CGA-Effective (Radial) Potential Energy

Mohamed E. Hassani[1]
Institute for Fundamental Research
BP.197, CTR, GHARDAIA 47800, ALGERIA

Abstract: The main conceptual foundation of previous work on the Combined Gravitational Action is briefly surveyed. Next, we derive a general expression for the CGA-effective (radial) potential energy in order to investigate the behavior and shape of the orbits of a test-body during its orbital motion inside the vicinity of the principal gravitational source. And as direct consequences, two expressions for the concepts of gravitational momentum and dynamic gravitational force are derived. Starting from the concept of the combined gravitational potential energy and using only the familiar tools of Newtonian mechanics, the classical Binet's orbital equation is *combgravactionalized* and its physico-mathematical expression is exactly identical to that already found in the context of general relativity theory, which enables us to calculate, among other things, the secular perigee precession of the Moon; the secular perihelion advance of the planets and the angular deflection of light passing near the massive object.

Keywords: combined gravitational action; combined gravitational potential energy; CGA-effective (radial) potential energy; Binet's orbital equation

1. Introduction

As it was repeatedly mentioned in a series of articles [1,2,3,4,5] relative to the Combined Gravitational Action (CGA) as an alternative gravity theory that should regard as a refinement and generalization of Newton's one. Also in the same papers we have shown that the CGA is very capable of investigating, explaining and predicting some old and new gravitational phenomena [1,2,3,4,5]. This characteristic is greatly due to the coherence and simplicity of the CGA-formalism that is exclusively based on the concept of combined gravitational potential energy (CGPE), which is actually a new form of velocity-dependent gravitational potential energy defined by the following expression

$$U \equiv U(r,v) = -\frac{k}{r}\left(1+\frac{v^2}{w^2}\right), \tag{1}$$

where $k = GMm$; G being the Newton's gravitational constant; M and m are the masses of the gravitational source A and the moving test-body B; $r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$ is the relative distance between A and B; $v = \sqrt{v_x^2+v_y^2+v_z^2}$ with $v_x = dx/dt$ etc, is the velocity of the test-body B relative to the inertial reference frame of source A; and w is a specific kinematical parameter having the physical dimensions of a constant velocity defined by

$$w = \begin{cases} c_0, \text{ if } B \text{ is in relative motion inside the vicinity of } A \\ v_{\text{esc}} = \sqrt{2GM/R}\,, \text{ if } B \text{ is in relative motion outside the vicinity of } A \end{cases}, \tag{2}$$

where c_0 is the light speed in local vacuum and v_{esc} is the escape velocity at the surface of the gravitational source A.

(1) E-mail: hassani641@gmail.com

The CGPE (1) ,which is conceptually the core of CGA and considered to be simpler and more useful than the usual velocity-dependent-GPSs because it contains several properties. This –as we know from [1,2,3,4,5]– has allowed us to carry out our calculations using only the familiar tools of classical mechanics.

2. Classical Effective(radial) potential Energy

As it is well established, the concept of effective (radial) potential energy (EPE) is mainly inherited from the classical and celestial mechanics and it is largely used, as a very important tool, in many branches of physics, particularly, gravitational physics and quantum physics to analyze, for example, the behavior and shape of orbits. Usually, the classical-EPE is defined like this

$$V_{\mathrm{CL}}(r) = \frac{\ell^2}{2mr^2} - \frac{k}{r}. \tag{3}$$

Therefore, in pure energetic terms, we can write the classical equation of motion in form of a one- dimensional equation in radial coordinate *r* as follows

$$E = \tfrac{1}{2} m\dot{r}^2 + V_{\mathrm{CL}}(r). \tag{4}$$

The importance of doing this is that we are usually very familiar with one-dimensional equation's solutions for a test-particle moving in a potential well determined by $V_{\mathrm{CL}}(r)$. Depending on the total energy *E*, the test-particle may possibly escape the well and travel to infinity, or it may be trapped in the well and oscillate backward and forward in the bottom of the well. For suitably small energy it may be at relative rest at the bottom. Thus, dimensionally, the use of the radial coordinate *r* as a one-dimension facilitates considerably the study of the classical orbits around a spherical material body.

The investigation of the behavior and shape of the orbits depending on $V_{\mathrm{CL}}(r)$ and also on the total energy *E*. For example, the radii of the circular orbits are found by finding the maximum and minimum of the EPE by solving $dV_{\mathrm{CL}}/dr = 0$.

In terms of the total energy (4), that is when the test-particle having $E < 0$, the test-particle oscillates around the bottom between $r_{\min}$ and $r_{\max}$. Hence, at these two radii, we have elliptical orbits. For the case $E > 0$ that is when the test-particle comes in from infinity, reach a point of closest approach and then return to infinity. The orbits are typically hyperbolic ones. Lastly, the test-particle with $E = 0$ should do a comparable course, but at this time the orbits are characteristically parabolic.

However, all these considerations and characteristics are in fact a special case of the CGA-EPE as we shall see soon.

