Stability of Hamiltonians.

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Abstract

Hamiltonians in particle interactions are subject to a number of constraints originating from locality, the cluster decomposition principle and Lorentz covariance of the scattering matrix. Moreover, it is always assumed that the Hamiltonian must be defined on a Hilbert space and be bounded from below, the latter would be a requirement following from stability of the system. In this paper we examine if these criteria are really mandatory, as it is well known that all of them taken together lead to the usual infinities of quantum field theory. In particular, we study a class of Hamiltonians unbounded from below and examine its stability. This leads us into the construction of novel statistics in three space dimensions; it is shown that for rigid strings the possibilities for statistics exceed those for standard anyons in two space dimensions.

1 Introduction.

It is often stated that Hamiltonians must be bounded from below, since otherwise physical states could decay and radiate an abritrary amount of energy[1]. An obvious reaction against such argument is that the opposite process can also happen and with the same amplitude; therefore, the issue needs to be examined with more care at a nonperturbative level. In particular, we will adress stability of the natural vacuum state in such formalism and work also towards multi-particle states. Explicit computations are laborious even for rather simple perturbations and interesting counting problems show up which we will approach from a few different angles. In this paper, we have nothing to say about Lorentz invariance, we work with Hamiltonians consisting of a (at most) denumerable set of oscillators which are not derived from a local Hamiltonian density. In a second part of this paper, we try to look for new building blocks to construct Hamiltonians in order to enhance stability; this is an investigation of particle statistics beyond the traditional Bose and Fermi rules. It is shown that for rigid strings in three space dimensions, the possibilities for statistics exceed those for anyons in two space dimensions. Also, the first homotopy group of configuration space for rigid strings offers a new way to look at the spin statistics connection. In the conclusions, we discuss our preliminary results and propose further investigations.

It would be interesting to perform the even more general analysis at the level of

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[1]Indeed, some authors take boundedness from below as the very definition of stability.

the Lagrangian in light of the path integral formulation of quantum mechanics which is more general as the Hamiltonian theory. In other words, which conditions does it have to satisfy in order to give a physically meaningful theory? An even more general question of this kind arises in the quantum measure formulation by Sorkin which goes beyond the Lagrangian. In that sense do we only adress this issue from the most conservative point of view, which is that of plain unitary quantum theory.

2 A general class of operators.

In this section, we study perturbations on a free Hamiltonian which is not bounded form below. The vacuum state is defined as the state with no particles and not the state with the lowest energy. All our computations are made with bosons, the implementation of fermions being obvious. Denote by $\mathcal{H}$ a standard Fock space and let $a^{\dagger}_n, b^{\dagger}_n$ denote two sets of creation operators where $n \in \mathbb{Z}_0$. Let

$$H_0 = \sum_{n \in \mathbb{Z}_0} \hbar \left(\omega_n a^{\dagger}_n a_n + \rho_n b^{\dagger}_n b_n \right)$$

where $\omega_{-n} = -\omega_n < 0$ and $\rho_{-n} = -\rho_n < 0$ for $n > 0$, be the free Hamiltonian consisting of pairs $(n, -n)$ of opposite energies. For future reference, denote by $T_n = a^{\dagger}_n a_n - a^{\dagger}_{-n} a_{-n}$ and $R_n = b^{\dagger}_n b_n - b^{\dagger}_{-n} b_{-n}$ then $H_0 = \sum_{n>0} (\omega_n T_n + \rho_n R_n)$. Obviously, T_n, R_m are mutually commuting and self adjoint. We are interested in the following class of perturbations

$$V = \sum_{n>m} \left(\alpha_{nm} a^{\dagger}_n a^{\dagger}_m + \overline{\alpha}_{nm} a_n a_m \right) + \sum_{n \neq m,k} \left(\gamma_{nmk} b^{\dagger}_k a^{\dagger}_n a_m + \overline{\gamma}_{nmk} b_k a^{\dagger}_m a_n \right).$$

Let $|0\rangle$ be the vacuum state annihilated by all a_n, b_m which represent annihilation operators of *particles*. We call a normalized state $|\psi\rangle$ ϵ-*stable* if and only if

$$|\langle \psi | U(t, t_0) | \psi \rangle| > 1 - \epsilon$$

for all $t > t_0$. Here $U(t, t_0) = e^{-\frac{i}{\hbar} H (t - t_0)}$ is the evolution operator for $H = H_0 + V$. Of course ϵ should depend upon the perturbations $\alpha_{nm}, \gamma_{nmk}$ and taken to zero if the latter converge in a suitable sense to zero as well. If $|\psi\rangle$ is ϵ-stable then for all perturbations $|\phi\rangle$ orthogonal to $|\psi\rangle$ we have that $|\langle \psi | U(t, t_0) | \phi \rangle|^2 \leq 1 - |\langle \psi | U(t, t_0) | \psi \rangle|^2 < 1 - (1 - \epsilon)^2 < 2\epsilon$ which means that $|\psi\rangle$ is stable against perturbations in another sense. There is a second, weaker, definition of stability which is the following: we call a normalized state ϵ- stable on a time scale T if and only if

$$|\langle \psi | U(t, t_0) | \psi \rangle| > 1 - \epsilon$$

for all $t_0 + T \geq t \geq t_0$. Here, it is understood that T goes to infinity and ϵ to zero in the limit of zero interaction and that T is in general large enough for sufficiently small ϵ. Indeed, it is this last requirement which makes this definition nontrivial since stability follows automatically for sufficiently small T. What we will show in this paper, is that under suitable conditions the vacuum state (and possibly excited states) are all stable in one sense or another against perturbations defined by V. As usual, it is convenient to work in the interaction picture defining an operator

$$W(t,t_0) = e^{-\frac{i}{\hbar}H_0 t}U(t,t_0)e^{\frac{i}{\hbar}H_0 t_0}$$

since it doesn't affect stability studies and as usual,

$$W(t,t_0) = T\left(e^{\frac{i}{\hbar}\int_{t_0}^{t} V(s)ds}\right)$$

where T denotes time ordering and $V(s) = e^{-\frac{i}{\hbar}H_0 s}Ve^{\frac{i}{\hbar}H_0 s}$. For a general state of definite particle number $a^\dagger_{n_1}\ldots a^\dagger_{n_k}b^\dagger_{m_1}\ldots b^\dagger_{m_l}|0\rangle$ one has that

$$\langle 0|b_{m_l}\ldots b_{m_1}a_{n_k}\ldots a_{n_1}U(t,t_0)a^\dagger_{n_1}\ldots a^\dagger_{n_k}b^\dagger_{m_1}\ldots b^\dagger_{m_l}|0\rangle =$$

$$e^{i\left(\sum_{\alpha=1}^{k}\omega_{n_\alpha}+\sum_{\alpha=1}^{l}\rho_{m_\alpha}\right)(t-t_0)}\langle 0|b_{m_l}\ldots b_{m_1}a_{n_k}\ldots a_{n_1}W(t,t_0)a^\dagger_{n_1}\ldots a^\dagger_{n_k}b^\dagger_{m_1}\ldots b^\dagger_{m_l}|0\rangle$$

which relates both pictures to one and another. It is obvious that $\langle 0|V(s)|0\rangle = 0$ so the first nontrivial contribution in perturbation theory comes from

$$\int_{t_0}^{t} ds_1\int_{t_0}^{s_1} ds_2\langle 0|V(s_1)V(s_2)|0\rangle = \sum_{n>m}|\alpha_{nm}|^2\left(\sum_{k=3}^{\infty}\frac{(i(\omega_n+\omega_m))^{k-2}(t-t_0)^k}{k!}+\frac{1}{2}(t-t_0)^2\right)$$

as the reader can immediately verify. In particular, the above formula simplifies in case $n=-m$ which is obvious since those parts in V commute with H_0. Since there is no harm in starting out from a more limited proposal, define $L_n = a^\dagger_n a^\dagger_{-n}$ and $K_n = a^\dagger_n a_n + a^\dagger_{-n}a_{-n}$, then $[L_n, L_m] = [K_n, K_m] = 0$, $\left[L^\dagger_n, L_m\right] = \delta_{nm}\left(1+K_n\right)$ and $[K_n, L_m] = 2L_n\delta_{nm}$. Moreover, $[T_n, L_m] = 0 = [T_n, K_m]$ so that $V' = \sum_{n>0}\left(\alpha_n L_n + \overline{\alpha}_n L^\dagger_n\right)$ commutes with H_0. First, let us make some computations with V' and switch later on to the case of only one mode L and T (obviously, this Hamiltonian is still unbounded from below). Let us show the result of some preliminary computations: define $\gamma_k = \sum_{n>0}|\alpha_n|^k$ for $k \geq 0$, then we will assume that the γ_{2k} are all finite. In particular, we have that

$$\begin{aligned}\langle 0|V'^2|0\rangle &= \gamma_2\\ \langle 0|V'^4|0\rangle &= 3\gamma_2^2+2\gamma_4\\ \langle 0|V'^6|0\rangle &= 15\gamma_2^3+30\gamma_2\gamma_4+16\gamma_6\end{aligned}$$

and in general $\langle 0|V'^{(2k+1)}|0\rangle = 0$ for all $k \geq 0$. However, these computations are laborious and show no clear systematic, so that we study the more simple case of a single mode (we drop all indices n henceforth) which already shows interesting numerics.

2.1 Fermionic harmonic oscillator.

There are two reasons for studying the toy model of a pair of fermionic oscillators: (a) the computations are much easier since the Hilbert space is finite dimensional (b) real matter is fermionic, so we aim to gain some physical insight here. Let T, L, K be defined as before, then the commutation relations for fermionic oscillators become $\left[L^\dagger, L\right] = 1-K$, $[K, L] = 2L$ and $[T, L] = [T, K] = 0$ so that effectively only the sign of K differs in the first expression. Moreover, we have that $L^2 = 0$, $L^\dagger|0\rangle = K|0\rangle = Lb^\dagger_\pm|0\rangle = L^\dagger b^\dagger_\pm|0\rangle = 0$ leading immediately to the stability of the exited states $b^\dagger_\pm|0\rangle$. In fact,

$$\langle 0|b_\pm U(t,t_0)b^\dagger_\pm|0\rangle = e^{\mp i\omega(t-t_0)}$$

so that we only need to adress vacuum stability. So, we are interested $(V')^n|0\rangle$ which equals $|\alpha|^n|0\rangle$ for n even and $\alpha|\alpha|^{n-1}L|0\rangle$ for n odd so that $\langle 0|U(t,t_0)|0\rangle = \cos\left(\frac{|\alpha|}{\hbar}(t-t_0)\right)$. Therefore, we must conclude that the vacuum is not ϵ-stable in the strict sense for any $\epsilon > 0$ but oscillates between $|0\rangle$ and $L|0\rangle$ as

$$U(t,t_0)|0\rangle = \cos\left(\frac{|\alpha|}{\hbar}(t-t_0)\right)|0\rangle - i\frac{\alpha}{|\alpha|}\sin\left(\frac{|\alpha|}{\hbar}(t-t_0)\right)L|0\rangle.$$

Obviously, for any $\epsilon > 0$ there exists a $T = \frac{\hbar\cos^{-1}(1-\epsilon)}{|\alpha|}$ such that $|0\rangle$ is ϵ stable on a timescale T which goes to infinity if $|\alpha| \to 0$; however $\hbar\cos^{-1}(1-\epsilon)$ is a ridiculously small number so that $|\alpha|$ must be much smaller than the free energy of the oscillator in order to give appreciable time scales; hence it is justified to think that this notion of stability does not give a reliable result. We will come now to a third notion of stability which is much more robust. One calculates that

$$U(t,t_0)L|0\rangle = \cos\left(\frac{|\alpha|}{\hbar}(t-t_0)\right)L|0\rangle - i\frac{\overline{\alpha}}{|\alpha|}\sin\left(\frac{|\alpha|}{\hbar}(t-t_0)\right)|0\rangle$$

so that the eigenstates are given by

$$|0,+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \frac{\alpha}{|\alpha|}L|0\rangle\right)$$

and

$$|0,-\rangle = \frac{1}{\sqrt{2}}\left(\frac{\overline{\alpha}}{|\alpha|}|0\rangle - L|0\rangle\right)$$

with eigenvalues for H of $\pm|\alpha|$. So, the vacuum breaks up into two vacua with opposite energies; indeed, the correct interpretation of $L|0\rangle$ is that of an alternative vacuum state since the free energy of both equals exactly zero. Indeed from the perspective of $H_0 = \hbar\omega T$, the states $|0,\pm\rangle$ both have zero enery; moreover, $b^\dagger|0,\pm\rangle \sim b^\dagger|0\rangle$ so that both vacua cannot be distinguished from the point of view of particle creation. Of course, $|0\rangle$ remains the only state which is annihilated by both $b_\pm$ and therefore represents the true vacuum. Summarizing, we propose that $|0\rangle$ and $L|0\rangle$ are indistinguishable experimentally, a pair of two particles with opposite energies should not evoque detection of any kind. In that sense, the vacuum is stable since the motion of $|0\rangle$ occurs in its *equivalence class*. A similar interpretation will be useful for the excited states in case multiple oscillators are included. Therefore, we generalize our notion of stability as follows: let P_ψ denote the Hermitian projection operator on the space of physically equivalent states of $|\psi\rangle$, then $|\psi\rangle$ is ϵ-stable within its equivalence class if and only if

$$\langle\psi|U^\dagger(t,t_0)P_\psi U(t,t_0)|\psi\rangle > (1-\epsilon)^2.$$

The reader notices that the new definition coincides with the old one in case $P_\psi = |\psi\rangle\langle\psi|$. With this knowledge, we now turn to the study of the bosonic case, the investigation of the multi-mode fermi theory being postponed to the next section.

