Conic and Cyclidic Sections in the
G8,2 Geometric Algebra, DCGA

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Abstract

The $G_{8,2}$ Geometric Algebra, also called the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA), has entities that represent conic sections. DCGA also has entities that represent planar sections of Darboux cyclides, which are called cyclidic sections in this paper. This paper presents these entities and many operations on them. Operations include reflection, projection, rejection, and intersection with respect to spheres and planes. Other operations include rotation, translation, and dilation. Possible applications are introduced that include orthographic and perspective projections of conic sections onto view planes, which may be of interest in computer graphics or other computational geometry subjects.

Keywords: conformal geometric algebra, conic sections, perspective projection

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1 Introduction

The $G_{8,2}$ Geometric Algebra, called the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA), is introduced in [1]. This paper\(^1\) presents some additional results in DCGA for representing conic and cyclidic sections and for operating on conic and cyclidic sections in DCGA.

The following sections of this paper elaborate on the conic and cyclidic sections with many illustrative figures produced using the Geometric Algebra Computing software called the Geometric Algebra ALgorithms Optimizer Gaalop that is introduced in [2]. In [1], it is explained how to use the Gaalop Visualizer to visualize DCGA entities. The Gaalop Visualizer is introduced in [4]. At the time of writing this paper, Gaalop appeared to be a very unique software for its ability to render visualizations of the DCGA conic and cyclidic section entities.

2 DCGA GIPNS section

The concepts and definitions of geometric inner product null space (GIPNS) and geo-
metric outer product null space (GOPNS) entities are introduced by Perwass in [3]. In this paper, all geometric entities are DCGA GIPNS entities.

The DCGA GIPNS 2-vector 2D-surface entity $\mathbf{Y}$ is defined in [1] as generally being an instance of the DCGA GIPNS 2-vector Darboux cyclide surface entity $\Omega$ or an instance of a degenerate form of $\Omega$. The first degenerate forms are the DCGA GIPNS 2-vector Dupin cyclide surface entity $\Phi$ and the DCGA 2-vector horned Dupin cyclide surface entity $\Gamma$. The next degenerate form is the DCGA GIPNS 2-vector parabolic cyclide surface entity $\Psi$. Further degenerates of the parabolic cyclide entity $\Psi$ are the DCGA 2-vector quadric surface entities. The Darboux cyclide $\Omega$ and Dupin cyclide $\Phi$, $\Gamma$ entities are generally quartic surfaces. The parabolic cyclide $\Psi$ entity is generally a cubic surface.

All of the DCGA GIPNS 2D-surface entities $\mathbf{Y}$ can be intersected with a standard DCGA GIPNS 2-vector plane entity $\Pi$ or with a standard DCGA GIPNS 2-vector sphere entity $S$. The DCGA GIPNS 4-vector intersection 1D-surface entity $\mathbf{Y} \wedge \Pi$ can also be called a DCGA GIPNS section entity $\psi = \mathbf{Y} \wedge \Pi$. A 1D-surface on a plane is also called a plane curve. The degree of a plane curve or section $\psi$ depends on the degree of the 2D-surface $\mathbf{Y}$ it is cut from. A section of a quartic surface $\mathbf{Y} = \Omega$, $\Phi$ or $\Gamma$ is a quartic plane curve. A section of a cubic surface, such as a parabolic cyclide $\mathbf{Y} = \Psi$, is a cubic plane curve. A section of a quadric surface, such as a cone $K$, is a quadratic plane curve, also called a conic section.

A DCGA GIPNS 4-vector section 1D-surface plane-curve entity $\psi$ is defined as the intersection

$$\psi = \mathbf{Y} \wedge \Pi$$

(1)

of a DCGA GIPNS 2-vector 2D-surface entity $\mathbf{Y}$ and a standard DCGA GIPNS 2-vector plane entity $\Pi$.

If $\mathbf{Y}$ is a Darboux cyclide $\Omega$ or Dupin cyclide $\Phi$, then $\psi$ is generally a cyclidic section that is a quartic plane curve. If $\mathbf{Y}$ is a parabolic cyclide $\Psi$, then $\psi$ is generally a cyclidic section that is a cubic plane curve. If $\mathbf{Y}$ is a degenerate parabolic cyclide or quadric surface, then $\psi$ is generally a conic section that is a quadratic plane curve.