3. CGA-Effective (Radial) Potential Energy

In what follows, our central goal is to derive a general expression for the effective (radial) potential energy (EPE) in the context of the CGA when the test-body *B* evolving inside the vicinity of the principal gravitational source *A*. This EPE is called CGA-EPE because it contains the CGA-terms. In other words, this means that we have *combgravactionalized* the classical-EPE (3) to facilitate the study of CGA-orbits, *i.e.*, as we know from[1,2,3], when the test-body orbiting the principal gravitational source, it is permanently under the action of the combined gravitational field.

3.1. Derivation

Considering the aforementioned test-body *B* orbiting the principal gravitational source *A* at the relative radial distance r with the velocity v. Under such conditions, we have in the CGA-context [1-5] the following expression for the total energy of the moving test-body *B*

$$\mathscr{E} = T + U = \tfrac{1}{2} m v^2 - \frac{k}{r}\left(1 + \frac{v^2}{c_0^2}\right), \quad k = GMm. \tag{5}$$

Now, rewriting (5) in spherical coordinate, that is, with $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$. Let's simplify by supposing that the motion taking place in the (*x*, *y*)-plan so $z = 0$; $\theta = \pi/2$ and $\dot{\theta} = 0$. Then $\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$; $\dot{y} = \dot{r}\sin\phi - r\dot{\phi}\cos\phi$. Substituting this in (5) and using $\cos^2\phi + \sin^2\phi = 1$, we find

$$\mathscr{E} = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{k}{r}\left(1 + \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c_0^2}\right). \tag{6}$$

Next defining the angular momentum $\ell = m v_\phi r = m\dot{\phi} r^2$, so $\dot{\phi} r = \ell / mr$, thus we can rewrite (6) as follows

$$\begin{aligned}\mathscr{E} &= \tfrac{1}{2} m\left(\dot{r}^2 + \frac{\ell^2}{m^2 r^2}\right) - \frac{k}{r}\left[1 + \left(\frac{\ell^2}{m^2 c_0^2 r^2} + \frac{\dot{r}^2}{c_0^2}\right)\right] \\ &= \tfrac{1}{2} m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{k}{r}\left[1 + \left(\frac{\ell^2}{m^2 c_0^2 r^2} + \frac{\dot{r}^2}{c_0^2}\right)\right]\end{aligned}, \tag{7}$$

from where we deduce the very expected expression for the CGA-EPE

$$V(r, \dot{r}) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} - \frac{k}{r}\left(\frac{\ell^2}{m^2 c_0^2 r^2} + \frac{\dot{r}^2}{c_0^2}\right). \tag{8}$$

Therefore, the total energy (7) may be finally rewritten in the form $\mathscr{E} = K(\dot{r}) + V(r, \dot{r})$ where $K(\dot{r}) = \tfrac{1}{2} m\dot{r}^2$. It is worthwhile to note that the CGA-EPE (8) is in fact a function of tow variables, *viz*., r and $\dot{r}$. Also, it is clear from expression (8), $V(r,\dot{r})$ may be reduced to $V_{\mathrm{CL}}(r)$ for the case $\left(\dot{r}^2 + \ell^2/m^2r^2\right)c_0^{-2} << 1$, that is when the CGA-terms are sufficiently smaller than unity. Moreover, since as we know $k = GMm$, thus the well-known general relativistic-EPE

$$V_{\mathrm{GR}}(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} - \frac{GM\ell^2}{mc_0^2r^3}\ , \tag{9}$$

is obviously included in the CGA-EPE (8). This inclusion implies, among other things, that the geometrization of gravity is not necessary at all and the (curved) space-time is a pure mathematical concept, an abstract entity and consequently it is not a tangible physical reality. Therefore, according to the CGA-context, this form of GR-EPE (9) is highly questionable since in the framework of general relativity theory (GRT), the gravity is not a fundamental force of Nature but it is supposed to be caused by the curvature of space-time. However, as it was mentioned, the space-time itself is no more than an abstract notion.

But unfortunately many theories depend on this mathematical concept, which *artificially* becomes a real entity that is capable of acting on matter and can be acted upon, therefore, curved space-time itself behaves like *a sort of matter* and logically speaking –we can ask the following naive question: what are the physical and chemical proprieties of space-time?! Of course, the answer is simply negative because phenomenologically speaking, the geometrization of gravity implies the materialization of curved space-time itself, and as a direct result the usual principle of causality is violated because the causal source of such *materialization* is absolutely without existence. The interested reader is referred to [6].

3.2. Circular orbits and their stability

As previously mentioned, the EPE is a pivotal tool of analysis and computation. Since we are presently concerned with the circular orbits and their stability, thus let us examine the expression (8): the first two terms are well-known classical energies, the first being the Newtonian gravitational potential energy and the second corresponding to the centrifugal potential energy; however, the third term is an energy unique to the CGA. As shown in [2,4], this extra energy causes elliptical orbits to precess gradually by a very small angle per revolution. Therefore, CGA-term is typically attractive and dominates at small r values, giving a critical inner radius which in fact a minimum possible radius $r_{\min}$ at which the test-body A is drawn inexorably inwards to $r = 0$; this minimum (inner) radius is a function of A's angular momentum ℓ and the velocity $\dot{r}$ as it is shown below.

