Logical independence of imaginary and complex numbers in Elementary Algebra

context: Theory of indeterminacy and quantum randomness

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Abstract As opposed to the classical logic of *true* and *false*, when Elementary Algebra is treated as a *formal axiomatised system*, formulae in that algebra are either *provable*, *disprovable* or otherwise, *logically independent* of axioms. This logical independence is well-known to Mathematical Logic. Here I show that the imaginary unit, and by extension, all complex numbers, exist in that algebra, logically independently of the algebra's axioms. The intention is to cover the subject in a way accessible to physicists. This work is part of a project researching logical independence in quantum mathematics, for the purpose of advancing a complete theory of quantum randomness. Elementary Algebra is a theory that cannot be completed and is therefore subject to Gödel's Incompleteness Theorems.

Keywords mathematical logic, formal system, axioms, mathematical propositions, Soundness Theorem, Completeness Theorem, logical independence, mathematical undecidability, foundations of quantum theory, quantum mechanics, quantum physics, quantum indeterminacy, quantum randomness.

1 Introduction

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. The 'randomness' stems from ignorance of *physical information* in the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [7], the inequalities of John Bell [3], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, Tomasz Paterek et al offer explanation in *mathematical information.* They demonstrate a link between quantum randomness and *logical independence* in Boolean propositions [8,9]. Logical independence refers to the null logical connectivity that exists between mathematical propositions (in the same language) that neither prove nor disprove one another. In experiments measuring photon polarisation, Paterek et al demonstrate statistics correlating *predictable* outcomes with logically dependent mathematical propositions, and *random* outcomes with propositions that are logically independent.

Whilst Paterek et al do demonstrate that quantum randomness correlates with Boolean propositions, in this *Boolean* context, any insight offered is obscure. To gain proper insight, quantum randomness must be understood in context of logical independence in *standard textbook quantum theory.* The natural place to begin is *Elementary Algebra* treated as a *formal axiomatised logical system.* This is a formal version of the very familiar algebra upon which applied mathematics and mathematical physics rest. Logical independence, in this system, is well-known to Mathematical Logic [10].

A *formal system* comprises: a precise language, rules for writing formulae (propositions) and further rules of deduction. Information is designated in two levels

A good reference for the physicists is Edward Stabler's book *An introduction to mathematical thought* [10].

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AXIOMS of INFINITE FIELDS

	ADDITIVE GROUP	
A0	$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma$	CLOSURE
A1	$\exists 0\forall\alpha \mid \alpha + 0 = \alpha$	IDENTITY 0
A2	$\forall\alpha\exists\beta \mid \alpha + \beta = 0$	INVERSE
A3	$\forall\alpha\forall\beta\forall\gamma \mid (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$	ASSOCIATIVITY
A4	$\forall\alpha\forall\beta \mid \alpha + \beta = \beta + \alpha$	COMMUTATIVITY
	MULTIPLICATIVE GROUP	
M0	$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta \times \gamma$	CLOSURE
M1	$\exists 1\forall\alpha \mid \alpha \times 1 = \alpha$	IDENTITY 1
M2	$\forall\beta\exists\alpha \mid \alpha \times \beta = 1 \wedge \beta \neq 0$	INVERSE
M3	$\forall\alpha\forall\beta\forall\gamma \mid (\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$	ASSOCIATIVITY
M4	$\forall\alpha\forall\beta \mid \alpha \times \beta = \beta \times \alpha$	COMMUTATIVITY
AM	$\forall\alpha\forall\beta\forall\gamma \mid \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma)$	DISTRIBUTIVITY
C0	$0 \neq 1; \quad 0 \neq p, \quad p = \text{any prime}$	CHARACTERISTIC 0

Table 1 AXIOMS of INFINITE FIELDS. These are written as sentences in *first-order logic*. They comprise the *standard field axioms* with added axioms that exclude modulo arithmetic. Variables: $\alpha, \beta, \gamma, 0, 1$ represent objects the axiom-set acts upon. Semantic interpretations of objects complying with AXIOMS are known as *scalars*. The fact that ALGEBRA is intrinsically existential is clearly seen in the general use of the 'there exists' quantifier: $\exists$.

of compulsion. *Propositions* assert information that is questionable. *Axioms* are propositions presupposed to be 'true'. These are postulates adopted *a priori*.

In such a formal system, any two propositions are **either** *logically dependent* — in which case, one proves, or disproves the other — **or otherwise** they are *logically independent*, in which case, neither proves, nor disproves the other. A helpful perspective on this is the viewpoint of Gregory Chaitin's information-theoretic formulation [6]. In that, logical independence is seen in terms of information content. If a proposition contains information, not contained in some given set of axioms, then those axioms can neither prove nor disprove the proposition.

