be coordinated at the outset A and then compared at the conclusion B. Notably absent from [1], however, was a similar geodesic development of the Lorentz force law $d^2x^\mu / ds^2 = (e/m)F^\mu{}_\alpha\left(dx^\alpha / ds\right)$. Subsequent papers by Kaluza [2] and Klein [3] did succeed in explaining the Lorentz force as a type of geodesic motion and even gave a geometric explanation for the electric charge itself, but only at the cost of adding a fifth dimension to spacetime and curling that dimension into a cylinder. To date, a century later, there still does not appear to have been any fully-successful attempt to obtain the Lorentz force from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime. In this letter, we show how this is done.

2. Basis and derivation

As the basis for obtaining the Lorentz force from a geodesic variation in four dimensions, we begin with the equation $m^2 = p_\sigma p^\sigma$ that describes the relativistic relationship between any mass m and its "kinetic" energy-momentum $p^\mu = mu^\mu = m\left(dx^\mu / ds\right)$. We then promote this kinetic momentum to a "canonical" momentum π^μ via the prescription $p^\mu \to \pi^\mu = p^\mu + eA^\mu$ taught by the local gauge (really, phase) theory of Hermann Weyl developed over 1918 to 1929 in [4], [5], [6], and so obtain $m^2 = p_\sigma p^\sigma \to m^2 = \pi_\sigma \pi^\sigma$. It will be appreciated that this prescription is the momentum space equivalent of $\partial_\mu = \partial / \partial x^\mu \to D_\mu = \partial_\mu + ieA_\mu$ which is the gauge-covariant derivative specified in a configuration space for which the metric tensor of the tangent flat Minkowski space is $\mathrm{diag}\left(\eta_{\mu\nu}\right) = (+1, -1, -1, -1)$. Consequently, deconstructing into a linear equation using the Dirac matrices $\frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} = \eta^{\mu\nu}$ in flat spacetime, one can employ $m^2 = \pi_\sigma \pi^\sigma$ to obtain Dirac's equation $\left(i\gamma^\mu D_\mu - m\right)\psi = 0$ for an electron wavefunction ψ in an electromagnetic potential A_μ, which equation Dirac first derived in [7] for a free electron in a form equivalent to $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$, i.e., without yet using $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$.

So to obtain the Lorentz force from a geodesic variation in spacetime, we backtrack from $m^2 = \pi_\sigma \pi^\sigma$ to a linear metric element:

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to ds^2 = g_{\mu\nu}d\chi^\mu d\chi^\nu = g_{\mu\nu}\left(dx^\mu + ds(e/m)A^\mu\right)\left(dx^\nu + ds(e/m)A^\nu\right) \\ = g_{\mu\nu}dx^\mu dx^\nu + 2(e/m)A_\sigma dx^\sigma ds + (e/m)^2 g_{\mu\nu}A^\mu A^\nu ds^2 \end{aligned}, \tag{2.1}$$

which uses a canonical gauge prescription for the spacetime coordinates themselves, namely:

$$dx^{\mu} \rightarrow d\chi^{\mu} = dx^{\mu} + ds\left(e/m\right)A^{\mu} . \tag{2.2}$$

This is just another variation of $p^{\mu} \rightarrow \pi^{\mu} = p^{\mu} + eA^{\mu}$ and $\partial_{\mu} \rightarrow D_{\mu} = \partial_{\mu} + ieA_{\mu}$. Indeed, it is easily seen that if one multiplies $ds^2 = \left(dx_{\sigma} + ds\left(e/m\right)A_{\sigma}\right)\left(dx^{\sigma} + ds\left(e/m\right)A^{\sigma}\right)$ in (2.1) through by m^2/ds^2, the result is identical to the canonical $m^2 = \pi_{\sigma}\pi^{\sigma}$. Now, all we need do is apply a variation $0=\delta\int_A^B ds$ to the linear element (2.1) and the Lorentz force naturally emerges as a geodesic equation of motion right alongside of the gravitational equation of motion.

Proceeding with this derivation which largely parallels that in the online [8], we first use (2.1) to construct the number

$$1 = \sqrt{g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}} , \tag{2.3}$$

which we then use to write the variation as:

$$0 = \delta\int_A^B ds = \delta\int_A^B ds\sqrt{g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}} . \tag{2.4}$$

Applying δ to the integrand and using (2.3) to clear the denominator, this yields:

$$0 = \delta\int_A^B ds = \frac{1}{2}\int_A^B ds\delta\left(g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{dx^{\sigma}}{ds} + \left(\frac{e}{m}\right)^2 g_{\mu\nu}A^{\mu}A^{\nu}\right). \tag{2.5}$$

Dropping the ½ and using the product rule, while assuming that there is no variation in the charge-to-mass ratio – i.e., that $\delta(e/m)=0$ – over the path from *A* to *B*, we now distribute δ using the product rule to obtain:

$$0 = \int_A^B ds\left(\begin{array}{l}\delta g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + g_{\mu\nu}\frac{d\delta x^{\mu}}{ds}\frac{dx^{\nu}}{ds} + g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{d\delta x^{\nu}}{ds} + 2\frac{e}{m}\delta A_{\sigma}\frac{dx^{\sigma}}{ds} + 2\frac{e}{m}A_{\sigma}\frac{d\delta x^{\sigma}}{ds} \\ +\left(e/m\right)^2\left(\delta g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\delta A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\delta A^{\nu}\right)\end{array}\right). \tag{2.6}$$