Physico-mathematically, the circular orbits are possible when the effective total force is zero:

$$F = \frac{\partial V(r,\dot{r})}{\partial r} = \frac{k}{r^2} - \frac{\ell^2}{mr^3} + \frac{k}{r^2}\left(\frac{\ell^2}{m^2c_0^2r^2} + \frac{\dot{r}^2}{c_0^2}\right) = 0\,, \tag{10}$$

i.e., when the two attractive forces — Newtonian gravity (first term) and the attraction unique to CGA (third term) — are exactly balanced by the repulsive centrifugal force (second term). There are two radii at which this balancing can occur, denoted here as $r_{\min}$ and $r_{\max}$:

$$r_{\min} \equiv r_{\min}(\ell, \dot{r}) = \frac{\ell}{2km}\left[\ell - \sqrt{\ell^2 - \frac{12k^2}{c_0^2}\left(1 + \frac{\dot{r}^2}{c_0^2}\right)}\right]\left(1 + \frac{\dot{r}^2}{c_0^2}\right)^{-1}, \tag{11}$$

$$r_{\max} \equiv r_{\max}(\ell, \dot{r}) = \frac{\ell}{2km}\left[\ell + \sqrt{\ell^2 - \frac{12k^2}{c_0^2}\left(1 + \frac{\dot{r}^2}{c_0^2}\right)}\right]\left(1 + \frac{\dot{r}^2}{c_0^2}\right)^{-1}, \tag{12}$$

which are obtained using the quadratic formula and supposing the quantity in the square root to be positive, *i.e.*, $\ell^2 > 12k^2 c_0^{-2}\left(1 + \dot{r}^2 c_0^{-2}\right)$. Thus, we have two circular orbits at $r_{\min}$ and $r_{\max}$, respectively. Mathematically, when there are two orbits we can investigate their behavior, *i.e.*, their stability or instability by taking the second derivative $\partial^2 V(r, \dot{r})/\partial r^2$, and physically we can also use the effective total force (10) for the same investigation as discussed below.

3.3. Discussion

i) At the minimum (inner) radius $r_{\min}$, the circular orbit is unstable, because the attractive third force (CGA-term) strengthens much faster than the other two forces when *r* becomes small.

ii) If the test-body falls slightly inwards from $r_{\min}$ (where all three forces are in balance), the third force dominates the other two and draws the test-body inexorably inwards to $r = 0$.

iii) At the maximum (outer) radius, however, the circular orbits are stable; the third term is less important and the system behaves more like the classical Kepler problem.

iv) Further, when $12k^2\ell^{-2}c_0^{-2}\left(1 + \dot{r}^2 c_0^{-2}\right)$ is much smaller than unity and $\dot{r} << c_0$,*i.e.* the classical case, the formulae (11) and (12) become approximately

$$r_{\min} \cong 3r_{\mathrm{G}}, \tag{13}$$

$$r_{\max} \cong \frac{\ell^2}{km}, \tag{14}$$

where $r_{\mathrm{G}} = GMc_0^{-2}$ is the well-known gravitational radius.

v) The quadratic solutions above ensure that $r_{\max}$ is always greater than $3r_{\mathrm{G}}$, whereas $r_{\min}$ is less than or equal to $3r_{\mathrm{G}}$ .In CGA-context, circular orbits smaller than $3r_{\mathrm{G}}$ are not possible.

vi) At the other extreme, when $\ell^2 = 12k^2 c_0^{-2}\left(1 + \dot{r}^2 c_0^{-2}\right)$ from (11) and (12), the two radii become equal to a single value

$$r_{\min} = r_{\max} = 6r_{\mathrm{G}}, \tag{15}$$

that is just exactly six times the gravitational radius. It is clear from this equality (15), the orbit at radius $6r_{\mathrm{G}}$ is neither stable nor unstable, but neutral, which may be interpreted as a radius of *transition*. Therefore, the total effective force (10) manifesting its neutrality just during the process of transition when the antagonism between the attractive forces (Newton's and CGA-term) and centrifugal force (repulsive-term) ceases at least momentarily for the reason that at the radius $r = 6r_{\mathrm{G}}$ the attractive forces together balance the repulsive force, *i.e.*, physico-mathematically, we have for the case $\ell^2 = 12k^2c_0^{-2}\left(1+\dot{r}^2c_0^{-2}\right)$:

$$\left[\left(1+\frac{\ell^2}{m^2c_0^2r^2}+\frac{\dot{r}^2}{c_0^2}\right)\frac{k}{r^2}\right]_{r=6r_{\mathrm{G}}} = \left[\frac{\ell^2}{mr^3}\right]_{r=6r_{\mathrm{G}}} . \tag{16}$$

In passing, we can determine the other different types of orbits (i.e., elliptical, hyperbolic and parabolic orbits) exactly as we did for the classical-EPE. However, it is worthwhile to note that close inspection should show that, in the CGA-context, the three orbits are not exactly comparable to those found in the framework of Newton's gravity. For example, the CGA-elliptical orbits are not close that's why the perihelion of each planet advances [2,3,4].