A good (efficient) axiom-set is a selection of propositions, all logically independent of one another. An important point to note is that there is no contradiction in a theory consisting of information whose source is some axiom-set, plus extra information whose source is a logically independent proposition. These might typically be axioms asserting the theory's set conditions, plus a proposition posing a question.

Elementary algebra is the abstraction of the familiar arithmetic used to combine numbers in the rational, real and complex number systems, through operations of *addition* and *multiplication*. These number systems are *infinite fields*. I denote this algebra — ALGEBRA of INFINITE FIELDS[1] (or just ALGEBRA) — as distinct from any other algebra or arithmetic, such as Peano arithmetic or the arithmetic of integers. At a fundamental level, in some form or other, quantum theory rests on mathematical rules of ALGEBRA.

Now, ALGEBRA may be treated as a formal system, based on axioms listed in Table 1. These, I denote – AXIOMS of INFINITE FIELDS (or just AXIOMS). Essentially, these are the conventional *field axioms* appended with additional axioms that exclude the finite fields by denying modulo arithmetic. This ensures that ALGEBRA covers only the *infinite fields*.

Collectively, AXIOMS assert a definite set of information, deriving a definite set of theorems. I denote these – THEOREMS. Any proposition (in the language) is either a THEOREM or is otherwise logically independent. And so, any given formula can be regarded as a proposition in ALGEBRA, that may prove to be a THEOREM, or may otherwise prove to be logically independent. Which of these is actually the case is decided in a process that compares information in that formula against information contained in the AXIOMS. In practice, that means deriving the formula from AXIOMS, to discover: that either it *is* a THEOREM, or otherwise, to discover, whatever extra information is needed to complete its derivation – that AXIOMS cannot provide.

[1] FIELDS should not be confused with the field concept commonly used in physics.

2 Language

The material of this paper spans *formal number theory* and formulae typically seen in *mathematical physics.* These do not share the same language; indeed the language of the former is far smaller. For example, there is no definition for the symbol: 4 and many statements are needed, typified by: $4 = 1 + 1 + 1 + 1$. In the interest of accessibility, these low-level definitions are left to intuition.

Logical connectives used are: *not* ($\neg$), *and* ($\wedge$) , or ($\vee$), *implies* ($\Rightarrow$) and *if-and-only-if* ($\Leftrightarrow$). Turnstile symbols are used: *derives* ($\vdash$) and *models* ($\models$). Also used are the quantifiers: *there-exists* ($\exists$) and *for-all* ($\forall$).

Use of Quantified formulae is crucial in the conveyance of full information. For instance, quantifiers eliminate ambiguities suffered by ordinary equations. To illustrate: the equation $y = x^2$ doesn't express whether $\forall y \exists x \left(y = x^2\right)$ or $\forall x \exists y \left(y = x^2\right)$ is intended. Yet, logically, these two are very different.

3 Examples of logic in the algebra of FIELDS

The propositions (1) – (5) are five examples illustrating the three distinct logical values possible under FIELDS AXIOMS. Notice that these formulae do not assert equality; they assert *existence.* Each is a proposition asserting existence for some instance of a variable α, complying with an equality, specifying a particular numerical value.

$$\exists \alpha \mid \alpha = 3 \tag{1}$$
$$\exists \alpha \mid \alpha^2 = 4 \tag{2}$$
$$\exists \alpha \mid \alpha^2 = 2 \tag{3}$$
$$\exists \alpha \mid \alpha^2 = -1 \tag{4}$$
$$\exists \alpha \mid \alpha^{-1} = 0 \tag{5}$$

Of the five examples, AXIOMS prove only (1) and (2). Proofs are given below in this section. Also, AXIOMS prove the negation of (5); in point of fact, (5) contradicts, and is inconsistent with AXIOM M2. The remaining two, (3) and (4), are neither proved nor negated, and are logically independent of AXIOMS.

