Beyond Set theory in Bell inequality

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Abstract

Feynman pointed out on a logic and mathematical paradox in particle physics. The paradox is that we get for the same entity only local dependence and global dependence at the time. This contradiction is coming from the dual nature of the particle viewed as a wave. In the first capacity it has only local dependence in the second (wave) capacity it has a global dependence. The classical logic has difficulties to resolve this paradox. Changing the classical logic to logic makes the paradox apparent. Particle has the local property or zero dependence with other particles, media has total dependence so is a global unique entity. Now, in set theory, any element is independent from the other so disjoint set has not element in common. With this condition we have that the true false logic can be applied and set theory is the principal foundation. Now with conditional probability and dependence by copula the long distance dependence has effect on any individual entity that now is not isolate but can have different type of dependence or synchronism (constrain) which effect is to change the probability of any particle. So particle with different degree of dependence can be represented by a new type of set as fuzzy set in which the boundary are not completely defined or where we cannot separate a set in its parts as in the evidence theory. In conclusion the Feynman paradox and Bell violation can be explained at a new level of complexity by many valued logic and new type of set theory.

1. Bell inequality

Let X = $\{x_1, x_2, ..., x_n\}$ be a set of elements with joint probabilities

$$p_j(x_j),\ p_{i,j}(x_i, x_j),\ p_{i,j,k}(x_i, x_j, x_k).......$$

Example, consider $X = \{x_1, x_2, x_3\}$ and a power set

$$2^X = \{\varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}$$

In the classical probability calculus this power set can be rewritten as

$$2^X = \left\{ \begin{array}{l} \varnothing, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\} = \{x_1\} \cup \{x_2\}, \{x_1, x_3\} = \{x_1\} \cup \{x_3\} \\ , \{x_2, x_3\} = \{x_2\} \cup \{x_3\}, \{x_1, x_2, x_3\} = \{x_1\} \cup \{x_2\} \cup \{x_3\} \end{array} \right\}$$

Also in the classical probability theory $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ and the probability of a set is the sum of probabilities of its elements:

$$\left\{ \begin{array}{l} p(\varnothing) = 0,\, p(\{x_1\}),\, p(\{x_2\}),\, p(\{x_3\}),\, p(\{x_1, x_2\}) = p(\{x_1\}) + p(\{x_2\}),\, p(\{x_1, x_3\}) = p(\{x_1\}) + p(\{x_3\}) \\ ,\, p(\{x_2, x_3\}) = p(\{x_2\}) + p(\{x_3\}),\, p(\{x_1, x_2, x_3\}) = p(\{x_1\}) + p(\{x_2\}) + p(\{x_3\}) \end{array} \right\}$$

because the intersection of elementary events is empty.

$$S = \begin{bmatrix} A \cap B \cap C \\ A^C \cap B \cap C \\ A \cap B^C \cap C \\ A \cap B \cap C^C \\ A^C \cap B^C \cap C \\ A^C \cap B \cap C^C \\ A \cap B^C \cap C^C \\ A^C \cap B^C \cap C^C \end{bmatrix}$$