4. Consequence of CGA-EPE

Generally speaking, the CGA-formalism [1,2,3,4,5] containing many new physical quantities defined as original concepts. For instance in the present work, the CGA-third term in the effective total force (10) is an additional attractive force to the main attractive force, *namely*, the gravitational force represented by the first term. Phenomenologically, the existence of this extra-attractive term is in fact caused/induced by the motion of the test-body *B* of mass *m* inside the vicinity of the principal gravitational source *A* of mass *M*, where in general $m/M << 1$. For detailed discussion on this theme, the reader may be referred to the papers [3,4]. Now, as a clarification, the emergence of the CGA-terms reflecting the influence of motion on the gravitation and in this sense, Einstein himself argued in 1912, "*The gravitation acts more strongly on a moving body than on the same body in case it is at rest.*"[7]. But Einstein's claim has been stated in 1912, that is to say, before the publication of the final version of GRT in 1915 in which, as we know, the very realistic concept of the gravitational force is abandoned and replaced by the concept of the curvature of space-time, and at the same time, Einstein claimed that GRT may be reduced to Newtonian gravity theory for low-velocities and weak-gravitational fields!

Curiously, Lorentz has already arrived at some conclusion very comparable to that of Einstein, but more than one decade before him. In his very influential work entitled 'Considerations on Gravitation' published in 1900, Lorentz wrote "*Every theory of gravitation has to deal with the problem of the influence, exerted on this force by the motion of the heavenly bodies.*" [8]. Again, Lorentz' claim clearly reinforcing the fact that the CGA-terms are really induced by the motion of massive test-body *B* in the gravitational field of the central body *A*.

4.1. Gravitational momentum

In the paper [4], Section 3 entitled consequence of potential equations, we have derived an expression for a new physical quantity called '*gravitational momentum*' which is in reality one of the several consequences of CGA-formalism. Thus, the conceptual existence and formulation of the gravitational momentum is a direct result of the influence of the motion on the gravitation as cleverly pointed out by Einstein and Lorentz. Here, our central object is to derive, once again, the expression of the gravitational momentum (vector) from the CGA-EPE (8) as follows:

$$\mathbf{P}_{\mathrm{G}} = -\frac{\partial V(r,\dot{r})}{\partial \dot{r}}\mathbf{e}_r\,, \tag{17}$$

and its magnitude $P_{\mathrm{G}} = \|\mathbf{P}_{\mathrm{G}}\|$ is given by the relation

$$P_{\mathrm{G}} = \varepsilon m\dot{r}\,, \tag{18}$$

where

$$\varepsilon \equiv \varepsilon(r) = 2r_{\mathrm{G}}/r\,, \quad r_{\mathrm{G}} = GM/c_0^2\,. \tag{19}$$

It is helpful to note that since ε in (18) is a dimensionless physical quantity and $m\dot{r}$ is the magnitude of the classical (linear) momentum vector, thus the gravitational momentum is proportional to the classical momentum, and the constant of this proportionality is exactly ε. Therefore, in the CGA-context, the gravitational momentum vector (17) should be associated to any moving material body inside the vicinity of the main gravitational source.

Since we are exclusively dealing with the orbital motion, thus for an elliptical orbit $\dot{r}$ should play the role of the average orbital velocity

$$\dot{r} \equiv v_{\mathrm{orb}} = \sqrt{\frac{GM}{r}} \tag{20}$$

of a test-body orbiting the principal gravitational source and r is the semi-major axis of the elliptical orbit. Here, without loss of generality, we have implicitly supposed $m/M \ll 1$.

4.2. Dynamic gravitational force

In the previous papers [1,2,3,4,5], we have seen that the main consequences of the CGA-formalism [1,2,3] is the dynamic gravitational field and the dynamic gravitational force, together represented by the couple $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$, which are gravito-dynamically induced due to the relative motion of the material test-body in the vicinity of the principal gravitational source. However, if historically, the GRT was capable of explaining the secular perihelion advance of Mercury this exploit is due in great part to the extra-field $\mathbf{\Lambda}$ or equivalently to the extra-force $\mathbf{F}_{\mathrm{D}}$ that may be deduced from Eq.(25) in Ref.[3] which is a direct consequence of GRT for a test-body orbiting the main gravitational source and coincided perfectly with CGA-Eq.(27) in Ref.[3]. Therefore, physically, the secular perihelion advance of Mercury and other planets of the Solar System is not caused by the curvature of space-time but causally is due to the couple $(\mathbf{\Lambda},\mathbf{F}_\mathrm{D})$ that acting on each planet as an extra field-force.

