

The initial evidence for M-theory: fractal nearly tri-bimaximal neutrino mixing and CP violation

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We propose an instructive possibility to generalize the tri-bimaximal neutrino mixing ansatz, such that leptonic CP violation and the fractal feature of the universe can naturally be incorporated into the resultant scenario of *fractal* nearly tri-bimaximal flavor mixing. The consequences of this new ansatz on the latest experimental data of neutrino oscillations are analyzed. This theory is perfectly matched with the current experimental data, and surprisingly, we find that the existing neutrino oscillation experimental data is the initial experimental evidence supporting one kind of high dimensional unified theories, such as M-theory. Besides, an interesting approach to construct lepton mass matrices in fractal universe under permutation symmetry is also discussed. This theory opens an unexpected window on the physics beyond the Standard Model.

It is well known that the mixing factors of solar, atmospheric and CHOOZ neutrino oscillations read: $\sin^2 2\theta_{sun} = 4|V_{e1}|^2|V_{e2}|^2$, $\sin^2 2\theta_{atm} = 4|V_{\mu3}|^2(1 - |V_{\mu3}|^2)$, $\sin^2 2\theta_{chz} = 4|V_{e3}|^2(1 - |V_{e3}|^2)$, where V is the 3×3 lepton flavor mixing matrix linking the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$. As current experimental data favor $\sin^2 2\theta_{chz} \ll \sin 2\theta_{sun} \sim \sin 2\theta_{atm} \sim o(1)$, two large flavor mixing angles can be concluded from the above equations in a specific parametrization of V : one between the 2nd and 3rd lepton families and the other between the 1st and 2nd lepton families.

In this article we pay close attention to the ansatz of “tri-bimaximal” flavor mixing pattern proposed by Harrison, Perkins and Scott^{1,2}. It predicts $\sin^2 2\theta_{sun} = 8/9$ and $\sin^2 2\theta_{atm} = 1$, consistent with the large-angle M-SW solution to the solar neutrino problem and the atmospheric neutrino oscillation data. However, it also leads to $\sin^2 2\theta_{chz} = 0$, implying the absence of both intrinsic CP violation and high-energy matter resonances in neutrino oscillations. Xing³ discussed two possibilities to modify the tri-bimaximal neutrino mixing pattern, such that CP violation can naturally be incorporated into the resultant scenarios of nearly tri-bimaximal flavor mixing. In Xing’s article there is one scenario whose predictions of $\sin^2 2\theta_{sun}$ and $\sin^2 2\theta_{atm}$ are consistent well with the current neutrino oscillation data, but the prediction of $\sin^2 2\theta_{chz} \approx 0.01$ is not consistent with the current neutrino oscillation data⁴: $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$.

The main purpose of this work is to discuss one simple but instructive possibility to modify the tri-bimaximal neutrino mixing pattern, such that CP violation and the fractal feature of the universe can naturally be incorporated into the resultant scenario of *fractal* nearly tri-bimaximal flavor mixing. One specific texture of the charged lepton mass matrix is taken into account, in order to obtain small but non-vanishing $|V_{e3}|$ or $\sin^2 2\theta_{chz}$. We find that when the current experimental

data⁴ $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$ is adopted to limit the value of parameter q , one has $10.46118470068419 \leq q \leq 12.931439345418308$; then substituting the q value into our theory, the predicted $\sin^2 2\theta_{atm}$ are highly consistent with the current data; and the current data limits the value of parameter φ , the ranges of which are much better than that in usual space-time. In addition, we also consider another way of thought. Given the q close to 11 and the intimate relation $q = d_f$ between q and fractal dimension d_f when the Euclidean dimension is one⁵, we assume $q = 11$, then our theory gives the predicted values of $\sin^2 2\theta_{chz}$ and $\sin^2 2\theta_{atm}$ which confirms to the current data; the range of parameter φ limited by current data is also much better than that in usual space-time. According to the mentioned two ways of thought, the calculations suggest that the universe is fractal, and its dimension is high; the consistent fact between the calculation according to the second thought and the current data suggests that some high dimensional space-time theories, such as M-theory, can be a theory in line with expectations. In addition, the predicted strength of CP or T violation in neutrino oscillations is given. We also discuss an interesting approach to construct lepton mass matrices in fractal universe under permutation symmetry, from which one may derive another *fractal* nearly tri-bimaximal neutrino mixing scenario with $|V_{e3}| \neq 0$ but with no intrinsic CP violation in neutrino oscillations.