One can use the chain rule in the small variation $\delta \rightarrow \partial$ limit to show that $\delta g_{\mu\nu} = \partial_{\alpha}g_{\mu\nu}\delta x^{\alpha}$ and $\delta A_{\sigma} = \partial_{\alpha}A_{\sigma}\delta x^{\alpha}$. So the bottom line equals $\delta x^{\alpha}\left(e/m\right)^2\left(\partial_{\alpha}g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\partial_{\alpha}A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\partial_{\alpha}A^{\nu}\right)$. Likewise, we may recondense $\partial_{\alpha}\left(g_{\mu\nu}A^{\mu}A^{\nu}\right) = \partial_{\alpha}g_{\mu\nu}A^{\mu}A^{\nu} + g_{\mu\nu}\partial_{\alpha}A^{\mu}A^{\nu} + g_{\mu\nu}A^{\mu}\partial_{\alpha}A^{\nu}$ via the product rule. Therefore, the entire integral on the bottom line contains a total derivative given by:

$$\int_A^B \delta x^\alpha \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial x^\alpha}\left(g_{\mu\nu}A^\mu A^\nu\right)ds = \delta x^\alpha \left(\frac{e}{m}\right)^2 \frac{\partial s}{\partial x^\alpha}\left(g_{\mu\nu}A^\mu A^\nu\right)\Big|_A^B = 0\,. \tag{2.7}$$

This equals zero, because the two worldlines intersect at the boundary events A and B but have a slight variational difference between A and B otherwise, so that $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$ while $\delta x^\sigma \neq 0$ elsewhere. Consequently, the bottom line of (2.6) zeros out, leaving us with:

$$0 = \int_A^B ds\left(\delta g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + g_{\mu\nu}\frac{dx^\mu}{ds}\frac{d\delta x^\nu}{ds} + 2\frac{e}{m}\delta A_\sigma\frac{dx^\sigma}{ds} + 2\frac{e}{m}A_\sigma\frac{d\delta x^\sigma}{ds}\right). \tag{2.8}$$

From here, again using $\delta g_{\mu\nu} = \partial_\alpha g_{\mu\nu}\delta x^\alpha$ and $\delta A_\sigma = \partial_\alpha A_\sigma \delta x^\alpha$, and also re-indexing and using the symmetry of $g_{\mu\nu}$ to combine the second and third terms above, we obtain:

$$0 = \int_A^B ds\left(\delta x^\alpha \partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + 2g_{\mu\nu}\frac{d\delta x^\mu}{ds}\frac{dx^\nu}{ds} + 2\delta x^\alpha\frac{e}{m}\partial_\alpha A_\sigma\frac{dx^\sigma}{ds} + 2\frac{e}{m}A_\sigma\frac{d\delta x^\sigma}{ds}\right). \tag{2.9}$$

Next, we integrate by parts. First, we use the product rule to replace $g_{\mu\nu}\left(d\delta x^\mu / ds\right)\left(dx^\nu / ds\right) = \left(d/ds\right)\left(\delta x^\mu g_{\mu\nu}\left(dx^\nu/ds\right)\right) - \delta x^\mu\left(d/ds\right)\left(g_{\mu\nu}\left(dx^\nu/ds\right)\right)$ and likewise $A_\sigma d\delta x^\sigma / ds = \left(d/ds\right)\left(A_\sigma \delta x^\sigma\right) - \left(dA_\sigma / ds\right)\delta x^\sigma$. But the terms containing the total derivatives will vanish for the same reasons that the terms in (2.7) vanished as a result of the boundary conditions $\delta x^\sigma(A) = \delta x^\sigma(B) = 0$. As a result, (2.9) now becomes:

$$0 = \int_A^B ds\left(\delta x^\alpha \partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu\frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) + 2\delta x^\alpha\frac{e}{m}\partial_\alpha A_\sigma\frac{dx^\sigma}{ds} - 2\delta x^\sigma\frac{e}{m}\frac{dA_\sigma}{ds}\right). \tag{2.10}$$

Applying the d/ds derivative contained in the second term above then yields:

$$0 = \int_A^B ds\left(\delta x^\alpha \partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{ds^2} - 2\delta x^\mu\frac{dg_{\mu\nu}}{ds}\frac{dx^\nu}{ds} + 2\delta x^\alpha\frac{e}{m}\partial_\alpha A_\sigma\frac{dx^\sigma}{ds} - 2\delta x^\sigma\frac{e}{m}\frac{dA_\sigma}{ds}\right), \tag{2.11}$$

for the first time revealing the acceleration d^2x^ν / ds^2 in the second term above.

Next, we use the chain rules $dg_{\mu\nu}/ds = \partial_\alpha g_{\mu\nu}\left(dx^\alpha/ds\right)$ and $dA_\sigma/ds = \partial_\alpha A_\sigma\left(dx^\alpha/ds\right)$ to rewrite the third and fifth terms above, thus obtaining:

$$0 = \int_A^B ds\begin{pmatrix}\delta x^\alpha \partial_\alpha g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu}\frac{d^2x^\nu}{ds^2} - 2\delta x^\mu \partial_\alpha g_{\mu\nu}\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} \\ +2\delta x^\alpha\frac{e}{m}\partial_\alpha A_\sigma\frac{dx^\sigma}{ds} - 2\delta x^\sigma\frac{e}{m}\partial_\alpha A_\sigma\frac{dx^\alpha}{ds}\end{pmatrix}. \tag{2.12}$$

