A Neutrosophic Description Logic

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Abstract—Description Logics (DLs) are appropriate, widely used, logics for managing structured knowledge. They allow reasoning about individuals and concepts, i.e., set of individuals with common properties. Typically, DLs are limited to dealing with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership: we may say that an individual is an instance of a concept only to a certain degree, depending on the individual’s properties. The DLs that deal with such fuzzy concepts are called fuzzy DLs. In order to deal with fuzzy, incomplete, indeterminate and inconsistent concepts, we need to extend the capabilities of fuzzy DLs further.

In this paper we will present an extension of fuzzy ACC, combining Smarandache’s neutrosophic logic with a classical DL. In particular, concepts become neutrosophic (here neutrosophic means fuzzy, incomplete, indeterminate and inconsistent), thus, reasoning about such neutrosophic concepts is supported. We will define its syntax, its semantics, describe its properties.

Index Terms—neutrosophic logic, description logic, neutrosophic description logic, fuzzy description logic

I. INTRODUCTION

The modelling and reasoning with uncertainty and imprecision is an important research topic in the Artificial Intelligence community. Almost all the real world knowledge is imperfect. A lot of works have been carried out to extend existing knowledge-based systems to deal with such imperfect information, resulting in a number of concepts being investigated, a number of problems being identified and a number of solutions being developed [1], [2], [3], [4].

Description Logics (DLs) have been utilized in building a large amount of knowledge-based systems. DLs are a logical reconstruction of the so-called frame-based knowledge representation languages, with the aim of providing a simple well-established Tarski-style declarative semantics to capture the meaning of the most popular features of structured representation of knowledge. A main point is that DLs are considered as to be attractive logics in knowledge based applications as they are a good compromise between expressive power and computational complexity.

Nowadays, a whole family of knowledge representation systems has been build using DLs, which differ with respect to their expressiveness, their complexity and the completeness of their algorithms, and they have been used for building a variety of applications [5], [6], [7], [8].

The classical DLs can only deal with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership. There are many works attempted to extend the DLs using fuzzy set theory [9], [10], [11], [12], [13], [14]. These fuzzy DLs can only deal with fuzzy concepts but not incomplete, indeterminate, and inconsistent concepts (neutrosophic concepts). For example, “Good Person” is a neutrosophic concepts, in the sense that by different subjective opinions, the truth membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.6, which is inconsistent, or the truth-membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.3, which is incomplete.

The set and logic that can model and reason with fuzzy, incomplete, indeterminate, and inconsistent information are called neutrosophic set and neutrosophic logic, respectively [15], [16]. In Smarandache’s neutrosophic set theory,a neutrosophic set $A$ defined on universe of discourse $X$, associates each element $x$ in $X$ with three membership functions: truth membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$, where $T_A(x)$, $I_A(x)$, $F_A(x)$ are real standard or non-standard subsets of $[0,1]$, and $T_A(x)$, $I_A(x)$, $F_A(x)$ are independent. For simplicity, in this paper, we will extend Straccia’s fuzzy DLs [9], [11] with neutrosophic logic, called neutrosophic DLs, where we only use two components $T_A(x)$ and $F_A(x)$, with $T_A(x) \in [0,1]$, $F_A(x) \in [0,1]$, $0 \leq T_A(x) + F_A(x) \leq 2$. The neutrosophic DLs is based on the DL ACC, a significant and expressive representative of the various DLs. This allows us to adapt it easily to the different DLs presented in the literature. Another important point is that we will show that the additional expressive power has no impact from a computational complexity point of view. The neutrosophic ACC is a strict generalization of fuzzy ACC, in the sense that every fuzzy concept and fuzzy terminological axiom can be represented by a corresponding neutrosophic concept and neutrosophic terminological axiom, but not vice versa.

II. A QUICK LOOK TO FUZZY ACC

We assume three alphabets of symbols, called atomic concepts (denoted by $A$), atomic roles (denoted by $R$) and individuals (denoted by $a$ and $b$). ¹

¹Through this work we assume that every metavariable has an optional subscript or superscript.
A concept (denoted by \( C \) or \( D \)) of the language \( \mathcal{ALC} \) is built out of atomic concepts according to the following syntax rules:

\[
\begin{align*}
& C, D \quad \rightarrow \quad T \quad \text{(top concept)} \\
& \bot \quad \text{(bottom concept)} \\
& A \quad \text{(atomic concept)} \\
& C \sqcap D \quad \text{(concept conjunction)} \\
& C \sqcup D \quad \text{(concept disjunction)} \\
& \neg C \quad \text{(concept negation)} \\
& \forall R.C \quad \text{(universal quantification)} \\
& \exists R.C \quad \text{(existential quantification)}
\end{align*}
\]