2.2 A single pair of bosonic modes.

In this section we will try to fully understand the physics behind the Hamiltonian operator $H = \hbar\omega T + \alpha L + \overline{\alpha} L^\dagger$, satisfying the bosonic algebra. We will attack this problem from two sides: (a) by means of the spectral decomposition of $V' = \alpha L + \overline{\alpha} L^\dagger$ and (b) by a more direct way in fully exploiting all algebraic properties. The spectrum of the former operator is continuous and so we have only improper eigenvectors (in contrast to the fermionic case), but first let us establish some useful formulae:

$$\left[L^\dagger, L^n\right] = n^2 L^{n-1} + nL^{n-1}K$$

so that the norm of a state $L^n|0\rangle$ equals $n!$ since $K|0\rangle = L^\dagger|0\rangle = 0$. In particular, the states $|n\rangle = \frac{1}{n!}L^n|0\rangle$ satisfy

$$\langle n|m\rangle = \delta_{nm}.$$

We first try to prove stability of the vacuum state by a direct method. Note from the outset that stability within the equivalence classes of states is an obvious property of the kind of perturbations we are studying here, so nothing can be learned in that sense (a state can only evolve within the eigenspace of $H_0 = \hbar\omega T$ since the perturbations commute with the free Hamiltonian). Therefore, the issue of stability we study here is the restricted one. Also, the techniques developed below will be of importance for the more general study in the next section.

2.2.1 A direct computation.

We are interested in calulating

$$\alpha V'^n L|0\rangle = \sum_{k=1}^{\left[\frac{n}{2}\right]+1} f_k(n)\alpha^{n+3-2k}|\alpha|^{2(k-1)}L^{n+3-2k}|0\rangle$$

where $[x]$ denotes the smallest integer greater or equal than x. hence, we must determine the $f_k(n)$ and in particular the $f_n(2n-2)$ since those correspond precisely to

$$\langle 0|V'^{2n}|0\rangle = |\alpha|^{2n} f_n(2(n-1)).$$

One immediately notices that

$$f_k(n) = f_k(n-1) + (n+4-2k)^2 f_{k-1}(n-1)$$

resulting in

$$f_k(n) = \sum_{i=2k-4}^{n-1} f_{k-1}(i)(i+5-2k)^2$$

for $k \geq 2$ and $f_1(n) = 1$ for $n \in \mathbb{N}$. Using these formula, one can derive that $f_k(n)$ is a polynomial in n of degree $3(k-1)$ with $2k-3$ zeroes at $0, 1, \ldots 2k-4$ leaving $k+1$ parameters to be determined. Alas, this is not a simple matter, we list here the result of some computations:

$$\begin{aligned}
f_2(n) &= \frac{1}{6}n(n+1)(2n+1) \\
f_3(n) &= \frac{1}{360}(n+1)n(n-1)(n-2)(20n^2-32n-9) \\
f_4(n) &= \frac{1}{136080}(n+1)n(n-1)(n-2)(n-3)(n-4)(840n^3-5292n^2+7638n+639)
\end{aligned}$$

and as the reader notices these coefficients become almost intractable. The reader checks that $f_2(2) = 5 = 3+2$ and $f_3(4) = 61 = 15+30+16$ as computed before. However, as mentioned, we are mainly interested in the $f_n(2(n-1))$ and we proceed now by putting up a general computational scheme. From above, or by direct computation, $f_4(6) = 1385$ and $f_5(8) = 50521$ both numbers which blow up rapidly. One notices from these numerics that

$$(2(n-1))! < f_n(2(n-1)) < 3(2n-2)!$$

and it remains to prove if this is really true or not[2]. There is another method to generate these functions and that is by making the Laplace transformation

$$f(z,w) = \sum_{k=1}^{\infty}\sum_{n=2k-3}^{\infty}\frac{w^k}{k!}z^{n+3-2k}f_k(n).$$

Our recursion relations then lead to the following partial differential equation

$$(1-z)\frac{d}{dw}f = z\frac{d^2}{dz^2}f + \frac{d}{dz}f + 1$$

which needs to be solved with initial condition $f(z,0) = 0$. The reader may prefer to solve this equation iteratively and calculate $\frac{d^n}{dw^n}f_{|w=0}$, the above equation gives directly

$$\frac{d}{dw}f_{|w=0} = \frac{1}{1-z}.$$

One immediately notices that $\frac{d^n}{dw^n}f_{|w=0} = \frac{P_n(z)}{(1-z)^{3n-2}}$, where $P_n(z)$ is a polynomial of degree $n-1$. Elementary calculations show that $P_1(z) = 1$, $P_2(z) = z+1$ and $P_3(z) = 9z^2+26z+5$ and as the reader notices, there seems to be a conservation of trouble here since it is hard to explicitely determine the coefficients of these polynomials. In principle it is sufficient to know $\frac{d^n}{dw^n}f(0,0) = f_n(2n-3) = f_{n-1}(2(n-1)-2)$, but even that appears to be too much to ask. Another way to go would be to notice that $f(z,w) = -z + g(z,w)$ where g is a solution of the homogeneous differential equation $(1-z)\frac{d}{dw}g = z\frac{d^2}{dz^2}g + \frac{d}{dz}g$ with initial condition $g(z,0) = z$. The reader may first wish to solve the eigenvalue problem

$$z\frac{d^2}{dz^2}h_\lambda(z) + \frac{d}{dz}h_\lambda(z) = \lambda(z-1)h_\lambda(z)$$

and then look for the appropriate solution in the space spanned by

$$f_\lambda(z,w) = e^{-\lambda w}h_\lambda(z).$$

[2]It would make the power series of $\langle 0|U(t,t_0)|0\rangle$ converge since we must divide $\gamma_n(2n-2)$ by $(2n)!$.

Clearly $\lambda = 1$ is an eigenvalue with $h_1(z) = e^{-z}$, also $\lambda = 0$ is with $h_0(z) = 1$. However, trying to determine more general eigenvectors in the form $h_\lambda(z) = \sum_{n\geq 0} h_n(\lambda)\frac{z^n}{n!}$ leads to problems as before; therefore, we know how to approach the counting problem from a couple of different sides but none of them gives a more simple answer. Let us try to proceed by another method.

2.2.2 A spectral decomposition.

In this subsection, we assume α to be real, therefore V' can be written as $\alpha(\partial_x\partial_y + xy)$ on $L^2(\mathbb{R}^2, dx \wedge dy)$ where we have identified the annihilation operators with $\frac{1}{\sqrt{2}}(\partial_x + x)$ and $\frac{1}{\sqrt{2}}(\partial_y + y)$ respectively. We study now the spectral decomposition of V'; as mentioned previously V' commutes with $T = \frac{1}{2}\left(-\partial_x^2 + x^2 + \partial_y^2 - y^2\right)$. Obviously, $\psi_0(x,y) = e^{\frac{i}{2}(x^2+y^2)}$ and its complex conjugate are improper eigenvectors of V' with eigenvalue zero. Actually, the "eigenspace" of zero is infinite dimensional as one can apply T^n to ψ_0; for example

$$T\psi_0 = \left(x^2 - y^2\right)\psi_0.$$

We knew this already to be the case for T itself, since the states $|n\rangle$ all satisfy $T|n\rangle = 0$. For the stability issue of the vacuum state, it is sufficient to look for the spectral decomposition of V' in the subspace spanned by all $|n\rangle$ since we know V' must leave that invariant. We shall actually do more than that in this section. The restricted problem is this, given a general state $|\psi\rangle = \sum_{n=0}^{\infty}\gamma_n|n\rangle$ with $\gamma_0 = 1$ (without restriction of generality), determine the condition for this to be an eigenstate

$$\alpha\left(L + L^\dagger\right)|\psi\rangle = \lambda|\psi\rangle.$$

Since $L|n\rangle = (n+1)|n+1\rangle$ and $L^\dagger|n\rangle = n|n-1\rangle$, we arrive at the equations

$$\begin{aligned} n\gamma_{n-1} + (n+1)\gamma_{n+1} &= \frac{\lambda}{\alpha}\gamma_n \;; n > 0 \\ \gamma_1 &= \frac{\lambda}{\alpha} \end{aligned}$$

which corresponds to the eigenvalue problem of the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}.$$

We will first show that the spectrum equals $\mathbb{R}$ and conjecture that the discrete spectrum is empty. First of all, it is obvious that λ belongs to the continuous spectrum if and only if there exists a sequence of unit vectors $|\psi_n\rangle$ such that $||(M-\lambda)|\psi_n\rangle||$ converges to zero for n to infinity and moreover, λ belongs not to the discrete spectrum. Let us show first that 0 belongs to the continuous spectrum: it is easy to calculate that $\gamma_{2n+1} = 0$ and $\gamma_{2n} = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}$. Hence, γ_{2n} satisfies the following bounds

$$\frac{\alpha}{\sqrt{n}} < |\gamma_{2n}| < \frac{\beta}{\sqrt{n}}$$

for some $\beta > \alpha > 0$ and therefore one concludes that the norm of the "eigenvector" equals infinity. More rigorously, take $k > 0$ and define

$$|\psi_k\rangle = \frac{\sum_{i=0}^{\infty} \frac{\gamma_{2i}}{(2i)^{\frac{1}{k}}} |2i\rangle}{\sqrt{\sum_{j=0}^{\infty} |\frac{\gamma_{2j}|^2}{(2j)^{\frac{2}{k}}}}}$$

then $||M|\psi_k\rangle||$ converges to zero for k to infinity as the reader may verify for himself. For general $\lambda \in \mathbb{R}$, it is easily checked that $\gamma_k(\lambda)$ is a polynomial of degree k which is even when k is even and odd when k is odd. Hence, $\gamma_k(\lambda)$ contains $\left[\frac{k}{2}\right] + 1$ coefficients. These however, resemble very much the very numbers $f_n(2n-3)$ we are trying to compute and all we can do again is work in a different method with isomorphic problems. Nevertheless, let us proceed; taking the m'th derivative at $\lambda = 0$ of the defining relation yields

$$n\gamma_{n-1}^{m} + (n+1)\gamma_{n+1}^{m} = \gamma_n^{m-1}$$

where $\gamma_n^m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\gamma_n(0)$. Hence, we get a recursive system of vectors γ^m where

$$M\gamma^m = \gamma^{m-1}$$

for $m > 0$ and γ^0 is given by the solution above. Trying to solve $M\gamma^1 = \gamma^0$ gives, as mentioned previously, a system which is hard if not impossible to solve explicitely; the first few solutions for γ^1 are $\gamma_{2k}^1 = 0$, $\gamma_1^1 = 1$, $\gamma_3^1 = -\frac{5}{3!}$, $\gamma_5^1 = \frac{89}{5!}$ and finally $\gamma_7^1 = -\frac{3429}{7!}$. For what it is worth the sign is alternating and the absolute value of the nominator of $\gamma_k^1 = \frac{\delta_k^1}{k!}$ appears to be a number between $(\left[\frac{k}{2}\right]+1)(k-1)!$ and $(\left[\frac{k}{2}\right]+2)(k-1)!$. This may help either in getting a grip upon those numbers or perhaps even determine them explicitely. We see that in general, any λ belongs to the spectrum since we can find exactly one (generalized?) eigenvector. This strongly suggests that the continuum spectrum is $\mathbb{R}$, the only thing which could happen here is that for a discrete number of λ the associated eigenvector actually has finite norm, so that the discrete spectrum lies in the closure of the continuous spectrum. Albeit this is possible in principle and it is easy to construct such operators[3] I conjecture it not to be the case for M. Certainly, a proof is lacking and for sure an explicit spectral decomposition appears out of range. This means we are not able to find a nonperturbative expression for our numbers $f_n(2n-3)$ and hence the only thing we can do is study vacuum stability perturbatively. This is not a very satisfying situation especially given that the perturbations are extremely simple. What would be neat however, is that the computation would be finite in the sense that the total Hilbert space is finite; this would mean that our particles don't satisfy Bose statistics but some nilpotent statistics of order n. This issue is introduced in sections four and five.