In the bottom line above, we may rename indexes $\alpha \leftrightarrow \sigma$ in the last term, to find that we may rewrite $\delta x^\alpha \partial_\alpha A_\sigma \left(dx^\sigma / ds \right) - \delta x^\sigma \partial_\alpha A_\sigma \left(dx^\alpha / ds \right) = \delta x^\alpha F_{\alpha\sigma} \left(dx^\sigma / ds \right)$ using the electromagnetic field strength tensor $F_{\alpha\sigma} = \partial_\alpha A_\sigma - \partial_\sigma A_\alpha$, which has now appeared as a result of the variation. So the above now simplifies to:

$$0 = \int_A^B ds \left(\delta x^\alpha \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu\nu} \frac{d^2 x^\nu}{ds^2} - 2\delta x^\mu \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} + 2\delta x^\alpha \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds} \right). \tag{2.13}$$

Now we rename indexes so that the δx terms all contain the index α, that is, so all of these terms are δx^α. We then factor this out and interchange the first and second terms, obtaining:

$$0 = \int_A^B ds \delta x^\alpha \left(-2 g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\partial_\mu g_{\alpha\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds} \right). \tag{2.14}$$

For material worldlines, $ds \neq 0$. Likewise, while $\delta x^\sigma (A) = \delta x^\sigma (B) = 0$ at the boundaries, between these boundaries where the variation occurs, $\delta x^\sigma \neq 0$. Thus, multiplying through by ½, for (2.14) to be true the integrand must be zero, and so we have:

$$0 = -g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \frac{1}{2} \partial_\alpha g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \partial_\mu g_{\alpha\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds}. \tag{2.15}$$

Now we move the acceleration term to the left, split the term with $\partial_\mu g_{\alpha\nu} = \frac{1}{2} \partial_\mu g_{\alpha\nu} + \frac{1}{2} \partial_\mu g_{\alpha\nu}$ into two halves, rename some indexes while using the symmetry of $g_{\alpha\nu}$, and finally multiply through by $g^{\beta\alpha}$ and then raise indexes. This all yields:

$$\frac{d^2 x^\beta}{ds^2} = \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F^\beta{}_\sigma \frac{dx^\sigma}{ds}. \tag{2.16}$$

But of course, we recognize that the Christoffel symbols $-\Gamma^\beta{}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha} \left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu} \right)$. As a consequence, the above reduces to:

$$\frac{d^2 x^\beta}{ds^2} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F^\beta{}_\sigma \frac{dx^\sigma}{ds}. \tag{2.17}$$

In the presence of gravitational and electromagnetic fields, this contains both the equations of gravitational motion and the Lorentz force law, obtained via the geodesic variation of the canonical invariant metric length element (2.1). In the absence of gravitation, i.e., for $g_{\mu\nu} = \eta_{\mu\nu}$ over the spacetime region being considered thus $\Gamma^\beta{}_{\mu\nu} = 0$, this reduces to the Lorentz force law.

As a result, we have proved that by using Weyl's canonical prescription in form of $dx^\mu \to d\chi^\mu = dx^\mu + ds\left(e/m\right)A^\mu$ from (2.2) to define the linear metric element by $ds^2 = g_{\mu\nu} d\chi^\mu d\chi^\nu$ as shown in (2.1), the Lorentz force law of electrodynamics may indeed be obtained from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime geometry.

3. Non-abelian gauge fields

It will be observed that the field strength tensor emerging in (2.17) is an abelian field strength $F_{\alpha\sigma} = \partial_{[\alpha} A_{\sigma]}$. It is natural to inquire whether one can use this same approach to also obtain the equation for the motion for charged particles in non-abelian fields $F_{\alpha\sigma} = \partial_{[\alpha} A_{\sigma]} + ie\left[A_\alpha, A_\sigma\right] = D_{[\alpha} A_{\sigma]}$.

It turns out that this is fairly simple to do. First, we again re-index (2.12) by interchanging $\alpha \leftrightarrow \sigma$ in the bottom line. Likewise, we re-index the top line so that all of the coordinate variations have the same index, namely, are of the form δx^α, which we then factor out as earlier. We also move the acceleration term to the first position, and we split the final term on the top line in half once again and re-index to show the cycling of the Christoffel indexes. We also restore the factor of ½ that we earlier dropped after (2.5), and restructure into what is clearly the form of the final result (2.17). So what we now have is:

$$0 = \int_A^B \delta x^\alpha ds \left(-g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \frac{1}{2}\left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\right) \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \frac{e}{m} \partial_{[\alpha} A_{\sigma]} \frac{dx^\sigma}{ds} \right). \tag{3.1}$$

From here, we perform a local gauge (phase) transformation $A_\sigma \to A'_\sigma = e^{i\Lambda(x^\mu)} A_\sigma$ on the gauge fields, and insist that this variation remain invariant under such transformation. Consequently, we must promote the derivative that acts on the gauge fields to the gauge-covariant $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ in the usual way. As a result (3.1) now becomes:

$$0 = \int_A^B \delta x^\alpha ds \left(-g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} + \frac{1}{2}\left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\right) \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \frac{e}{m} D_{[\alpha} A_{\sigma]} \frac{dx^\sigma}{ds} \right). \tag{3.2}$$

The integrand must still be zero because $\delta x^\alpha \neq 0$ except at the boundaries. Therefore, this yields the exact same result as was found in (2.17) with no change whatsoever in form. The only difference is that the field strength is now the non-abelian $F_{\alpha\sigma} = D_{[\alpha} A_{\sigma]}$.