Fractal nearly tri-bimaximal neutrino mixing

In the picture of neutrino as Majorana particle, the light (left-handed) neutrino mass matrix M_ν must be symmetric and can be diagonalized by a single unitary transformation: $U_\nu^\dagger M_\nu U_\nu^* = \text{Diag}\{m_1, m_2, m_3\}$. In general, the charged lepton mass matrix M_l is non-Hermitian; hence, the diagonalization of M_l needs a special bi-unitary transformation: $U_l^\dagger M_l \tilde{U}_l = \text{Diag}\{m_e, m_\mu, m_\tau\}$. The lepton flavor mixing matrix V , defined to link the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$, measures the mismatch between the

diagonalization of M_l and that of M_ν : $V = U_l^\dagger U_\nu$. It is worth noting that (m_1, m_2, m_3) and (m_e, m_μ, m_τ) are physical (real and positive) masses of light neutrinos and charged leptons, respectively.

In the flavor basis where M_l is diagonal (i.e., $U_l = 1$ being a unity matrix), the flavor mixing matrix is simplified to $V = U_\nu$. The tri-bimaximal neutrino mixing pattern $U_\nu = V_0$ can then be constructed from the product of two modified Euler rotation matrices: $R_{12}(\theta_x) = (c_x, s_x, 0; -s_x, c_x, 0; 0, 0, 1)$, and $R_{23}(\theta_y) = (1, 0, 0; 0, c_y, s_y; 0, -s_y, c_y)$, where $s_x \equiv \sin_q \theta_x, c_y \equiv \cos_q \theta_y$, and so on. Functions $\sin_q u$ and $\cos_q u$ can be defined with $\exp_q(u)$ which is the one-dimensional q-exponential function that naturally emerges in nonextensive statistics⁶. For a pure imaginary iu , one de-

finies $\exp_q(iu)$ as the principal value of $\exp_q(iu) = [1 + (1-q)iu]^{1/(1-q)}$, $\exp_1(iu) \equiv \exp(iu)$. The above function satisfies⁷: $\exp_q(\pm iu) = \cos_q(u) \pm i \sin_q(u)$, $\cos_q(u) = \rho_q(u) \cos\left\{\frac{1}{q-1} \arctan[(q-1)u]\right\}$, $\sin_q(u) = \rho_q(u) \sin\left\{\frac{1}{q-1} \arctan[(q-1)u]\right\}$, $\rho_q(u) = [1 + (1-q)^2 u^2]^{1/[2(1-q)]}$, $\exp_q(iu) \exp_q(-iu) = \cos_q^2(u) + \sin_q^2(u) = \rho_q^2(u)$. Notice that $\exp_q[i(u_1 + u_2)] \neq \exp_q(iu_1) \exp_q(iu_2)$ for $q \neq 1$ ⁶. Then we obtain: $V_0 = R_{23}(\theta_y) \otimes R_{12}(\theta_x) = (c_x, s_x, 0; -s_x c_y, c_x c_y, s_y; s_x s_y, -s_y c_x, c_y)$. The vanishing of the (1, 3) element in V_0 assures an exact decoupling between solar ($\nu_e \rightarrow \nu_\mu$) and atmospheric ($\nu_\mu \rightarrow \nu_\tau$) neutrino oscillations. The general form of the corresponding neutrino mass matrix M_ν is

$$M_\nu = V_0 \otimes \text{Diag}\{m_1, m_2, m_3\} \otimes V_0^T = \begin{pmatrix} c_x^2 m_1 + s_x^2 m_2 & -c_x c_y s_x (m_1 - m_2) & c_x s_x s_y (m_1 - m_2) \\ -c_x c_y s_x (m_1 - m_2) & c_y^2 s_x^2 m_1 + c_x^2 c_y^2 m_2 + s_y^2 m_3 & -c_y s_y (s_x^2 m_1 + c_x^2 m_2 - m_3) \\ c_x s_x s_y (m_1 - m_2) & -c_y s_y (s_x^2 m_1 + c_x^2 m_2 - m_3) & s_x^2 s_y^2 m_1 + c_x^2 s_y^2 m_2 + c_y^2 m_3 \end{pmatrix}.$$

Taking $q = 1$, $\theta_x = \arctan(1/\sqrt{2}) \approx 35.3^\circ$ and $\theta_y = 45^\circ$, the results in usually space-time are reproduced, and M_ν might have a meaningful interpretation in an underlying theory of neutrino masses with specific flavor symmetries³.