3 Multi mode perturbation theory.

The previous section made it clear that the easiest bosonic systems can lead to (insurmountable) computational complications, an issue which we will come

[3]For example, consider a finite dimensional Hilbert space $\mathcal{H}$ and take the direct sum $\mathcal{H} \oplus L^2(\mathbb{R}, dx)$. Define the operator $A = 1 \oplus x$, then A has as continuous spectrum $\mathbb{R} \setminus \{1\}$ and 1 is a discrete eigenvalue.

back to in a short while. Fermions, on the other hand allow sometimes[4] for explicit computations and it is in this section that we will try to generalize our results in two ways. First, we consider a system with a finite number of fermi modes (with positive and negative energy) and we consider Fermi-Fermi couplings of the kind defined in section two. We will adress stability of the vacuum and particle states (within their equivalence classes). Later on, we include a coupling to a finite number of Bose modes (the total Hamiltonian is then unbounded from below) and try to perform a similar study.

3.1 A finite number of coupled Fermi modes.

The Hamiltonian we will study in this section is given by

$$H = \sum_{k=1}^{N} \hbar \omega_k T_k + \sum_{N \geq k > l \geq -N} \left(\alpha_{kl} b_k^\dagger b_l^\dagger + \overline{\alpha}_{kl} b_l b_k \right)$$

and first, we will try to gain some insight in the case $N = 2$ on 16 dimensional Hilbert space.

3.1.1 The case $N = 2$.

It is easily seen that this time the one particle states are not invariant under evolution since the potential V or $V(s)$ acts nontrivially on them, what remains to study is stability of the vacuum (within its equivalence class) as well as of the nontrivial particle states. There are two ways to go, either we calculate explicitely the spectral decomposition of H and then the time evolution easily follows, or we do perturbation theory as usual (which amounts to the same since the Hamiltonian is bounded). We will start with the latter and return to the former at a later stage[5]; it is convenient to work in the interaction picture and

$$V(s) = \sum_{2 \geq k > l \geq -2} \left(\alpha_{kl} e^{-i(\omega_k + \omega_l)s} b_k^\dagger b_l^\dagger + \overline{\alpha}_{kl} e^{i(\omega_k + \omega_l)s} b_l b_k \right)$$

with $\omega_{-l} = -\omega_l$ for $l = 1, 2$. Clearly, $V(s)$ leaves the eight dimensional space spanned by the even particle states invariant as well as the eight dimensional space of the odd particle states. We shall first adress vacuum stability[6] so we calculate the operation of time ordered products

$$V(s_1)V(s_2)\ldots V(s_n)|0\rangle$$

with $s_1 \geq s_2 \geq \ldots \geq s_n$. Let us treat first the case n even since for n odd this expression is a two particle state, while for n even it is a superposition of the vacuum state and the four particle state $L_2 L_1 |0\rangle = b_2^\dagger b_1^\dagger b_{-1}^\dagger b_{-2}^\dagger |0\rangle$ and it is the latter expression which has a nice geometric significance. Define the functions $\Gamma(s,t)_{(++)}$ and $\Gamma(s,t)_{(+-)}$ as follows:

$$\Gamma(s,t)_{(++)} = \sum_{2 \geq k > l \geq -2; 2 \geq r > s \geq -2} e^{-i(\omega_r + \omega_s)s - i(\omega_k + \omega_l)t} \epsilon_{rskl} \alpha_{rs} \alpha_{kl}$$

[4] It is not because a system is finite dimensional that you can explicitely calculate it!

[5] The reason is that in general no explicit spectral decomposition can be made beyond 3 dimensions!

[6] Within its four dimensional equivalence class spanned by $|0\rangle$, $L_i|0\rangle$ and $L_1 L_2|0\rangle$ where $i = 1, 2$ and the L_i are defined as before.

and

$$\Gamma(s,t)_{(-+)} = \sum_{2\geq k>l\geq -2} e^{i(\omega_k+\omega_l)(s-t)}|\alpha_{kl}|^2$$

then the geometric interpretation of $\Gamma(s,t)_{++}$ is that of a path consisting out of two moves forwards where t coincides with the first move and s with the second one. The interpretation of $\Gamma(s,t)_{-+}$ is that of a path defined by two moves, the first being upward corresponding to t, the second being downwards given by s. One can now take the inverses of these moves which corresponds to complex conjugation; that is a path going downwards twice is given by $\Gamma(s,t)_{(--)} = \overline{\Gamma(s,t)}_{(++)}$ while a path going down first and upwards second is given by $\Gamma(s,t)_{(+-)} = \overline{\Gamma(s,t)}_{(-+)} = \Gamma(t,s)_{(-+)}$. To make this more precise, the endpoints of the paths are $0,1,2$ and the only paths we are interested in have an even number of steps and start at 0 so the only possible endpoints are $0,2$. Obviously, for the number of steps greater or equal than two, there are precisely as many paths which end up in 0 as those which end up in 2, the only exception being $n=0$ where of course one path ends up at 0 and none at 2. The number of allowed paths of length $2n$ equals 2^{n-1} as the reader may easily show by induction. Denoting by $\Gamma_{(2n,0)}$ the set of paths of $2n$ moves ending at 0 and $\Gamma_{(2n,2)}$ likewise the set of paths of $2n$ moves ending at 2 where a path is an allowed sequence of $+$ and $-$, we arrive that the formula for the time ordered product $V(s_1)V(s_2)\ldots V(s_{2n})|0\rangle$ is given by:

$$\begin{aligned} V(s_1)V(s_2)\ldots V(s_{2n})|0\rangle &= \sum_{\gamma\in\Gamma_{(2n,0)}}\prod_{i=1}^{n}\Gamma(s_{2n+1-2i},s_{2n+2-2i})_{(\gamma(2i),\gamma(2i-1))}\,|0\rangle + \\ &\quad \sum_{\gamma\in\Gamma_{(2n,2)}}\prod_{i=1}^{n}\Gamma(s_{2n+1-2i},s_{2n+2-2i})_{(\gamma(2i),\gamma(2i-1))}\,L_2L_1|0\rangle. \end{aligned}$$

We will also use the shorthand $\Gamma(s_1,s_2,\ldots,s_{2n})_0$ and $\Gamma(s_1,s_2,\ldots,s_{2n})_2$ for the coefficients of $|0\rangle$ and $L_2L_1|0\rangle$ in the above expression; in particular $\Gamma_0 = 1$ while $\Gamma_2 = 0$. The case of an odd number $2n+1$ of steps cannot be neglected since 2 out of 6 two particle states are within the equivalence class of $|0\rangle$, that is $L_1|0\rangle$ and $L_2|0\rangle$. The reader can easily deduce that the relevant formula is given by

$$\begin{aligned} V(s_1)V(s_2)\ldots V(s_{2n})V(s_{2n+1})|0\rangle &= \Gamma(s_2,s_3,\ldots,s_{2n+1})_0 \sum_{2\geq k>l\geq -2} e^{-i(\omega_k+\omega_l)s_1}\alpha_{kl}b_k^\dagger b_l^\dagger|0\rangle + \\ &\quad \Gamma(s_2,s_3,\ldots,s_{2n+1})_2 \sum_{2\geq k>l\geq -2;2\geq r>s\geq -2} \overline{\alpha}_{kl}e^{i(\omega_k+\omega_l)s_1}\epsilon_{klrs}b_r^\dagger b_s^\dagger|0\rangle. \end{aligned}$$

We are now left to explicitely calculate the projection of the integration of these expressions on the four dimensional equivalence class of $|0\rangle$. Let us first start by calculating $\int_{t_0}^t du\int_{t_0}^u dv\,\Gamma(u,v)_0$

$$\int_{t_0}^t du\int_{t_0}^u dv\,\Gamma(u,v)_0 = \sum_{2\geq k>l\geq -2}|\alpha_{kl}|^2\left(\sum_{r=0}^{\infty}\frac{(i(\omega_k+\omega_l))^r(t-t_0)^{r+2}}{(r+2)!}\right)$$

which is the best form to consider since $\omega_k+\omega_l$ can be zero so that dividing through it is forbidden. $\int_{t_0}^t du\int_{t_0}^u dv\,\Gamma(u,v)_2$ can be solved by noticing that

$\Gamma(u,v)_2 = \Gamma(v,u)_2$ so that the ordered integral equals $\frac{1}{2}\int_{t_0}^{t} du \int_{t_0}^{t} dv\, \Gamma(u,v)_2$ and this is calculated to be

$$\frac{1}{2}\sum_{2\geq k>l\geq -2;2\geq r>s\geq -2} \epsilon_{rskl}\alpha_{rs}\alpha_{kl}\left(\sum_{p=0}^{\infty}\frac{(-i(\omega_r+\omega_s))^p(t^{p+1}-t_0^{p+1})}{(p+1)!}\right)\left(\sum_{p=0}^{\infty}\frac{(-i(\omega_k+\omega_l))^p(t^{p+1}-t_0^{p+1})}{(p+1)!}\right)$$

which looks even more messy. Therefore, it appears hopeless that we will get any nonperturbative result at this level of generality and we need to make additional simplifications. To get an idea what a suitable reduction is, it is convenient to write the restriction of the full Hamiltonian on the subspace of even particle states and express it as a matrix with respect to the orthonormal basis $|0\rangle, b_2^\dagger b_{-2}^\dagger|0\rangle, b_1^\dagger b_{-1}^\dagger|0\rangle, b_2^\dagger b_1^\dagger|0\rangle, b_2^\dagger b_{-1}^\dagger|0\rangle, b_1^\dagger b_{-2}^\dagger|0\rangle, b_{-1}^\dagger b_{-2}^\dagger|0\rangle$ and $b_2^\dagger b_1^\dagger b_{-1}^\dagger b_{-2}^\dagger|0\rangle$ respectively:

$$H = \begin{pmatrix} 0 & \overline{\alpha}_{2,-2} & \overline{\alpha}_{1,-1} & \overline{\alpha}_{2,1} & \overline{\alpha}_{2,-1} & \overline{\alpha}_{1,-2} & \overline{\alpha}_{-1,-2} & 0 \\ \alpha_{2,-2} & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\alpha}_{1,-1} \\ \alpha_{1,-1} & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\alpha}_{2,-2} \\ \alpha_{2,1} & 0 & 0 & \hbar(\omega_1+\omega_2) & 0 & 0 & 0 & \overline{\alpha}_{-1,-2} \\ \alpha_{2,-1} & 0 & 0 & 0 & \hbar(\omega_2-\omega_1) & 0 & 0 & -\overline{\alpha}_{1,-2} \\ \alpha_{1,-2} & 0 & 0 & 0 & 0 & \hbar(\omega_1-\omega_2) & 0 & -\overline{\alpha}_{2,-1} \\ \alpha_{-1,-2} & 0 & 0 & 0 & 0 & 0 & -\hbar(\omega_1+\omega_2) & \overline{\alpha}_{2,1} \\ 0 & \alpha_{1,-1} & \alpha_{2,-2} & \alpha_{-1,-2} & -\alpha_{1,-2} & -\alpha_{2,-1} & \alpha_{2,1} & 0 \end{pmatrix}.$$

