Energy and Equations of Motion

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Abstract. From the total time derivative of energy an equation is obtained, this equation gives same results with the Lagrange equations under certain conditions for holonomic systems with potentials; velocity independent and linear velocity dependent. This method does not include Lagrangian, it is used for only comparison.

1. Introduction

The energy is one of the most basic quantity in physics, and in basic physics we can use it for understanding physical systems. Though we use Lagrangian to analyze physical systems in classical mechanics. In this work, we will try to answer whether for problems of the classical mechanics we can define changes in physical systems in terms of energy or not. We will use a relation obtained from the total time derivative of energy to analyze changes in the physical systems. We will push the possibility of the usage of energy, especially its total time derivative, and see whether it is successful or not. To do that we will test this idea for some well-known cases and compare them with the Lagrange equations.

Usage of the total time derivative of energy to study motion of a particle can be seen in different places, but it is not used to obtain the equations of the motion with a theoretical generalization till recent years. Recently different articles, using total time derivative of energy to find equations of motion, were published in different journals by Carlson and Vinokurov [1, 2]. Carlson by using the total time derivative of energy and equations similar to Hamilton’s equation, \[ \frac{\partial E}{\partial v_i} = P_i \] and \[ \frac{\partial E}{\partial q_i} = -\dot{P}_i \], reaches the equation

\[ \frac{\partial E}{\partial q_i} + \frac{d}{dt} \left( \frac{\partial E}{\partial v_i} \right) = 0 \] (1)

just like the Lagrange equations. Carlson considers some velocity independent potential cases and reaches same results with Lagrange equations in these. He also considers electromagnetic interaction and free relativistic particle as examples, and he could not reach the same results with the Lagrange equations by using Eq.(1). However, for electromagnetic interaction, he could be able to reach the consistent results when he used a different equation, similar to one that we will obtain and use for studying, instead of Eq.(1). At the end, he claims that the usage of energy is not enough for classical mechanics considering results related with electromagnetic interaction and relativistic free particle. Vinokurov also starts with the total differential of energy to his derivation. He uses equivalence of energy with Hamiltonian, and by using differentials of the Hamiltonian he reaches Hamilton equations. He obtains a differential equation for Lagrangian from energy, and gives a solution in terms of energy, and mentions about a "nonuniqueness" in the presence of "gyroscopic forces". In this work, we will study similar situations and get consistent results in gravitational rotation and electromagnetic interaction against Carlson’s claim that energy is not enough to describe
classical mechanics and Vinokurov’s claim that in the presence of ”gyroscopic forces” there is a ”nonuniqueness” between energy and Lagrangian. According to Vinokurov’s claim the Lorentz force is a ”gyroscopic force” since it does no work.

This paper is organized as follows: quick review of the equations of motion and Lagrangian formalism, derivation of the equation from the total time derivative of energy, comparison of the Lagrange equations with obtained equation for two different cases: velocity independent potential and velocity dependent potential, each followed by examples, and finally summary and conclusion part. The people with knowledge of equations of motion can pass review part. Examples are needed to see how the mentioned equation works. The first example includes a logical difference. The second one shows that one can obtain same results for electromagnetic interaction from the Lagrange equations and equation obtained from the total time derivative of energy.

2. Equations of motion

The equations of motion are required to analyze a physical system, and can be obtained in different ways. The first method involves writing the equations directly from Newton’s second law $\vec{F} = \frac{d\vec{p}}{dt}$, where $\vec{F}$ is the force, $\vec{p}$ is the momentum and $t$ is the time. If one writes force and time derivative of the momenta explicitly in this equation, then can get equations of motion of the system. The second way is to get them from the Lagrangian, we will give a summary (if reader wants to learn it in detail see e.g. [3]). The third way is to obtain them from the Hamiltonian.