Fuzzy DL extends classical DL under the framework of Zadeh’s fuzzy sets [17]. A fuzzy set \( S \) with respect to an universe \( U \) is characterized by a membership function \( \mu_S : U \rightarrow [0, 1] \), assigning an \( S \)-membership degree, \( \mu_S(u) \), to each element \( u \) in \( U \). In fuzzy DL, (i) a concept \( C \), rather than being interpreted as a classical set, will be interpreted as a fuzzy set and, thus, concepts become fuzzy; and, consequently, (ii) the statement “\( a \) is \( C \)”, i.e. \( C(a) \), will have a truth-value in \( [0, 1] \) given by the degree of membership of being the individual \( a \) a member of the fuzzy set \( C \).

A. Fuzzy Interpretation

A fuzzy interpretation is now a pair \( \mathcal{I} = (\Delta^\mathcal{I}, \mathcal{I}) \), where \( \Delta^\mathcal{I} \) is, as for the crisp case, the domain, whereas \( \mathcal{I} \) is an interpretation function mapping

1) individual as for the crisp case, i.e. \( a^\mathcal{I} \neq b^\mathcal{I} \) if \( a \neq b \);
2) a concept \( C \) into a membership function \( C^\mathcal{I} : \Delta^\mathcal{I} \rightarrow [0, 1] \);
3) a role \( R \) into a membership function \( R^\mathcal{I} : \Delta^\mathcal{I} \times \Delta^\mathcal{I} \rightarrow [0, 1] \).

If \( C \) is a concept then \( C^\mathcal{I} \) will naturally be interpreted as the membership degree function of the fuzzy concept (set) \( C \) w.r.t. \( \mathcal{I} \), i.e. if \( d \in \Delta^\mathcal{I} \) is an object of the domain \( \Delta^\mathcal{I} \) then \( C^\mathcal{I}(d) \) gives us the degree of being the object \( d \) an element of the fuzzy concept \( C \) under the interpretation \( \mathcal{I} \). Similarly for roles. Additionally, the interpretation function \( \mathcal{I} \) has to satisfy the following equations: for all \( d \in \Delta^\mathcal{I} \),

\[
\begin{align*}
\top^\mathcal{I}(d) &= 1 \\
\bot^\mathcal{I}(d) &= 0 \\
(C \sqcap D)^\mathcal{I}(d) &= \min\{C^\mathcal{I}(d), D^\mathcal{I}(d)\} \\
(C \sqcup D)^\mathcal{I}(d) &= \max\{C^\mathcal{I}(d), D^\mathcal{I}(d)\} \\
\neg C^\mathcal{I}(d) &= 1 - C^\mathcal{I}(d) \\
\forall R.C^\mathcal{I}(d) &= \inf_{d' \in \Delta^\mathcal{I}} \{\max\{1 - R^\mathcal{I}(d, d'), C^\mathcal{I}(d')\}\} \\
\exists R.C^\mathcal{I}(d) &= \sup_{d' \in \Delta^\mathcal{I}} \{\min\{R^\mathcal{I}(d, d'), C^\mathcal{I}(d')\}\}.
\end{align*}
\]

We will say that two concepts \( C \) and \( D \) are said to be equivalent (denoted by \( C \equiv D \)) when \( C^\mathcal{I} = D^\mathcal{I} \) for all interpretation \( \mathcal{I} \). As for the crisp non fuzzy case, dual relationships between concepts hold: e.g. \( \top \equiv \bot \), \( (C \sqcap D) \equiv (\neg C \sqcup \neg D) \) and \( (\forall R.C) \equiv (\exists R.\neg C) \).