The tri-bimaximal neutrino mixing pattern will be modified, if U_l deviates somehow from the unity matrix. This can certainly happen, provided that the charged lepton mass matrix M_l is not diagonal in the flavor basis. As $U_\nu = V_0$ describes a product of two special Euler rotations in the real (2, 3) and (1, 2) planes, the simplest form of U_l that allows $V = U_l^\dagger U_\nu$ to cover the whole 3×3 space should be $U_l = R_{12}(\theta_x, q = 1)$ or $U_l = R_{31}(\theta_z, q = 1)$ (see Ref.^{8,9} for a detailed discussion). Because when $U_l = R_{31}(\theta_z, q = 1)$ is adopted, the calculated result³ $0.873 \leq \sin^2 2\theta_{sun}^{(z)} \leq 0.903$ is not consistent with the experimental data⁴ $\sin^2 2\theta_{12} = 0.846 \pm 0.021$, we focus on the calculation of case $U_l = R_{12}(\theta_x)$. For convenience, θ_x will be replaced by θ in the following context.

To make CP violation and the fractal feature of the universe be naturally incorporated into V , we adopt the following complex rotation matrices: $R_{12}(\theta, \varphi) = (c, se_q^{i\varphi}, 0; -se_q^{-i\varphi}, c, 0; 0, 0, 1)$, where $c \equiv \cos_q \theta, s \equiv \sin_q \theta$, and $e_q^{i\varphi} = \exp_q(i\varphi)$. In this case, we obtain the lepton flavor mixing of the pattern:

$$V = R_{12}^\dagger(\theta, \varphi) \otimes V_0 = \begin{pmatrix} \frac{1}{\sqrt{6}}(2c + se_q^{i\varphi}) & \frac{1}{\sqrt{3}}(c - se_q^{i\varphi}) & -\frac{1}{\sqrt{2}}se_q^{i\varphi} \\ -\frac{1}{\sqrt{6}}(c - 2se_q^{-i\varphi}) & \frac{1}{\sqrt{3}}(c + se_q^{-i\varphi}) & \frac{1}{\sqrt{2}}c \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

It is obvious that V represents a fractal nearly tri-bimaximal flavor mixing scenario, if the rotation angle θ is small in magnitude. The parameters q and φ in

V are the source of leptonic CP violation in neutrino oscillations.

Constraints on mixing factors and CP violation

The mixing angle θ is expected to be a simple function of the ratios of charged lepton masses due to the fact that it arises from the diagonalization of M_l . Then the strong mass hierarchy of charged leptons naturally assures the smallness of θ as one can see lately.

Indeed, a proper texture of M_l which may lead to the flavor mixing pattern V is $M_l = (0, C_l, 0; C_l^*, B_l, 0; 0, 0, A_l)$, where $A_l = m_\tau, B_l = m_\mu - m_e$, and $C_l = \sqrt{m_e m_\mu} e^{i\varphi}$. Then the mixing angle θ in V reads $\tan_q(\theta) = \frac{\sin_q \theta}{\cos_q \theta} = \sqrt{\frac{m_e}{m_\mu}}$. It is easy to prove that when $q \rightarrow 1$, the results in usual space-time are recovered, namely³, $C_l = \sqrt{m_e m_\mu} e^{i\varphi}$, $\tan(2\theta) = 2\sqrt{\frac{m_e m_\mu}{m_\mu - m_e}}$. Given the hierarchy of three charged lepton masses (i.e., $m_e \ll m_\mu \ll m_\tau$) and $q \sim o(1)$, we have $\tan_q \theta \approx \tan \theta \approx \sin \theta \approx \sqrt{m_e/m_\mu}$ to a good degree of accuracy. Numerically, we find $\theta \approx 3.978^\circ$ with the inputs $m_e = 0.511\text{MeV}$ and $m_\mu = 105.658\text{MeV}$ ⁴.

Now let us calculate the mixing factors of solar, atmospheric and reactor neutrino oscillations. According to this theory, one obtains $\sin^2 2\theta_{sun} = \frac{8}{9}(1 - \frac{3}{4}s^2 - sc \cos_q \varphi + \frac{3}{2}s^3 c \cos_q \varphi - 2s^2 c^2 \cos_q^2 \varphi)$, $\sin^2 2\theta_{atm} = 1 - s^4, \sin^2 2\theta_{chz} = 1 - c^4$. It is noted that when $q \rightarrow 1$, the results in usual space-time are recovered³: $\sin^2 2\theta_{sun} = \frac{8}{9}\left(1 - \frac{3}{4}\sin^2 \theta - \sin \theta \cos \theta \cos \varphi + \frac{3}{2}\sin^3 \theta \cos \theta \cos \varphi - 2\sin^2 \theta \cos^2 \theta \cos^2 \varphi\right)$, $\sin^2 2\theta_{atm} = 1 - \sin^4 \theta, \sin^2 2\theta_{chz} = 1 - \cos^4 \theta$. In this scenario, adopting experimental data⁴