To obtain the equations of motion from the Lagrangian, one needs the Lagrange equations. Those, the Lagrange equations, can be derived in two different ways. In the first method, one starts with Newton’s second law $\vec{F} = \frac{d\vec{p}}{dt}$, and use D’Alembert’s principle[4]

$$ (F_i - \dot{p}_i)\delta r_i = 0 \tag{2} $$

where $\delta r_i$ represents the variation of the position, $i$ shows the $i$th component, e.g. $x, y$ or $z$-component, the dot over $p$ represents time derivative, i.e. $\dot{p} = \frac{dp}{dt}$, and Einstein summation convention is used, i.e. repeated indices are summed up. It will be used in the following equations too. D’Alembert’s principle can be defined as describing the statement that applied force and inertial forces do equal amount of work for reversible processes. D’Alembert’s principle in the above form can be considered as the total of actual and virtual works in all directions is equal to zero. Here, the virtual work corresponds the work defined for inertial forces, and the actual work corresponds the work done by applied forces. Only if each component is linearly independent, then it is possible to say that each multiplier of $\delta r_i$ separately is equal to zero. To be able to obtain such an equation, holonomic systems should be considered, and the Eq.(2) should be written in terms of the generalized coordinates[4]. Holonomic systems are the systems that has only position dependent constraints[5]. First, the position should be defined in terms of the generalized coordinates

$$ r_i = r_i(q_1, q_2, ...q_n, t) \tag{3} $$

where $q_i$’s are the generalized coordinates. The acting forces and geometry of the medium are primary things for deciding which generalized coordinates will be used. In Eq.(2), $\dot{p}_i\delta r_i$ is one of the terms that should be written in terms of the generalized coordinates. By using the definition of $r_i$ in terms of generalized coordinates and $p_i = m\frac{dr_i}{dt}$, it can be written as

$$ (m\ddot{r}_i) \frac{\partial r_i}{\partial q_j} \delta q_j = \left[ \frac{d}{dt} \left( m\dot{r}_i \frac{\partial r_i}{\partial q_j} \right) - m\dot{r}_i \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_j} \right) \right] \delta q_j. \tag{4} $$
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Here to go further one need the definition of velocity in generalized coordinates, \( v_i = \dot{\gamma}_i = \frac{\partial \gamma_i}{\partial \theta_j} \dot{\theta}_j + \frac{\partial \gamma_i}{\partial \theta_j} \dot{\theta}_j \) and from this, one can see \( \frac{\partial \theta_i}{\partial \gamma_j} = \frac{\partial \gamma_i}{\partial \gamma_j} \). Then by using those in the first term and \( \frac{\partial \gamma_i}{\partial \gamma_j} \) in the second term, Eq.(4) becomes

\[
(m \ddot{\gamma}_i) \frac{\partial \gamma_i}{\partial \theta_j} \delta \theta_j = \left[ \frac{d}{dt} \left( m v_i \frac{\partial v_i}{\partial \theta_j} \right) - m v_i \frac{\partial v_i}{\partial \theta_j} \right] \delta \theta_j.
\]  

(5)

With the definition of the kinetic energy, \( T = \frac{1}{2} m v_i^2 \), it can be written as

\[
(m \ddot{\gamma}_i) \frac{\partial \gamma_i}{\partial \theta_j} \delta \theta_j = \left[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_j} \right) - \frac{\partial T}{\partial \theta_j} \right] \delta \theta_j.
\]  

(6)

The other term in Eq.(2) is \( F_i \delta \gamma_i \), it should also be written in terms of generalized coordinates. The force \( F_i \) for the conserved systems can be written as \( F_i = -\frac{\partial U}{\partial \gamma_i} \), where \( U \) is the potential energy. Then \( F_i \delta \gamma_i \) can be written as

\[
F_i \delta \gamma_i = -\frac{\partial U}{\partial \gamma_i} \delta \gamma_i = -\frac{\partial U}{\partial \gamma_i} \frac{\partial \gamma_i}{\partial \gamma_j} \delta \gamma_j.
\]  

(7)