In a graphic way it is shown in Fig. 1

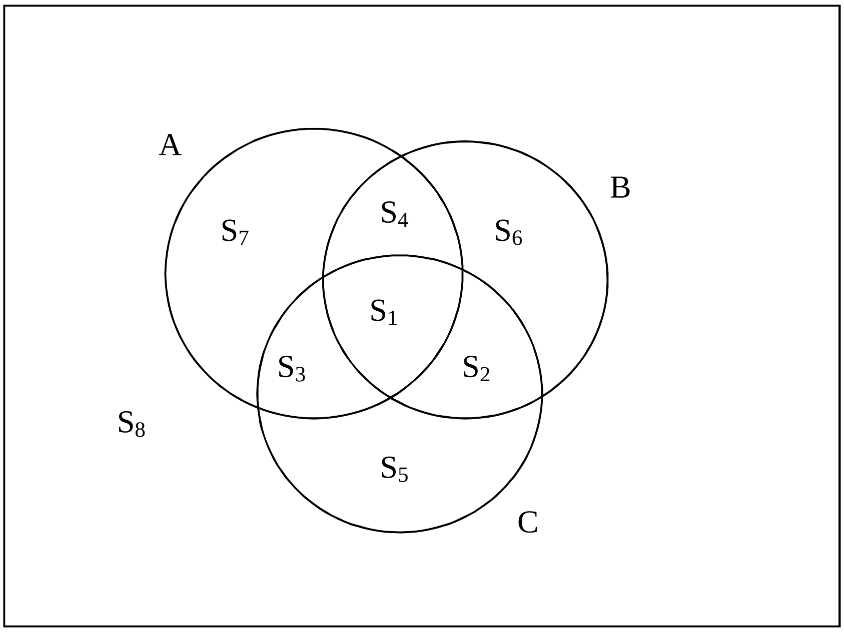


Figure 1. Set theory intersections or elements

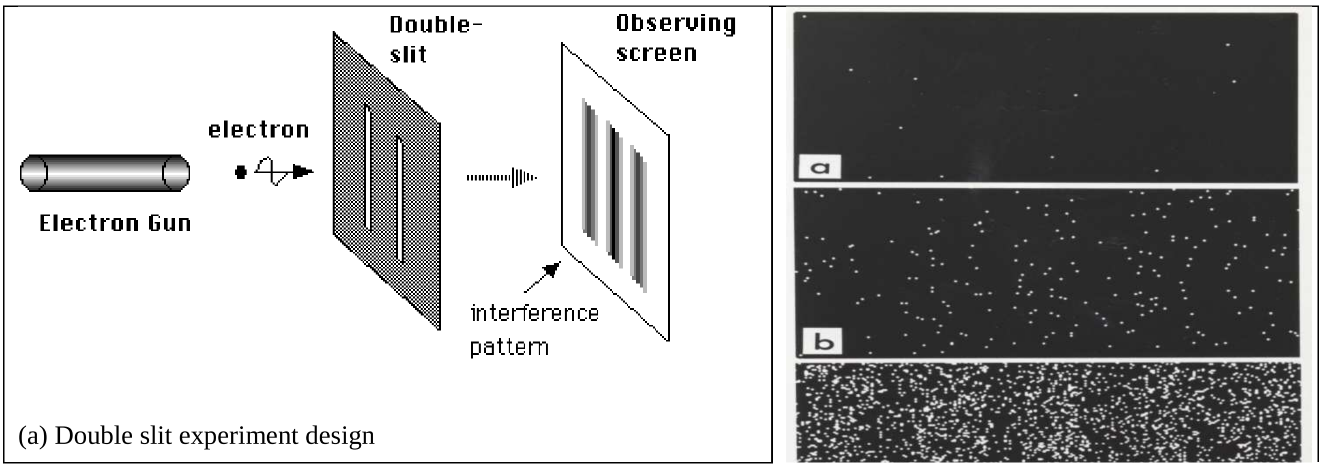
These sets have the following Bell inequality.

$$(A \cap B^C) \cup (B \cap C^C) = S_1 \cup S_7 \cup S_4 \cup S_6$$
$$A \cap C^C = S_7 \cup S_4$$
$$A \cap C^C \subseteq (A \cap B^C) \cup (B \cap C^C)$$
and
$$\left| A \cap C^C \right| \le \left| (A \cap B^C) \cup (B \cap C^C) \right|$$