$\sin^2 2\theta_{chz} = (8.5 \pm 0.5) \times 10^{-2}$, one obtains $10.46118470068419 \leq q \leq 12.931439345418308$; thus there is $0.999987 \leq \sin^2 2\theta_{atm} \leq 0.999999$, which is extremely consistent with the experimental data⁴: $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy and $\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy; in addition, to make $\sin^2 2\theta_{sun} \leq 0.867$ to accord with the experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$, one needs only $-1537.79 \leq \varphi_{q=10.46} \leq 1537.79$ or $-6372.47 \leq \varphi_{q=12.93} \leq 6372.47$, which are much better than the usual space-time case ($0.485366 \leq \varphi_{q=1} \leq 1.27256$).

Additionally, given the q close to 11 and the intimate relation $q = d_f$ between q and fractal dimension d_f when the Euclidean dimension is one⁵, we assume $q = 11$, then this scenario gives the predicted values of $\sin^2 2\theta_{chz} = 0.082456$ and $\sin^2 2\theta_{atm} = 0.999987$ which amazingly fit in with the current data⁴ $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$ and $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy ($\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy), respectively; the range of parameter $-2162.81 \leq \varphi_{q=11} \leq 2162.81$ limited by current data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ is also much better than that in usual space-time ($0.485366 \leq \varphi_{q=1} \leq 1.27256$). According to above calculations, we come to the following conclusions: i) the universe is fractal with high dimension; ii) some high dimensional space-time theories, such as M-theory, can be a theory in line with expectations. A numerical illustration of $\sin^2 2\theta_{sun}$ as the function of q

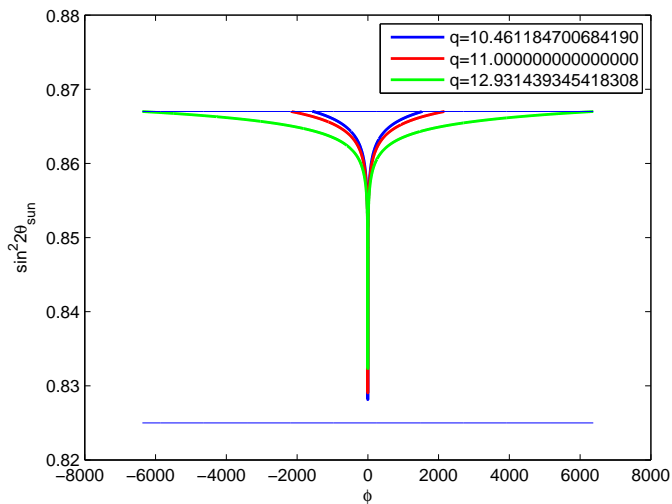


FIG. 1. The mixing factors $\sin^2 2\theta_{sun}$ against parameter φ under different values of q in fractal nearly tri-bimaximal neutrino mixing patterns.

and φ is shown in Fig. 1, and two blue lines are the top and bottom limits of experimental data.

The strength of CP or T violation in neutrino oscillations, no matter whether neutrinos are Dirac or Majorana particles, is measured by a universal parameter J which

is defined as¹⁰: $Im(V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*) = J \sum_{\gamma, k} (\varepsilon_{\alpha\beta\gamma} \varepsilon_{ijk})$,

in which the Greek subscripts run over (e, μ, τ) , and the Latin subscripts run over (e, μ, τ) . Considering the lepton mixing scenario proposed above, one has $J = \frac{1}{6} sc \sin_q \varphi (c^2 + s^2 \rho_q^2(\varphi))$. Obviously, when $q \rightarrow 1$, the result in usual space-time is recovered³: $J = \frac{1}{6} sc \sin \varphi$. Based on Figs. 1 and 2 as well as the numerical calcula-