Here \( \frac{\partial \gamma_i}{\partial \gamma_j} = \frac{\partial \gamma_i}{\partial \gamma_j} \) and it is equal to 1 only if \( k = j \), and otherwise it is equal to zero. By the definition of the generalized force \( Q_i = -\frac{\partial U}{\partial \gamma_i} \), one can write

\[
F_i \delta \gamma_i = Q_j \delta \gamma_j
\]

\[
= -\frac{\partial U}{\partial \gamma_j} \delta \gamma_j.
\]  

(8)

Here \( i \) and \( j \) are dummy indices. Since they are summed up, we can replace \( j \)'s with \( i \)'s. So, from D’Alembert’s principle one can get

\[
\left( -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}_j} \right) - \frac{\partial T}{\partial \theta_j} - \frac{\partial U}{\partial \theta_j} \right) \delta \theta_j = 0.
\]  

(9)

If \( \delta \theta_j \)s do not depend on each other, then \( \delta \theta_j \) from the Eq.(9) can be dropped. Having holonomic constraints and writing in terms of generalized coordinates provide this. This means that each multiplier of \( \delta \gamma_i \) should be separately equal to zero. With the definition of Lagrangian function, \( L = T - U \), one can write

\[
-\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\gamma}_i} \right) + \frac{\partial L}{\partial \gamma_i} = 0.
\]  

(10)

By considering velocity independent potential energy, one can subtract \( U \) from \( T \) in the first term, since its derivative with respect to \( \dot{\gamma}_i \) will be equal to zero, and then one can get

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\gamma}_i} \right) - \frac{\partial L}{\partial \gamma_i} = 0,
\]  

(11)

which are the Lagrange equations.

In the above derivation, we assumed that the potential energy does not depend on the velocity. As being not defined for the velocity dependent potentials, the Lagrange equations still can give equations of the motion for such a system. If the generalized force can be obtained from

\[
Q_i = -\frac{\partial U}{\partial \gamma_i} + \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{\gamma}_i} \right)
\]  

(12)

then the Lagrange equations are still valid for velocity dependent cases. One known example of this situation is the Lorentz force[6]. Interestingly, the Lorentz force and the potential
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energy defined for the magnetic and electric fields in terms of the vector and scalar fields do satisfy Eq.(12).

The second method of deriving the Lagrange equations utilizes defined Lagrangian to obtain them from an action integral[7]

$$I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt$$  \hspace{0.5cm} (13)

The Hamilton’s principle can be stated as an action integral has a stationary value for the actual path of the motion through $t_1$ to $t_2$. Then, action integral’s variation should be equal to zero due to this steadiness. Here, we should mention that this procedure valid for holonomic systems, and potentials should depend only on the coordinates. There are some extensions to the more general cases, which can be found again in the classical mechanics books (see e.g. [3]). For variation, we will consider a dependence of generalized coordinate on a parameter $\alpha$, $q_i(t, \alpha)$, here $\alpha$ represents small changes from the actual path in a linear way and if it is equal to zero, then $q_i(t, 0)$ should be on the actual path. Then, the variation of action integral is

$$\delta I = \frac{\partial I}{\partial \alpha} \delta \alpha = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_i} \frac{\partial \dot{q}_i}{\partial \alpha} d\alpha + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha} d\alpha \right) dt.$$  \hspace{0.5cm} (14)

Using integration by parts for the second term and the variation at the end points are equal to zero since we are keeping the end points fixed, we can replace it by $-\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \frac{\partial q_i}{\partial \alpha} d\alpha$, and after some simplification we get

$$\delta I = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \right) \frac{\partial q_i}{\partial \alpha} d\alpha dt.$$  \hspace{0.5cm} (15)

Since the variation of action should be equal to zero for a real path, the terms in the parenthesis should be equal to zero, which gives the Lagrange equations.