A first reduction consists in putting $\alpha_{k,l}$ equal to zero for $k \neq -l$ so that our perturbation commutes again with the free Hamiltonian: this is an obvious generalization of the theory in section 2.1. In this case, denoting $\alpha = \alpha_{2,-2}$ and $\beta = \alpha_{1,-1}$ and by a permutation of the above basis, H can be written as

$$H = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix} \oplus \begin{pmatrix} \hbar(\omega_1+\omega_2) & 0 & 0 & 0 \\ 0 & \hbar(\omega_2-\omega_1) & 0 & 0 \\ 0 & 0 & \hbar(\omega_1-\omega_2) & 0 \\ 0 & 0 & 0 & -\hbar(\omega_1+\omega_2) \end{pmatrix}$$

where

$$L = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}.$$

This shows that all two particle states $b_k^\dagger b_l^\dagger|0\rangle$ with $k \neq -l$ are stable as well that the vacuum is stable within its equivalence class. A further spectral decomposition is obtained by calculating the eigenvalues and eigenvectors of $L^\dagger L$: these turn out to be $(|\alpha| \pm |\beta|)^2$ and

$$v_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm\frac{\alpha\beta}{|\alpha||\beta|} \end{pmatrix}.$$

Hence, one has that the full normalized eigenvectors are given by:

$$X_\pm = \frac{1}{2}\begin{pmatrix} \pm\frac{\alpha}{|\alpha|} \\ \pm\frac{\beta}{|\beta|} \\ 1 \\ \frac{\alpha\beta}{|\alpha||\beta|} \end{pmatrix} \quad Y_\pm = \frac{1}{2}\begin{pmatrix} \pm\frac{\alpha}{|\alpha|} \\ \mp\frac{\beta}{|\beta|} \\ 1 \\ -\frac{\alpha\beta}{|\alpha||\beta|} \end{pmatrix}$$

where $X_\pm$ corresponds to $\pm(|\alpha|+|\beta|)$ and $Y_\pm$ to $\pm(|\alpha|-|\beta|)$. In our particle basis this reads:

$$X_\pm = \frac{1}{2}\left(\pm\frac{\alpha}{|\alpha|}L_2|0\rangle \pm \frac{\beta}{|\beta|}L_1|0\rangle + |0\rangle + \frac{\alpha\beta}{|\alpha||\beta|}L_2L_1|0\rangle\right)$$

and

$$Y_\pm = \frac{1}{2}\left(\pm\frac{\alpha}{|\alpha|}L_2|0\rangle \mp \frac{\beta}{|\beta|}L_1|0\rangle + |0\rangle - \frac{\alpha\beta}{|\alpha||\beta|}L_2L_1|0\rangle\right)$$

which are the correct expressions as the reader may verify. We will now enter the next stage of dimensional reduction and turn on $\alpha_{2,1}$; the reader then notices that we can as well take $\alpha_{-1,-2}$ to be nonzero and we will denote them by γ and δ respectively. Obviously $b_2^\dagger b_{-1}^\dagger|0\rangle$ and $b_1^\dagger b_{-2}^\dagger|0\rangle$ remain stable (invariant) and we are left to examine a 6×6 Hermitian matrix, which by a change of basis, can be written as:

$$H' = \begin{pmatrix} 0 & L & M \\ L^\dagger & 0 & 0 \\ M^\dagger & 0 & \kappa\sigma_3 \end{pmatrix}$$

where $\kappa = \hbar(\omega_1+\omega_2)$,

$$L = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \beta & \alpha \end{pmatrix},\quad M = \begin{pmatrix} \overline{\gamma} & \overline{\delta} \\ \delta & \gamma \end{pmatrix}$$

and

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the third Pauli matrix. Denoting by (a,b,c) a general eigenvector, the eigenvalue equation can be rewritten as

$$\begin{aligned} L^\dagger Lb + L^\dagger Mc &= \lambda^2 b \\ M^\dagger Lb + (M^\dagger M + \kappa\lambda\sigma_3)c &= \lambda^2 c \\ L^\dagger a &= \lambda b \end{aligned}$$

and we will use the notation $\mu = \kappa\lambda$. Hence, we will solve the more general eigenvalue equation of the 4×4 Hermitian matrix

$$H'' = \begin{pmatrix} L^\dagger L & L^\dagger M \\ M^\dagger L & M^\dagger M + \mu\sigma_3 \end{pmatrix}$$

and later substitute $\mu = \kappa\lambda$ again. Notice that $\lambda = 0$ implies $\mu = 0$ and the eigenvectors of H' then correspond to solutions of the equation $Lb + Mc = 0$, $L^\dagger a = 0$ and $M^\dagger a + \kappa\sigma_3 c = 0$; for an invertible L, the solution space is empty while for a singular nonzero L it is zero, one or two dimensional depending upon whether $M\sigma_3 M^\dagger \mathrm{Ker}(L^\dagger) \subseteq \mathrm{Im}(L)$ or not. If it is not, then the kernel of H' is still empty, while if it is, equality will imply it is two dimensional, otherwise it is one dimensional. In the following, we must take μ nonzero and look for strictly positive eigenvalues; this is in general an impossible task. It is here that we will make a further simplification, which is that $L^\dagger M = 0$; the most general solution being

$$L = r\begin{pmatrix} e^{-i\psi} & e^{-i\phi} \\ e^{i\phi} & e^{i\psi} \end{pmatrix} \quad M = t\begin{pmatrix} -e^{i(\zeta-\phi)} & e^{-i(\zeta+\psi)} \\ e^{i(\zeta+\psi)} & -e^{i(\phi-\zeta)} \end{pmatrix}$$

where $r, t \geq 0$ and $\psi, \phi, \zeta \in [0, 2\pi)$. A straightforward computation yields that the spectrum of H' is given by $\pm 2r, \pm\sqrt{4t^2+\kappa^2}, 0, 0$ with respective eigenvectors

$$V_{\pm 2r} = \frac{1}{2}\begin{pmatrix} \pm e^{-i\psi} \\ \pm e^{i\phi} \\ 1 \\ e^{i(\phi-\psi)} \\ 0 \\ 0 \end{pmatrix}, \quad W_{\pm\sqrt{4t^2+\kappa^2}} = \frac{1}{2\sqrt{4t^2+\kappa^2}}\begin{pmatrix} \mp 2te^{i(\zeta-\phi)} \\ \pm 2te^{i(\zeta+\psi)} \\ 0 \\ 0 \\ \sqrt{4t^2+\kappa^2} \pm \kappa \\ -(\sqrt{4t^2+\kappa^2} \mp \kappa)e^{2i\zeta}e^{i(\psi-\phi)} \end{pmatrix}$$

and finally

$$Y_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ e^{i\psi} \\ -e^{i\phi} \\ 0 \\ 0 \end{pmatrix}, \quad Z_0 = \frac{1}{\sqrt{8t^2+2\kappa^2}}\begin{pmatrix} \kappa \\ -\kappa e^{i(\psi+\phi)} \\ 0 \\ 0 \\ 2te^{i(\phi-\zeta)} \\ 2te^{i(\psi+\zeta)} \end{pmatrix}.$$

The components of these vectors correspond to the coefficients of $|0\rangle, L_2L_1|0\rangle, L_2|0\rangle, L_1|0\rangle, b_2^\dagger b_1^\dagger|0\rangle$ and $b_{-1}^\dagger b_{-2}^\dagger|0\rangle$ respectively. Hence, the vacuum state can be written as

$$|0\rangle = \frac{e^{i\psi}}{2}V_{+2r} - \frac{e^{i\psi}}{2}V_{-2r} - \frac{t}{\sqrt{4t^2+\kappa^2}}e^{i(\phi-\zeta)}W_{+\sqrt{4t^2+\kappa^2}} + \frac{t}{\sqrt{4t^2+\kappa^2}}e^{i(\phi-\zeta)}W_{-\sqrt{4t^2+\kappa^2}} + \frac{\kappa}{\sqrt{8t^2+2\kappa^2}}Z_0$$

and time evolution $U(t_0+T, t_0)|0\rangle$ equals therefore

$$U(t_0{+}T, t_0)|0\rangle = \frac{e^{i\psi}}{2}e^{-i\frac{2rT}{\hbar}}V_{+2r} - \frac{e^{i\psi}}{2}e^{i\frac{2rT}{\hbar}}V_{-2r} - \frac{t}{\sqrt{4t^2+\kappa^2}}e^{i(\phi-\zeta)}e^{-i\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}}W_{+\sqrt{4t^2+\kappa^2}}$$

$$+ \frac{t}{\sqrt{4t^2+\kappa^2}}e^{i(\phi-\zeta)}e^{i\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}}W_{-\sqrt{4t^2+\kappa^2}} + \frac{\kappa}{\sqrt{8t^2+2\kappa^2}}Z_0.$$

For sake of simplicity we project this on the orthogonal complement of the equivalence class of $|0\rangle$; this gives:

$$P_\perp U(t_0{+}T, t_0)|0\rangle = \frac{2t}{8t^2+2\kappa^2}\begin{pmatrix} e^{i(\phi-\zeta)}\left(i\sqrt{4t^2+\kappa^2}\sin\left(\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}\right) + \kappa\left(1-\cos\left(\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}\right)\right)\right) \\ e^{i(\zeta+\psi)}\left(-i\sqrt{4t^2+\kappa^2}\sin\left(\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}\right) + \kappa\left(1-\cos\left(\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}\right)\right)\right) \end{pmatrix}$$

and the norm squared of this equals

$$\frac{8t^2}{(8t^2+2\kappa^2)^2}\left(4t^2\sin^2\left(\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}\right) + 2\kappa^2\left(1-\cos\left(\frac{\sqrt{4t^2+\kappa^2}T}{\hbar}\right)\right)\right)$$

which is smaller than

$$\frac{t^4+\kappa^2t^2}{2t^4+\kappa^2t^2+\frac{1}{8}\kappa^4} < 1$$

and this number is much smaller than one for t a few orders of magnitude smaller than κ. Hence, the vacuum is stable within its equivalence class against the perturbations studied in this section.

3.1.2 A remark on the case of general N.

From the previous subsection, it must be clear that explicit computations are impossible; one can of course always reduce the dimension but what happens if multiple couplings are turned on simultaneously is in general not predictable. Here, computer simulations will be of great value by approximating the time evolution operator $U(t, t_0)$ by $\left(1 - i\frac{H(t-t_0)}{n\hbar}\right)^n$ for n large enough. We leave this open for future work and remain with the knowledge that stability is a fact for certain low dimensional perturbations.