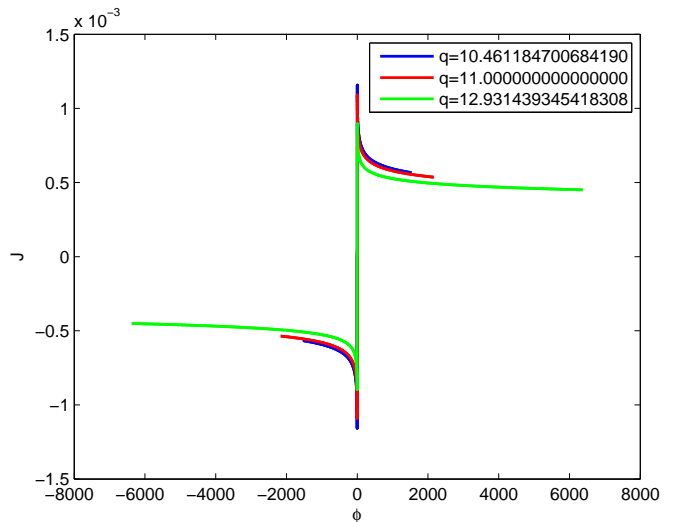


FIG. 2. The strength of CP or T violation J against parameter φ under different values of q in fractal nearly tri-bimaximal neutrino mixing patterns.

tions, one obtains the table I. The strength of CP or T

TABLE I. The range of φ and strength of CP or T violation

q	φ_{\min}	φ_{\max}	J_{\min}	J_{\max}
1	0.485366	1.27256	0.0054	0.011
10.46118470068419	-1537.79	1537.79	-0.0012	0.0012
11	-2162.81	2162.81	-0.0011	0.0011
12.931439345418308	-6372.47	6372.47	-0.00089595	0.00089595

violation J in fractal nearly tri-bimaximal neutrino mixing patterns is predicted as: $-0.0011 \leq J_{q=11} \leq 0.0011$. The experimental data of strength of CP or T violation may limit the range of parameter φ , but it is a pity that at present no experimental information on the Dirac and Majorana CP violation phases in the neutrino mixing matrix is available⁴. The former could be determined by the T-violating asymmetry between $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transitions or by the CP-violating asymmetry between $\nu_\mu \rightarrow \nu_e$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ transitions in a long-baseline neutrino oscillation experiment, when the terrestrial matter effects are under control or insignificant.

Further discussions and remarks

We have discussed a possibility to construct the charged lepton and neutrino mass matrices, from which a fractal nearly tri-bimaximal neutrino mixing pattern can nat-

urally emerge. This scenario is compatible with the large-angle MSW solution to the solar neutrino problem. The fact that the predictions of the fractal nearly tri-bimaximal neutrino mixing pattern are perfectly consistent with the current experimental data strongly suggests that some high dimensional unified theories, such as M-theory, could be the theory in line with expectations, and neutrino oscillation experiment is the first robust evidence of M-theory, broken the spell that the M-theory has no experimental evidence, and opened an unexpected window on the physics beyond the Standard Model. The allowed range width of φ by the fractal nearly tri-bimaximal neutrino mixing pattern is 3 orders of magnitude larger than that of the usual theory. This theory also yields an prediction of CP- or T-violating asymmetry in long-baseline neutrino oscillation experiments.

The nearly tri-bimaximal neutrino mixing pattern, as Xing³ expected, serves as the leading-order approximation of a more complicated flavor mixing matrix, which is the $q \rightarrow 1$ limit case of the fractal nearly tri-bimaximal neutrino mixing pattern under discussion, though its prediction on $\sin^2 2\theta_{chz}$ is not consistent with the experimental data. There are certainly other possibilities to

modify the tri-bimaximal neutrino mixing ansatz, such that nonvanishing $|V_{e3}|$ (and CP violation) can be naturally incorporated into the resultant scenario of fractal nearly tri-bimaximal mixing. Note that our scenario predicts that $-0.0011 \leq J_{q=11} \leq 0.0011$, and when $\varphi = 0$, $J_{q=11} = 0$, namely, there is no CP violation. Therefore, our theory can be adapted to the CP broken or not at the same time.

Finally, let us remark that the fractal nearly tri-bimaximal mixing pattern and its possible extensions require some peculiar flavor symmetries to be imposed on the charged lepton and neutrino mass matrices. It is likely that the fractal nearly tri-bimaximal neutrino mixing pattern under discussion serves as the more complicated flavor mixing matrix that scientists are looking for³, and one of the nearly tri-bimaximal neutrino mixing patterns is its leading-order approximation. We expect that more delicate neutrino oscillation experiments in the near future can verify the fractal nearly tri-bimaximal mixing pattern, from which one may get some insight into the underlying flavor symmetry and its breaking mechanism responsible for the origin of both lepton masses and leptonic CP violation.

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