4. Einstein's Equation and Maxwell's Equations

Because the metric length $ds^2 = g_{\mu\nu} d\chi^\mu d\chi^\nu$ of (2.1) under a variation $0 = \delta\int_A^B ds$ simultaneously provides a geodesic description of motion in a gravitational field and in an electromagnetic field, and because the prescription $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ is no more than a variant of Weyl's gauge prescriptions $p^\mu \to \pi^\mu = p^\mu + eA^\mu$ in momentum space and $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ in configuration space and leads directly as well to Dirac's equation $(i\gamma^\mu D_\mu - m)\psi = 0$ for an interacting fermion, this may fairly be regarded as a classical metric-level unification of electrodynamics with gravitation, using four spacetime dimensions only. But the equations of motion in a field are only half the matter. We also need to know the equations for the fields themselves in relation to their sources. Thus we now ask, can the field equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ which specifies the gravitational field, be shown to relate in some direct fashion to Maxwell's field equations for electric and (the absence of) magnetic sources?

Because the Lorentz force (2.17) is obtained by dilating or contracting the differential coordinate elements via $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ without in any way altering the metric tensor $g_{\mu\nu}$ as is done, for example, in Kaluza-Klein theory, one might incorrectly conclude that the electromagnetic interaction does not affect spacetime curvature as represented by the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ with the field dynamics specified by $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. However, one must keep in mind that the Riemann tensor may be defined via $R^\alpha{}_{\beta\mu\nu} V_\alpha \equiv [\partial_{;\nu}, \partial_{;\mu}] V_\beta$ as a measure of the extent to which the gravitationally-covariant derivatives $\partial_{;\mu} V_\beta = \partial_\mu V_\beta - \Gamma^\alpha{}_{\mu\beta} V_\alpha$ operating on a vector V_β do not commute. Likewise, the field strength tensor $F_{\nu\mu}$ may be defined via $ieF_{\nu\mu} V_\beta \equiv [D_\nu, D_\mu] V_\beta$ as a measure of the extent to which the gauge-covariant derivatives $D_\mu V_\beta = (\partial_\mu + ieA) V_\beta$ do not commute when operating on this same vector V_β. Indeed, this latter definition results in $F_{\nu\mu} = D_{[\nu} A_{\mu]} = \partial_{[\nu} A_{\mu]} + ie[A_\nu, A_\mu]$ for a *non-abelian* gauge theory defined such that $[A_\nu, A_\mu] \neq 0$, which simplifies to $F_{\nu\mu} = \partial_{[\nu} A_{\mu]}$ for an abelian theory such as electrodynamics in which $[A_\nu, A_\mu] = 0$.

Therefore, let us now apply Weyl's canonical prescription to the gravitationally-covariant derivatives by employing:

$$\partial_{;\mu} V_\beta = \partial_\mu V_\beta - \Gamma^\alpha{}_{\mu\beta} V_\alpha \to D_{;\mu} V_\beta = (\partial_\mu + ieA_\mu) V_\beta - \Gamma^\alpha{}_{\mu\beta} V_\alpha, \tag{4.1}$$

for vectors V_β a.k.a. first-rank tensors, and likewise extended for second and higher-rank tensors. This is the same prescription that in the form $dx^\mu \to d\chi^\mu = dx^\mu + ds(e/m)A^\mu$ of (2.2) yielded the Lorentz force law in (2.17). If we then use these derivatives (4.1) to define a gauge-enhanced canonical Riemann tensor $\Re^\alpha{}_{\beta\mu\nu}$ as:

$$\Re^\alpha{}_{\beta\mu\nu} V_\alpha \equiv [D_{;\nu}, D_{;\mu}] V_\beta, \tag{4.2}$$

it can be expected as a consequence of $ieF_{\nu\mu}V_\beta \equiv \left[D_\nu, D_\mu\right]V_\beta$ that the electrodynamic fields $F_{\nu\mu}$ and potentials A_μ will appear in this Riemann tensor. Further, because $F_{\nu\mu} = D_{[\nu}A_{\mu]}$ encompasses both abelian and non-abelian field strengths, one would expect that the gravitational field equations using $\Re_{\beta\mu} = \Re^\alpha{}_{\beta\mu\alpha}$ and $\Re = \Re^\sigma{}_\sigma$ can be related not only to abelian electrodynamics, *but also to non-abelian such as weak and strong interactions*. So let us expressly calculate this enhanced canonical $\Re^\alpha{}_{\beta\mu\nu}$ using (4.2) and see what results.

We first calculate:

$$\begin{aligned} D_{;\nu}\left(D_{;\mu}V_\beta\right) &= \left(\partial_\nu + ieA_\nu\right)\left(\left(\partial_\mu + ieA_\mu\right)V_\beta - \Gamma^\alpha{}_{\mu\beta}V_\alpha\right) \\ &\quad -\Gamma^\tau{}_{\mu\nu}\left(\left(\partial_\tau + ieA_\tau\right)V_\beta - \Gamma^\alpha{}_{\tau\beta}V_\alpha\right) - \Gamma^\tau{}_{\nu\beta}\left(\left(\partial_\mu + ieA_\mu\right)V_\tau - \Gamma^\alpha{}_{\mu\tau}V_\alpha\right) \end{aligned} \tag{4.3}$$

as well as the like expression interchanging $\mu \leftrightarrow \nu$, then subtract the latter from the former and reduce using index renaming and the symmetries of the objects in the resulting equations. Many terms cancel, but with the vector V_α still attached as the operand on the right, what remains is:

$$\begin{aligned} &\Re^\alpha{}_{\beta\mu\nu}V_\alpha \equiv \left[D_{;\nu}, D_{;\mu}\right]V_\beta = D_{;\nu}\left(D_{;\mu}V_\beta\right) - D_{;\mu}\left(D_{;\nu}V_\beta\right) \\ &= \left(-\partial_\nu\Gamma^\alpha{}_{\mu\beta} + \partial_\mu\Gamma^\alpha{}_{\nu\beta} + \Gamma^\tau{}_{\nu\beta}\Gamma^\alpha{}_{\mu\tau} - \Gamma^\tau{}_{\mu\beta}\Gamma^\alpha{}_{\nu\tau} - ie\delta^\alpha{}_\beta F_{\mu\nu}\right)V_\alpha \end{aligned}, \tag{4.4}$$

including a *non-abelian* field strength:

$$F_{\mu\nu} = T^a F^a{}_{\mu\nu} = \partial_{[\mu}A_{\nu]} + ie\left[A_\mu, A_\nu\right] = T^a\partial_{[\mu}A^a{}_{\nu]} + ie\left[T^b, T^c\right]A^b{}_\mu A^c{}_\nu = T^a\left(\partial_{[\mu}A^a{}_{\nu]} - ef^{abc}A^b{}_\mu A^c{}_\nu\right), \tag{4.5}$$

which becomes abelian in the event $\left[A_\mu, A_\nu\right] = 0$. When we explicitly display the group structure constants f^{abc} for the non-abelian Hermitian generators T^a via $if^{abc}T^a = \left[T^b, T^c\right]$, we see that $F^a{}_{\nu\mu} = \partial_{[\nu}A^a{}_{\mu]} - ef^{abc}A^b{}_\nu A^c{}_\mu$ is real and so $ieF_{\mu\nu} = ieT^aF^a{}_{\mu\nu}$ in (4.4) is a *complex* Hermitian field owing to the T^a. With V_α removed and some index renaming and lowered to covariant form, the canonical Riemann tensor in (4.4) is then seen to be:

$$\begin{aligned} \Re_{\alpha\beta\mu\nu} &= g_{\alpha\tau}\Re^\tau{}_{\beta\mu\nu} = -g_{\alpha\tau}\partial_\nu\Gamma^\tau{}_{\beta\mu} + g_{\alpha\tau}\partial_\mu\Gamma^\tau{}_{\beta\nu} + g_{\alpha\tau}\Gamma^\sigma{}_{\beta\nu}\Gamma^\tau{}_{\sigma\mu} - g_{\alpha\tau}\Gamma^\sigma{}_{\beta\mu}\Gamma^\tau{}_{\sigma\nu} - ieg_{\alpha\beta}F_{\mu\nu} \\ &= R_{\alpha\beta\mu\nu} - ieg_{\alpha\beta}F_{\mu\nu} \end{aligned} \tag{4.6}$$

As expressed by $\Re_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - ieg_{\alpha\beta}F_{\mu\nu}$, the terms containing Christoffels are no different from the usual in $R_{\alpha\beta\mu\nu}$. But the new term $-ieg_{\alpha\beta}F_{\mu\nu}$ resulting from the same gauge prescription $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$ that likewise brought the Lorentz force law into (2.17) changes several aspects of $\Re_{\alpha\beta\mu\nu}$ in relation to the ordinary $R_{\alpha\beta\mu\nu}$. First, while for the last two indexes

$\Re_{\alpha\beta\mu\nu} = -\Re_{\alpha\beta\nu\mu}$ as with $R_{\alpha\beta\mu\nu}$, for the first two indexes $\Re_{\alpha\beta\mu\nu} \neq -\Re_{\beta\alpha\mu\nu}$ due to the presence of the symmetric $g_{\alpha\beta}$ next to the antisymmetric $F_{\mu\nu}$ in the term $g_{\alpha\beta}F_{\mu\nu}$. Thus, $\Re_{\alpha\beta\mu\nu}$ is *non-symmetric* in α,β. Second, noting that $F_{\mu\nu} = T^a F^a{}_{\mu\nu}$ is Hermitian, the term $ieg_{\alpha\beta}F_{\mu\nu}$ provides an similar *complex* aspect to $\Re_{\alpha\beta\mu\nu}$, so that overall, this enhanced $\Re_{\alpha\beta\mu\nu}$ is a *complex* object. Third, as a consequence of both these matters, the *real* part of $\Re_{\alpha\beta\mu\nu}$ has the usual symmetries of $R_{\alpha\beta\mu\nu}$, while the new *complex* part has the mixed symmetry of $g_{\alpha\beta}F_{\mu\nu}$.

It is readily seen from (4.6) after some re-indexing that the canonical Ricci tensor:

$$\Re_{\mu\nu} = \Re^{\alpha}{}_{\mu\nu\alpha} = -\partial_\alpha \Gamma^\alpha{}_{\mu\nu} + \partial_\nu \Gamma^\alpha{}_{\mu\alpha} + \Gamma^\sigma{}_{\mu\alpha}\Gamma^\alpha{}_{\sigma\nu} - \Gamma^\sigma{}_{\mu\nu}\Gamma^\alpha{}_{\sigma\alpha} + ieF_{\mu\nu} = R_{\mu\nu} + ieF_{\mu\nu}, \tag{4.7}$$

concisely $\Re_{\mu\nu} = R_{\mu\nu} + ieF_{\mu\nu}$, is likewise non-symmetric, with the usual gravitational terms being real and symmetric, and the new, Hermitian electrodynamic term being antisymmetric in μ,ν. Finally, because $F^\sigma{}_\sigma = 0$, the canonical Ricci scalar is the usual:

$$\Re = g^{\mu\nu}\Re_{\mu\nu} = -g^{\sigma\tau}\partial_\alpha \Gamma^\alpha{}_{\sigma\tau} + g^{\sigma\tau}\partial_\tau \Gamma^\alpha{}_{\sigma\alpha} + g^{\sigma\tau}\Gamma^\beta{}_{\sigma\alpha}\Gamma^\alpha{}_{\beta\sigma} - g^{\sigma\tau}\Gamma^\beta{}_{\sigma\tau}\Gamma^\alpha{}_{\beta\alpha} = R \tag{4.8}$$

with no residual terms from electrodynamics, that is, $\Re = R$.