Accordingly, instances of α, in (1) and (2), are numbers consistent with AXIOMS and accepted as scalars, proved to *necessarily* exist; the instance of α in (5) is inconsistent with AXIOMS and rejected as *necessarily* non-existent; and instances of α in (3) and (4) are numbers consistent with AXIOMS and accepted as scalars whose existences are not provable, and not necessary, but *possible.*

In the cases of propositions (1) and (2), logical dependence, on AXIOMS, is established by the fact that these propositions (syntactically) derive, directly from AXIOMS. Likewise for the negation of (5). In contrast, however, logical independence of (3) and (4) is not provable by direct derivation because AXIOMS do not assert such information. In essence, that is the whole point of the discussion. What *does* confirm logical independence is a proposition's *truth-table*, viewed from the context of the Soundness Theorem and its converse, the Completeness Theorem. Briefly, Soundness says: if a formula is provable, it will be true, irrespective of whether variables are understood as rational, real or complex (or any other field). Completeness says: if a formula is true, irrespective of how variables are understood, then it will be provable. Consequently, if there is disagreement in a truth-table, jointly, Soundness and Completeness except an *excluded middle* whose formulae are neither provable nor disprovable. This is the predicament of Propositions (3) and (4). Sections 4 – 7 explain the detail.

Proof of (1): that $\exists \alpha \mid \alpha = 3$

$$\forall \beta \forall \gamma \exists \alpha \mid \alpha = \beta + \gamma \qquad \text{AXIOM A0} \tag{6}$$
$$\forall \beta \exists \alpha \mid \alpha = \beta + \beta \qquad \gamma \curvearrowright \beta\,(6) \tag{7}$$
$$\forall \gamma \exists \alpha \mid \gamma = \beta + \beta \qquad \alpha \curvearrowright \gamma\,(7) \tag{8}$$
$$\forall \beta \exists \alpha \mid \alpha = \beta + \beta + \beta \qquad \text{Subst.}\,(8),(6) \tag{9}$$
$$\exists 1 \forall \alpha \mid \alpha \times 1 = \alpha \qquad \text{AXIOM M1} \tag{10}$$
$$\exists \beta \mid \beta = 1 \qquad \text{by}\,(10) \tag{11}$$
$$\exists \alpha \mid \alpha = 1 + 1 + 1 \qquad \text{Subst.}\,(11),(9)) \tag{12}$$

Substitution involving quantifiers

$$\begin{aligned} \forall \beta \underline{\forall \gamma} \exists \alpha \mid\ & \alpha = \beta + \gamma \\ \forall \beta \underline{\exists \gamma} \mid\ & \gamma = \beta + \beta \\ \Rightarrow\ \forall \beta \exists \alpha \mid\ & \alpha = \beta + \beta + \beta \end{aligned}$$

In the example above, an existential quantifier of one proposition must be matched with a universal quantifier of the other. These are highlighted by underlining.

Notation

$\gamma \curvearrowright \beta$ indicates swapping to different bound variable. This is always allowed under the quantifier, so long as all instances are swapped

Proof of (2): that $\exists\alpha \mid \alpha^2 = 4$

$$\forall\beta\forall\gamma\exists\alpha \mid \alpha = \beta + \gamma \qquad \text{AXIOM A0} \quad (13)$$
$$\forall\beta\exists\alpha \mid \alpha = \beta + \beta \qquad \gamma \curvearrowright \beta\,(13) \quad (14)$$
$$\forall\alpha \mid \alpha \times \alpha = \alpha \times \alpha \qquad \text{identity rule} \quad (15)$$
$$\forall\beta\exists\alpha \mid \alpha \times \alpha = (\beta + \beta) \times (\beta + \beta) \qquad \text{Subst.}\,(14),(15) \quad (16)$$
$$\forall\alpha\forall\beta\forall\gamma \mid \alpha \times (\beta + \gamma) = (\alpha \times \beta) + (\alpha \times \gamma) \qquad \text{AXIOM D} \quad (17)$$
$$\forall\beta\exists\alpha \mid \alpha \times \alpha = \beta \times (\beta + \beta) + \beta \times (\beta + \beta) \qquad \text{by}\,(17),(16) \quad (18)$$
$$\forall\beta\exists\alpha \mid \alpha \times \alpha = (\beta \times \beta) + (\beta \times \beta) + (\beta \times \beta) + (\beta \times \beta) \qquad \text{by}\,(17),(18) \quad (19)$$
$$\exists 1\forall\alpha \mid \alpha \times 1 = \alpha \qquad \text{AXIOM M1} \quad (20)$$
$$\exists\beta \mid \beta = 1 \qquad \text{by}\,(20) \quad (21)$$
$$\exists\alpha \mid \alpha \times \alpha = (1 \times 1) + (1 \times 1) + (1 \times 1) + (1 \times 1) \qquad \text{Subst.}\,(21),(19) \quad (22)$$
$$\exists\alpha \mid \alpha \times \alpha = 1 + 1 + 1 + 1 \qquad \text{by}\,(20),(22) \quad (23)$$

4 Soundness and Completeness

Model theory is a branch of Mathematical Logic applying to all first-order theories, and hence to ALGEBRA [4,5]. Our interest is in two standard theorems: the Soundness Theorem and its converse, the Completeness Theorem, and theorems that follow from them. These theorems formalise the link connecting the truth (semantic information) of a formula and its provability (syntactic information). Together, their combined action identifies an *excluded middle*, comprising the set of all non-provable, non-negatable propositions — those that are logically independent of AXIOMS.

