

**Non-stationary helical flows for
incompressible 3D Navier-Stokes equations**

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In fluid mechanics, a lot of authors have been executing their researches to obtain the analytical solutions of Navier-Stokes equations, even for 3D case of *compressible* gas flow. But there is an essential deficiency of non-stationary solutions indeed.

In our derivation, we explore the case of non-stationary *helical* flow of the Navier-Stokes equations for incompressible fluids at *any* given initial conditions for velocity fields (*it means an open choice for the space part of a solution*).

Keywords: Navier-Stokes equations, non-stationary helical flow, Arnold-Beltrami-Childress (ABC) flow.

1. Introduction, the Navier-Stokes system of equations.

In accordance with [1-3], the Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented in the Cartesian coordinates as below (*under the proper initial conditions*):

$$\nabla \cdot \vec{u} = 0, \quad (1.1)$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F}, \quad (1.2)$$

- where \mathbf{u} is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, ν is the kinematic viscosity, and \mathbf{F} represents external force (*per unit of mass in a volume*) acting on the fluid. Let us also choose the Ox axis coincides to the main direction of flow propagation; notation \mathbf{u} or \vec{u} means a vector field.

Besides, we assume here external force \mathbf{F} above to be the force, which has a potential ϕ represented by $\mathbf{F} = -\nabla \phi$.

2. The originating system of PDE for Navier-Stokes Eqs.

Using the identity $(\mathbf{u} \cdot \nabla) \mathbf{u} = (1/2) \nabla(\mathbf{u}^2) - \mathbf{u} \times (\nabla \times \mathbf{u})$, we could present the Navier-Stokes equations (1.1)-(1.2) for incompressible inviscid flow $\mathbf{u} = \{u_1, u_2, u_3\}$ as below [4-5]:

$$\nabla \cdot \vec{u} = 0, \quad (2.1)$$

$$\frac{\partial \vec{u}}{\partial t} = \vec{u} \times \vec{w} + \nu \cdot \nabla^2 \vec{u} - \left(\frac{1}{2} \nabla(\vec{u}^2) + \frac{\nabla p}{\rho} + \nabla \phi \right)$$

- here we denote *the curl field* \mathbf{w} , a pseudovector *time-dependent* field [6]; besides, let us denote: $-\{(\nabla p/\rho) + \nabla \phi\} = \{f_x, f_y, f_z\}$.

Vorticity, associated with the curl field, is assumed to be arising due to the proper sources of vorticity in the flow of fluids [4-5]. For example, such a sources could be associated with the solid surface or pressure gradient in case of non-barotropic compressible fluids, influence of viscous forces, Coriolis forces (when one's reference frame is rotating rigidly) or curving shock fronts when speed is supersonic.

3. The presentation of time-dependent solution.

Let us search for solutions of the system (2.1) in a form of *helical* flow below:

$$\vec{w} = \alpha \cdot \vec{u} \Rightarrow \vec{u} \times \vec{w} = \vec{0}, \quad \nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) = -\alpha^2 \cdot \vec{u} \quad (3.1)$$

- here α is the constant coefficient, given by the initial conditions ($\alpha \neq 0$).

References:

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Appendix

Let us explore the fixed (stationary) points of ABC -flow dynamical system [4-5], which could be associated with the local *stability* of the dynamical trajectories for such a system. It means that there are valid appropriate conditions $dx/dt = dy/dt = dz/dt = 0$; so, we obtain from the equations of ABC -flow dynamical system [4-5]:

$$\begin{cases} C \cos y = -A \sin z, \\ B \sin x = -A \cos z, \\ B \cos x = -C \sin y . \end{cases} \quad (\text{A.1})$$

System of Eqs. (A.1) yields *three* types of solutions, depending on the meanings of parameters $\{A, B, C\}$, as presented below

$$C^2(\cos^2 y - \sin^2 y) + B^2 = A^2 \Rightarrow y = \frac{1}{2} \arccos\left(\frac{A^2 - B^2}{C^2}\right), \quad -C^2 \leq A^2 - B^2 \leq C^2 \quad (\text{A.2})$$

$$A^2(\cos^2 z - \sin^2 z) + C^2 = B^2 \Rightarrow z = \frac{1}{2} \arccos\left(\frac{B^2 - C^2}{A^2}\right), \quad -A^2 \leq B^2 - C^2 \leq A^2 \quad (\text{A.3})$$

$$B^2(\cos^2 x - \sin^2 x) + A^2 = C^2 \Rightarrow x = \frac{1}{2} \arccos\left(\frac{C^2 - A^2}{B^2}\right), \quad -B^2 \leq C^2 - A^2 \leq B^2 \quad (\text{A.4})$$

For solution of a type (A.2) we could obtain from the system Eqs. (A.1) as below

$$x = \pm \arccos\left(-\frac{C}{B} \sin\left(\frac{1}{2} \arccos\left(\frac{A^2 - B^2}{C^2}\right)\right)\right), \quad z = -\arcsin\left(\frac{C}{A} \cos\left(\frac{1}{2} \arccos\left(\frac{A^2 - B^2}{C^2}\right)\right)\right) \quad (\text{A.5})$$

If we assume $A = B = 1$, the last expressions for x, z could be simplified as below

$$x = \pm \arccos\left(-\frac{C}{\sqrt{2}}\right), \quad z = -\arcsin\left(\frac{C}{\sqrt{2}}\right), \quad -\sqrt{2} \leq C \leq \sqrt{2} \quad (\text{A.6})$$

Solutions (A.2), (A.6) could be substituted under assumptions $A = B = 1$ to the system (A.1) for checking of such a solution.

So, if there exists a fixed (stationary) point of ABC -flow dynamical system [4-5] (for the case $A = B = 1$), the condition below should be valid for the range of meanings of parameter C :

$$\pm \sin\left(\arccos\left(-\frac{C}{\sqrt{2}}\right)\right) = -\cos\left(\arcsin\left(\frac{C}{\sqrt{2}}\right)\right)$$

- which is obviously valid for $C = 0$ for example (with the sign “-” in a left part of expression above), or for $C = 1$ (with the resulting sign “-” in a left part of expression above).

In general case (A.5), if there exist a fixed (stationary) points of ABC -flow dynamical system [4-5], the condition below should be valid for the appropriate ranges of meanings of parameters $\{A, B, C\}$:

$$\begin{aligned} & \pm B \sin\left(\arccos\left(-\frac{C}{B} \sin\left(\frac{1}{2} \arccos\left(\frac{A^2 - B^2}{C^2}\right)\right)\right)\right) = \\ & = -A \cos\left(\arcsin\left(\frac{C}{A} \cos\left(\frac{1}{2} \arccos\left(\frac{A^2 - B^2}{C^2}\right)\right)\right)\right), \end{aligned} \quad (\text{A.7})$$