3.2 A pair of Fermionic modes coupled to a pair of Bosonic ones.

The Hamiltonian we will study in this section is given by

$$H = \hbar\omega T + \hbar\rho R + \gamma_+ a_+^\dagger b_+^\dagger b_- + \gamma_- a_-^\dagger b_+^\dagger b_- + \overline{\gamma_+} a_+ b_-^\dagger b_+ + \overline{\gamma_-} a_- b_-^\dagger b_+$$

where $T = b_+^\dagger b_+ - b_-^\dagger b_-$ and $R = a_+^\dagger a_+ - a_-^\dagger a_-$. The a's are Bosonic and the b's Fermionic; obviously this Hamiltonian is unbounded from below and above. The Hilbert space is $\mathcal{H}_f \otimes \mathcal{H}_b$ where $\mathcal{H}_f$ is spanned by the states $|0,0\rangle, |1,1\rangle, |0,1\rangle$ and $|1,0\rangle$ where the first index corresponds to the positive energy Fermion and second to the negative energy Fermion. The states in $\mathcal{H}_b$ may likewise be written in the form $|n,m\rangle$ with $n, m \in \mathbb{N}$; H can then be written in this tensor product form as:

$$H = \begin{pmatrix} F & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & -\hbar\omega 1 \otimes 1 + F & \overline{\gamma_+} D \otimes 1 + \overline{\gamma_-} 1 \otimes D \\ 0 & 0 & \gamma_+ D^\dagger \otimes 1 + \gamma_- 1 \otimes D^\dagger & \hbar\omega 1 \otimes 1 + F \end{pmatrix}$$

where D denotes the annihilation operator

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and F is the bosonic energy operator given by

$$F = \hbar\rho \left(D^\dagger D \otimes 1 - 1 \otimes D^\dagger D\right).$$

Hence, the vacuum and two fermion states are all stable and it remains to investigate the one fermion states. Notice first that for $\gamma_+ \neq 0 \neq \gamma_-$, 0 belongs to the residual spectrum[7] of the operator $A = \gamma_+ D^\dagger \otimes 1 + \gamma_- 1 \otimes D^\dagger$; therefore, an eigenvector v of the operator $B = \hbar\omega 1 \otimes 1 + F$ corresponds only to an eigenvector (w, v) of the operator

$$H' = \begin{pmatrix} -\hbar\omega 1 \otimes 1 + F & \overline{\gamma_+} D \otimes 1 + \overline{\gamma_-} 1 \otimes D \\ \gamma_+ D^\dagger \otimes 1 + \gamma_- 1 \otimes D^\dagger & \hbar\omega 1 \otimes 1 + F \end{pmatrix}$$

[7]The reader shows first that 0 does not belong to the discrete spectrum and then simply notices that the vacuum state does not belong to the closure of the image of A so that the latter is not dense in $\mathcal{H}_b$.

with the same eigenvalue if and only if $w = 0$ and

$$\text{Ker}\,(B - \lambda 1 \otimes 1) \cap \text{Ker}(A^\dagger) \neq \{0\}.$$

One can easily show that this is only the case for $\lambda = \hbar\omega$ corresponding to the vacuum state $|0,0\rangle$; hence, $\hbar\omega$ belongs to the discrete spectrum of H' which is encouraging in showing that the one particle Fermion state $|1,0\rangle$ is stable. We will now examine the case that λ is not in the spectrum of B so that $B - \lambda 1 \otimes 1$ is invertible; therefore, the (generalized) eigenvector equation for H' reduces to the (generalized) kernel of the operator

$$C = -(\hbar\omega+\lambda)1\otimes 1+F-(\overline{\gamma_+}D \otimes 1 + \overline{\gamma_-}1 \otimes D)\left((\hbar\omega - \lambda)1 \otimes 1 + F\right)^{-1}\left(\gamma_+ D^\dagger \otimes 1 + \gamma_- 1 \otimes D^\dagger\right)$$

defined on $\mathcal{H}_b$. The latter operator, C, can be computed explicitely since

$$(\hbar\omega - \lambda)1 \otimes 1 + F$$

is just a diagonal matrix, so that taking its inverse is pretty easy. A vector $\sum_{n,m} \alpha_{n,m}|n,m\rangle$ belongs to the kernel of C if and only if

$$a_{n,m}\alpha_{n,m} + b_{n,m}\alpha_{n-1,m+1} + c_{n,m}\alpha_{n+1,m-1} = 0$$

where

$$a_{n,m} = (-\hbar\omega + \lambda + \hbar\rho(n-m)) + |\gamma_+|^2(n+1)\left[\hbar\omega - \lambda + \hbar\rho(n-m+1)\right]^{-1} +$$
$$|\gamma_-|^2(m+1)\left[\hbar\omega - \lambda + \hbar\rho(n-m-1)\right]^{-1}$$

and

$$b_{n,m} = \gamma_+\overline{\gamma_-}\sqrt{n}\sqrt{m+1}\left[\hbar\omega - \lambda + \hbar\rho(n-m-1)\right]^{-1}$$

and finally

$$c_{n,m} = \gamma_-\overline{\gamma_+}\sqrt{m}\sqrt{n+1}\left[\hbar\omega - \lambda + \hbar\rho(n-m+1)\right]^{-1}.$$

One has that $b_{n,m} = 0$ if and only if $n = 0$ and likewise $c_{n,m} = 0$ if and only if $m = 0$; moreover for each n, m there exist only three λ such that $a_{n,m} = 0$. Solving for the kernel is no easy matter: for $(0,0)$ the equation reduces to $a_{0,0}\alpha_{0,0} = 0$ which generically implies that $\alpha_{0,0} = 0$. The $(0,m)$ equations for $m > 0$ can be written as $a_{0,m}\alpha_{0,m} + c_{0,m}\alpha_{1,m-1}$ which implies the $\alpha_{0,m}$ can be freely chosen and all $\alpha_{1,m}$ are fixed (also for $m = 0$); the (n,m) equations with $n, m > 0$ fix all $\alpha_{n,m}$ with $n > 0$ and $m \geq 0$. They are all linear functions of the $\alpha_{0,n}$; more specifically $\alpha_{n,m}$ only depends upon $\alpha_{0,n+m}$. Remains the equations $(n,0)$ with $n > 0$; these are given by $a_{n,0}\alpha_{n,0} + b_{n,0}\alpha_{n-1,1} = 0$ and they can be rewritten in the form $x_n\alpha_{0,n} = 0$. This means that some of the $x_n(\lambda) \neq 0$ which only happens for an $\aleph_0$ of λ's since each x_n can be viewed as a polynomial of finite degree in λ; hence, the spectrum of H' is a countable subset of the real line (which doesn't mean that the spectrum is discrete however) which strongly suggests stability of the one Fermi states (this is by no means a proof however). An explicit spectral decomposition is again beyond our reach and we postpone a further analysis of these operators to future research. We notice for the second time that computations with Bosonic systems are impossible and we treat this issue now further in the following two sections.

4 Infinities in physics.

This section lies somewhat out of the main development of this article but intends to comment upon the things which went wrong previously, in particular regarding our computations made in the bosonic sector. We make a temporary split (which may change in the future) between kinematical and dynamical infinities and between those we would expect and those we would not expect to occur. Indeed, one infinity I have no objection to would be the infinite extend of the universe or associated to it, its infinite energy. Likewise, I would envision quantum states with an infinite number of particles enlarging the dimension of Fock space to be $\aleph_1$ instead of $\aleph_0$. Let us proceed with the kinematical infinities first.

4.1 Kinematical infinities.

The most famous infinity here, even if just thought as an untestable mathematical assumption, leads to many *physical* infinities which we will list also. Of course I am speaking about the spacetime continuum which gives a local infinity in the sense that the cardinality of each open set equals $\aleph_1$. Even if just thought as a mathematical convenience, it directly implies that in a finite region of space, an infinite number of physical states can be constructed which means an infinite number of particles. This type of infinity leads in general relativity to the formation of black holes and dynamical infinities, known as singularities. Indeed, it looks like nature only allows for a critical finite density of states which is also suggested by black hole entropy where the number of states scales as the exponential of the Hawking Bekenstein entropy. This does not imply that one needs to give up the real numbers all together, but that the real number continuum may not serve as a basis for spacetime. If one assumes this continuous substratum to exist as well as a vierbein on it, then one arrives at a second infinity which is the noncompactness of the (local) Lorentz group. This symmetry probably cannot hold exactly in nature (albeit it might exist in a statistical sense) for the same reason since it would lead to one particle states (which can be localized in space) for which the (expectation value of) the energy blows up to infinity. This is at least so in the representations considered in standard quantum field textbooks and it is very well possible that nonstandard representations do not suffer from this problem as explained in section 5.2. Now, we arrive at some infinity which is almost never mentioned but is as unphysical as all the others and which is deeply rooted into the very postulates of quantum mechanics. This is the infinity of bosons where it is possible to have an infinite number of particles in the same state. One can think of particle statistics as being not related to the standard postulates of quantum mechanics (the commutation relations) but then the dynamical origin behind the spin statistics theorem would fade away. What is done in most textbooks is to simply quantize field theory and notice that bosons show up naturally and that fermions must exist because of a so called "consistent quantization procedure" of the Dirac action which amounts to demanding that the Hamiltonian must be bounded from below. As we have learned in this paper, this may very well turn out to be a bogus motivation and I certainly prefer the more axiomatic approach; then, however, we should change quantum mechanics at a very profound level. Indeed, even in finite causal set theory, where there exists only a finite number of events (points), one obtains an infinite Fock space in the bosonic sector which amounts to an arbitrary high stress energy, which is clearly unphysical. The fermi theory on a causal set does, of course, not suffer from such drawbacks. Usually, in three space dimensions it is "derived" that Bose and Fermi statistics are the only possibilities on grounds of the cluster decomposition principle [1]; the latter, however is too strong since it demands the S matrix elements to factorize for experiments which are done far away from one and another. What should factorize is the modulus squared of the S matrix elements since those are actually measured in nature, not the S matrix elements themselves. This implies, in particular, that other particles in a state intervene, as spectators, when two particles are swapped leading to nontrivial phases depending upon particle species and particle numbers. The last kinematical infinity we meet is, as mentioned previously, the (possibly) infinite extend of the universe. Naively, this would imply an infinite number of particles as well as infinite energies since energy is an extensive quantity which would be a disaster for quantum mechanics since it would not be well defined anymore. Here, however, the principle of general covariance *may* come to our rescue since it states that the energy and momentum densities should vanish. Classically, this is expressed by the constraints $H_\alpha(x) = 0$. Quantum mechanically, however, it must be possible to write $H_\alpha(x) = L_\alpha(x) + L^\dagger_\alpha(x)$ and demand states $|\psi\rangle$ to be physical if and only if $L_\alpha(x)|\psi\rangle = 0$. An implication of this is that for any two physical states

$$\langle\phi|H_\alpha(x)|\psi\rangle = 0$$

which is only meaningful if and only if the $H_\alpha(x)$ are unbounded from below. There are some constraints on the $L_\alpha(x)$: in order for the time evolution of a physical state to be a physical state[8], it is natural to impose the condition that

$$[H(N,N^a)(t), L_\alpha(\vec{x},t)] = \int d^3\vec{y}\, c^\gamma_\alpha(N,N^a,h,\pi,\vec{x},\vec{y},t)\, L_\gamma(\vec{y},t)$$

where $H(N,N^a)(t) = \int d\vec{x}\, H_\alpha(\vec{x},t)N^\alpha(\vec{x},t)$ denotes the total Hamiltonian[9] and h, π the canonical coordinates[10]. This, of course, implies that

$$[H_\beta(\vec{y},t), L_\alpha(\vec{x},t)] = \int d\vec{y}\, c^\gamma_{\beta\alpha}(\vec{x},\vec{y},t)\, L_\gamma(\vec{y},t)$$

meaning that $H_\beta(\vec{y},t)$ maps physical states to physical states which implies that

$$H_\beta(\vec{y},t)|\psi\rangle = 0$$

for every physical state $|\psi\rangle$ unless the physical Hilbert space is degenerate, something which we can only achieve if the full Hilbert space is degenerate or contains negative norm states. In principle, the physical Hilbert space could

[8]One can of course sacrify unitarity and allow for the time evolution to map physical states to nonphysical states, in that case the problem of time would evaporate. This is not as crazy as it sounds since the constraints are not a gauge condition as is the case for the Lorentz constraint in electromagnetism. It would make time evolution dependent upon the lapse and shift vector however which is undesirable as they are pure gauge.