As a special case, if $\mathbf{Y}$ is a standard DCGA GIPNS 2-vector sphere $S$ or plane $\Pi$, then $\psi = \mathbf{Y} \wedge \Pi$ is either a standard DCGA GIPNS 4-vector circle entity $\mathbf{C}$ or a standard DCGA GIPNS 4-vector line entity $\mathbf{L}$. These entities are special since they can be intersected with each other, while other entities cannot be intersected with each other. These are special conic sections of a standard sphere or plane.

2.1 DCGA GIPNS conic section

Conic sections are planar cuts through a quadric cone surface. A conic section is an ellipse, parabola, hyperbola, line, non-parallel lines pair, or the cone vertex point. Planar cuts through other quadric surfaces, such as ellipsoid, hyperboloid, and paraboloid, can produce some of the conic section quadratic plane curves, but not all of them.

The DCGA GIPNS 4-vector conic section quadratic 1D-surface plane-curve entity $\kappa$
is defined as the intersection

\[ \kappa = K \land \Pi \]  

(2)

defined as the intersection of a DCGA GIPNS 2-vector cone quadric 2D-surface entity \( K \) and a standard DCGA GIPNS 2-vector plane entity \( \Pi \). The cone entity \( K \) is a degenerate form of the parabolic cyclide entity \( \Psi \).

Figure 1. Conic sections: quadratic plane curves

Figure 1 shows some DCGA conic section entities \( \epsilon, \rho, \eta, \chi \) rendered by the Gaalop Visualizer. The conic section entities are intersections \( K \land \Pi \) of various standard DCGA plane entities \( \Pi \) with a DCGA cone entity \( K \).

In [1], the ellipse \( \epsilon || xy = H || z \land \Pi || z = 0 \), parabola \( \rho || xy = B || z \land \Pi || z = 0 \), and hyperbola \( \eta || xy = J || z \land \Pi || z = 0 \) in the \( xy \)-plane (\( z = 0 \)) are defined as the intersection of the \( xy \)-plane \( \Pi || z = 0 \) with a \( z \)-axis aligned elliptic cylinder \( H || z \), parabolic cylinder \( B || z \), and hyperbolic cylinder \( J || z \), respectively.

2.2 DCGA GIPNS cyclidic section

Cyclidic sections are planar cuts through Darboux cyclide, Dupin cyclide, and parabolic cyclide surfaces. Since quadric surfaces are degenerate cyclides, the cyclidic sections are a generalization of the conic sections. Cyclidic sections through Darboux cyclides and Dupin cyclides are quartic plane curves. Cyclidic sections through parabolic cyclides are
cubic plane curves. Conic sections through degenerate parabolic cyclides are quadratic or linear plane curves. The general cyclidic section can be represented by the intersection of a Darboux cyclide and plane.

The DCGA GIPNS 4-vector cyclidic section quartic 1D-surface plane-curve entity $\mathbf{\omega}$ is defined as the intersection

$$\mathbf{\omega} = \mathbf{\Omega} \wedge \mathbf{\Pi}$$

of a DCGA GIPNS 2-vector Darboux cyclide quartic 2D-surface entity $\mathbf{\Omega}$ and a standard DCGA GIPNS 2-vector plane entity $\mathbf{\Pi}$.

The entity $\mathbf{\Omega}$ may be a degenerate form that includes Dupin cyclides $\Phi$, $\Gamma$ and parabolic cyclides $\Psi$, but not degenerate parabolic cyclides or quadric surfaces. A cyclidic section entity $\mathbf{\omega}$ represents either a quartic or cubic plane curve. A quadratic or linear plane curve is a conic section entity $\mathbf{\kappa}$, which may also be called a degenerate cyclidic section.

Figure 2. Cyclidic section: quartic plane curve

Figure 2 shows an example of a cyclidic section that is a quartic plane curve. The DCGA ring Dupin cyclide $\Phi$ ($R = 3, r_1 = 1, r_2 = 2$) is cut by a DCGA plane $\mathbf{\Pi}$, ($n = \mathbf{e}_3, d = 0$) that is rotated 25° around the $x$-axis $\mathbf{e}_1$ using a DCGA rotor $R$ operation as $\mathbf{\Pi} = R\mathbf{\Pi}_i R^{-1}$. The cyclidic section entity is the intersection entity $\mathbf{\omega} = \Phi \wedge \mathbf{\Pi}$. 
Figure 3 shows an example of a cyclidic section that is a cubic plane curve. The DCGA GIPNS 2-vector parabolic cyclide $\Psi = \text{SOS}^-$ is the DCGA GIPNS 2-vector toroid $O$ ($R = 3, r = 2$) reflected in the standard DCGA 2-vector sphere $S$ ($p = e_1, r = 2$). The sphere $S$ center point $P_D = D(p)$ is on the toroid $O$ surface, and the parabolic cyclide $\Psi$ is the inversion of the toroid $O$ in the sphere $S$. The cyclidic section $\Psi \land \Pi$ is the intersection of the parabolic cyclide $\Psi$ and a standard DCGA plane $\Pi$ ($n = e_3, d = 0$) that is rotated $25^\circ$ around the $x$-axis $e_1$.

