AN ASYMPTOTIC ROBIN INEQUALITY

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Abstract. The conjectured Robin inequality for an integer $n > 7!$ is $\sigma(n) < e^{\gamma} n \log\log n$, where γ denotes Euler constant, and $\sigma(n) = \sum_{d|n} d$. We prove that Robin inequality holds up to terms vanishing for large n. The main ingredients of the proof are an estimate for Chebyshev first summatory function, and an effective version of Mertens third theorem due to Rosser and Schoenfeld. As a byproduct, by an oscillation theorem of Robin, the truth of RH follows.

1. Introduction

1.1. **History.** The conjectured Robin inequality for an integer $n > 7! = 5040$ is $\sigma(n) < e^{\gamma} n \log\log n$, where $\gamma \approx 0.577\cdots$ denotes Euler constant, and σ is the sum-of-divisors functions $\sigma(n) = \sum_{d|n} d$. This inequality has been shown to hold unconditionally for families of integers that are

- odd > 9 [3]
- square-free > 30 [3]
- a sum of two squares and > 720 [1]
- not divisible by the fifth power of a prime [3]
- not divisible by the seventh power of a prime [10]
- not divisible by the eleventh power of a prime [2]

Ramanujan showed that Riemann Hypothesis implied that conjecture [7]. Robin proved the converse statement [8], thus making that conjecture a criterion for RH. This criterion was made popular by [5] which derives an alternate criterion involving Harmonic numbers.

1.2. **Contribution.** Denote the difference between the right hand side and the left hand side of Robin inequality by $D(n) = e^{\gamma} n \log\log n - \sigma(n)$. The main result of this note is

Theorem 1. *For large n we have* $\liminf_{n\to\infty} D(n) \geq 0$.

The main ingredients of the proof are a combinatorial inequality between arithmetic functions (Lemma 1), an effective version of Mertens third theorem due to Rosser and Schoenfeld (Lemma 2), and an asymptotic estimate of Chebyshev first summatory function (Lemma 4). An immediate but important Corollary is

Corollary 1. *RH is true.*

Its proof will *not* result from Robin criterion, but from an oscillation theorem of Robin [8] for $\sigma(n)$ at so-called Colossally abundant numbers, modelled after and depending upon an oscillation theorem of Nicolas [6] for the Euler totient function.

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Thus while our proof of the 156-year old RH might seem too short to be true it relies in fact on the long references [6, 8, 9].

1.3. **Organization.** The material is arranged as follows. The next section contains the proof of Theorem 1, and Section 3 that of Corollary 1. Section 4 concludes and gives some open problems.

2. Proof of Theorem 1

We will show that $\liminf_{n\to\infty} \frac{D(n)}{n} \geq 0$, a stronger result since

$$\liminf_{n\to\infty} D(n) \geq \liminf_{n\to\infty} \frac{D(n)}{n}.$$

For any integer n write its decomposition into prime powers as

$$n = \prod_{i=1}^{m} q_i^{a_i},$$

where the q_i's are prime numbers, indexed by increasing order, and a_i's are positive integers. Denote by p_i the i^{th} prime number, and for any integer n, let

$$\bar{n} = \prod_{i=1}^{m} p_i^{a_i}.$$

Note that, by definition, for each $i = 1, 2, \cdots, m$ we have $q_i \geq p_i$, and that, therefore, $n \geq \bar{n}$. With this notation observe that

$$\sigma(\bar{n}) = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i - 1} = \bar{n} \prod_{i=1}^{m} \frac{p_i - p_i^{-a_i}}{p_i - 1}.$$

In particular

$$\frac{\sigma(\bar{n})}{\bar{n}} \leq \prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq 2^m,$$

and, likewise, $\frac{\sigma(n)}{n} \leq 2^m$. Thus, if m is bounded and $n \to \infty$, we see that $\frac{D(n)}{n} \to \infty$. We can thus assume when considering $\liminf_{n\to\infty} \frac{D(n)}{n}$ that $m \to \infty$. We prepare for the proof by a series of Lemmas. The first Lemma is a combinatorial bound between arithmetic functions.

Lemma 1. *For any integer $n \geq 1$, we have $\frac{D(n)}{n} \geq \frac{D(\bar{n})}{\bar{n}}$.*

Proof. Let $\frac{D(n)}{n} = f_1(n) - f_2(n)$, with $f_1(n) = e^\gamma \log\log n$, and $f_2(n) = \frac{\sigma(n)}{n}$. The monotonicity of the log and $n \geq \bar{n}$ yields $f_1(n) \geq f_1(\bar{n})$. Write $f_2(n) = \prod_{i=1}^m g(a_i, q_i)$, where $g(a, x) = \frac{x - x^{-a}}{x-1}$. Writing

$$g(a, x) = \frac{1 + x + \cdots + x^a}{x^a} = \sum_{i=0}^{a} \frac{1}{x^i},$$

we see that, for fixed a, the function $x \mapsto g(a, x)$ is nonincreasing in x as a sum of a non decreasing functions. This implies that $g(a_i, q_i) \leq g(a_i, p_i)$ for each $i = 1, 2, \cdots, m$ and, therefore, multiplying m inequalities between nonnegative numbers, that $f_2(n) \leq f_2(\bar{n})$. The result follows then by $\frac{D(n)}{n} = f_1(n) - f_2(n)$.

□

The second Lemma is an effective bound related to Mertens third Theorem.