In the present work, the dynamic gravitational force, as a new concept proper to the CGA-formalism, is a direct consequence of CGA-EPE, therefore, in order to preserve the simplicity and uniformity of the original expression of $\mathbf{F}_\mathrm{D}$, it is very convenient to define this dynamic gravitational force as follows

$$\mathbf{F}_\mathrm{D} = \frac{1}{2}\frac{d\mathbf{P}_\mathbf{G}}{dt} = -\frac{1}{2}\frac{d}{dt}\left(\frac{\partial V(r,\dot{r})}{\partial \dot{r}}\right)\mathbf{e}_r\,, \tag{22}$$

from where we get the following expression for the dynamic gravitational field

$$\mathbf{\Lambda} = \frac{\mathbf{F}_\mathrm{D}}{m}\,. \tag{23}$$

The magnitude $F_\mathrm{D} = \|\mathbf{F}_\mathrm{D}\|$ is, after computation, defined by the relation

$$F_\mathrm{G} = \frac{1}{2}\varepsilon m\ddot{r}\,. \tag{21}$$

Like before, *i.e.*, for an elliptical orbit $\ddot{r}$ should play the role of the centripetal acceleration

$$\ddot{r} = \frac{v_\mathrm{orb}^2}{r} \tag{22}$$

of a test-body orbiting the principal gravitational source. Finally, to be sure that the relation (21) is the correct definition of the magnitude of $\mathbf{F}_\mathrm{D}$, the reader –who is already familiarized with the CGA-formalism– can verify this easily. To this end, it suffices to combine the relations (20) and (22), and after substitution in (21) and by taking into account the relation (19), we get

$$F_\mathrm{G} = \frac{m}{r}\left[\frac{GM}{c_0 r}\right]^2\,, \tag{23}$$

which is exactly the usual expression. The detailed study of the couple $(\mathbf{\Lambda},\mathbf{F}_\mathrm{D})$ is published in Ref.[3].

To my great surprise, when I read the Lorentz article [8] for the first time, that is, after having written the CGA, I found, among other interesting things, that Lorentz had arrived at an extra-gravitational force (Eqs.(24) in Ref.[8]) whose components are very similar to those of $\mathbf{F}_\mathrm{D}$. Also, Broginsky, Caves and Thorn, in their seminal paper [9] entitled 'Laboratory experiments to test relativistic gravity' published in 1977, they found an extra-gravitational acceleration called by them post-Newtonian gravitational acceleration (Eq.(2.1) in Ref.[9]) whose magnitude is also comparable to that of $\mathbf{\Lambda}$.

Now, as an illustration, let us apply the formula (18) to the solar system. Our goal is to calculate the magnitude of the gravitational momentum vector associated to each planet. But before this, we prefer beginning with the Earth-Moon system in order to facilitate the comprehension and familiarize the reader with the calculations. Since in this system, the Earth playing the role of principal gravitational source A and the Moon has the role of test-body B. We have for the Moon's orbital and physical parameters: semi-major axis $r = 3.844\times10^{8}\ \mathrm{m}$, average orbital velocity $v_{\mathrm{orb}} = 1.022\times10^{3}\ \mathrm{ms}^{-1}$, $m = 7.3420\times10^{22}\ \mathrm{kg}$; while for the values of the Earth's mass and of the physical constants, we take $M = 5.97260\times10^{24}\ \mathrm{kg}$, $G = 6.67384\times10^{-11}\ \mathrm{m}^{3}\,\mathrm{kg}^{-1}\mathrm{s}^{-2}$, $c_0 = 299\,792\,458\ \mathrm{m\,s}^{-1}$. After substituting all these quantities in (18), we find the magnitude of the gravitational momentum vector associated to the Moon

$$P_{\mathrm{G}} = 1.731450\times10^{15}\ \mathrm{kg\,ms}^{-1}. \tag{24}$$

Now, like above, from the formula (18), the predicted average magnitude of the gravitational momentum vector associated to each planet is computed and listed in Table 1.

Planet	r (m)	m (kg)	v_{orb} (ms^{-1})	P_{G} ($\mathrm{kg\ ms}^{-1}$)
Mercury	57.92×10^{9}	3.30100×10^{23}	47.40×10^{3}	7.980242×10^{20}
Venus	108.25×10^{9}	4.86760×10^{24}	35.03×10^{3}	4.653160×10^{21}
Earth	149.60×10^{9}	5.97260×10^{24}	29.80×10^{3}	3.514543×10^{21}
Mars	227.95×10^{9}	6.41740×10^{23}	24.12×10^{3}	2.005938×10^{20}
Jupiter	778.60×10^{9}	1.8980×10^{27}	13.06×10^{3}	9.404717×10^{22}
Saturn	1.433×10^{12}	5.68360×10^{26}	9.64×10^{3}	1.129471×10^{22}
Uranus	2.872×10^{12}	8.68160×10^{25}	6.80×10^{3}	6.072185×10^{20}
Neptune	4.495×10^{12}	1.02420×10^{26}	5.43×10^{3}	3.654901×10^{20}
Pluto	5.906×10^{12}	1.30300×10^{22}	4.73×10^{3}	3.082711×10^{16}

Table 1. Above, column 1 gives the planet's name; column 2 gives the semi-major axis of each planet; columns 3 and 4 give, respectively, the mass and average orbital velocity of each planet; column 5 gives the value of P_{G} for each planet.