3 Operations on DCGA GIPNS section entities

Operations on DCGA GIPNS section entities $\psi$ include:

- reflection of any section entity $\psi$ in a sphere $S$ or plane $\Pi$
- projection of any section entity $\psi$ onto a sphere $S$ or plane $\Pi$
- rejection of any section entity $\psi$ from a sphere $S$ or plane $\Pi$
• intersection of any section entity \( \psi \) with a sphere \( S \) or plane \( \Pi \)

• and general rotation, translation, and dilation.

The reflection in a sphere is also known as inversion in a sphere. Inversion of a curve in a sphere produces the curve with points that are at an inverse displacement from the sphere center point when the sphere radius is \( r = 1 \), which is as expected. Reflection of a curve in a plane produces the reflected image of the curve on the far side of the reflection plane, which is also as expected.

The projection is a spherical or orthographic projection onto a sphere or plane surface, respectively. The inversion and projection of a curve point in and onto a sphere are collinear, on a line from the center of the sphere to the curve point. A curve point projects onto a sphere where the line through the curve point and its inverse point intersects the sphere surface. The projection line also passes through the sphere center point. A curve point projects onto a plane where the line through the curve point and its reflected point intersects the plane surface, and the projected point is midway between the curve point and its reflected point.

The rejection is a perpendicular projection of a curve or surface from a sphere or plane, and it emerges at 90° or normal to the sphere or plane when the curve or surface and the sphere or plane have an intersection. The rejection produces a projected curve on the surface of a perpendicular plane or perpendicular sphere through intersection points. The rejection of a surface from a plane or sphere produces a rejected surface that is perpendicular. For example, an ellipsoid that intersects a plane can be rejected from the plane to produce an elliptic cylinder representing the ellipse cut through the ellipsoid by the plane.

The intersection of the section entities \( \psi \) with each other in general, not just with standard DCGA planes, spheres, lines, and circles, would be a very useful operation but unfortunately it does not work. The DCGA intersections rule and formula is given in [1], which states that only a single entity that is not a standard DCGA plane \( \Pi \), sphere \( S \), line \( L \), or circle \( C \) can be included in a wedge that forms an intersection entity. This rule or limitation still applies when forming intersections that include a DCGA GIPNS section entity \( \psi \). This means that it is possible to intersect any section entity \( \psi = Y \wedge \Pi \) with other coplanar circles \( C = S \wedge \Pi \) and lines \( L = \Pi_2 \wedge \Pi \) as, for example, an intersection entity such as \( Y \wedge S \wedge \Pi_2 \wedge \Pi \), which is an 8-vector. When the common plane \( \Pi \) is known, it can be contracted out of entities, and the example intersection could be written as \((\psi \cdot \Pi) \wedge (C \cdot \Pi) \wedge (L \cdot \Pi) \wedge \Pi\).

Although the operations are limited to working against only spheres and planes, they still allow for many interesting possibilities to produce transformed curves and surfaces that may have scientific or artistic uses. The following subsections explore these operations in more detail, with many illustrative figures rendered by the Gaalop Visualizer.

### 3.1 Reflection

The reflection \( \psi' \) of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity \( \psi \)
in a standard DCGA GIPNS 2-vector plane entity \( \Pi \) is defined as

\[
\psi' = \Pi \psi \Pi^\sim.
\] (4)

The reflection, also called the inversion, \( \psi' \) of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity \( \psi \) in a standard DCGA GIPNS 2-vector sphere entity \( S \) is defined as

\[
\psi' = S \psi S^\sim.
\] (5)

![Figure 4. Reflection (inversion) of cyclidic section in sphere](image)

Figure 4 shows the reflection (inversion) of a cyclidic section in a sphere. The Gaalop Visualizer CLUscript for this figure is:

```
:Green;:D=DupinCyclide(3,2,1);
:Black;:P=Plane(0,0,1/2,0);
:Blue;:S=Sphere(0,0,7/2,5/2);
:Black;:DP=D^P/16;
:Black;:R=S*DP*S/1024;
```