The third method for obtaining the equations of motion requires the definition of Hamiltonian and also uses Hamilton’s equations. Firstly, we define momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and use Legendre transformations to obtain the Hamiltonian. Then by writing the differentials for the Hamiltonian and Lagrangian and by comparing them with each other, we can obtain Hamilton’s equations[8].

3. An equation from the total time derivative of the energy

Let us consider a system whose energy is written as a function of generalized position, generalized velocity and time $E(q_i, \dot{q}_i, t)$. Its total time derivative is

$$\frac{dE}{dt} = \frac{\partial E}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial E}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial E}{\partial t}.$$  \hspace{0.5cm} (16)

If the Hamiltonian is equal to the energy, by using Poisson brackets, it can be shown that $\frac{dE}{dt} = \frac{\partial E}{\partial t}$ [9]. If energy is independent of time, it is always valid and below we will consider only time independent cases. If energy is time independent, then energy is conserved, and in this work we studied such cases. Then from Eq.(16), we can write

$$\frac{\partial E}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial E}{\partial q_i} \frac{dq_i}{dt} = 0.$$  \hspace{0.5cm} (17)

We will use this equation to obtain equations of motion.
Now let us try to understand: What does this equation imply? If we look at the dimension of Eq.(17), it is seen that the units of the terms are power. Power defines change in energy with respect to time, which means that this equation describes how energy changes within the system. We know that after taking derivatives of the energy, the terms in the Eq.(17) will be a function of position and velocity in general. Then, as position or velocity changes, these terms change. Hence Eq.(17) states that any change in one form of energy can result in a change in the other form(s) of energy since total of them is equal to zero. It also explicitly gives these changes. Since it gives these changes, it may be called as the equation of energy conversion.

Now, let us try to compare this with Lagrangian and the Lagrange equations. The dimension of the Lagrangian is energy, and it is a function that is defined to obtain equations of motion. It does not give any information on physical systems alone, it is useful with the Lagrange equations. In employing Lagrange equations, the main steps of the method as follows: we write the Lagrangian in an appropriate coordinate system and then use Lagrange equations to study the motion in the considered system. The Lagrange equations give consistent results in classical mechanics, and we use them to understand motion. The dimension of the Lagrange equation is force, and force and acceleration are the main physical quantities for studying the motion.

Differently from Lagrange equations, the equation obtained from the total time derivative of energy has the dimension of power since we obtained it from the total time derivative of energy. Then, we will obtain terms like force times velocity when we apply Eq.(17). In the situations, that we will be able to get common velocity multiplier, by canceling this common velocity multiplier, we will get an equation in the same dimension with the Lagrange equations.

To see whether the statements above are valid, and these two formalism give consistent results or not, we will study on different holonomic cases.

4. The equation obtained from the total time derivative of energy and the Lagrange equations

In this section, we will consider different types of potential; velocity independent and velocity dependent. Then, we will calculate equations of motion from the equation obtained from the total time derivative of energy and from the Lagrange equations, and compare them. Finally, we will employ the equation obtained from the total time derivative of energy to some examples to see how it works.

4.1. Velocity independent potential

Let us now look at whether Eq.(17) gives consistent result with the Lagrange equations for the velocity independent potential or not.

At the beginning, it is hard to see the Lagrange equations, Eq.(11), and equation obtained from the total time derivative of energy, Eq.(17), are implying same results. Let us start with the simplest case where the kinetic energy is given by $T(\dot{\vec{q}}) = \frac{1}{2}m\dot{q}_i^2$ and potential energy is only a function of generalized coordinates $U(\vec{q})$; this is the most common case in the classical mechanics.

Now let us employ Eq.(17). By using the definition of energy $E = T + U$, we can write

$$\left( m\ddot{q}_i + \frac{\partial U}{\partial \dot{q}_i} \right) \dot{q}_i = 0. \quad (18)$$

In this equation, the total of terms in the parenthesis is equal to zero for each $i$th component since we are studying on the holonomic systems. Then we can drop common velocity
multiplier.