Note: The values in columns 1, 2 and 3 are from Planetary Fact Sheet-NASA

5. Combgravactionalization of Binet's Orbital Equation

Historically, the Binet's orbital equation, derived by the French scientist Jacques Philippe Marie Binet (1786 –1856), provides the form of a central force given the shape of the orbital motion in plane polar coordinates

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}. \tag{25}$$

This equation can also be used to derive the shape of the orbit for a given force law, but this usually involves the solution to a second order nonlinear ordinary differential equation. Actually, we have already combgravactionalized this classical form (25) in earlier work [2,4],*i.e.*, rewriting the Binet's orbital equation in the CGA-context.

Presently, we are so motivated by the fact that through the previous papers [1,2,3,4,5] we have seen that the CGA as an alternative post-Newtonian gravity theory is very capable of exploring, predicting and explaining some old and new gravitational phenomena. For example, in [2] we have derived two important formulae one for the perihelion advance of Mercury and the other for the angular deflection of starlight. Indeed, the two formulae had been deduced from the CGA-Binet's orbital equation, which has exactly the same physico-mathematical structure as the general relativistic Binet's orbital equation developed in the framework of curved space-time and Schwarzschild metric [10,11,12]. The fact seemed a pure coincidence at first sight, but when one analyzes the paper [3] with fully open mind, he/she will find that in spite of the concept of curved space-time there is a certain compatibility between CGA and GRT reflected by Eq.(25) deduced from Eq.(24) which itself is an expression of the gravitational force derived by Ridgely [13] in the framework of GRT. Also, from the same Eq.(25) in Ref.[3] we can deduce the basic result of CGA, namely, the dynamic gravitational field-force $\left(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}}\right)$.

Again, from all that, we can logically assert that the concept of curved space-time is nothing but only a mathematical artifact and the existence of such compatibility signifies, among other things, the CGA is a counterexample to GRT because, before the advent of the CGA as an alternative gravity theory, it was constantly stressed that the study of the compact stellar objects is exclusively belonging to GRT-domain because their strong compactness is enough to bend the local space-time in such a way that some observable GRT-effects should occur. However, as we have already seen in [3], the CGA is also able to investigate, predict and explain the same type of effects in compact stellar objects and all that in the framework of Euclidean geometry and Galilean relativity principle. This is a sign of a tangible fact that the propagation of gravitational field and the action of gravitational force both are independent of the topology of space-time.

Furthermore, in order to make the cited counterexample more lucid, more localizable and more understandable, we shall combgravactionalize once again the Binet's orbital equation as follows. Let us consider the test-body *B* of mass *m* orbiting the gravitational source *A* of mass *M*. Therefore *B* evolving in the combined gravitational field. Moreover, supposing that the orbital motion takes place in the polar plan $(0, \mathbf{e}_r, \mathbf{e}_\phi)$ inside the vicinity of *A*; so exactly like before (see Section 3), that is in such case, the CGPE-function (1) takes the form

$$U \equiv U(r,\dot{r}) = -\frac{k}{r}\left[1+\left(\frac{\ell^2}{m^2 c_0^2 r^2}+\frac{\dot{r}^2}{c_0^2}\right)\right]. \tag{26}$$

We can now write directly the force due to the CGPE as follows

$$F = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) - \frac{\partial U}{\partial r} = -\frac{k}{r^2}\left[1+\left(\frac{3\ell^2}{m^2 c_0^2 r^2}+\frac{\dot{r}^2}{c_0^2}+\frac{2\ddot{r}r}{c_0^2}\right)\right]. \tag{27}$$

We have also for a velocity-dependent central force $\mathrm{f}(r,\dot{r})$ for an orbit in the polar plane $(0,\mathbf{e}_r,\mathbf{e}_\phi)$:

$$\mathbf{F} = \mathrm{f}(r,\dot{r})\mathbf{r}, \tag{28}$$

and according to Newton's second law

$$\mathbf{F} = m\mathbf{a}, \tag{29}$$

or more explicitly

$$\mathbf{F} = \mathrm{f}(r,\dot{r})\mathbf{r} = m\left(\ddot{r} - r\dot{\phi}^2\right)\mathbf{e}_r + m\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\mathbf{e}_\phi \ . \tag{30}$$

The differential equations relative to the directions $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are:

$$\frac{F}{m} = \frac{\mathrm{f}(r,\dot{r})}{m} = \ddot{r} - r\dot{\phi}^2, \tag{31}$$

$$m\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0. \tag{32}$$

That implies

$$r^2\dot{\phi} = h = \text{constant}. \tag{33}$$

Let us put

$$r = \frac{1}{u}. \tag{34}$$

By differentiating relation (34) with respect to time, we get

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\left(\frac{du}{d\phi}\right)\frac{d\phi}{dt}, \tag{35}$$

thus

$$\dot{r} = -\frac{1}{u^2}\dot{\phi}\,\frac{du}{d\phi} \ . \tag{36}$$

We have from (33) and (34)

$$\dot{\phi} = \frac{h}{r^2} = hu^2, \tag{37}$$

By substituting (37) in (36), we obtain

$$\dot{r} = -h\,\frac{du}{d\phi}\,. \qquad (38)$$

From (39), the second time derivative is

$$\ddot{r} = -\,h\,\frac{d}{dt}\left(\frac{du}{d\phi}\right) = -\,h\frac{d}{dt}\left(\frac{du}{d\phi}\right)\frac{d\phi}{d\phi} = -\,h\frac{d}{d\phi}\left(\frac{du}{d\phi}\right)\frac{d\phi}{dt}\,,$$