[9]Which may explicitely depend upon time by means of the lapse and shift vector

[10]Strictly speaking, the Hamiltonian should only leave the null eigenspace of $L_\alpha(x)$ invariant.

have an orthogonal basis of ghosts[11] but then observables[12] have no probability interpretation anymore. As is well known, there is no time in this formalism in the naive sense, a way to reinstate it would be to take an observable H as the physical energy and state that the change in time of an observable O is defined as

$$i\hbar\frac{d}{dt}O = [H, O]\,.$$

The problem is that in general such H is not even found classically, so that also this escape route seems unplausible[13]. For quantum electrodynamics in the Lorentz gauge $\mathcal{C}(x) = \partial_\mu A^\mu(x) = 0$, one writes $\mathcal{C}(x) = L(x) + L^\dagger(x)$ where $[H, L(\vec{x}, t)] = -\frac{1}{(2\pi)^3}\int d\vec{y}\, c(\vec{x}, \vec{y})\, L(\vec{y}, t)$ where $c(\vec{x}, \vec{y}) = \int d\vec{l}\, e^{-i\vec{l}.(\vec{x}-\vec{y})}\, |\vec{l}|$ which is a distribution. If this condition would not hold, then unitarity would be jeopardized. In any case, the above strongly suggests that Hamiltonians unbounded from below should occur in a theory of quantum gravity.

4.2 Dynamical infinities.

The most famous infinity here is the so called *locality* assumption in building physical Hamiltonians (as well classically and quantum mechanically) which means that particles can interact in *points*. Taken together with "causality", this implies that Hamiltonian densities at spacelike separated points commute. In my opinion, it is this postulate which leads to the infinities in quantum field theory as well as general relativity; moreover, it appears to be a generic feature of field theories. Now, it might be possible that on a discrete spacetime the locality condition can be maintained since one point could equally be thought of as a fundamental region, but in the continuum it appears to be totally unphysical (as is most likely the continuum). Another infinity, which is deeply rooted in relativity, is the infinity of relativistic mass as the velocity of a particle approaches the velocity of light. This is intimately connected to the noncompactness of the Lorentz group which in a particular sense had some pretty bad implications. Most likely, this infinity also, should dissapear from physics allowing for tunneling through the light cone "potential".

5 New building blocks to devise Hamiltonians: generalized statistics.

First, let me start with the exposition of statistics in [1], especially regarding the existence of anyons in 2 space-dimensions. Weinberg presents an argument which basically boils down to the statement that one needs to look for representations of the first homotopy group of configuration space of indistinguishable particles in determining statistical properties[14]. This argument, based upon the path integral, recognizes that each topologically distinct configuration of paths

[11] That is, to be fully degenerate.

[12] Here, observables are operators on the physical Hilbert space.

[13] One could of course have a one parameter class of observables and interpret that parameter as time, such as is the case for point particles added to gravity.

[14] This configuration space for N indistinguishable particles is defined as the space of N d-vectors, excluding d vectors that coincide (or are within a limiting distance) with another, and identifying configurations that differ only by a permutation.

comes with its own weight factor; in 2 space dimensions, this group is the Braid group, while in 3 or more space dimensions this group is the permutation group. One way to build generalized bosons would be to start from parafermions

$$a^n_q = \sum_{k=1}^{n} b^k_q$$

where n, k, l denote the "generation" and p, q are a shorthand for all other quantum numbers. The b^k_q satisfy the following properties $\left[b^k_q, b^l_p\right] = 0 = \left[b^k_q, (b^l_p)^\dagger\right]$ for $k \neq l$ and $\{b^k_q, b^k_p\} = 0$, $\{b^k_q, (b^k_p)^\dagger\} = \delta(q-p)$. An immediate consequence is that $(a^n_q)^{n+1} = 0$ which is good, but unfortunately the $(a^k_q)^\dagger$ have no nice symmetry properties among one and another. Standard derivations of statistics all start from a few assumptions which are taken for granted: (a) we know we are dealing with a system of N particles, this is a measurable property (b) it makes sense to speak about the state of N-particles exclusively in terms of single particle states (c) a constraint on the representations. The second assumption translates itself mathematically into the statement that a N-particle Hilbert[15] space $\mathcal{H}_N = \otimes_{i=1}^{N} \mathcal{H}_1$ and one is left now with studying unitary representations of S_N. This leads to a statistics based upon Young tableau called parastatistics; it has been shown that field theories with parastatistics can be mapped onto field theories with standard Bose and Fermi representations carrying an extra quantum number[16]. As Wilczek pointed out, a way to go beyond parastatistics would be to look for *projective* unitary representations of S_N which we will study in more detail later on. Another thing to try out would be to consider representations by means of anti-unitary operators (at least for transpositions) and the author is not aware if this has been done. Let me first comment upon (c), what I mean by this is the following: it is *assumed* that for N particles denoted by ψ_i, the *physical* N-particle state $|\psi_1, \ldots, \psi_N, \alpha\rangle$ can be directly expressed in $\mathcal{H}_N$ in terms of the $|\psi_i\rangle$ (and possibly some other non-local quantum numbers α) *and* the state varies continuously if the single particle states $|\psi_i\rangle$ vary continuously in $\mathcal{H}_1$. This assumption has to my knowledge never been explicitely acknowledged in the literature. Indeed, the only thing one can derive for N particles in three or more space dimensions (modulo certain assumptions) is that $|\psi_1, \ldots, \psi_l, \ldots, \psi_k, \ldots, \psi_N\rangle = \pm|\psi_1, \ldots, \psi_k, \ldots, \psi_l, \ldots, \psi_N\rangle$ but in case of the plus sign, this doesn't need to imply that $|\psi_1, \ldots, \psi_l, \ldots, \psi_k, \ldots, \psi_N\rangle$, or more in particular $|\psi_1, \ldots, \psi, \ldots, \psi, \ldots, \psi_N\rangle$ is different from zero! This cannot be inferred from any reasoning of any kind and it is just assumed to be the case. In particular, as this author noticed in [2], this leaves open the possibility of discontinuous statistics where states with $m \geq n$ identical particles vanish for $n > 2$ which is a generalization of Fermi statistics. Another possibility is that $|\psi_k, \ldots, \psi_l, \ldots, \psi_l, \ldots, \psi_N\rangle$ cannot be expressed directly in terms of the $|\psi_i\rangle$ but are defined by means of the projections of the latter on some preferred orthonormal basis $|\phi_j\rangle$ where the $|\phi_{i_1}, \ldots, \phi_{i_N}\rangle$ can be zero if some of the $|\phi_{i_j}\rangle$ are too close to one and another in a spatial sense. This would meet my two objections against particle statistics in the previous section and, moreover, such representations could be continuous. We shall give examples of both ways to

[15]For N anyons, $\mathcal{H}_N$ could be chosen as $\left(\otimes_{i=1}^{N} \mathcal{H}_1\right) \otimes \mathcal{H}_{(b,N)}$ where $\mathcal{H}_{(b,N)}$ denotes the Hilbert space of all N-braids. Other choices are possible and we shall be more explicit later on.

[16]Physical states are then defined as eigenstates under the respective permutations.

violate condition (c) in a short while. More radically, we may assume that (a) holds[17] but that (b) goes wrong, just like for nontrivial anyons, for the very fundamental reason that particles themselves have a nontrivial topology (unlike a point or a small ball). For example, two strings states go way beyond $\mathcal{H}_1 \otimes \mathcal{H}_1$ and it is this we will study first in the following subsection.

5.1 Abelian anyons and statistics for strings.

Before we come to the strings, let us first gain some insight into abelian anyons by directly constructing the relevant physical states for N anyons on $\left(\otimes_{i=1}^{N} \mathcal{H}_1\right) \otimes \mathcal{H}_{(b,N)}$. Let us first give an explicit treatment for $N = 2$; a faithful representation of the braid group is constructed by taking a manifold equipped with a volume form such that the braid group has a nonlinear action on it by means of measure preserving diffeomorphisms. The simplest such space for two particles is $\mathcal{M} = \mathbb{R}^2 \times \left(\mathbb{R}^+_0 \times \mathbb{R}\right)$ with measure $d\mu = r\, d^2\vec{x} \wedge dr \wedge d\theta$ where $\vec{x}$ represents $\vec{r}_1 + \vec{r}_2$, $r = |\vec{r}_1 - \vec{r}_2|$ and θ equals the oriented angle between $\vec{r}_1 - \vec{r}_2$ and the x-axis plus $2\pi n$ for some $n \in \mathbb{Z}$. Obviously, $\vec{r}_i$ denotes the position vector of particle i in $\mathbb{R}^2$; the action of σ on $L^2(\mathcal{M}, d\mu)$ is given by $T(\sigma)\Psi(\vec{x}, r, \theta) = \Psi(\vec{x}, r, \theta + \pi)$. *Physical* vectors are now defined to satisfy

$$T(\sigma)\Psi(\vec{x}, r, \theta) = e^{i\alpha}\Psi(\vec{x}, r, \theta).$$

Without limitation of generality Ψ can be written as

$$\Psi(\vec{x}, r, \theta) = e^{i\frac{\alpha}{\pi}\theta}\Phi(\vec{r}_1, \vec{r}_2)$$

where Φ is defined on $\mathcal{N} = \mathbb{R}^4 \setminus \{(\vec{r}, \vec{r}) | \vec{r} \in \mathbb{R}^2\} = \mathbb{R}^2 \times \left(\mathbb{R}^2 \setminus \{0\}\right)$. Hence, one notices that

$$\int_{\mathcal{M}} |\Psi(\vec{x}, r, \theta)|^2 \; d\mu = \infty \int_{\mathcal{N}} d^2\vec{r}_1\, d^2\vec{r}_2\, |\Phi(\vec{r}_1, \vec{r}_2)|^2$$

and therefore Ψ cannot be a vector in $L^2(\mathcal{M}, d\mu)$. Fortunately,

$$\int_{\mathcal{N}} d^2\vec{r}_1\, d^2\vec{r}_2 \overline{\Psi_1}(\vec{r}_1 + \vec{r}_2, |\vec{r}_1 - \vec{r}_2|, \theta)\Psi_2(\vec{r}_1 + \vec{r}_2, |\vec{r}_1 - \vec{r}_2|, \theta) = \int_{\mathcal{N}} d^2\vec{r}_1\, d^2\vec{r}_2 \overline{\Phi_1}(\vec{r}_1, \vec{r}_2)\Phi_2(\vec{r}_1, \vec{r}_2)$$

so that the physical vectors have a natural interpretation on $L^2\left(\mathcal{N}, d^2\vec{r}_1\, d^2\vec{r}_2\right) \otimes \mathbb{C}\{e^{i\frac{\alpha}{\pi}\theta}\}$. So, although we did not start from it, our physical states live in a tensor product construction where $\mathcal{H}_{(b,2)}$ is represented one dimensionally. For $\alpha = 0, \pi$ the physical wave Ψ reduces automatically to a wave in $L^2\left(\mathcal{N}, d^2\vec{r}_1\, d^2\vec{r}_2\right)$ which is symmetric if $\alpha = 0$ and anti-symmetric if $\alpha = \pi$. Note that in higher space dimensions, an identical construction supplemented with the condition that $T(\sigma)^2 = 1$ leads to ordinary Bose and Fermi statistics[18]. For B_3 the

[17] Also this is nontrivial and I would be curious to see the consequences of its negation being examined.

[18] For example in 3 space dimensions, anyonic statistics could be found on $\mathbb{M} = \mathbb{R}^3 \times \left(\mathbb{R}^+_0 \times [0, \pi] \times \mathbb{R}\right)$ where again a splitting in $\vec{r}_1 + \vec{r}_2$ and $\vec{r}_1 - \vec{r}_2$ has been made. The coordinates are $(\vec{r}_1 + \vec{r}_2, |\vec{r}_1 - \vec{r}_2|, \theta, \psi)$ where ψ runs from $-\infty$ to $+\infty$ instead of from 0 to 2π. The action $T(\sigma)$ then reads $T(\sigma)\Psi(\vec{r}, r, \theta, \psi) = \Psi(\vec{r}, r, \pi - \theta, \psi + \pi)$ which is clearly measure preserving since $d\mu = r^2 \sin(\theta) dr d\theta d\psi d^3\vec{r}$. Again, we define Ψ to be physical if and only if $T(\sigma)\Psi(\vec{r}, r, \theta, \psi) = e^{i\alpha}\Psi(\vec{r}, r, \theta, \psi)$. In principle, one can represent B_2 in this way in three

situation gets somewhat more complicated to write down since one needs to implement two generators σ and δ satisfying the relation

$$\sigma\delta\sigma = \delta\sigma\delta$$

but the construction above can essentially be generalized. Our analysis sofar assumed that (a) and (c) hold and we leave it up to the reader to study what happens if (c) were violated.