Here the Lagrangian is \( L = \frac{1}{2} m \dot{q}_i^2 - U(\dot{q}) \), then the Lagrange equations are

\[
m\ddot{q}_i + \frac{\partial U}{\partial q_i} = 0.
\]

(19)

This is the same result with Eq.(18). Hence, we obtained consistent results in the considered case from Eq.(17) and the Lagrangian. Now let us give an example to see whether they have similar properties or there are differences between them.

**Example 1: Gravitational rotation**

Let us see how it works for gravitational rotation. We can write the energy as

\[
E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{G M m}{r}.
\]

(20)

where \( G \) is the gravitational constant, \( M \) is the mass of the central body, and \( m \) is the mass of the rotating body. This equation is independent of \( \theta \). Then, like the Lagrangian formalism, the corresponding conserved momentum can be written as \( p_\theta = \frac{\partial E}{\partial \dot{\theta}} = m r^2 \dot{\theta} \). After stating conserved quantity, we can rewrite Eq.(20) as

\[
E = \frac{1}{2} m \dot{r}^2 + \frac{p_\theta^2}{2 m r^2} - \frac{G M m}{r}.
\]

(21)

Now, we have obtained a 1-dimensional equation since the only variables are \( r \) and \( \dot{r} \). So we can write the Eq.(17) for this case as

\[
m\ddot{r} - \frac{p_\theta^2}{m r^3} + \frac{G M m}{r^2} \dot{r} = 0.
\]

(22)

So, we have either \( \dot{r} = 0 \) or \( m \ddot{r} - \frac{p_\theta^2}{m r^3} + \frac{G M m}{r^2} = 0 \). \( \dot{r} = 0 \) is valid only for circular orbits, and in other orbits it is different than zero. Since we can drop \( \dot{r} \), we have \( m \ddot{r} = \frac{p_\theta^2}{m r^3} - \frac{G M m}{r^2} \). This is the same equation as the one obtained from the Lagrange equations[11].

Here there is a difference between the Lagrangian formalism and the derivation above. Although in Lagrangian formalism, one need to insert conserved momenta in the equation of motion after obtaining it for \( r \)[12], in the derivation above, we used it before obtaining equation of motion. What does this imply? First let us go over the situation. In Lagrangian formalism, at the beginning the equation for \( r \), \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \), is calculated then conserved momenta, \( p_\theta \), is used. If we look at the calculated conserved momenta \( p_\theta = m r^2 \dot{\theta} \), it is seen that \( \dot{\theta} \) is changing as \( r \) changes. So we can say that while \( \dot{\theta} \) is changing with respect to \( r \), we considered as if it is independent of \( r \). This can bring the thought that it is fixed during the derivation. Do we have any good reason for it? Or we just need to do it to get the consistent result? On the other hand, conserved momenta is substituted in the equation obtained from the total time derivative of energy before finding the equation of motion. So it can be considered that the equation of motion for \( r \) has to be restricted before finding how it changes. In other words, the change in \( r \), which is described by equation of motion, should include information that \( p_\theta \) is constant before derivation. Although we obtained same results from both derivation, logically these two are different. Which way makes more sense? If we follow the considerations above it seems that the derivation for the equation obtained from the total time derivative of energy makes more sense since we did not consider \( \dot{\theta} \) as fixed during the derivation. We need to study more to see other aspects of the situation, and then we can evaluate in a better way.

Finding conserved momentum and equation of motion for \( r \) is not the whole solution of the planetary rotation, the Kepler problem, the change of \( r \) with respect to \( \theta \) should also
be found. To find it one need to derive energy from Lagrangian formalism or imply it. On
the other hand, for the equation obtained from the total time derivative of energy we already
have energy and used it. In Lagrangian formalism, we need the Lagrangian, the Lagrange
equations and the energy; in the employed derivation we need energy and Eq.(17) to solve the
Kepler problem.

4.2. Linear velocity-dependent potential

Now, we will test the equation obtained from the total time derivative of energy for linear
velocity dependent potentials.