thus

$$\ddot{r} = -\,h\,\frac{d^2u}{d\phi^2}\,\dot{\phi}\,. \qquad (39)$$

Taking account of (37), Eq.(39) becomes

$$\ddot{r} = -\,h^2u^2\,\frac{d^2u}{d\phi^2}\,, \qquad (40)$$

From (37) and (40), Eq.(31)

$$\frac{F}{m} = \frac{\mathrm{f}\left(r,\dot{r}\right)}{m} = -\,h^2u^2\,\frac{d^2u}{d\phi^2} - h^2u^3\,. \qquad (41)$$

Taking into consideration the relations (34) and (38), we find from (27) and (41) the following equation

$$\left(1+\frac{2ku}{mc_0^2}\right)\frac{d^2u}{d\phi^2} - \frac{k}{mc_0^2}\left(\frac{du}{d\phi}\right)^2 + u = \frac{k}{h^2m}\left[1+\frac{3\ell^2u^2}{m^2\,c_0^2}\right]\,. \qquad (42)$$

Recall $k = GMm$, thus for the case of planetary motion, the dimensionless quantity $\frac{2ku}{mc_0^2}$ and the term $\frac{k}{mc_0^2}\left(\frac{du}{d\phi}\right)^2$ in the above equation may be practically neglected , and without loss of generality, Eq.(42) reduces to

$$\frac{d^2u}{d\phi^2} + u - \left[1+\frac{3GM\,\ell^2}{m^2\,c_0^2h^2}\right]u^2 = \frac{GM}{h^2}\,. \qquad (43)$$

Remark since $\ell = m\dot{\phi}r^2$ and $h = \dot{\phi}r^2$, thus the quantity in square brackets on the left-hand side of Eq.(43), becomes $\left[\frac{3GM\,\ell^2}{m^2\,c_0^2\,h^2}\right] = \left[\frac{3GM}{c_0^2}\right]$ and finally after substitution in (43), we get the very expected CGA-Binet's orbital equation

$$\frac{d^2u}{d\phi^2} + u - \left[\frac{3GM}{c_0^2}\right]u^2 = \frac{GM}{h^2}\,. \qquad (44)$$

Eq.(44) has exactly the physico-mathematical structure of the general relativistic Binet's orbital equation developed in the context of curved space-time and Schwarzschild metric [10,11,12]. This equation is highly important because as it was already shown in [2] and also in many well written pedagogical textbooks on GRT, particularly Refs.[10,11,12]. For example, as previously reported, we have already derived from Eq.(32) of Ref.[2], which is identical to Eq.(44), the formula:

$$\Delta\varphi = \frac{6\pi GM}{ac_0^2\left(1-e^2\right)} \qquad \text{(rad/rev)}, \tag{45}$$

for planetary orbital precession, which is identical to that derived from GRT. Moreover, in view of the fact that the perihelion advance by $\Delta\varphi$ *per* revolution, thus in this case the resultant equation for the elliptical orbit should be

$$r = \frac{a\left(1-e^2\right)}{1-e\cos\left(\varphi+\Delta\varphi\right)} \ . \tag{46}$$

Eq.(46) may be regarded as a generalization of the classical one $r = a\left(1-e^2\right)/\left(1-e\cos\varphi\right)$ when the test-body evolving in the combined gravitational field $\mathbf{g} = \boldsymbol{\gamma} + \boldsymbol{\Lambda}$ as well detailed in Refs.[3,4,5]. Also in Refs.[2] and [10], a very important formula –for the angular deflection of light ray passing nearby a massive body– was derived from Eq.(44):

$$\theta = \frac{4GM}{c_0^2 r} \ . \tag{47}$$

The formula (47) playing a central role in the gravitational lensing. It is judged imperative to note that before the advent of CGA as a post-Newtonian gravity theory, such an effect was taken as an important astrophysical consequence of GRT, however, it is also a direct consequence of the CGA-formalism. There is another natural consequence of the same CGA-formalism, namely, the gravitational time dilation. According to CGA, this gravitational phenomenon occurring in close proximity to a near-spherical massive body in orbital motion, such as the Earth. In the CGA-context, this phenomenon is called CGA-time dilation and defined by the expression

$$\frac{\delta t}{\delta t'} = \left(1+\frac{GM}{c_0^2 r}\right)^{-1/2} , \tag{48}$$

where δt is the elapsed time for an observer within the gravitational field; $\delta t'$ is the elapsed time for an observer sufficiently distant from the massive object (and therefore outside of the gravitational field); *r* is the radial distance from the center of the gravitational source of mass *M*. The formula (48) may be easily deduced from Eq.(10) of Ref.[4]. Therefore, CGA-time dilation is a phenomenon whereby time runs slower in a strength combined gravitational field . Put simply, the closer you are to a large body like the Earth the slower time runs, thus time runs slower for someone on the surface of the earth compared to someone in orbit around the earth.