B. Fuzzy Assertion

A fuzzy assertion (denoted by \( \psi \)) is an expression having one of the following forms \( (\alpha \geq n) \) or \( (\alpha \leq m) \), where \( \alpha \) is an \( \mathcal{ALC} \) assertion, \( n \in (0, 1] \) and \( m \in [0, 1] \). From a semantics point of view, a fuzzy assertion \( (\alpha \leq n) \) constrains the truth-value of \( \alpha \) to be less or equal to \( n \) (similarly for \( \geq \)). Consequently, e.g. \( (\forall v \in \mathcal{V} \exists \mathcal{A}.\mathcal{B} \in \mathcal{V})(\forall v1 \geq 0.8) \) states that video \( v1 \) is likely about basket. Formally, an interpretation \( \mathcal{I} \) satisfies a fuzzy assertion \( (C(a) \geq n) \) (resp. \( (R(a, b) \geq n) \)) iff \( C\mathcal{I}(a) \geq n \) (resp. \( R\mathcal{I}(a, b) \geq n \)). Similarly, an interpretation \( \mathcal{I} \) satisfies a fuzzy assertion \( (C(a) \leq n) \) (resp. \( (R(a, b) \leq n) \)) iff \( C\mathcal{I}(a) \leq n \) (resp. \( R\mathcal{I}(a, b) \leq n \)). Two fuzzy assertion \( \psi_1 \) and \( \psi_2 \) are said to be equivalent (denoted by \( \psi_1 \equiv \psi_2 \)) iff they are satisfied by the same set of interpretations. An atomic fuzzy assertion is a fuzzy assertion involving an atomic assertion (assertion of the form \( A(a) \) or \( R(a, b) \)).

C. Fuzzy Terminological Axiom

From a syntax point of view, a fuzzy terminological axiom (denoted by \( \tau \)) is either a fuzzy concept specialization or a fuzzy concept definition. A fuzzy concept specialization is an expression of the form \( A \prec C \), where \( A \) is an atomic concept and \( C \) is a concept. On the other hand, a fuzzy concept definition is an expression of the form \( A \equiv C \), where \( A \) is an atomic concept and \( C \) is a concept. From a semantics point of view, a fuzzy interpretation \( \mathcal{I} \) satisfies a fuzzy concept specialization \( A \prec C \) iff

\[
\forall d \in \Delta^\mathcal{I}, A^\mathcal{I}(d) \leq C^\mathcal{I}(d), \quad (1)
\]

whereas \( \mathcal{I} \) satisfies a fuzzy concept definition \( A \equiv C \) iff

\[
\forall d \in \Delta^\mathcal{I}, A^\mathcal{I}(d) = C^\mathcal{I}(d). \quad (2)
\]

D. Fuzzy Knowledge Base, Fuzzy Entailment and Fuzzy Subsumption

A fuzzy knowledge base is a finite set of fuzzy assertions and fuzzy terminological axioms. \( \Sigma_A \) denotes the set of fuzzy assertions in \( \Sigma \), \( \Sigma_T \) denotes the set of fuzzy terminological axioms in \( \Sigma \) (the terminology), if \( \Sigma_T = \emptyset \) then \( \Sigma \) is purely assertional, and we will assume that a terminology \( \Sigma_T \) is such that no concept \( A \) appears more than once on the left hand side of a fuzzy terminological axiom \( \tau \in \Sigma_T \) and that no cyclic definitions are present in \( \Sigma_T \).

An interpretation \( \mathcal{I} \) satisfies (is a model of) a set of fuzzy \( \Sigma \) iff \( \mathcal{I} \) satisfies each element of \( \Sigma \). A fuzzy KB \( \Sigma \) fuzzy entails a fuzzy assertion \( \psi \) (denoted by \( \Sigma \models \psi \)) iff every model of \( \Sigma \) also satisfies \( \psi \).

Furthermore, let \( \Sigma_T \) be a terminology and let \( C, D \) be two concepts. We will say that \( D \) fuzzy subsumes \( C \) w.r.t. \( \Sigma_T \) (denoted by \( C \preceq_{\Sigma_T} D \)) iff for every model \( \mathcal{I} \) of \( \Sigma_T \), \( \forall d \in \Delta^\mathcal{I}, C^\mathcal{I}(d) \leq D^\mathcal{I}(d) \) holds.

III. A NEUTROSOPHIC DL

Our neutrosophic extension directly relates to Smarandache’s work on neutrosophic sets [15], [16]. A neutrosophic set \( S \) defined on universe of discourse \( U \), associates each element \( u \) in \( U \) with three membership functions: truth-membership function \( T_S(u) \), indeterminacy-membership function \( I_S(u) \), and falsity-membership function \( F_S(u) \), where \( T_S(u), I_S(u), F_S(u) \) are real standard or non-standard subsets.
of $]^{-1}, 1^{+}[$, and $T_{S}(u), I_{S}(u), F_{S}(u)$ are independent. For simplicity, here we only use two components $T_{S}(u)$ and $F_{S}(u)$, with $T_{S}(u) \in [0,1]$, $F_{S}(u) \in [0,1]$, $0 \leq T_{S}(u) + F_{S}(u) \leq 2$. It is easy to extend our method to include indeterminacy-membership function $T_{S}(u)$ gives us an estimation of degree of $u$ belonging to $U$ and $F_{S}(u)$ gives us an estimation of degree of $u$ not belonging to $U$. $T_{S}(u) + F_{S}(u)$ can be 1 (just as in classical fuzzy sets theory). But it is not necessary. If $T_{S}(u) + F_{S}(u) < 1$, for all $u \in U$, we say the set $S$ is incomplete, if $T_{S}(u) + F_{S}(u) > 1$, for all $u \in U$, we say the set $S$ is inconsistent. According to Wang [16], the truth-membership function and falsity-membership function has to satisfy three restrictions: for all $u \in U$ and for all neurospheric sets $S_{1}, S_{2}$ with respect to $U$

