Non-stationary helical flows for incompressible 3D Navier-Stokes equations

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In fluid mechanics, a lot of authors have been executing their researches to obtain the analytical solutions of Navier-Stokes equations, even for 3D case of *compressible* gas flow. But there is an essential deficiency of non-stationary solutions indeed.

In our derivation, we explore the case of non-stationary *helical* flow of the Navier-Stokes equations for incompressible fluids at *any* given initial conditions for velocity fields (*it means an open choice for the space part of a solution*).

Keywords: Navier-Stokes equations, non-stationary helical flow, Arnold-Beltrami-Childress (ABC) flow.

1. Introduction, the Navier-Stokes system of equations.

In accordance with [1-3], the Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented in the Cartesian coordinates as below (*under the proper initial conditions*):

$$\nabla \cdot \vec{u} = 0 , \tag{1.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F} , \tag{1.2}$$

- where ***u*** is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, ν is the kinematic viscosity, and ***F*** represents external force (*per unit of mass in a volume*) acting on the fluid. Let us also choose the Ox axis coincides to the main direction of flow propagation; notation ***u*** or $\vec{u}$ means a vector field.

Besides, we assume here external force ***F*** above to be the force, which has a potential ϕ represented by ***F*** = -∇ ϕ.

2. The originating system of PDE for Navier-Stokes Eqs.

Using the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (1/2)\nabla(\boldsymbol{u}^2) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$, we could present the Navier-Stokes equations (1.1)-(1.2) for incompressible inviscid flow $\boldsymbol{u} = \{u_1, u_2, u_3\}$ as below [4-5]:

$$\begin{aligned} &\nabla \cdot \vec{u} = 0 , \\ &\frac{\partial \vec{u}}{\partial t} = \vec{u} \times \vec{w} + \nu \cdot \nabla^2 \vec{u} - \left(\frac{1}{2}\nabla(\vec{u}^2) + \frac{\nabla p}{\rho} + \nabla \phi \right) \end{aligned} \tag{2.1}$$

- here we denote *the curl field* ***w***, a pseudovector *time-dependent* field [6]; besides, let us denote: $-\{(\nabla\ p/\rho) + \nabla\ \phi\} = \{f_x, f_y, f_z\}$.

Vorticity, associated with the curl field, is assumed to be arising due to the proper sources of vorticity in the flow of fluids [4-5]. For example, such a sources could be associated with the solid surface or pressure gradient in case of non-barotropic compressible fluids, influence of viscous forces, Coriolis forces (when one's reference frame is rotating rigidly) or curving shock fronts when speed is supersonic.

3. The presentation of time-dependent solution.

Let us search for solutions of the system (2.1) in a form of *helical* flow below:

$$\vec{w} = \alpha \cdot \vec{u} \;\Rightarrow\; \vec{u} \times \vec{w} = \vec{0}, \quad \nabla^2 \vec{u} \;=\; \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u}) = -\alpha^2 \cdot \vec{u} \qquad (3.1)$$

- here α is the constant coefficient, given by the initial conditions ($\alpha \neq 0$).

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Appendix

Let us explore the fixed (stationary) points of *ABC*-flow dynamical system [4-5], which could be associated with the local *stability* of the dynamical trajectories for such a system. It means that there are valid appropriate conditions *dx/dt* = *dy/dt* = *dz/dt* = 0; so, we obtain from the equations of *ABC*-flow dynamical system [4-5]:

$$\begin{cases} C\cos y = -A\sin z\,, \\ B\sin x = -A\cos z\,, \\ B\cos x = -C\sin y\ . \end{cases} \qquad (A.1)$$

System of Eqs. (A.1) yields *three* types of solutions, depending on the meanings of parameters {*A*, *B*, *C*}, as presented below

$$C^2(\cos^2 y - \sin^2 y) + B^2 = A^2 \Rightarrow y = \frac{1}{2}\arccos\left(\frac{A^2 - B^2}{C^2}\right), \quad -C^2 \le A^2 - B^2 \le C^2 \quad (A.2)$$

$$A^2(\cos^2 z - \sin^2 z) + C^2 = B^2 \Rightarrow z = \frac{1}{2}\arccos\left(\frac{B^2 - C^2}{A^2}\right), \quad -A^2 \le B^2 - C^2 \le A^2 \quad (A.3)$$

$$B^2(\cos^2 x - \sin^2 x) + A^2 = C^2 \Rightarrow x = \frac{1}{2}\arccos\left(\frac{C^2 - A^2}{B^2}\right), \quad -B^2 \le C^2 - A^2 \le B^2 \quad (A.4)$$

For solution of a type (A.2) we could obtain from the system Eqs. (A.1) as below

$$x = \pm\arccos\left(-\frac{C}{B}\sin\left(\frac{1}{2}\arccos\left(\frac{A^2 - B^2}{C^2}\right)\right)\right), \quad z = -\arcsin\left(\frac{C}{A}\cos\left(\frac{1}{2}\arccos\left(\frac{A^2 - B^2}{C^2}\right)\right)\right) \quad (A.5)$$

If we assume $A = B = 1$, the last expressions for x, z could be simplified as below

$$x = \pm\arccos\left(\mp C\sin\left(\frac{\pi}{2}\right)\right), \quad z = 0, \quad -1 \le C \le 1 \qquad (A.6)$$

Solutions (A.2), (A.6) could be substituted under assumptions $A = B = 1$ to the system (A.1) for checking of the resulting solutions above:

$$\begin{cases} C\cos\left(\pm\frac{\pi}{2}\right) = -\sin 0, \\ \sin\left(\pm\arccos\left(\mp C\sin\left(\frac{\pi}{2}\right)\right)\right) = -\cos 0, \\ \mp C\sin\left(\frac{\pi}{2}\right) = \mp C\sin\left(\frac{\pi}{2}\right). \end{cases}$$

So, if there exists a fixed (stationary) point of dynamical system (4.2) (for the case $A = B = 1$), the condition below should be valid for the range of meanings of parameter C:

$$\sin\left(\pm\arccos\left(\mp C\sin\left(\frac{\pi}{2}\right)\right)\right) = -\cos 0,$$

$$\Rightarrow \quad \mp C\sin\left(\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) \quad \Rightarrow \quad C = 0$$

- which is obviously valid only for $C = 0$.

In general case (A.5), if there exist a fixed (stationary) points of dynamical system (4.2), the condition below should be valid for the appropriate ranges of meanings of parameters $\{A, B, C\}$:

$$\pm B\sin\left(\arccos\left(-\frac{C}{B}\sin\left(\frac{1}{2}\arccos\left(\frac{A^2-B^2}{C^2}\right)\right)\right)\right) =$$

$$= -A\cos\left(\arcsin\left(\frac{C}{A}\cos\left(\frac{1}{2}\arccos\left(\frac{A^2-B^2}{C^2}\right)\right)\right)\right), \qquad (A.7)$$