We now try to generalize these thoughts to the case that particles are strings in 3 space dimensions (in 4 or more space dimensions the theory is trivial again because any knot is). We will treat strings on a number of distinct levels starting with the most restricted one: for now, we will assume for simplicity that strings are rigid oriented circles which can change of radius over time. That is, they are *geometric* circles α which lie in a two plane P_α and determine an inner disc D_α both which inherit an orientation from the string so that we can speak about a positive outward direction $\vec{n}_\alpha$. There is one important subtlety here which consists in whether we assume the *images* of the string to vary continuously or for the strings to vary continuously as conformal embeddings (with a constant conformal factor) of $\mathcal{S}^1$ in $\mathbb{R}^3$. We will call these of type I and type II respectively; let us first treat type I first. Configuration space $\mathcal{C}^I_N$ of N strings of type I is then the space of N unbraided oriented circles in three dimensions excluding configurations with intersections of the circles and identifying configurations which differ only by a permutation. As the reader can easily notice, the space $\mathcal{C}^I_N$ is connected so that its first homotopy group equals the homotopy group at any base point $(\alpha_i)_{i=1}^N$; for technical convenience, we will choose the latter such that all P_{α_i} coincide but that all inner disks are disjoint. It turns out that the first homotopy group of $\mathcal{C}^I_N((\alpha_i)_{i=1}^N)$ has a much richer structure than the Braid group B_N; we now turn to the case $N = 1, 2$ first. Define a congruence α between $a < b$ of geometrical circles as a continuous[19] function $t \to \alpha(t)$ for any $a \leq t \leq b$ where $\alpha(t)$ is a geometric circle in $\mathbb{R}^3$; any continuous path $\gamma : [a, b] \to \mathcal{C}_N$ defines precisely N non-intersecting congruences (meaning that $\alpha(t) \cap \beta(t) = \emptyset$ for any $t \in [a, b]$) between $a < b$ and conversely any set of N non-intersecting congruences defines a continuous path in $\mathcal{C}_N$. Given a congruence between $a < b$, it is obvious we can speak about the inner discs $D_\alpha(t)$, planes $P_\alpha(t)$ and outward direction $\hat{n}_\alpha(t)$ at time t. Define the interior of a congruence α between $s < t$ as the union of all inner discs $D_\alpha(r)$ for $s \leq r \leq t$. Let us first examine the case $N = 1$ where there is a distinction between $\pi_1(\mathcal{C}^I_1)$ and $\pi_1(\mathcal{C}^{II}_1)$; in the former case $\mathcal{C}^I_1 = \mathbb{R}^3 \times R^+_0 \times S^2$ where $\mathbb{R}^3$ stands for the coordinate in 3 space of the centrum of the string, $\mathbb{R}^+_0$ for its radius and S^2 for the oriented unit vector perpendicular to the string's plane. This space is simply connected and therefore its fundamental group is trivial; therefore, the natural

space dimensions, but now we need to take into account that $T(\sigma)^2 = 1$ implying that $\alpha = 0, \pi$ and $\Psi(\vec{r}, r, \theta, \psi + 2\pi) = \Psi(\vec{r}, r, \theta, \psi)$ meaning that Ψ is well defined on $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$. Moreover, it is symmetric or antisymmetric for $\alpha = 0$ or π respectively. Perhaps gravity might play a roll here in the sense that a distinguished third dimension reduces particle exchange again to two dimensions; by this I want to suggest that a *physical* exchange of two particles in a curved spacetime does not only depend upon the topological equivalence class of the respective paths but also on the absolute value of the amount of work you have to perform on *one* particle to switch position (obviously the total amount of work equals zero).

[19] Continuity is here meant in the Vietoris topology on $\mathbb{R}^3$.

unitary action of $SO(3)$ on S^2 can only be decomposed into irreducible representations of integer spin. The configuration space $\mathcal{C}_1^{II}$, on the other hand, equals $\mathbb{R}^3 \times \mathbb{R}_0^+ \times SO(3)$ where the first two factors have the same interpretation and $SO(3)$ equals the space of ordered pairs $(\vec{v}, \vec{w})$ of orthornormal vectors[20] where v defines the oriented unit vector to the strings plane and w is the oriented unit vector between the strings centre and a preffered point on the string. The latter space has fundamental group $\mathbb{Z}_2$ since $SO(3) = \frac{S^3}{\mathbb{Z}_2}$; this means that strings of type II have an intrinsic spin which is half integer. Indeed, in this case, the wave function is defined on the universal cover of configuration space which is $\mathbb{R}^3 \times \mathbb{R}_0^+ \times S^3$ where S^3 is equipped with the Haar measure μ of $SU(2)$, with a global coordinate system given by $(\vec{r}, \lambda, xe^{i(\frac{\theta}{2}+\psi)}, \sqrt{1-x^2}e^{i\frac{\theta}{2}})$ where $0 \leq x \leq 1$ and $\psi \in [0, 2\pi]$ and finally $\theta \in [0, 4\pi]$. The reason why we have chosen θ to run from 0 to 4π is that a rotation by 4π is homotopic to the identity in $\frac{S^3}{\mathbb{Z}_2}$. $L^2(S^3, d\mu)$ carries a natural action of $SU(2)$ which is just given by left multiplication on $SU(2)$; as is well known, this decomposes into a direct sum of *all* finite dimensional irreducible representations. Identifying the action of a rotation of 2π around the three axis, given by $\theta \to \theta + 2\pi$, with the nontrivial element of the homotopy group $\mathbb{Z}_2$ gives that eigenstates of J_3 with z-component of spin given by $\frac{n}{2}$ are of the form

$$\psi(x, \theta, \psi) = e^{i\frac{\theta n}{2}} \phi(x, \theta, \psi)$$

where $\phi(x, \theta + 2\pi, \psi) = \phi(x, \theta, \psi)$. Hence, for both type I and II, we have found a geometrical origin of spin by means of internal degrees of freedom of a *classical* particle, something which has been imported into physics only by symmetry arguments. Our reasoning on the other hand, appears to be open to generalization of a purely topological nature although we have used some metrical information so far.

We now come first to $\mathcal{C}_2^I$; in general, take $a \leq t \leq b$, then any congruence α of type I with $\alpha(a) = \alpha(b)$ fixed corresponds to a congruence of conformal isometries (with a constant conformal factor) of $\mathbb{R}^3$; indeed for any t there exists a unique conformal isometry $\kappa(t)$ mapping $\alpha(a)$ to $\alpha(t)$:

$$\kappa(t) = T(\vec{a}(t))R(\hat{n}(t), \psi(t))C(\lambda(t))$$

where $\lambda(t) > 0$ and $C(\lambda(t))$ is defined as the mapping $\vec{r} \to \lambda(t)(\vec{r} - \vec{c}) + \vec{c}$ where $\vec{c}$ is the center of the circle $\alpha(a)$. $\hat{n}(t)$ is a unit vector in $P_\alpha(a)$ such that $\hat{n}_\alpha(a) \times \hat{n}_\alpha(t)$ has the same orientation as $\hat{n}(t)$ and $\psi(t) \in [0, \pi]$. This definition is only ambiguous in case $\psi = 0, \pi$ but we choose them such that $R : t \to R(\hat{n}(t), \psi(t))$ defines a continuous mapping from $[a, b]$ into the rotation group. Finally $T(\vec{a}(t))$ defines a translation with $\vec{a}(t)$. At $t = a, b$ we have that $\psi(t) = 0$ as well as $\lambda(t)$ and $\vec{a}(t)$ so that R defines a closed loop in S^2. Let α and β be non-intersecting congruences between times $a < b$; then we can perform an homotopy in $\mathcal{C}_2^I$ holding the intial and endconfigurations fixed such that α is a straight tube from $\alpha(a)$ to $\alpha(b)$ without any twists (it is easy to prove this using the above arguments). In case $\alpha(a) = \alpha(b)$, α remains stationary for all times $a \leq t \leq b$. In $\mathcal{C}_2^{II}$ a similar argument holds with the exception that α could make a twist of 2π around $\hat{n}_\alpha(a) = \hat{n}_\alpha(t)$, which simplifies our analysis a

[20]Indeed, any ordered pair of two orthonormal vectors define uniquely a rotation.

lot. In general, we say that α crosses β in the positive direction between times $a < s < t < b$ if and only if α is on the positive side of $P_\beta(s)$ at time s and on the negative side of $P_\beta(t)$ at time t and, moreover, the intersection of $\alpha(r)$ with $P_\beta(r)$ lies entirely in $D_\beta(r)$ for all $s < r < t$. Likewise, one can speak about a crossing in the negative direction; we say that a crossing in the positive or negative direction is irreducible if and only if it cannot be decomposed in such crossings. Denote by $\sigma^+_{\alpha\beta}$ an irreducible crossing of β by α in the positive direction and likewise by $\sigma^-_{\alpha\beta}$ an irreducible crossing in the negative direction, then clearly $(\sigma^+_{\alpha\beta})^{-1} = \sigma^-_{\alpha\beta}$ meaning that a successive application of both is an operation which can be trivially reduced to the identity. As a shorthand, we will drop the $+$ on $\sigma^+_{\alpha\beta}$ and just speak about $\sigma_{\alpha\beta}$; consider α to make a loop from $\alpha(a)$ to $\alpha(b) = \alpha(a)$ and β to remain stationary at $\beta(a)$; moreover, hold the initial and final configurations at $t = a$ and $t = b$ respectively fixed and consider that α just crosses β once in the positive direction. Then, any continuous deformation in $\mathcal{C}_2$ of this closed path is represented by $\sigma_{\alpha\beta}$; we will now first prove this to be true in $\mathcal{C}_2^{II}$.

Note first that in $\mathcal{C}_2^{II}$, $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ carry no relationship and the following equalities hold: $R_\alpha \sigma_{\alpha\beta} = \sigma_{\alpha\beta} R_\alpha$, $R_\alpha R'_\beta = R'_\beta R_\alpha$ and $R'_\beta \sigma_{\alpha\beta} = \sigma_{\alpha\beta} R'_\beta$ for any rotations R_α, R'_β so that in the end rotations of strings are represented by $\mathbb{Z}_2$ in case of type II strings and by the unit for type I strings. Note that these relations are symmetric in α and β and in $\sigma_{\alpha\beta}$ and $\sigma^{-1}_{\alpha\beta}$. However, $\pi_1(\mathcal{C}_1)$ is *not* in the center of $\pi_1(\mathcal{C}_2)$ for type II strings. This can be seen as follows: denote by F_2 the free group in two generators $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ and S_2 the permutation group in two elements, then both groups can be merged by means of the relationship $\sigma_{\alpha\beta} \circ (\alpha\beta) = (\alpha\beta) \circ \sigma_{\beta\alpha}$ where $\circ$ denotes the group decomposition. In case of type II strings, one has furthermore the generators R_α and R_β representing a rotation by 2π of the string α and β respectively satisfying $R_\alpha \circ (\alpha\beta) = (\alpha\beta) \circ R_\beta$. Taking this into account, we have two groups V^I and V^{II} and we prove that $\pi_1(\mathcal{C}_2^I) = V^I$ and $\pi_1(\mathcal{C}_2^{II}) = V^{II}$. Although we will need a more general argument than this, it is instructive to notice that any operator $\sigma^n_{\alpha\beta}$ defines a different equivalence class: we may assume that β is fixed and that α crosses β n-times; then, in particular we have that the closed paths $s \to \alpha(s)\,[\theta]$ wind n-times around β. This number is a topological invariant for any homotopy of these paths[21] and therefore also of any homotopy on $\mathcal{C}_2^{II}$ holding β fixed. This argument also holds in $\mathcal{C}_2^I$ since any homotopy there defines many homotopies in $\mathcal{C}_2^{II}$. Hence, we are left to show that $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ have no relationship; we will split the proof in four parts. First, we leave it up to the reader to show that any path in $\mathcal{C}_2^{II}$ decomposes into irreducible crossings and rotations around 2π; second, any path in $\mathcal{C}_2^{II}$ is homotopic to a path where β is just undergoing a permutation $(\alpha\beta)$ or is kept fixed (composed with an eventual rotation of 2π around the unit vector $\hat{n}_\beta$) and that as well $\alpha(t)$ as $\beta(t)$ determine a constant unit vector $\hat{n}_\beta(t) = \hat{n}_\alpha(t) = \hat{n}_\alpha(a)$. Moreover, the sequence of irreducible crossings is identical for the deformed path as for the original one: we will call this a standard representation. Third, we leave it up to the reader to show that