Though linear velocity dependence of the potential is excluded in the derivation of the
Lagrange equations, as mentioned previously it can still give consistent results in some special
cases. To be able to get comparable results for equation obtained from the total time derivative
of energy and the Lagrange equations, we need to separate consistent in two parts as the
coordinate dependent and velocity dependent parts; $U(\vec{q}, \vec{\dot{q}}) = U_1(\vec{q}) + U_2(\vec{q}, \vec{\dot{q}})$. We know
how to handle generalized coordinate dependent part, and we should develop a technique for
the velocity dependent part. We know that potential energy should be a scalar quantity, and
since we consider linear dependence of potential on the velocity, one way to obtain such a
potential energy is to use dot product of the generalized velocity with a vector function of $q_i$;
$U_2(\vec{q}, \vec{\dot{q}}) = \dot{q}_i f_i(\vec{q})$. Here, this function can be some scalar function $f_1(\vec{q}^2)$ multiplied with
the position vector $f_i(\vec{q}) = f_1(\vec{q}^2)q_i$.

Now, we will calculate the equations of motion by using the equation obtained from the
total time derivative of energy. In this case the energy is $E = \frac{1}{2}m\vec{q}^2 + U_1(\vec{q}) + \dot{q}_i f_i(\vec{q})$, and
for this case Eq.(17) can be written as

$$m\ddot{q}_i \dot{q}_i + \dot{q}_i f_i(\vec{q}) + \dot{q}_i \dot{q}_j \frac{\partial f_j(\vec{q})}{\partial q_i} + \dot{q}_i \frac{\partial U_1}{\partial q_i} = 0. \tag{23}$$

Here we have an unfamiliar term, $\dot{q}_i f_i(\vec{q})$, and we should write it in a familiar form. Let
us consider that the total time derivative of the velocity-dependent part of the potential
energy is equal to a function, $\frac{\partial}{\partial t}(\dot{q}_i f_i(\vec{q})) = g(q_i, \ddot{q}_i)$. Then, by using it, we can write
$\ddot{q}_i f_i(\vec{q}) = -\dot{q}_i \dot{q}_j \frac{\partial f_j(\vec{q})}{\partial q_i} + g(q_i, \ddot{q}_i)$. When we used this in Eq.(23), we obtain

$$\dot{q}_i \left[m\ddot{q}_i + \ddot{q}_j \left(\frac{\partial f_j(\vec{q})}{\partial q_i} - \frac{\partial f_i(\vec{q})}{\partial q_j}\right) + \frac{\partial U_1}{\partial q_i}\right] + g(q_i, \ddot{q}_i) = 0. \tag{24}$$

Now, we can write the Lagrangian as $L = \frac{1}{2}m\dot{q}_i^2 - U_1(\vec{q}) - \dot{q}_i f_i(\vec{q})$. Then, we can obtain
the equations of motion as

$$m\ddot{q}_i + \ddot{q}_j \left(\frac{\partial f_j(\vec{q})}{\partial q_i} - \frac{\partial f_i(\vec{q})}{\partial q_j}\right) + \frac{\partial U_1}{\partial q_i} = 0. \tag{25}$$

Now let us compare this one with Eq.(24). The terms in the square parenthesis are the
same with the equations of motion obtained from Lagrangian. If $g(q_i, \ddot{q}_i) = 0$, we can obtain
consistent results since we can drop the multiplier $\dot{q}_i$ in front of the square parenthesis for
holonomic cases.

So we can say that the Eq.(17) for the velocity dependent potentials gives consistent
results only if the total time derivative of the velocity dependent potential is equal to zero. Let
us see with an example, the Lorentz force, whether $g(q_i, \ddot{q}_i) = 0$ is acceptable or not.