Illustrative example: Let P_1 and P_2 be two idealized identical simple (gravity) pendulum, that is each one is composed of the same point mass $(m_1 = m_2)$ on the end of a massless cord of the same length $(l_1 = l_2)$ suspended from a pivot, without friction. When given an initial push, it will swing back and forth at a constant amplitude. The first is supposed at rest on Earth's surface, then $g_0 = GM_{\mathrm{E}}/R_{\mathrm{E}}^2$ and the second is placed above the earth's surface, *e.g.*, at certain altitude *z*, thus $g = GM_{\mathrm{E}}/(R_{\mathrm{E}} + z)^2$. So, the question to be addressed is: what is the value of each pendulum's period? The answer is: we have for P_1 and P_2, respectively, $T_1 = 2\pi\sqrt{l_1/g_0}$ and $T_2 = 2\pi\sqrt{l_2/g}$. However, since $l_1 = l_2$ and $g < g_0$ this implies $T_1 < T_2$, that is P_1 swing back and forth more slower than P_2.

In passing, as a natural consequence, the conceptual existence of CGA-time dilation implies the existence of CGA-gravitational redshift. Of course, this gravitational phenomenon may be easily deduced from classical mechanics and before ending this passage, it is best to mention another phenomenon usually called Lense-Thirring effect [14,15,16], which supposed to be a direct result of GRT: an orbiting test-body around a rotating central body will have its orbital plane dragged around the spinning body in the same sense as the rotation of the central body. Also this phenomenon is called 'dragging of inertial frame' or more simply 'frame dragging' as Einstein named it. Exactly, the same phenomenon exists in CGA-context and it is called 'CGA-dragging effect'. It is a natural consequence of the gravitorotational acceleration field (GRAF) $\boldsymbol{\lambda}$, which is derived from the dynamic gravitational field $\boldsymbol{\Lambda}$ [17].

Finally, as the reader (who is too familiar with CGA-formalism) can remark the following evidence: the aforementioned gravitational phenomena are conventionally and generally attributed to GRT alone as a direct result of space-time curvature. However, as we have seen, the CGA is also able to investigate, predict and explain the same type of the gravitational phenomena and all that was done in the context of the usual Euclidean geometry and the Galilean relativity principle. This reflects a tangible reality that the propagation of gravitational field and the action of gravitational force both are independent of the *topology* of space-time. In this sense, CGA should be regarded not only as a refinement and generalization of Newton's gravity theory but also as a counterexample to GRT because according to CGA, it is the couple $(\boldsymbol{\Lambda}, \mathbf{F}_{\mathrm{D}})$ that is physically responsible for the above mentioned gravitational phenomena rather than to be due to the curved space-time.

6. Conclusion

Based solely on the CGA-formalism, we have derived a general expression for the CGA-effective (radial) potential energy and investigated the behavior and shape of the orbits of a test-body during its orbital motion inside the vicinity of the principal gravitational source. As direct consequences, two expressions for the concepts of gravitational momentum and the dynamic gravitational force are derived and the classical Binet's orbital equation is generalized in the CGA-context, and its physico-mathematical expression is exactly identical to that already found in the framework of general relativity theory (GRT). Also it is shown that the CGA is very capable of predicting and investigating the same gravitational phenomena that are previously attributed to GRT uniquely.

REFERENCES

[1] M.E. Hassani, Galilean Electrodynamics **20**, Special Issues 3, 54 (2009)
[2] M.E. Hassani, Galilean Electrodynamics **22**, Special Issues 3, 43 (2011)
[3] M.E. Hassani, viXra.org/pdf/1303.0112 (2013)
[4] M.E. Hassani, viXra.org/pdf/1306.0001 (2013)
[5] M.E. Hassani, viXra.org/pdf/1503.0143 (2015)
[6] M.E. Hassani, viXra.org/pdf/1505.0189 (2015)
[7] J. D. Norton, Archive for History of Exact Sciences, 15. XII, Volume **45**, Issue1, pp 17-94 (1992)
[8] H.A. Lorentz, Proceedings of the Royal Netherlands Academy of Arts and Sciences, **2**, 559 (1900)
[9] V.B. Braginsky, M.C. Caves and K.S. Thorn, Phys. Rev D. **15**, 2047 (1977)
[10] J.L. Lopes, Théorie Relativiste de la Gravitation (Masson, Paris, 1993)
[11] J. Weber, General Relativity and Gravitational wave (Interscience, New York, 1961)
[12] R d'Inverno, Introducing Einstein's Relativity (Oxford U. P., New York, 2001)
[13] C.T. Ridgely, Eur. J. Phys. **31**, 949 (2010)
[14] J. Lense and H. Thirring , Phys. Z. **19**, 156 (2015)
[15] C.W. Misner, K.S. Thorne and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973)
[16] I. Ciufolini and J. A. Wheeler, Gravitation and Inertia (Princeton Univ. Press, Princeton, NJ, 1995)
[17] M.E. Hassani, viXra.org/pdf/1409.0017 (2015)