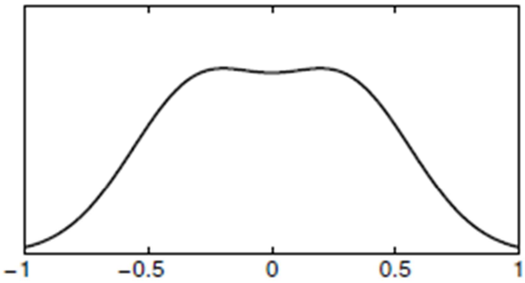
Now we introduce a *dependence* between events. Consider an event with property A and another event with a negated property A^C. These events can be called dependent (correlated). This dependence takes place for particles. Consider an event with property $A \cap C^C$ that is with both properties A and C^C at the same time. We cannot measure the two properties by using one instrument at the same time, but we can use the correlation to measure the second property if two properties are correlated. We can also view an event with property $A \cap C^C$ as two events: event e_A with property A and e_{C^C} with the property C in the opposite state (negated). The number of pairs of events (e_A, e_{C^C}) is the same as the number of events with the superposition of A and C^C, $A \cap C^C$. In this d'Espagnat explains the connection between the set theory and Bell's inequality. It is known that the Bell's inequality that gives us the reality condition is violated . Conclusion. The Bell inequality is based on the classical set theory that is connected with the classical logic. The set theory assumes empty overlap (as a form of independence) of elementary elements which is the basis for the Bell inequality. Thus the logic of dependence can differ from the logic of independence. Thus we must use a theory beyond the classical set theory.

2. Dependence and independence in the double slit experiment as physical image of copula and fuzzy

The goal of this section is to analyze the double slits experiment [Feynman, 1988] as a demonstration of the need to build a separate theory to deal with dependent/related evens under uncertainty. The design and results of the double slits experiment is outlined in Fig. 5a,b [Double-slit experiment, 2015], where points in Fig 2b show particles (elementary probability event) that pass slits.



(a) Double slit experiment design



(c) Distribution of independent particles (events)

(b) Result of double slit experiment: electron buildup over time

Figure 2.

Fig. 5c shows theoretical result of the double slit expreiment when only the set theory is used to combine events: one event e_1 for one slid and another event e_2 for the second slid. In this set-theoretical approach it is assumed that events e_1 and e_2 are *elementary events* that do not overlap (have empty intersection, “incompatible”, completely independent). In this case, the probability that either one of these two events will occur is

$$p(e_1 \cup e_2) = p(e_1) + p(e_2)$$

In classical logic it is always true that variable is self-dependent (that is the repeat of the process produces the same result). In the probability calculus it is not the case. The random factors can change the output when the situation is repeated. Quite often the probabilistic approach is applied to study frequency of independent phenomena. In the case of dependent variables we cannot derive $p(x_1,x_2)$ as a product of independent probabilities, $p(x_1)p(x_2)$ and must use multidimensional probability distribution with dependent valuables. The common technique for modeling it is a Bayesian network. In the Bayesian approach the evidence about the true state of the world is expressed in terms of degrees of belief in the form of Bayesian conditional probabilities. The conditional probability is the main element to express the dependence or inseparability of the two states x_1 and x_2 in the probability theory. The joint probability $p(x_1,x_2,\ldots,x_n)$ is represented via multiple conditional probabilities to express the dependence between variables. The *copula approach* introduces a *single* function $c(u_1,u_2)$ denoted as *density of copula* as a way to model the *dependence* or *inseparability* of the variables with the following property in the case of two variables. The copula allows representing the joint probability $p(x_1,x_2)$ as a combination (product) of single dependent part $c(u_1,u_2)$ and independent parts: probabilities $p(x_1)$ and $p(x_2)$.The investigation of copulas and their applications is a rather recent subject of mathematics. From one point of view, copulas are functions that join or 'couple' one-dimensional distribution functions u_1 and u_2 and the corresponding joint distribution function.

3. Conditional probability ,dependence in probability calculus and copula

A ***joint probability distribution***

$$p(x_1 x_2,.....x_n) = p(x_1)\,p(x_2 \mid x_1)\,p(x_3 \mid x_2, x_1).......p(x_n \mid x_1, x_2,...., x_{n-1})\ ,$$

e.g., for two variables $p(x_1, x_2) = p(x_1)\,p(x_2 \mid x_1)$. A function $c(u_1, u_2)$ is a ***density of copula***

if $p(x_1,x_2) = c(u_1,u_2)p(x_1)p(x_2) = p(x_1)p(x_2|x_1)$

where $u_1=\int p(x_1)\, dx_1$ and $u_1=\int p(x_2)\, dx_2$.