**Example 2: Lorentz force from time independent potentials**

There is a case in which we have a velocity dependent potential: motion of a charged
particle in magnetic field.
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For a particle with charge $q$ and mass $m$ moving under the magnetic field, we can write the total energy as total of the kinetic energy and magnetic energy, $-q\dot{x}_iA_i$ [13], as

$$E = \frac{1}{2}m\dot{x}_i^2 - q\dot{x}_iA_i$$  \hspace{1cm} (26)

where $\vec{A}$ is the vector potential. In the time-independent case, it is [14]

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3\vec{x}'.$$  \hspace{1cm} (27)

In this case, Eq.(17) can be written as

$$m\ddot{x}_i - q\dot{x}_iA_i - q\dot{x}_i\dot{x}_j\frac{\partial A_i}{\partial x_j} = 0.$$  \hspace{1cm} (28)

Here we have an unusual term $q\dot{x}_iA_i$. Before proceeding further, it is better to first consider the total time derivative of magnetic energy; in general we can write $\frac{d(q\dot{x}_iA_i)}{dt} = g(q_i, \dot{q}_i)$. Now let us calculate the left hand side

$$\frac{d(q\dot{x}_iA_i)}{dt} = q\dot{x}_iA_i + q\dot{x}_i\dot{x}_j\frac{\partial A_i}{\partial x_j}.$$  \hspace{1cm} (29)

As mentioned previously, the dimension of the time derivative of the energy is power, and it defines the work done per unit time. We know that magnetic fields do no work [15]. Since time derivative of the energy defines rate of work done, $g(q_i, \dot{q}_i)$ should be equal to zero. It is seen that the previous mathematical necessity for equivalence of two formalism turn out to be related with the fact that magnetic fields do no work. Then, we can write $q\dot{x}_iA_i = -q\dot{x}_i\dot{x}_j\frac{\partial A_i}{\partial x_j}$, which shows that the unusual term is related with the rotational acceleration in the motion of charged particles under magnetic field. After its usage, we obtain

$$m\ddot{x}_i + q\dot{x}_i\dot{x}_j\frac{\partial (A_i)}{\partial x_j} - q\dot{x}_i\dot{x}_j\frac{\partial (A_i)}{\partial x_j} = 0.$$  \hspace{1cm} (30)

In Eq.(30), the last two terms cancel each other. These two terms are related with the magnetic energy and the change in the forms of the energy is described by Eq.(30). This means that there is not any change in the form of the magnetic energy due to these two magnetic interaction terms, which is an expected result. Since canceling can lead to loss of knowledge, we will not let cancel those two terms with each other. Now, in the second term, we can replace the indices $i$ and $j$ with one another, since they are dummy. Then from Eq.(30) we can obtain

$$\dot{x}_j \left[ m\ddot{x}_j + q\dot{x}_i \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \right] = 0.$$  \hspace{1cm} (31)

We can write last two terms as follows $\dot{x}_j\dot{x}_i \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\dot{x}_i\dot{x}_j\frac{\partial A_m}{\partial x_i} = \dot{x}_i\epsilon_{ijk}\dot{x}_j\epsilon_{klm}\frac{\partial A_m}{\partial x_i}$. If we write it in the vector form, we obtain $\vec{v} \cdot (\vec{x} \times \vec{B})$ by using $\vec{B} = \nabla \times \vec{A}$.

We clearly see that it is equal to zero and this is an expected result [16].

Now to obtain a more familiar equation for Eq.(31), by eliminating $\dot{x}_j$ and using the double cross product property, we can write the equation as

$$m\ddot{\vec{x}} = q\vec{v} \times (\nabla \times \vec{A})$$

$$= q\vec{v} \times \vec{B}.$$  \hspace{1cm} (32)

We see that the right hand side of the equation is the Lorentz force.
If we add electric energy, $q\phi$, we can write the total energy in the presence of electric and magnetic fields as

$$E = \frac{1}{2} m \dot{x}_i^2 - q \dot{x}_i A_i + q\phi$$  \hspace{1cm} (33)

where $\phi$ is the electric potential, and given by

$$\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'.$$ \hspace{1cm} (34)