### 3.2 Projection

The projection \( \psi' \) of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity \( \psi \) onto a standard DCGA GIPNS 2-vector plane entity \( \Pi \) is defined as

\[
\psi' = (\psi \cdot \Pi)\Pi^{-1} = (\psi \cdot \Pi)\Pi^\sim.
\] (6)

The projection \( \psi' \) of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity \( \psi \)
onto a standard DCGA GIPNS 2-vector sphere entity $S$ with radius $r$ is defined as

$$\psi' = (\psi \cdot S)S^{-1} = \frac{1}{r^2}(\psi \cdot S)S^\wedge.$$  \hfill (7)

Figure 5 shows the projection of a cyclidic section onto a sphere. The Gaalop Visualizer CLUscript for this figure is:

:Green;:D=DupinCyclide(3,2,1);
:Blue;:S=Sphere(1,1,7/2,5/2);
:Black;:P=Plane(0,0,1/2,0);
:Black;:DP=D^P/16;
:Black;:PRJ=(DP.S)*~S/2048;

3.3 Rejection

The rejection $\psi'$ of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity $\psi$ from a standard DCGA GIPNS 2-vector plane entity $\Pi$ is defined as

$$\psi' = (\psi \wedge \Pi)\Pi^{-1} = (\psi \wedge \Pi)\Pi^\wedge.$$  \hfill (8)
The rejection $\psi'$ of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity $\psi$ from a standard DCGA GIPNS 2-vector sphere entity $S$ with radius $r$ is defined as

$$
\psi' = (\psi \wedge S) S^{-1} = \frac{1}{r^2} (\psi \wedge S) S^{-1}.
$$

Figure 6. Rejection of cyclidic section from sphere

Figure 6 shows the rejection of a cyclidic section from a sphere. The Gaalop Visualizer CLUscript for this figure is:

:Green;:D=DupinCyclide(3,2,1);
:Blue;:S=Sphere(-1,-1,1,5/2);
:Black;:P=Plane(0,0,1/2,0);
:Black;:DP=D^P/16;
:Red;:REJ=(DP^S)*~S/1024;
:Black;:SP=S^P;

It may be difficult to see from the figure, but the rejection curve (red) intersects through the sphere at a $90^\circ$ angle to the surface.
Figure 7. Rejection of cyclidic section from plane

Figure 7 shows the rejection of a cyclidic section from a plane. The Gaalop Visualizer CLU script for this figure is:

:Blue; D=DupinCyclide(3,2,1);
:Green; P2=Plane(1,0,2,0);
:Black; P=Plane(0,0,1/2,0);
:Black; DP=D^P/16;
:Red; REJ=(DP^P2)*~P2;

3.4 Commutator and anti-commutator projections

The geometric product $AB$ can be written as the sum of the anti-symmetric commutator product $\times$ and the symmetric anti-commutator product $\times$ as

$$AB = \frac{1}{2}(AB - BA) + \frac{1}{2}(AB + BA)$$

$$= A \times B + A \times B.$$  (10)

Given a cyclidic section $\psi$ and a sphere $S$ (or a plane), then

$$\psi = \psi S S^{-1}$$

$$= \left( \frac{1}{2} (\psi S - S \psi) + \frac{1}{2} (\psi S + S \psi) \right) S^{-1}$$

$$= (\psi \times S) S^{-1} + (\psi \times S) S^{-1}. $$  (13)

The commutator projection of $\psi$ onto $S$ can be defined as

$$\psi' = (\psi \times S) S^{-1}$$  (15)
and the anti-commutator rejection of $\psi$ from $S$ can be defined as

$$\psi' = (\psi \times S)S^{-1}. \quad (16)$$

As an example, the next figure looks at a parabola and its reflection, projection, rejection, commutator projection, and anti-commutator rejection with respect to a sphere.