If we now construct $\partial_{;\sigma}\Re_{\alpha\beta\mu\nu} + \partial_{;\mu}\Re_{\alpha\beta\nu\sigma} + \partial_{;\nu}\Re_{\alpha\beta\sigma\mu}$, then because (4.6) informs us that $\Re_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - ieg_{\alpha\beta}F_{\mu\nu}$, all of the Christoffel terms will zero out as a result of the second Bianchi identity $\partial_{;\sigma}R_{\alpha\beta\mu\nu} + \partial_{;\mu}R_{\alpha\beta\nu\sigma} + \partial_{;\nu}R_{\alpha\beta\sigma\mu} = 0$, simply due to the inherent structure of the Riemannian geometry itself. All that will remain are terms containing the field strength, so that:

$$\begin{aligned} &\partial_{;\sigma}\Re_{\alpha\beta\mu\nu} + \partial_{;\mu}\Re_{\alpha\beta\nu\sigma} + \partial_{;\nu}\Re_{\alpha\beta\sigma\mu} = -ieg_{\alpha\beta}\left(\partial_{;\sigma}F_{\mu\nu} + \partial_{;\mu}F_{\nu\sigma} + \partial_{;\nu}F_{\sigma\mu}\right) \\ &= -ieg_{\alpha\beta}\left(\partial_{;\sigma}\left[A_\mu, A_\nu\right] + \partial_{;\mu}\left[A_\nu, A_\sigma\right] + \partial_{;\nu}\left[A_\sigma, A_\mu\right]\right) \end{aligned} \tag{4.9}$$

Specifically: We see here that the Hermitian part of $\partial_{;\sigma}\Re_{\alpha\beta\mu\nu} + \partial_{;\mu}\Re_{\alpha\beta\nu\sigma} + \partial_{;\nu}\Re_{\alpha\beta\sigma\mu}$ contains the terms $\partial_{;\sigma}F_{\mu\nu} + \partial_{;\mu}F_{\nu\sigma} + \partial_{;\nu}F_{\sigma\mu}$ which specify magnetic charges. Because exterior calculus teaches that the differential forms $dF = ddA = 0$, this set of terms must be equal to zero for any *abelian* gauge theory with $\left[A_\mu, A_\nu\right] = 0$. And because this set of terms must be zero if $\left[A_\mu, A_\nu\right] = 0$, this means that for an abelian interaction the canonical Riemann tensor obeys the analogous Bianchi identity $\partial_{;\sigma}\Re_{\alpha\beta\mu\nu} + \partial_{;\mu}\Re_{\alpha\beta\nu\sigma} + \partial_{;\nu}\Re_{\alpha\beta\sigma\mu} = 0$ as a consequence of $dF = ddA = 0$ which is Maxwell's magnetic charge equation.

Next, given the identity (4.9), let us double-contract two pairs of indexes to find the canonical extension of the contracted Bianchi identity $\partial_{;\mu}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0$ which is used to ensure local energy conservation in the Einstein equation via $\partial_{;\mu} T^{\mu\nu} = \partial_{;\mu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 0$, for these canonically-extended $\Re_{\alpha\beta\mu\nu}$. Further, let us consider an abelian interaction $\left[A_\mu, A_\nu\right] = 0$, so that (4.9) will clearly be zero:

$$\partial_{;\sigma}\Re_{\alpha\beta\mu\nu} + \partial_{;\mu}\Re_{\alpha\beta\nu\sigma} + \partial_{;\nu}\Re_{\alpha\beta\sigma\mu} = -ieg_{\alpha\beta}\left(\partial_{;\sigma}F_{\mu\nu} + \partial_{;\mu}F_{\nu\sigma} + \partial_{;\nu}F_{\sigma\mu}\right) = 0 . \tag{4.10}$$

We want to contract indexes carefully for reasons that will momentarily become apparent, so let's go step by step and pay close attention to the indexes and their symmetries. First, let us raise the index α. Given that $\Re^{\alpha}{}_{\beta\sigma\mu}$ is antisymmetric in the last two indexes but unlike $R^{\alpha}{}_{\beta\sigma\mu}$ *not in the first two indexes*, let us interchange $\sigma \leftrightarrow \mu$ and reverse the sign in the final term $\partial_{;\nu}\Re_{\alpha\beta\sigma\mu}$. Finally, let's contract α with σ. As in (4.7), and as for any fourth rank tensor, we obtain the Ricci tensor extension $\Re_{\beta\nu} \equiv \Re^{\sigma}{}_{\beta\nu\sigma}$ and the trace tensor generally by contracting the first and last indexes together. In the term with the field strengths after this we will have a $\delta^{\sigma}{}_{\beta}$ so we pass β through σ into the field strength expression to yield:

$$\partial_{;\sigma}\Re^{\sigma}{}_{\beta\mu\nu} + \partial_{;\mu}\Re_{\beta\nu} - \partial_{;\nu}\Re_{\beta\mu} = -ie\left(\partial_{;\beta}F_{\mu\nu} + \partial_{;\mu}F_{\nu\beta} + \partial_{;\nu}F_{\beta\mu}\right) = 0 . \tag{4.11}$$

This now expresses Maxwell's magnetic charge equation very directly.