Lemma 2. *For any n large enough we have* $\frac{\sigma(\bar{n})}{\bar{n}} < e^\gamma \log p_m (1 + \frac{1}{\log^2 p_m})$.

Proof. Note that, with the notation of the proof of Lemma 1, we have $g(a, x) \leq \frac{x}{x-1}$, for $x \geq 2$ and $a \geq 1$, and, therefore

$$f_2(n) = \prod_{i=1}^{m} g(a_i, q_i) \leq \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

The result follows then by [9, Th. 8, (39)]. □

Recall Chebyshev summatory function $\vartheta(x) = \sum_{p \leq x} \log(p)$.

Lemma 3. *For all $n \geq 1$, we have* $\log \bar{n} \geq \vartheta(p_m)$.

Proof. By definition

$$\log \bar{n} = \sum_{i=1}^{m} a_i \log p_i \geq \sum_{i=1}^{m} \log p_i = \vartheta(p_m).$$

□

A classical result, related to the Prime Number Theorem, is

Lemma 4. *For large x we have* $\vartheta(x) = x + O(\frac{x}{\log x})$.

Proof. An effective version is in [9, Th. 4]. See for instance [4, Th 4.7] for a sharper error term in $O(x \exp(-\frac{\sqrt{\log x}}{15}))$. □

We are now ready for the proof of Theorem 1.

Proof. By Lemma 1 $\frac{D(n)}{n} \geq \frac{D(\bar{n})}{\bar{n}}$. By Lemma 2 we have

$$-\frac{\sigma(\bar{n})}{\bar{n}} > -e^\gamma \log p_m (1 + \frac{1}{\log^2 p_m}). \tag{1}$$

By Lemma 3 and 4 we have

$$e^\gamma \log \log \bar{n} \geq e^\gamma \log \vartheta(p_m) = e^\gamma \log \left(p_m + O(\frac{p_m}{\log p_m}) \right) = e^\gamma \left(\log(p_m) + O(\frac{1}{\log p_m}) \right), \tag{2}$$

where the last equality results from $\log(1+u) \sim u$ for $u \to 0$. Adding up inequations 1 and 2, after cancellation of the terms in $\log p_m$, we obtain the chain of inequalities

$$\frac{D(n)}{n} \geq \frac{D(\bar{n})}{\bar{n}} = e^\gamma \log \log \bar{n} - \frac{\sigma(\bar{n})}{\bar{n}} \geq O(\frac{1}{\log p_m}) - \frac{e^\gamma}{\log p_m},$$

the rightmost hand side of which goes to zero for large n. The result follows upon extracting a convergent subsequence from the left most handside and passing to the limit. □

3. Proof of Corollary 1

Recall the standard notation for oscillation theorems [4, p. 194]. If f, g are two real valued functions of a real variable x, with $g > 0$, then we write

- $f(x) = \Omega_+(g(x))$, if $\limsup_{x\to\infty} f(x)/g(x) > 0$
- $f(x) = \Omega_-(g(x))$, if $\liminf_{x\to\infty} f(x)/g(x) < 0$
- $f(x) = \Omega_\pm(g(x))$, if both $f(x) = \Omega_+(g(x))$, and $f(x) = \Omega_-(g(x))$ hold

We refer the reader to [8] for the definition of Colossally Abundant (CA) numbers. By [8, Proposition,§4] if RH is false then, for CA numbers we have

$$D(n) = \Omega_\pm\left(\frac{n \log\log n}{(\log n)^b}\right),$$

for some $b \in (0, 1)$. This would imply, using the infinitude of CA numbers [8], that $\liminf_{n\to\infty} D(n) = -\infty$, contradicting Theorem 1.

4. Conclusion and open problems

In this note, we have proved an asymptotic version of Robin inequality, that is strong enough to confirm Riemann Hypothesis. This latter fact, in turn, by Robin criterion proves that Robin inequality holds for every integer $> 7!$. Our proof of RH is not elementary in the sense that complex analysis is essential to the oscillation theorems of [6]. It is, however, much longer than this note if one takes into account the needed results in [6, 8, 9]. The main open problem at this point would be to generalize our approach to Dedekind zeta functions and $L-$series and other zeta functions. Deriving RH following our approach and by elementary means might be even harder.

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References

[1] W. D., Banks, D., Hart, P. Moree, C. W. Nevans, The Nicolas and Robin inequalities with sums of two squares. Monatsh. Math. 157, no. 4,(2009) 303–322.
[2] K, Broughan, T., Trudgian, Robin's inequality for 11-free integers. Integers 15 (2015), Paper No. A12, 5 pp.
[3] Y-J. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis. J. Théor. Nombres Bordeaux 19 (2007), no. 2, 357–372.
[4] W. J. Ellison, M. Mendès-France, *Les Nombres Premiers*, Hermann (1975).
[5] J. C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, American Mathematical Monthly 109 (2002), 534–543.
[6] J-L. Nicolas, Petites valeurs de la fonction d'Euler, J. Number Theory 17 (1983), no. 3, 375–388.
[7] S. Ramanujan. Highly composite numbers.The Ramanujan Journal, 1 (1997)119–153
[8] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. Pures Appl. (9), 63(2) (1984) 187–213.
[9] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers. Illinois J. Math. 6 (1962) 64–94.
[10] P. Solé, M. Planat, The Robin inequality for $7-$free integers. Integers 12, no. 2,(2012) 301–309.

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