![Figure 8. Spherical operations on a parabola](image)

Figure 8 shows spherical operations on a parabola. The sphere $S$, not rendered, is enclosed in the green circle and is shaded magenta. The parabola $\rho$ is constructed as the intersection of a $z$-axis aligned parabolic cylinder $B^{1z}$ and the $xy$-plane $\Pi^{z=0}$. The Gaalop Visualizer CLUscript for this figure is:

```
:Cyan;:C=P CylinderZ(0,0,0,1,1,1);
:Green;:P=Plane(0,0,1,0);
:Magenta;:S=Sphere(-2,0,0,5/2);
:Yellow;:CP=C^P;
```
If the plane $\Pi^z=0$ is contracted out of any of the projection or rejection plane curves, then the result is a surface that contains the plane curve. For example, the anti-commutator rejection plane curve can be turned into a DCGA GIPNS 2D-surface $\Upsilon$ as

$$\Upsilon = \Pi^z=0 \cdot (\rho \times S) S^{-1}$$

(17)

$$= \Pi^z=0 \cdot (\rho \times S) S^{-1}.$$  

(18)

### 3.5 Intersection

As explained in [1], all DCGA entities can be intersected only with standard DCGA spheres $S$ and planes $\Pi$. The standard DCGA line $L = \Pi_2 \wedge \Pi$ and circle $C = S \wedge \Pi$ entities are constructed from standard sphere $S$ and plane $\Pi$ entities. The set $S = \{S, \Pi\}$ includes all instances of the standard bi-DGAGIPNS 2-vector entities, which are spheres and planes. All of the plane-curve section entities $\psi = Y \wedge \Pi$ can be intersected with coplanar lines $L = \Pi_2 \wedge \Pi$ and circles $C = S \wedge \Pi$, but not with any other types of coplanar curves. Coplanar curves may intersect in four or less points in the plane $\Pi$.

The DCGA GIPNS 6-vector intersection $X$ of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity $\psi = Y \wedge \Pi$ and a coplanar standard DCGA GIPNS 4-vector line $L = \Pi_2 \wedge \Pi$ in the plane $\Pi$ is defined as

$$X = (\psi \cdot \Pi) \wedge (L \cdot \Pi) \wedge \Pi$$

(19)

$$= Y \wedge \Pi \wedge \Pi_2 \wedge \Pi.$$  

(20)

The DCGA GIPNS 6-vector intersection $X$ of a DCGA GIPNS 4-vector section 1D-surface plane-curve entity $\psi = Y \wedge \Pi$ and a coplanar standard DCGA GIPNS 4-vector circle $C = S \wedge \Pi$ in the plane $\Pi$ is defined as

$$X = (\psi \cdot \Pi) \wedge (C \cdot \Pi) \wedge \Pi$$

(21)

$$= Y \wedge S \wedge \Pi.$$  

(22)

If $\gamma_1 = (B_1 \in S) \wedge \Pi$ and $\gamma_2 = (B_2 \in S) \wedge \Pi$ are standard DCGA GIPNS 4-vector circle $C$ or line $L$ entities in the plane $\Pi$, then their DCGA GIPNS 8-vector intersection $X$ with a coplanar DCGA GIPNS 4-vector section 1D-surface plane-curve entity $\psi = Y \wedge \Pi$ can be defined as

$$X = (\psi \cdot \Pi) \wedge (\gamma_1 \cdot \Pi) \wedge (\gamma_2 \cdot \Pi) \wedge \Pi$$

(23)

$$= Y \wedge B_1 \wedge B_2 \wedge \Pi.$$  

(24)
4 Applications

4.1 Orthographic projection of a conic section

The orthographic projection $\kappa_{\text{ortho}}$ of a DCGA GIPNS 4-vector conic section entity $\kappa = K \land \Pi^\kappa$ onto a standard DCGA GIPNS 2-vector plane $\Pi$ is defined as

$$\kappa_{\text{ortho}} = (\kappa \cdot \Pi)^{-1}. \quad (25)$$

Figure 9 shows an orthographic projection $\kappa_{\text{ortho}}$ of an ellipse $\kappa = \epsilon$ onto a view plane $\Pi = \Pi^*=0$. The ellipse $\epsilon$, which is constructed as the intersection of an ellipsoid and a plane, is rotated by $45^\circ$ around the $y$-axis. The Gaalop Visualizer CLU script for this figure is:

```plaintext
E=Ellipsoid(0,0,-2,3,2,1)^Plane(0,0,-1,2); :Blue;:ER=Rotor(0,1,0,45)*E*~Rotor(0,1,0,45); :Black;:P=Plane(0,0,1,0); :Black;:ORTHO=(ER.P)*~P;
```