A ***cumulative function*** C with inverse functions $x_i(u_i)$ as arguments:

$$C(x_1(u_1), x_2(u_2)) = \int p(x_1, x_2)dx_1 dx_2 = \int c[u_1(x_1), u_2(x_2)]\,p(x_1)\,p(x_2)dx_1 dx_2 = \int p(x_1)\,p(x_2 \mid x_1)dx_1 dx_2$$

where $p(x_1) = \dfrac{du_1}{dx_1}.p(x_2) = \dfrac{du_2}{dx_2}$

and respectively ***inverse functions*** $u_1(x_1) = \int p(x_1)dx_1, u_2(x_2) = \int p(x_2)dx_2$

An alternative representation of a ***cumulative function*** C

$$C(x_1(u_1), x_2(u_2)) = C(u_1, u_2) = \int c(u_1, u_2)\,p(x_1)dx_1\,p(x_2)dx_2 = \int c(u_1, u_2)du_1 du_2$$

and

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}$$

Copula properties [13.14.15.16.17].

2-D case

$$p(x_1)\,p(x_2 \mid x_1) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}\,p(x_1)\,p(x_2)\ , \qquad p(x_2 \mid x_1) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}\,p(x_2)$$

3-D case

$$p(x_1, x_2, x_n) = p(x_1)\,p(x_2 \mid x_1)\,p(x_3 \mid x_2, x_1) = c(u_1, u_2, u_3)\,p(x_1)\,p(x_2)\,p(x_3)$$

$$C(u_1,u_2,u_3)=\int p(x_1,x_2,x_3)dx_1dx_2dx_3=\int c(u_1,u_2,u_3)\,p(x_1)\,p(x_2)\,p(x_3)dx_1dx_2dx_3=$$
$$\int p(x_1)\,p(x_2\mid x_3)\,p(x_3\mid x_2,x_3)dx_1dx_2dx_3$$

for $p(x_1)=\dfrac{du_1}{dx_1}.p(x_2)=\dfrac{du_2}{dx_2},\,p(x_3)=\dfrac{du_3}{dx_2}$

$$C(u_1,u_2,u_3)=\int c(u_1,u_2,u_3)\,p(x_1)dx_1\,p(x_2)dx_2\,p(x_3)dx_3=\int c(u_1,u_2,u_3)du_1du_2du_3$$

so

$$c(u_1,u_2,u_3)=\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}$$

$$p(x_1,x_2,x_n)=p(x_1)\,p(x_2\mid x_1)\,p(x_3\mid x_2,x_1)=\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}\,p(x_1)\,p(x_2)\,p(x_3)$$

and

$$p(x_2\mid x_1)\,p(x_3\mid x_2,x_1)=\frac{\partial^2 C(u_1,u_2,u_3)}{\partial u_1\partial u_2}\,p(x_2)\,p(x_3\mid x_2,x_1)=\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}\,p(x_2)\,p(x_3)$$

and

$$\frac{\partial^2 C(u_1,u_2,u_3)}{\partial u_1\partial u_2}\,p(x_3\mid x_2,x_1)=\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}\,p(x_3)$$

and

$$p(x_3\mid x_2,x_1)=\frac{\partial^3 C(u_1,u_2,u_3)}{\partial u_1\partial u_2\partial u_3}\,\frac{1}{\dfrac{\partial^2 C(u_1,u_2,u_3)}{\partial u_1\partial u_2}}\,p(x_3)$$

General n-D case

$$p(x_n\mid x_1,x_2,\ldots.,x_{n-1})=\frac{\partial^n C(u_1,u_2,\ldots,u_n)}{\partial u_1,\ldots,\partial u_n}\,\frac{1}{\dfrac{\partial^{n-1} C(u_1,u_2,\ldots.,u_n)}{\partial u_1,\ldots,\partial u_{n-1}}}\,p(x_n)$$