Now we move to the next contraction. In the term with $F_{\nu\beta} = -F_{\beta\nu}$ we flip indexes and sign. Then, we raise β and contract it with μ. Setting $\Re^{\beta}{}_{\beta} = \Re$ we rename the second term $\partial_{;\beta}\Re^{\beta}{}_{\nu} = \partial_{;\sigma}\Re^{\sigma}{}_{\nu}$ and in the third term we rename $\partial_{;\nu} = \delta^{\sigma}{}_{\nu}\partial_{;\sigma}$. Finally, we take what is now $\partial_{;\sigma}$ in front of all three extended Riemann terms and factor it all the way to the left, while multiplying everything through by ½. The result is:

$$\partial_{;\sigma}\left(\tfrac{1}{2}\Re^{\sigma\beta}{}_{\beta\nu} + \tfrac{1}{2}\Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}\Re\right) = -ie\tfrac{1}{2}\left(\partial^{;\beta}F_{\beta\nu} - \partial_{;\beta}F^{\beta}{}_{\nu} + \partial_{;\nu}F^{\beta}{}_{\beta}\right) = 0 . \tag{4.12}$$

The term $\partial^{;\beta}F_{\beta\nu} - \partial_{;\beta}F^{\beta}{}_{\nu} + \partial_{;\nu}F^{\beta}{}_{\beta}$ is clearly identical to zero given the antisymmetry of $F_{\beta\nu}$ (and will likewise be zero even for non-abelian fields). So the first-rank identity we obtain for the canonical Riemann extension is $\partial_{;\sigma}\left(\tfrac{1}{2}\Re^{\sigma\beta}{}_{\beta\nu} + \tfrac{1}{2}\Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}\Re\right) = 0$. One may be inclined to further simplify $\Re^{\sigma\beta}{}_{\beta\nu} \to \Re^{\sigma}{}_{\nu}$ to arrive at $\partial_{;\sigma}\left(\Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}\Re\right) = 0$ which is precisely the analog of $\partial_{;\sigma}\left(R^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}R\right) = 0$, *but that would be a subtle but important mistake*. Precisely because $\Re^{\sigma\beta}{}_{\beta\nu}$ is non-symmetric (neither symmetric nor antisymmetric) in its first two indexes as discussed at (4.6), we cannot contract via the second and third indexes, but only via the first and fourth indexes. That is, $\Re^{\beta\sigma}{}_{\nu\beta} = \Re^{\sigma}{}_{\nu}$ is correct, but $\Re^{\sigma\beta}{}_{\beta\nu} = \Re^{\sigma}{}_{\nu}$ is not.

This brings us back to in (4.7) which we write as $\Re^{\mu\nu} = R^{\mu\nu} + ieF^{\mu\nu}$ in contravariant form. Applying the four-gradient $\partial_{;\mu}$ operator throughout yields:

$$\partial_{;\mu}\Re^{\mu\nu} = \partial_{;\mu}R^{\mu\nu} + ie\partial_{;\mu}F^{\mu\nu} = \partial_{;\mu}R^{\mu\nu} + ieJ^{\nu}, \tag{4.13}$$

which enables us to pinpoint the electric source current $J^{\nu} = \partial_{;\mu}F^{\mu\nu}$. Given that $\Re = R$ as found in (4.8), let us next subtract $\frac{1}{2}\partial_{;\mu}\left(g^{\mu\nu}\Re\right)$ throughout from the above (note that $g_{\mu\nu;\sigma} = 0$ always because of the metricity of $g_{\mu\nu}$), and then apply the ordinary contracted Bianchi identity $\partial_{;\mu}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = 0$, to obtain:

$$\partial_{;\mu}\left(\Re^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\Re\right) = \partial_{;\mu}\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right) + ieJ^{\nu} = ieJ^{\nu}. \tag{4.14}$$

So, very importantly, we see that the current ieJ^{ν} is identified with the *non-zero* $\partial_{;\mu}\left(\Re^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\Re\right) \neq 0$ and that the actual "zero" is the one in $\partial_{;\sigma}\left(\frac{1}{2}\Re^{\sigma\beta}{}_{\beta\nu} + \frac{1}{2}\Re^{\sigma}{}_{\nu} - \frac{1}{2}\delta^{\sigma}{}_{\nu}\Re\right) = 0$ of (4.12). Indeed, we may subtract (4.12) from (4.14) with renaming $\mu \to \sigma$ and lowering ν to find that:

$$ieJ_{\nu} = \partial_{;\sigma}\left(\tfrac{1}{2}\Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\Re^{\sigma\beta}{}_{\beta\nu}\right). \tag{4.15}$$

This shows us not only that $\Re^{\sigma\beta}{}_{\beta\nu}$ cannot be contracted to $\Re^{\sigma}{}_{\nu}$ using the second and third indexes, but that the electric current itself is a measure of the non-symmetry of these first two $\alpha\beta$ indexes in $\Re_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - ieg_{\alpha\beta}F_{\mu\nu}$ from (4.6). This is why it was so important to be very careful with the index contractions leading to (4.12). However, absent electric charge sources, that is, for the source-free $J_{\nu} = 0$, the above will yield $\Re^{\sigma}{}_{\nu} = \Re^{\sigma\beta}{}_{\beta\nu}$ and then this contraction is permitted.