4.2 Perspective projection of a conic section

The perspective projection $\kappa_{\text{persp}}$ of a DCGA GIPNS 4-vector conic section entity $\kappa = K \land \Pi^\kappa$ onto a standard DCGA GIPNS 2-vector plane $\Pi$ from the view point $p = xe_1 + ye_2 + ze_3$ represented by a standard DCGA GIPNS 2-vector sphere $S$ with center $P_D = D(p)$ can be defined as

$$\kappa_{\text{persp}} = (((\kappa \cdot S)^{-1}) \cdot S) \land \Pi$$
$$= (S \cdot ((\kappa \cdot S)^{-1})) \land \Pi$$
$$= K_p \land \Pi. \quad (28)$$

The conic section $\kappa$ is projected onto the sphere $S$ as $\kappa' = (\kappa \cdot S)S^{-1}$. The sphere $S$ is contracted out of $\kappa'$ to form a cone $K_p = \kappa' \cdot S = S \cdot \kappa'$. The cone $K_p$ has vertex point $P_D = D(p)$ and it contains both curves $\kappa$ and $\kappa'$. The cone $K_p$ is intersected with any view plane $\Pi$ to form the perspective projection. The radius $r \neq 0$ of sphere $S$ is arbitrary, but $r = 1$ could be assumed.
It is also possible to use the reflection $S\kappa S^{-1}$ and define $\kappa_{\text{persp}}$ as

\begin{align}
\kappa_{\text{persp}} &= ((S\kappa S^{-1}) \cdot S) \land \Pi \\
 &= (S \cdot (S\kappa S^{-1})) \land \Pi \\
 &= K_p \land \Pi.
\end{align}

Figure 10 shows a perspective projection $\kappa_{\text{persp}}$ of a parabola $\kappa = \rho$ onto a view plane $\Pi^z = -1$ from a view point (or eye point) $p = e_1 + e_2 + e_3$ at the center $P_D = D(p)$ of a sphere $S$. The sphere $S$, not rendered, is shaded. The parabola $\rho$ is constructed as the intersection of a $z$-axis aligned parabolic cylinder $B_{||z}$ and the $xy$-plane $\Pi^z = -2$. The Gaalop Visualizer CLUscript for this figure is:

```
:Red;:C=PCylinderZ(1,-2,-2,5,1)^Plane(0,0,-1,2)/8;
:Blue;:S=Sphere(1,1,1,1);
:Red;:PRJS=(C.S)*~S;
:Green;:K=PRJS.S;
:Black;:P=Plane(0,0,-1,1);
:Black;:PERSP=K^P;
```
Figure 11. Perspective projection of ellipse using reflection

Figure 11 shows a perspective projection $\kappa_{\text{persp}}$ of an ellipse $\kappa = \epsilon$ onto a view plane $\Pi^{z=-1}$ from a view point (or eye point) $p = e_1 + e_2 + e_3$ at the center $P_D = D(p)$ of a sphere $S$ with radius $r = 2$. The ellipse $\epsilon$ is constructed as the intersection of an ellipsoid $E$ and plane $\Pi^{z=-2}$. The ellipse $\kappa = \epsilon$ is reflected in the sphere $S$ as $\kappa' = S\kappa S^{-1}$. The cone $K_p$ is formed by contraction as $K_p = \kappa' \cdot S = S|\kappa'$. The cone $K_p$ is intersected with a view plane $\Pi^{z=-1}$ to form the perspective projection $\kappa_{\text{persp}} = K_p \wedge \Pi^{z=-1}$. The Gaalop Visualizer CLUscript for this figure is:

```CLU
:Red;:E=Ellipsoid(1,-2,-2,3,2,1)^Plane(0,0,-1,2)/8;
:Blue;:S=Sphere(1,1,1,2);
:Red;:REFS=S*E*-S;
:Green;:K=REFS.S;
:Black;:P=Plane(0,0,-1,1);
:Black;:PERSF=K^P;
```

5 Conclusion

This paper has presented the DCGA conic section and DCGA cyclidic section entities, which are generally called DCGA section entities. The section entities are quartic, cubic, quadratic, or linear plane curves formed as the intersection of a DCGA plane with another DCGA surface, which can generally be any Darboux cyclide surface. This paper also presented many of the operations that are available on the DCGA section entities. The operations include reflections, projections, rejections, and intersections against spheres and planes. The DCGA operations for rotation, translation, and dilation on all DCGA entities are also valid on the DCGA section entities. As possible applications, orthographic and perspective projections of conic sections were presented.
This paper has presented, or introduced, only some of the most basic results on the DCGA section entities. Much additional research could be done on these entities to develop many more results and applications that could be discussed in future papers and books.

References