$$
T_{S_{1} \cap S_{2}}(u) = \min\{T_{S_{1}}(u), T_{S_{2}}(u)\},
$$

$$
F_{S_{1} \cap S_{2}}(u) = \max\{F_{S_{1}}(u), F_{S_{2}}(u)\},
$$

$$
T_{S_{1} \cup S_{2}}(u) = \max\{T_{S_{1}}(u), T_{S_{2}}(u)\},
$$

$$
F_{S_{1} \cup S_{2}}(u) = \min\{F_{S_{1}}(u), F_{S_{2}}(u)\}
$$

where $\overline{S_{1}}$ is the complement of $S_{1}$ in $U$. Wang [16] gives the definition of $N$-norm and $N$-conorm of neurospheric sets, $\min$ and $\max$ is only one of the choices. In general case, they may be the simplest and the best.

When we switch to neurospheric logic, the notion of degree of truth-membership $T_{S}(u)$ of an element $u \in U$ w.r.t. the neurospheric set $S$ over $U$ is regarded as the truth-value of the statement “$u$ is $S$”, and the notion of degree of falsity-membership $F_{S}(u)$ of an element $u \in U$ w.r.t. the neurospheric set $S$ over $U$ is regarded as the falsity-value of the statement “$u$ is $S$”. Accordingly, in our neurospheric DL, (i) a concept $C$, rather than being interpreted as a fuzzy set, will be interpreted as a neurospheric set and, thus, concepts become imprecise (fuzzy, incomplete, and inconsistent); and, consequently, (ii) the statement “$u$ is $C$”, i.e. $C(u)$ will have a truth-value in $[0,1]$ given by the degree of truth-membership of being the individual $a$ a member of the neurospheric set $C$ and a falsity-value in $[0,1]$ given by the degree of falsity-membership of being the individual $a$ not a member of the neurospheric set $C$.

A. Neurospheric Interpretation

A cm neurospheric interpretation is now a tuple $I = (\Delta^{\mathcal{I}}, (\cdot)^{\mathcal{I}}, \cdot^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is, as for the fuzzy case, the domain, and

1) $(\cdot)^{\mathcal{I}}$ is an interpretation function mapping
   a) individuals as for the fuzzy case, i.e. $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$;
   b) a concept $C$ into a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0,1] \times [0,1]$;
   c) a role $R$ into a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1] \times [0,1]$.

2) $\cdot^{\mathcal{I}}$ and $\cdot^{\mathcal{I}}$ are neurospheric valuation, i.e. $\cdot^{\mathcal{I}}$ and $\cdot^{\mathcal{I}}$ map

We will say that two concepts $C$ and $D$ are said to be equivalent (denoted by $C \equiv^{\mathcal{I}} D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $I$. As for the fuzzy case, dual relationships between concepts hold: e.g. $\top \equiv^{\mathcal{I}} \bot$, $(C \cap D) \equiv^{\mathcal{I}} (\neg C \cup \neg D)$ and $(\forall R.C) \equiv^{\mathcal{I}} (\exists R.\neg C)$.
B. Neurosets