Briefly: any given (first-order) axiom-set is *modelled* by particular mathematical structures. That is to say, there are certain structures, consistent with each individual axiom of that axiom-set. In the case of ALGEBRA, these modelling structures are the *infinite fields.* These are closed structures consisting of numbers known as *scalars.* In practical terms, if a proposition is logically independent of AXIOMS, this independence may be diagnosed by demonstrating disagreement on whether the proposition is true – between any two *models.* Of relevance to quantum theory is Proposition (4); this is true in the complex plane, but false in the real line.

Theorem 1 *The Soundness Theorem:*

$$\Sigma \vdash \mathcal{S} \Longrightarrow \forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S})\,. \tag{24}$$

If structure $\mathcal{M}$ models axiom-set Σ and Σ derives sentence $\mathcal{S}$, then every structure $\mathcal{M}$ models $\mathcal{S}$.

Alternatively**:** *If a sentence is a theorem, provable under an axiom-set, then that sentence is true for every model of that axiom-set.*

A *sentence* is a formula where there is no occurrence of any variable not *bound* by a quantifier. For example: $\forall\alpha\forall\beta\,(\alpha + \beta = \beta + \alpha)$.

Theorem 2 *The Completeness Theorem:*

$$\Sigma \vdash \mathcal{S} \Longleftarrow \forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S})\,. \tag{25}$$

If structure $\mathcal{M}$ models axiom-set Σ and every structure $\mathcal{M}$ models sentence $\mathcal{S}$, then *Σ derives sentence $\mathcal{S}$.*

Alternatively**:** *If a sentence is true for every model of an axiom-set, then that sentence is a theorem, provable under that axiom-set.*

5 Logical Dependence

Jointly, Theorems 1 and 2 imply the 2-way implications, in Theorem 3:

Theorem 3 *Soundness And Completeness:*

$$\Sigma \vdash \mathcal{S} \Longleftrightarrow \forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S}) \tag{26}$$
$$\Sigma \vdash \neg\,\mathcal{S} \Longleftrightarrow \forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \neg\,\mathcal{S})\,. \tag{27}$$

*If structure $\mathcal{M}$ models axiom-set Σ, then axiom-set Σ derives sentence $\mathcal{S}$ $(\neg\,\mathcal{S})$, if-and-only-if, all structures $\mathcal{M}$ model sentence $\mathcal{S}$ $(\neg\,\mathcal{S})$***.**

Alternatively**:** *A sentence is provable (disprovable) under an axiom-set, if-and-only-if, that sentence is true (false) for all models of that axiom-set.*

Remark For every provable sentence $\mathcal{S}$ there is a provable sentence $\neg\,\mathcal{S}$. We might normally think of these as the disprovable or negatable sentences. Both types of sentence are covered by Theorem 3 because the set $\{\mathcal{S}\}$ already includes all sentences $\neg\,\mathcal{S}$.

6 Logical Independence

Of special interest are those sentences, not covered by Theorem 3. They are the sentences to which the Soundness and Completeness Theorems do not apply. They constitute an *excluded middle,* not included in (26) nor (27), comprising sentences, *neither provable nor disprovable.*

Happily, whereas there is no suggestion of any excluded middle in the left hand sides of (26) and (27), the right hand sides jointly define one. It is the set of sentences $\mathcal{S}$ excluded by the right hand sides of both (26) and (27), thus:

$$\{\mathcal{S} \mid \neg\forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S}) \wedge \neg\forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \neg\,\mathcal{S})\}\,. \tag{28}$$

By writing the negations of (26) and (27), thus:

$$\neg\,(\Sigma \vdash \mathcal{S}) \iff \neg\forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S}) \tag{29}$$

$$\neg\,(\Sigma \vdash \neg\,\mathcal{S}) \iff \neg\forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \neg\,\mathcal{S}) \tag{30}$$

we may match the set of sentences specified in (28) with its corresponding left side, so as to construct:

$$\begin{aligned} &\neg\,(\Sigma \vdash \mathcal{S}) \wedge \neg\,(\Sigma \vdash \neg\,\mathcal{S}) \\ \iff &\neg\forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \mathcal{S}) \wedge \neg\forall\mathcal{M}\,(\mathcal{M} \models \Sigma \Rightarrow \mathcal{M} \models \neg\,\mathcal{S}) \end{aligned} \tag{31}$$

This includes all sentences excluded by (26) and (27). It limits sentences that are neither provable nor negatable (on the left) to those that are neither true nor false, across all structures that model the axiom-set (on the right).