Conditional copula:

$$c(x_n\mid x_1,x_2,\ldots.,x_{n-1})=\frac{\partial^n C(u_1,u_2,\ldots,u_n)}{\partial u_1,\ldots,\partial u_n}\,\frac{1}{\dfrac{\partial^{n-1} C(u_1,u_2,\ldots.,u_n)}{\partial u_1,\ldots,\partial u_{n-1}}}$$

where $\dfrac{\partial^n C(u_1,u_2,\ldots,u_n)}{\partial u_1,\ldots,\partial u_n}=c(u_1,u_2,\ldots.,u_n)$

In literature commonly $C(u_1,u_2,...,u_n)$ is denoted as *copula* and $c(u_1,u_2,....,u_n)$ is denoted as a *density of copula*.

4. Examples of Copula and dependence

When u(x) is a marginal probability F(x), u(x) = F(x) and u is uniformly distributed then the inverse function x(u) is not uniformly distributed, but has values concentrated in the central part as the Gaussian distribution. The inverse process is represented graphically in Figures1-3.

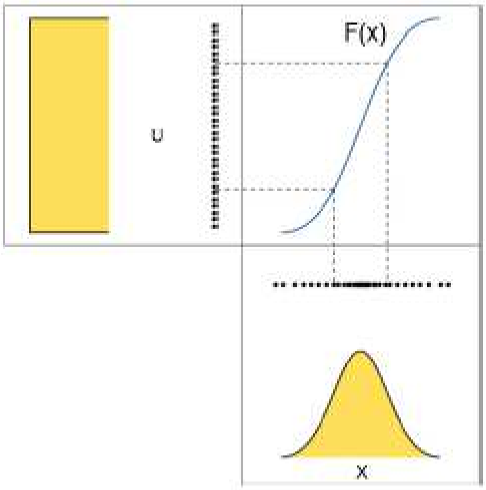


Figure 1. Relation between marginal probability F(x) and the random variable x

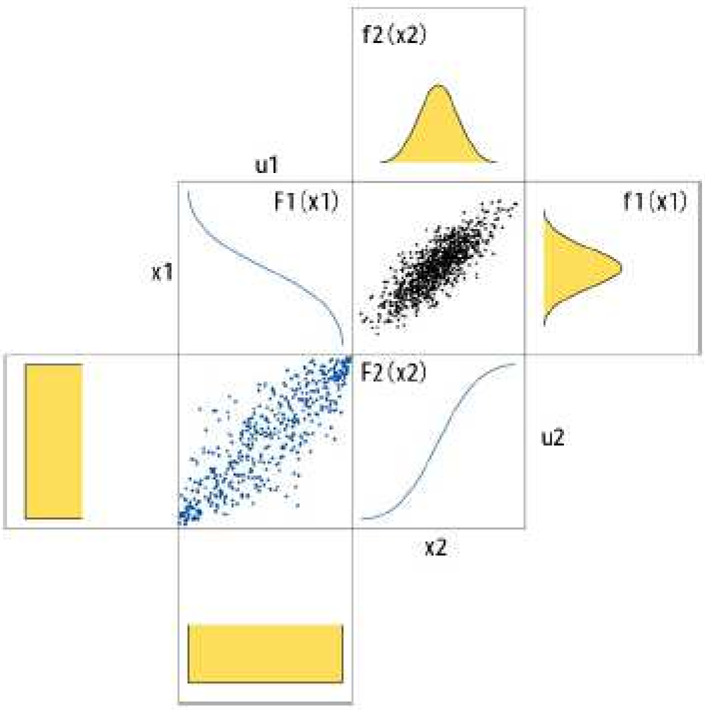


Figure 2. Symmetric joint probability and copula

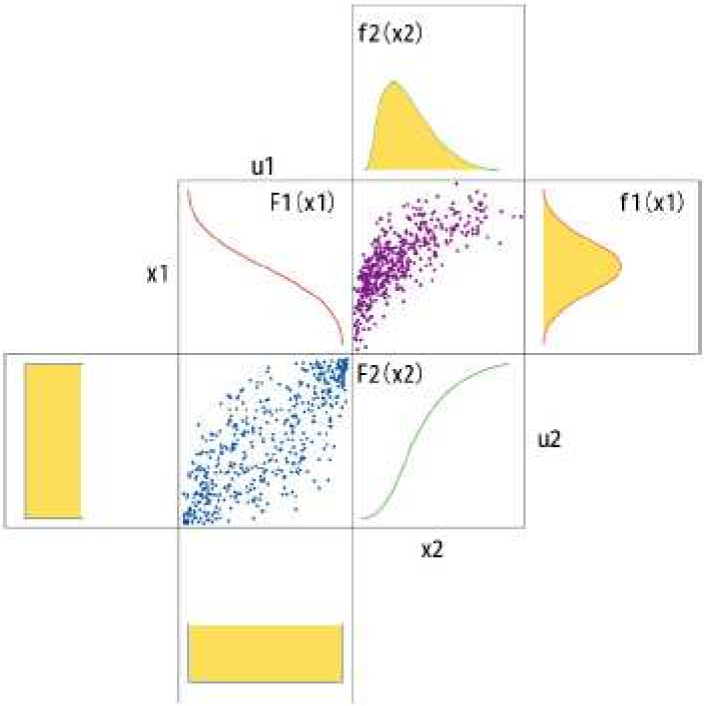


Figure 3 Asymmetric joint probability and symmetric copula