In this case, there will be an extra term in Eq.(28), $q \dot{x}_i \frac{\partial \phi}{\partial x_i}$. With this extra term, if we follow similar steps with above derivation, we obtain

$$m\ddot{\vec{x}} = -q\nabla \phi + q\vec{x} \times (\nabla \times \vec{A})$$

$$= q\vec{E} + q\vec{x} \times \vec{B}.$$ \hspace{1cm} (35)

While getting last equality, we used $\vec{E} = -\nabla \phi$. Right hand side is equal to the Lorentz force together with the electric force.

5. Summary

In this work, by using total time derivative of energy, we obtained an equation, Eq.(17), and we studied some physical systems with it. Since it is different than the Lagrange equations, to emphasize this difference, it may be named the equation of energy conversion. To be able to test this equation, we compared the results of the Lagrange equations with it by considering different potential energies for holonomic cases. Most basic and easy potentials depend only on position, and in this type of potentials, we obtained the same results with the Lagrange equations using Eq.(17). However, during the application of it to the gravitational rotation, we saw that there is a difference between these two techniques. In Eq.(17), one need to use conserved momenta before obtaining equation of motion although it is used after getting equations of motion in Lagrangian formalism. Then, we tested linear velocity dependent potentials. In this case, to be able obtain consistent results, we needed to take the total time derivative of the velocity dependent part of the potential as zero. When we tried to use it in the magnetic interaction, we saw that it corresponds to the fact that magnetic fields do no work. By using it, we obtained consistent results.

6. Discussion and Conclusion

In the introduction, we mentioned that there are other works using the total time derivative of the energy; Carlson’s and Vinokorov’s works. There are some differences between in this work and in the others. Differently from Carlson’s work, in this work Eq.(1) is not used as the main equation. Due to this difference, Carlson followed a different way for electromagnetic interaction from our derivation. In this work, a well-known experimental result is used: magnetic fields do no work, which is absent in his work, and we obtained consistent results and he could not by using Eq.(1). Carlson also studied on relativistic free particle, and in that case from mechanical energy equations of motion are not derivable. Though it is possible, if one uses Lagrangian instead of energy. In Vinokorov’s work, he claims that there is ”nonuniqueness” between energy and Lagrangian formalism. Nevertheless, in this work, the same results are obtained from both for the electromagnetic fields.

We mentioned previously about some of differences between the Lagrange equations and equation obtained from the total time derivative of energy. The main differences are the
dimensions, usage of conserved momenta, usage of the fact that magnetic fields do no work. Let us go over these differences. The dimension of the Lagrange equations is force, whereas it is power for Eq.(17). We obtained equations of motion from Eq.(17) by eliminating a common velocity multiplier. Though we do not know whether it is always possible or not. If we can not eliminate it, they are different equations. However, in the studied cases, even though at the beginning the equation obtained from the total time derivative of energy does not seem to be taken a common velocity multiplier parenthesis, we managed to obtain consistent results with the Lagrange equations. Another difference is related with usage of conserved momentum. We used conserved momentum before obtaining equations of motion contrary to the Lagrange equations. Using before seems to be more consistent if one considers dependence of the variables. Usage of the fact that magnetic fields do no work is another issue, and it is an extra statement for the obtaining equations of motion from Eq.(17). Though similar argument works in the reverse direction; one needs three statements in solving Kepler problem from Lagrange equations, whereas two statements for solving from energy. We should mention from another point also; in this work we used energy and an equation obtained from its total time derivative. We did not used Lagrangian and Lagrange equations to obtain equations of motion, we used them only for comparison. It is also important to underline that energy is a physical quantity, whereas Lagrangian is a function defined to obtain Lagrange equations. Eq.(17) provides a relatively simple technique in the logical manner. If we consider calculations, Lagrange equations are simpler. It seems that for a better understanding studying from different aspects are needed.

References