With all of this, we now move to Einstein's equation itself. We know that local conservation of the energy tensor is expressed by $-\kappa\partial_{;\sigma}T^{\sigma}{}_{\nu} = 0$, and we ordinarily enforce this by setting $-\kappa\partial_{;\sigma}T^{\sigma}{}_{\nu} = \partial_{;\sigma}\left(R^{\sigma}{}_{\nu} - \frac{1}{2}\delta^{\sigma}{}_{\nu}R\right) = 0$ because of the Bianchi identity of $\partial_{;\sigma}\left(R^{\sigma}{}_{\nu} - \frac{1}{2}\delta^{\sigma}{}_{\nu}R\right)$ with zero. But (4.12) gives us a different identity to zero (enforced by the antisymmetry of $F_{\beta\nu}$ which is inviolate even for non-abelian gauge theories) which can serve the same purpose and also has some very direct electrodynamic connections. So we instead use the identity (4.12) to write the conservation equation:

$$-\kappa\partial_{;\sigma}T^{\sigma}{}_{\nu} = \partial_{;\sigma}\left(\tfrac{1}{2}\Re^{\sigma\beta}{}_{\beta\nu} + \tfrac{1}{2}\Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}\Re\right) = 0 \tag{4.16}$$

which in turn integrates sans cosmological constant into a field equation:

$$-\kappa T^{\sigma}{}_{\nu} = \tfrac{1}{2}\Re^{\sigma\beta}{}_{\beta\nu} + \tfrac{1}{2}\Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}\Re . \tag{4.17}$$

Were it possible to contract $\Re^{\sigma\beta}{}_{\beta\nu} = \Re^{\sigma}{}_{\nu}$, (4.17) would take the exact form $-\kappa T^{\sigma}{}_{\nu} = \Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}\Re$ of the Einstein equation. We have shown at (4.15) that this contraction is only permitted for source-free fields, $J_{\nu} = 0$.

Finally, if we start from (4.16) and use (4.15) to replace $\tfrac{1}{2}\partial_{;\sigma}\Re^{\sigma\beta}{}_{\beta\nu} = \tfrac{1}{2}\partial_{;\sigma}\Re^{\sigma}{}_{\nu} - ieJ_{\nu}$ and then move the current to the same side of the equation so that all sources (gravitational and electric) are together, we may write the local energy conservation law in the form:

$$-\kappa\partial_{;\sigma}T^{\sigma}{}_{\nu} + ieJ_{\nu} = \partial_{;\sigma}\left(\Re^{\sigma}{}_{\nu} - \tfrac{1}{2}\delta^{\sigma}{}_{\nu}\Re\right) = ieJ_{\nu} . \tag{4.18}$$

Summarizing, the three main field equations we have found, written exclusively in terms of the canonically-extended $\Re_{\alpha\beta\mu\nu}$, from (4.15), (4.11) and (4.17) respectively, with some simple reindexing, are:

$$\begin{cases} ieJ^{\nu} = \partial_{;\mu}\left(\Re^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\Re\right) \\ 0 = \partial_{;\sigma}\Re^{\sigma\beta\mu\nu} + \partial^{;\mu}\Re^{\beta\nu} - \partial^{;\nu}\Re^{\beta\mu} \\ -\kappa T^{\mu}{}_{\nu} = \tfrac{1}{2}\Re^{\mu\beta}{}_{\beta\nu} + \tfrac{1}{2}\Re^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu}\Re \end{cases} . \tag{4.19}$$

The first two are Maxwell's electric and magnetic charge equations, while the final equation is a canonical gauge extension of the Einstein equation for gravitation which by identity will always produce a locally-conserved energy $\partial_{;\sigma}T^{\sigma\nu} = 0$.

5. Conclusion

It has been shown how the Lorentz force law (2.17) can be obtained from a geodesic variation confined exclusively to the four dimensions of ordinary spacetime geometry as a consequence, at bottom, of simply applying Weyl's gauge prescription $\partial_{\mu} \to D_{\mu} = \partial_{\mu} + ieA_{\mu}$ to dilate or contract the spacetime coordinate elements by $dx^{\mu} \to d\chi^{\mu} = dx^{\mu} + ds\left(e/m\right)A^{\mu}$. It has also been shown how this same prescription embeds Maxwell's equations a canonically-extended version (4.19) of the gravitational field equation. None of this appears to have been found before.

As a consequence of what has been shown here, it may well be possible to unify gravitation not only with electrodynamics, but – because the $F^{\sigma\tau}$ obtained in (3.2) and (4.4) encompass a non-abelian field strength $F^{a}{}_{\nu\mu} = \partial_{[\nu}A^{a}{}_{\mu]} - ef^{abc}\left[A^{b}{}_{\nu}, A^{c}{}_{\mu}\right]$ – with the remaining weak and strong interactions as well, because the canonical gauge prescriptions $p^{\mu} \to \pi^{\mu} = p^{\mu} + eA^{\mu}$ and $\partial_{\mu} \to D_{\mu} = \partial_{\mu} + ieA_{\mu}$ and now $dx^{\mu} \to d\chi^{\mu} = dx^{\mu} + ds\left(e/m\right)A^{\mu}$ remain at the root of the entire development. The main questions that would remain following such a unification, would be as to the specific non-abelian gauge groups that operate physically at any given energy ranging up to the Planck mass, and how the symmetry of those groups becomes broken at lower energies down to the phenomenological group $SU(3)_C \times SU(2)_W \times U(1)_Y \to SU(3)_C \times U(1)_{em}$ and the fermions on which these groups act. The author has previously published on these questions, and even shown how the three generations of quarks and leptons originate, and why their left-chiral projections engage in CKM mixing, at [9].

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