Consider another example where a joint probability density function p is defined in the two dimensional interval $(0,2)\times(0,1)$ as follows,

$$p(x, y) = \frac{x+y}{3}$$

Then the marginal function in this interval is

$$C(x, y) = \int p(x, y)dxdy = \int \frac{x+y}{3}dxdy = \frac{xy(x+y)}{6} \qquad (1)$$

Next we change the reference

$$p(x_1) = \frac{du_1}{dx_1}.p(x_2) = \frac{du_2}{dx_2}$$

and use the marginal probabilities

$$u_1 = \int p(x)dx, u_2 = \int p(y)dy$$

to get

$$u_1(x) = C(x,1) = \frac{x(x+1)}{6}, \quad x \in [0,2]$$

$$u_2 = C(2, y) = \frac{y(y+2)}{6} \quad y \in [0,1]$$

This allows us computing the inverse function to identify variables x and y as functions of the marginal functions u_1 and u_2:

$$x(u_1) = \frac{\sqrt{1+24u_1}-1}{2}$$
$$y(u_2) = \sqrt{1+3u_2}-1$$

Then these values are used to compute the copula C in function (1),

$$C(u_1,u_2) = \frac{\sqrt{1+24u_1}-1}{2}(\sqrt{1+3u_2}-1)\frac{\frac{\sqrt{1+24u_1}-1}{2}+(\sqrt{1+3u_2}-1)}{6} \quad (2)$$
$$= \frac{\sqrt{1+24u_1}-1}{2}(\sqrt{1+3u_2}-1)\frac{\frac{\sqrt{1+24u_1}-1}{2}+(\sqrt{1+3u_2}-1)}{6}$$