A neuroset (denoted by $\varphi$) is an expression having one of the following form $(\alpha \geq n, \leq m)$ or $(\alpha \leq n, \geq m)$, where $\alpha$ is an ALC assertion, $n \in [0, 1]$ and $m \in [0, 1]$. From a semantics point of view, a neuroset assertion $(\alpha \geq n, \leq m)$ constrains the truth-value of $\alpha$ to be greater or equal to $n$ and falsity-value of $\alpha$ to be less or equal to $m$ (similarly for $(\alpha \leq n, \geq m)$). Consequently, e.g., $(\text{Poll} \cap \text{Support.War})(p1) \geq 0.8, 0.1$ states that poll $p1$ is close to support War.$x$. Formally, an interpretation $I$ satisfies a neurosets assertion $(\alpha \geq n, \leq m)$ (resp. $(\alpha \leq n, \geq m)$) if $|C|^I(d) \geq n$ and $|C|^I(d) \leq m$ (resp. $|R|^I(d, b') \geq n$ and $|R|^I(d, b') \leq m$). Similarly, an interpretation $I$ satisfies a neurosets assertion $(\alpha \leq n, \leq m)$ (resp. $(\alpha \leq n, \geq m)$) if $|C|^I(d) \leq n$ and $|C|^I(d) \geq m$ (resp. $|R|^I(d, b') \leq n$ and $|R|^I(d, b') \geq m$). Two fuzzy assertion $\varphi_1$ and $\varphi_2$ are said to be equivalent (denoted by $\varphi_1 \equiv \varphi_2$) if they are satisfied by the same set of interpretations. Notice that $(\neg C(a) \geq n, \leq m) \equiv \neg (C(a) : \leq n, \geq m)$ and $(\neg C(a) : \leq n, \geq m) \equiv \neg (C(a) : \geq n, \leq m)$. An atomic neuroset assertion is a neurosets assertion involving an atomic assertion.

C. Neurosets Terminological Axiom

Neurosets terminological axioms we will consider are a natural extension of fuzzy terminological axioms to the neurosets case. From a syntax point view of a neurosets terminological axiom (denoted by $\tau$) is either a neurosets concept specialization or a neurosets concept definition. A neurosets concept specialization is an expression of the form $A <^n C$, where $A$ is an atomic concept and $C$ is a concept. On the other hand, a neurosets concept definition is an expression of the form $A : =^n C$, where $A$ is an atomic concept and $C$ is a concept. From a semantics point of view, we consider the natural extension of fuzzy set to the neurosets case [15], [16]. A neurosets interpretation $I$ satisfies a neurosets concept specialization $A <^n C$ iff

$$\forall d \in \Delta, |A|^I(d) \leq |C|^I(d)$$

whereas $I$ satisfies a neurosets concept definition $A : =^n C$ iff

$$\forall d \in \Delta, |A|^I(d) = |C|^I(d)$$

D. Neurosets Knowledge Base, Neurosets Entailment and Neurosets Subsumption

A neurosets knowledge base is a finite set of neurosets assertions and neurosets terminological axioms. As for the fuzzy case, with $\Sigma_A$ we will denote the set of neurosets assertions in $\Sigma$, with $\Sigma_T$ we will denote the set of neurosets terminological axioms in $\Sigma$ (the terminology), if $\Sigma_T = 0$ then $\Sigma$ is purely assertional, and we will assume that a terminology $\Sigma_T$ is such that no concept $A$ appears more than once on the left hand side of a neurosets terminological axiom $\tau \in \Sigma_T$ and that no cyclic definitions are present in $\Sigma_T$.

An interpretation $I$ satisfies (is a model of) a neurosets $\Sigma$ iff $I$ satisfies each element of $\Sigma$. A neurosets KB $\Sigma$ neurosetsically entails a neurosets assertion $\varphi$ (denoted by $\Sigma \models \varphi$) iff every model of $\Sigma$ also satisfies $\varphi$.

Furthermore, let $\Sigma^T$ be a terminology and let $C, D$ be two concepts. We will say that $D$ neurosetsically subsumes $C$ w.r.t. $\Sigma^T$ (denoted by $C \preceq^T \Sigma^T D$) iff for every model $I$ of $\Sigma^T$, $\forall d \in \Delta, |C|^I(d) \leq |D|^I(d)$ and $|C|^I(d) \geq |D|^I(d)$ holds.

Finally, given a neurosets KB $\Sigma$ and an assertion $\alpha$, we define the greatest lower bound of $\alpha$ w.r.t. $\Sigma$ (denoted by $\text{glb}(\Sigma, \alpha)$) to be $\{\sup\{n : \Sigma \models (\alpha \geq n, \leq m)\}, \inf\{m : \Sigma \models (\alpha \leq n, \geq m)\} \}$. Similarly, we define the least upper bound of $\alpha$ with respect to $\Sigma$ (denoted by $\text{lub}(\Sigma, \alpha)$) to be $\{\inf\{n : \Sigma \models (\alpha \leq n, \geq m)\}, \sup\{m : \Sigma \models (\alpha \leq n, \geq m)\} \}$. Determining the lub and the glb is called the Best Truth-Value Bound (BTVB) problem.

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