[21] The reader may easily see this by projecting the three dimensional configuration on $\mathbb{R} \times \mathbb{R}^+$ by forgetting the radial coordinate in P_β around the centre of β. Then β becomes a point and the projection of α winds n times around this point. By a well known theorem in two dimensions, this winding number is a topological invariant.

any homotopy defined between two standard representations can be deformed into an homotopy between the same standard representations such that at any instant of the deformation parameter, the homotopy defines a standard representation; we will call such homotopy a a standard homotopy. In proving those steps it will be useful to notice the following: let z be a parameter running from 0 to 1 and $\psi(z, r, \theta)$, where $a \leq r \leq b$ be a homotopy in $\mathcal{C}_2^{II}$ holding β fixed (so that we can effectively surpress its radial coordinate), such that for each z, $\psi(z, a, \cdot) = \alpha(a) = \psi(z, b, \cdot)$ and moreover, $\psi(0, r, \cdot) = \alpha(r)$. One notices that for any r, z, the intersection points of $\psi(z, r, \cdot)$ with P_β move continuously with z and r meaning that if such points exist for r_n, z_n converging to r, z then any accumulation point of this sequence is an intersection point for r, z. Remark also that if for some $\tilde{r}$ and z, the intersection of $\psi(z, \tilde{r}, \cdot)$ with P_β has both a point outside D_β and inside D_β, then by varying r continuously, we obtain the existence of an r' such that the intersection of $\psi(z, r', \theta)$ with P_β equals a point on β which is forbidden. The fourth and final step will be proven explicitely and we will use dimensional reduction as in the previous, more limited, case. We will prove that any irreducible word ρ containing only $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ determines an equivalence class; the extension including permutations and rotations then easily follows and is left to the reader. As proven in the third step, we may limit ourselves to standard homotopies holding β fixed; choose then as before a two dimensional coordinate system with the origin in the center of β, r denotes the distance of the projection of a point on P_β with respect to the origin and z is the coordinate corresponding to the axis defined by $\hat{n}_\beta$. As before, we pick any θ and consider the projection of the homotopy $(z, t) \to \psi(z, t, \theta)$, which we denote by ϕ and where $z \in [0, 1]$ and $a \leq t \leq b$, on the (r, z) halfplane $\mathbb{R}^+ \times \mathbb{R}$. We then have the following facts:

- the string β is mapped to the point $(1, 0)$ and $\alpha(a)\,[\theta]$ to $(x, 0)$ with $x > 1$;
- the path $t \to \psi(z, t, \cdot)$ crosses the plane P_β at deformation parameter z and parameter t_0 if and only if $t \to \phi(z, t)$ crosses the r-axis at $t = t_0$;
- we have have three different kinds of crossings of P_β; a crossing by α of β which we will denote by a black dot, a crossing of α by β which we will denote by a blue dot and finally a crossing of P_β such that the disc defined by $\psi(z, t_0, \cdot)$ is disjoint from D_β, which we will write by white dots;
- hence we have that the intersections of any path $t \to \phi(z, t)$ with the r-axis correspond to a black, blue or white dot; black dots all live in the interval $[0, 1)$ while blue or white dots live in $(1, \infty)$;
- we say that a traversal occurs if and only if the path $t \to \phi(z, t)$ traverses the r-axis once as z changes; without restriction, we may assume that any traversal produces two novel crossings which have the same color;
- we say that two crossings collide at z_0 if and only if their coordinates $(z, t(z))$ converge to each other at $z = z_0$; collisions may only occur between crossings of the same color;

- new crossings can only appear if a traversal occurs and they can only dissapear if and only if they collide; between their moment of birth and dissapearance, they move continuously on the r-axis without ever crossing the point $(1, 0)$.

Consider now our original word ρ and the two dimensional projection of its standard representation; this corresponds to an ordered sequence of blue, white and black dots where no two dots of the same color appear next to each other. Then it is obvious, by means of the above rules for crossing points, that any new ordered sequence defined by the action of a standard homotopy must contain the original one as an *ordered* subsequence; this proves that the word is irreducible. Hence, V^{II} is a subgroup of $\pi_1(\mathcal{C}_2^{II})$ and obviously, it is the entire group itself since, as proven before, any path in $\mathcal{C}_2^{II}$ can be expressed in terms of rotations, permutations and crossings. One can now generalize these thoughts to $\mathcal{C}_N^{II}$ which is obviously much bigger than the braid group; we leave this as an exercise for the reader.

Hence, we get nontrivial statistics once we identify a *geometrical* operation in space (that is the swapping of two particles in a particular way) with a nontrivial element of the first homotopy group. For example, we may identify the operation $s_{\alpha\beta}$ of swapping two strings α and β with $(\alpha\beta) \circ \sigma_{\alpha\beta}$ where $(\alpha\beta) \in S_2$ is the standard transposition. Then, $s^2_{\alpha\beta} = \sigma_{\beta\alpha} \circ \sigma_{\alpha\beta}$ and there is no relationship between the $s^n_{\alpha\beta}$. One can furthermore require that the swapping operation itself depends upon the distance of the centers of the strings with respect to one and another so that on large distances the swapping operation is given by $(\alpha\beta)$; a spin statistics connection is then provided by identifying the representation of $(\alpha\beta)$ with that of R_α and R_β something which is logical from a group theoretical point of view since for *abelian* statistics $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ and $R_\alpha = R_\beta$ so that the group theoretical role of $(\alpha\beta)$ is identical to the one of R_α. Hence, it is only natural that they get identified; this provides a topological argument for the spin-statistics connection something which is standard only derived in field theory using other assumptions. Abelian representations of $s_{\alpha\beta}$ gives any fractional statistics one likes and the reader notices there is in general *no* relationship between spin and statistics at this level. Hence, I hold it entirely plausible that the spin-statistics connection evaporates at small distance scales due to the nonlocal effects visible in $\sigma_{\alpha\beta}$.

One can make matters more complicated than this by considering less rigid strings and braided configurations as well; also one could allow for topology change so that the exchange of two strings is an ill defined operation (for example two strings could merge and give one string or one string could split giving three strings). We leave these more exotic possibilities for future considerations.

5.2 More general tensor product representations.

We now come to a way to violate assumption (c) given previously; as mentioned before, it is possible to effectively subdivide the world into “discrete units” such that one “unit” can only appear a finite number of times in physical states as do units with a large overlap. In such a formalism, the number operator does not commute with the multi-particle Hamiltonian which excludes free field theories and therefore our particle notion is *not* defined by a single particle irreducible representation of the Poincaré group but rather the multi particle space should carry a (irreducible?) representation. This is very physical as it will forbid us to probe spacetime at arbitrary short distances and any theory based upon such statistics cannot be a field theory. More concretely, let $\mathcal{H}_1 \subset L^2(\mathbb{R}^3, d^3x)$ be an infinite dimensional subspace of the Hilbert space of square integrable functions which is locally finite meaning that there exists a countable orthonormal basis $|\psi_n\rangle$ of compact support K_n such that any compact set K in $\mathbb{R}^3$ intersects a finite number, different from zero, of K_n. Such a basis will be called a locally finite basis and the reader may see that it determines in general a resolution with which space is probed. So, our model is not that of a particle continuously moving in space, but rather one of a particle hopping from one particle site to another. In the nonrelativistic limit, one may still assume that it is possible to move a definite number of particles around so that one may still make the usual statistical considerations regarding the permutation group. As previously mentioned, there is no guarantuee why a multiparticle state satisfying "Bose" statistics (meaning the physical state should be symmetric under exchange of identical particles) should not vanish. We will make an explicit construction here based upon the locally finite basis; in particular we will demand the property that $|\phi_1, \phi_2, \ldots, \phi_N\rangle$ is a multilinear mapping in terms of the ϕ_j, something which is also standard the case. But now, we will set some states of the kind $|\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_N}\rangle$ to zero; to make our life easy, we will simply assume that all N-particle states with more than M of the same states in the locally finite basis vanish. Denote by $a_n^\dagger$ the creation operator of $|\psi_n\rangle$ defined as usual, then $\left[a_n^\dagger, a_m^\dagger\right] = 0 = \left(a_n^\dagger\right)^{M+1}$. As usual, a_n eats a $|\psi_n\rangle$ away and $\left[a_n, a_m^\dagger\right] = 0$ for all $n \neq m$. What does not apply anymore is the rule that $\left[a_n, a_n^\dagger\right] = 1$ but this expression is rather changed to $\left[a_n, a_n^\dagger\right] = 1 - (M+1)P_n$ where P_n is the Hermitian projection operator on the subspace of states with M basis vectors of type n. This correction should occur because for one mode systems the trace of the right hand side must vanish[22].

This can be generalized to the continuum, as noted in [2] it is possible to define operators $a_k^\dagger$, where k is a continuum label denoting for example the energy momentum, such that the usual bosonic algebra holds, supplemented with the condition that $\left(a_k^\dagger\right)^{M+1} = 0$ which is obviously a discontinuity. However, this has no real physical significance since these relations are of measure zero (obviously $M \geq 1$) and they don't influence the results of calculations. Probably, discontinuities in general have no real physical meaning at least not on this level of simplicity.

5.2.1 A comment on Poincaré invariance.

Strictly speaking, we should not be too much worried about Poincaré covariance since it is not a symmetry of nature if gravity is turned on, but for sake of completeness we shall comment on it anyway. In particular, I hold it entirely possible that an irreducible representation of the Poincaré group on the space

[22]This commutation relation alone makes it clear we are not dealing with a field theory here.

of physical states with an arbitrary number of particles can be constructed. Irreducible unitary representations come with a quantum number of mass and spin which are then to be interpreted as the total mass (say the Bondi mass) and spin in the universe. Since spin has not been included in our previous analysis, we can put it to zero and work with the mass only; the only thing one should do is to explicitely define the momentum eigenstates in terms of the locally finite basis, the action of the Lorentz group is then uniquely determined by the group relations, see [1]. We leave such explicit constructions for future work however.

6 Conclusions.

In this paper, we made an attempt to study systems with negative energies and more in general Hamiltonians which are unbounded from below and above. Although much research remains to be done before any definite conclusion can be reached, I think it is fair to say that we have gathered some evidence that such Hamiltonians do not necessarily produce nonsensical results. Moreover, as argued in section four, they seem to be necessary in any future theory of quantum gravity. However, it appears to me that in order for such program to succeed, we need new building blocks for Hamiltonians; that is a novel *non-local* particle statistics such that Bose and Fermi are good approximations at atomic scales but fail entirely at much smaller scales between say 10^{-20} and 10^{-35} meters. We have formulated three main objections against the usual derivation of particle statistics and have provided examples for two of the three specific ways of violating the standard conclusions. One was based upon the assumption that elementary particles themselves have a non-trivial topology, the other was motivated by the idea of a spacetime grid (compatible with Poincaré invariance). Concerning the "topological" approach, I have used strings since they are the most easy to consider; it may be clear that our conclusions go way beyond strings so that many more constructions are possible here. It might even be that the topological fabric of spacetime itself provides for fermions and bosons as has been argued several times in the literature. This would mean a final and radical break with the idea that spacetime is just an ordinary topological space such as is the case for Minkowski spacetime. An intruiging possibility is uttered at the end of section 5.1: what happens if the swapping operation of particles is ill defined? Surely, we cannot speak about statistics anymore at that level, but what replaces it? These are all avenues for future research.

References

[1] S. Weinberg, The quantum theory of fields, foundations, Cambridge University Press.

[2] J. Noldus, Foundations of a theory of quantum gravity, ArXiv:1101.5113.

[3] O. Weinberger, The braid group, representations and non-abelian anyons, Bachelor's thesis KTH.