5. Physical Paradox and physical meaning of Copula and fuzzy theory

Feynman's argument [25] involves the idea that classically we think in terms of two distinct and incompatible concepts , particles and waves. These concepts are incompatible because particles are localized and waves are not. To see this, let us start with a point particle or *elementary event*. In classical mechanics, particles are objects localized in space, and therefore, can only interact with systems that local for them. If a particle then collides with another particle, say constituent of a wall placed in the way of the original particle, an interaction will occur. However, as soon as the particle loses contact with the wall, the interaction ceases to exist. In other words, particles interact locally or have *local* not *global dependence*. The second basic concept is the concept of waves. Historically, the physics describing a point particle was extended to include the description of *continuous media*, and, more importantly to our current discussion, the vibrations of such media in the form of waves. Therefore, *waves* were considered *vibrations of a medium* made out of *several* point particles, and the local interactions between two neighboring particles would allow for a perturbation in one point of the medium to be propagated to another point of the medium. More importantly, such effect depends not only on the position of the particle, but also possibly of *all other* particles or elementary events that make up the medium, and also on all interactions or boundary conditions that such particles need to satisfy. In other words, waves interact *non-locally*.
Thus, a media and the *wave* give an example of *total (global) dependence* in contrast with the particle. The paradox is that an element (a particle) has a property (global dependence) of the whole media. This is impossible in the classical logic. The global dependence (non-local interaction of the whole system) is a property of the structure of the media. An element cannot have such a property of the whole system because an element has no structure. To explain why the paradox is only apparent we start from Kolmogorov's probability measure that is defined at the level of propositional classical logic and set theory.

5.1 Probability Space

Let Ω be a finite set, F be an algebra over Ω and p be a real-valued function, p : $F \rightarrow R$. Then (Ω, F, p) is a *probability space* [Kolmogorov, 1950], and p a *probability measure*, if and only if:

K1. $0 \leq p(\{\omega_i\}) \leq 1, \forall \omega_i \in \Omega$

K2. $p(\Omega) = 1$

K3. $p(\{\omega_i, \omega_j\}) = p(\{\omega_i\}) + p(\{\omega_j\})$

The elements ω_i of Ω are called elementary probability events or simply elementary events. The elementary events are disjoint sets. Given two sets of elementary probability events A and B the intersection of the two events is given by the expression

$$p(A \cap B) = p(A, B) = P(A)P(B \mid A)$$

When the two sets of events are independent we have

$$p(A \cap B) = p(A, B) = P(A)P(B)$$

with a trivial density of copula, c(A,B) = 1.

$p(e_1 \cup e_2)) = p(\text{Aor B})$ = p(e1)+p(e2) + p(e1 and e2)

Now when the events ar dijoint one with the other we have

$p(e_1 \cup e_2)) = p(\text{Aor B})$ = p(e1) + p(e2)

The real joint probability for double slit experiment by quantum mechancis is

$$p(\alpha_1, \alpha_2) = k \cos^2(\alpha_1 - \alpha_2)$$

for which copula is

$$C(F(\alpha_1),F(\alpha_2)) = k\sin^2[\frac{1}{2}(\frac{\frac{\pi}{2} \pm \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_1(\alpha_1)})}}{2})^2 \frac{\frac{\pi}{2} \mp \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_2(\alpha_2)})}}{2} +$$

$$-(\frac{\frac{\pi}{2} \mp \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_2(\alpha_2)})}}{2})^2 \frac{\frac{\pi}{2} \pm \sqrt{(\frac{\pi}{2})^2 + \frac{16}{\pi}\arccos(\frac{1}{k}\sqrt{F_1(\alpha_1)})}}{2})] = C(u_1,u_2) \qquad (16)$$

This copula is tabulated as follows:

$$\mathrm{M1} = \begin{pmatrix} 1 & 0.846 & 0.7 & 0.561 & 0.43 & 0.307 & 0.192 & 0.088 & 0 \\ 0.846 & 1 & 0.969 & 0.899 & 0.805 & 0.692 & 0.56 & 0.401 & 0.125 \\ 0.7 & 0.969 & 1 & 0.979 & 0.923 & 0.839 & 0.725 & 0.572 & 0.25 \\ 0.561 & 0.899 & 0.979 & 1 & 0.982 & 0.93 & 0.843 & 0.708 & 0.375 \\ 0.43 & 0.805 & 0.923 & 0.982 & 1 & 0.982 & 0.927 & 0.82 & 0.5 \\ 0.307 & 0.692 & 0.839 & 0.93 & 0.982 & 1 & 0.98 & 0.909 & 0.625 \\ 0.192 & 0.56 & 0.725 & 0.843 & 0.927 & 0.98 & 1 & 0.973 & 0.75 \\ 0.088 & 0.401 & 0.572 & 0.708 & 0.82 & 0.909 & 0.973 & 1 & 0.875 \\ 0 & 0.125 & 0.25 & 0.375 & 0.5 & 0.625 & 0.75 & 0.875 & 1 \end{pmatrix}.$$