For theories whose axiom-set is modelled by more than one single structure – where $\mathcal{M}_1$ and $\mathcal{M}_2$ are distinct, (31) implies:

Theorem 4 *The logically independent, excluded middle:*

$$\begin{aligned} &\neg\,(\Sigma \vdash \mathcal{S}) \wedge \neg\,(\Sigma \vdash \neg\,\mathcal{S}) \\ \iff &\exists\mathcal{M}_1\,(\mathcal{M}_1 \models \Sigma \wedge \mathcal{M}_1 \models \mathcal{S}) \wedge \exists\mathcal{M}_2\,(\mathcal{M}_2 \models \Sigma \wedge \mathcal{M}_2 \models \neg\,\mathcal{S}) \end{aligned} \tag{32}$$

Axiom-set Σ derives neither sentence $\mathcal{S}$ nor its negation, if-and-only-if, there exist structures $\mathcal{M}_1$ and $\mathcal{M}_2$ *which each model axiom-set Σ, such that $\mathcal{M}_1$ models $\mathcal{S}$, and $\mathcal{M}_2$ models the negation of $\mathcal{S}$.*

Alternatively**:** *A sentence is true for some but not all models of an axiom-set, if-and-only-if, that sentence is logically independent of that axiom-set.*

7 Independence in Elementary Algebra

Here, the above Theorems are applied specifically to Elementary Algebra. The result is Theorem 5, a practical test, performed — *by inspection* — telling us when a proposition is logically independent. The test is performed by examining a proposition's truth-table. To illustrate, Table 2 lists the five truth-tables for propositions (1) to (5). The T and F entries are answers to the question: is the adjacent proposition, True or False, for the different interpretations on the variable α?

Theorem 5 ***Logical Independence*** *(of* AXIOMS*) is demonstrated if a proposition is* True *while its variables are interpreted as members of one infinite-field, but* False *when interpreted as members of a different infinite-field.*

Proof Sructures modelling AXIOMS include the field of complex numbers $\mathbb{C}$, the field of real numbers $\mathbb{R}$ and the field of rational numbers $\mathbb{Q}$. Hence, by Theorem 4, disagreement between those fields implies logical independence.

Proposition	Interpretations		
	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha \mid \alpha = 3$	T	T	T
$\exists\alpha \mid \alpha \times \alpha = 4$	T	T	T
$\exists\alpha \mid \alpha \times \alpha = 2$	T	T	F
$\exists\alpha \mid \alpha \times \alpha = -1$	T	F	F
$\exists\alpha \mid \alpha^{-1} = 0$	F	F	F

Table 2 Truth-tables for some existential propositions. T and F denote True and False. The T and F entries are answers to the question: is the proposition to the left, True or False, for the interpretation above? Disagreement along a row confirms the proposition's independence.

Conclusion

The premise of this paper is that mathematical physics rests on a foundation of Elementary Algebra, and in doing so, inherits the information it contains or conveys. The approach taken in this paper, treats Elementary Algebra as a formal axiomatised system, in order to expose logical information that might be passed into quantum mathematics.

Applying the Soundness and Completeness Theorems to this axiomatised system, the paper shows that the imaginary unit within Elementary Algebra exists logically independently of the algebra's axioms. By extension, any imaginary or complex scalar is also logically independent.

Elementary Algebra is a theory that cannot be completed and is therefore subject to Gödel's Incompleteness Theorems.

Ongoing research

Standard quantum theory has a further axiom, on top of Elementary Algebra, which imposes unitarity (or self-adjointness) – *by Postulate*. This postulate is syntactical information that conflicts with semantical information, already allowed by Elementary Algebra. It might be said to obliterate certain of semantical information of significance. Importantly, it blocks and destroys logical independence of imaginary scalars.

There is no suggestion that unitarity (or self-adjointness) is not needed. The suggestion is that it should be semantical, rather than syntactical information. If unitary information can be shown to emerge semantically out of *quantum mathematics,* without being imposed as a Physical Principle, without the need for it being imposed *by Postulate* — rendering redundant, this unitarity *by Postulate* — then logical independence from Elementary Algebra would freely enter quantum mathematics.

Further, if that were to be possible, the prospect would open up of finding a theoretical link, directly connecting logical independence in Elementary Algebra, with logical independence in Boolean propositions, used by Tomasz Paterek et al [8, 9] and hence, a link with quantum randomness.

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