$$y = x_1 \equiv x_2 = \begin{bmatrix} & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (17)$$

Now for the dependence element as copula we have that set theory is not sufficient because two disjoint sets can have a probability (evidence) different from the traditional formula

In a graphic way we see the traditional set theory with dependences by arrows

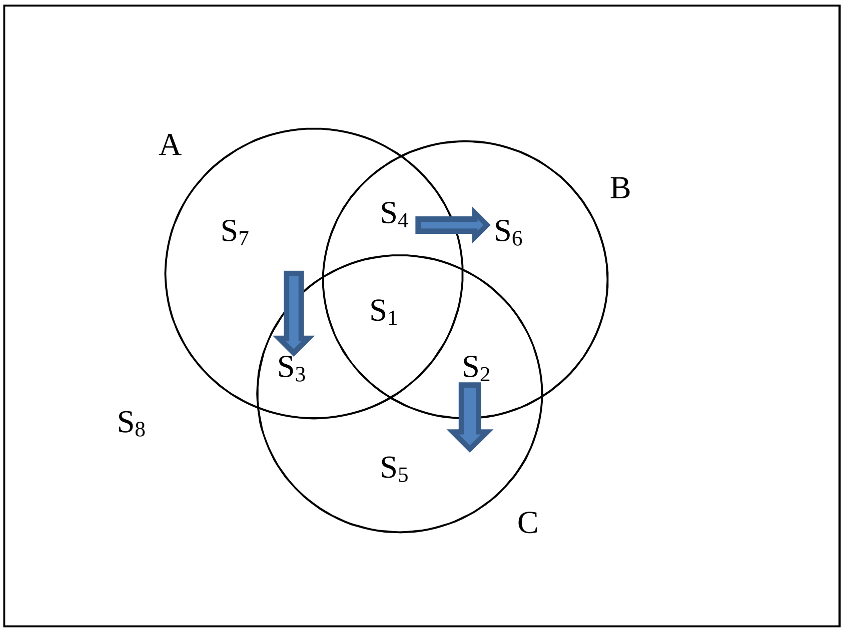


Figure 4. Set theory intersections or elements with dependence

Extension of the set theory by evidence theory in quantum mechanics can be found in the paper of

Germano Resconi and others International Journal of Modern Physics C. Vol. 10 No 1 (1999) 29-62

Conclusion

Feynman pointed out on a logic and mathematical paradox in particle physics [1]. The paradox is that we get for the same entity only local dependence and global dependence at the time.

This contradiction is coming from the dual nature of the particle viewed as a wave. In the first capacity it has only local dependence in the second (wave) capacity it has a global dependence. The classical logic has difficulties to resolve this paradox. Changing the classical logic to logic makes the paradox apparent. Particle has the local property or zero dependence with other particles, media has total dependence so is a global unique entity. Now, in set theory, any element is independent from the other so disjoint set has not element in common. With this condition we have that the true false logic can be applied and set theory is the principal foundation. Now with conditional probability and dependence by copula the long distance dependence has effect on any individual entity that now is not isolate but can have different type of dependence or synchronism (constrain) which effect is to change the probability of any particle. So particle with different degree of dependence can be represented by a new type of set as fuzzy set in which the boundary are not completely defined or where we cannot separate a set in its parts as in the evidence theory. In conclusion the Feynman paradox and Bell violation can be explained at a new level of complexity by many valued logic and new type of set theory.

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