Report

An Exact Solution of Riccati Form of Boltzmann Equation with Mathematica

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ABSTRACT

The Boltzmann equation is often used to calculate relic density of non-baryonic dark matter. The main idea is that particles were in thermal equilibrium with the early universe. There are many solutions for Boltzmann equation in the form of software packages, such as MadDM. This paper discusses an exact computational solution of Riccati form of Boltzmann equation with Mathematica. While in literature there is already another presentation of Mathematica routine to compute relic density, to our best knowledge, this solution has never been presented elsewhere before.

Key Words: approximate solution, Riccati form, Boltzmann equation, Mathematica.

Introduction

The Boltzmann equation is often used to calculate relic density of non-baryonic dark matter. The main idea is that particles were in thermal equilibrium with the early universe. The Boltzmann equation can be written as follows (see Murayama [1], and Tanedo [2]):

$$a^{-3}\frac{d(na^3)}{dt} = \langle \sigma \upsilon \rangle \left[(n_{EQ})^2 - n^2 \right], \tag{1}$$

Where a is the scale factor, n is the dark matter number density, n_{EQ} is the equilibrium number. After some variables are changed, equation (1) reduces to Riccati equation as follows [2]:

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}(Y^2 - Y_{EQ}^2), \tag{2}$$

Where the parameter λ relates the annihilation rate to the expansion rate of the universe [2],

$$\lambda = \frac{m^3 \langle \sigma \upsilon \rangle}{H(m)}. \tag{3}$$

For s-wave processes λ is a constant.

There are many solutions for Boltzmann equation in the form of software packages, such as MadDM [7]. This paper discusses an exact computational solution of Riccati form of Boltzmann equation with Mathematica rel. 9. While in literature there is already another presentation of

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Mathematica routine to compute relic density [6], to our best knowledge, the method outlined in this paper has never been presented elsewhere before.

According to Bender and Sarkar [3], Riccati form of Boltzmann equation can be written more precisely as follows:

$$\frac{dY}{dx} = -\frac{\lambda}{x^{2+n}}(Y^2(x) - Y_{EQ}^2(x)), \tag{4}$$

Where the integer n=0,1,2,… comes from a partial-wave analysis of the scattering of DM particles, n=0 refers to S-wave scattering.

The most direct approach is to convert this Riccati equation into a Schrodinger-type equation as follows [3]:

$$\upsilon''(x) - \frac{n(n+2)}{4x^2}\upsilon(x) - \lambda^2 x^{-2n-4} Y_{EQ}^2(x)\upsilon = 0. \tag{5}$$

Now, if we set n=0, we obtain the standard time-independent Schrodinger equation (TISE) in which 1/λ plays the role of $\hbar$. A method to solve TISE can be found in Zimmerman & Olness [8].

Nonetheless, we will not use this approach in this paper, instead we will solve the Riccati equation (4) using Mathematica rel. 9. First we will set n=0 in eq. (4). We will also assume that Y_{EQ} is constant.

Standard solution of Riccati equation

Based on Mathematica software, the standard solution of Riccati equation is obtained as follows:[4, p.178]

```
Clear["Global'*"]
ode=y'[x]+a y[x]^2+y[x]/x+1/a□0
sol=DSolve[ode,y,x]
```

$$\frac{1}{a} + \frac{y[x]}{x} + ay[x]^2 + y'[x] == 0 \tag{1}$$

Applying DSolve, we get:

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \frac{-\text{BesselY}[1,x] - \text{BesselJ}[1,x]C[1]}{a(\text{BesselY}[0,x] + \text{BesselJ}[0,x]C[1])}\right]\right\}\right\}$$

And

y=y[x]/.sol[[1]]

$$\frac{-\text{BesselY}[1,x] - \text{BesselJ}[1,x]C[1]}{a(\text{BesselY}[0,x] + \text{BesselJ}[0,x]C[1])}$$

Solution of Riccati form of Boltzmann equation

The solution of Riccati equation is notoriously difficult to find, so this author decides to use Mathematica software in order to get an exact analytical solution.

The Mathematica code to solve this problem is not quite straightforward. First we express equation (4) as follows:

```
Clear["Global'*"]
de = y'[x] == −α[y[x]^2 − β^2]/x^2;
soln=DSolve[de,y[x],x];
soln=y[x]/.(soln/.{(x^4)^(1/2)->x^2})//First
```

The result is given below:

$$\mathrm{InverseFunction}[\int_1^{\#1} \frac{1}{\alpha[-\beta^2 + K[1]^2]} \mathrm{d}K[1]\ \&][\frac{1}{x} + C[1]]$$

To “get rid of” the Bessel functions of order $\pm 5/4$ we need to apply the reduction rules:

$$\mathrm{rule} = \mathrm{BesselJ}[p + s, x] \to (2p/x)\mathrm{BesselJ}[p, x] - \mathrm{BesselJ}[p - s, x];$$
$$\mathrm{rule1} = \mathrm{rule}/.\{p \to 1/4, s \to 1, x \to x\text{^}2/2\}$$
$$\mathrm{rule2} = \mathrm{rule}/.\{p \to -(1/4), s \to -1, x \to x\text{^}2/2\}$$

Then the results are as follows:

$$\mathrm{BesselJ}[\frac{5}{4}, \frac{x^2}{2}] \to -\mathrm{BesselJ}[-\frac{3}{4}, \frac{x^2}{2}] + \frac{\mathrm{BesselJ}[\frac{1}{4}, \frac{x^2}{2}]}{x^2}$$

$$\mathrm{BesselJ}[-\frac{5}{4}, \frac{x^2}{2}] \to -\frac{\mathrm{BesselJ}[-\frac{1}{4}, \frac{x^2}{2}]}{x^2} - \mathrm{BesselJ}[\frac{3}{4}, \frac{x^2}{2}]$$

When we make these substitutions and adjust the arbitrary constant notation, we get the following simple form:

$$\mathrm{soln} = \mathrm{soln}/.\{\mathrm{rule1}, \mathrm{rule2}, C[1] \to 1/c\}//\mathrm{Simplify}$$

Then we get the following solution of equation (4):

$$\mathrm{InverseFunction}[\int_1^{\#1} \frac{1}{\alpha[-\beta^2+K[1]^2]} \mathrm{d}K[1]\ \&][\frac{1}{c} + \frac{1}{x}]$$

which is an exact computational solution of Riccati expression of Boltzmann equation. To our best knowledge, this solution has never been presented elsewhere before.

And then we can make a graphical plot:

$$\text{Manipulate}[\text{Plot}[-\frac{\sqrt{\beta}\text{Tanh}[\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}]}{\sqrt{\alpha}}, \{x,-8,8\}], \{c,-8,8\}, \{\alpha,-2,2\}, \{\beta,-2,2\}]$$

The plot is shown below for certain values of c, alpha and beta:

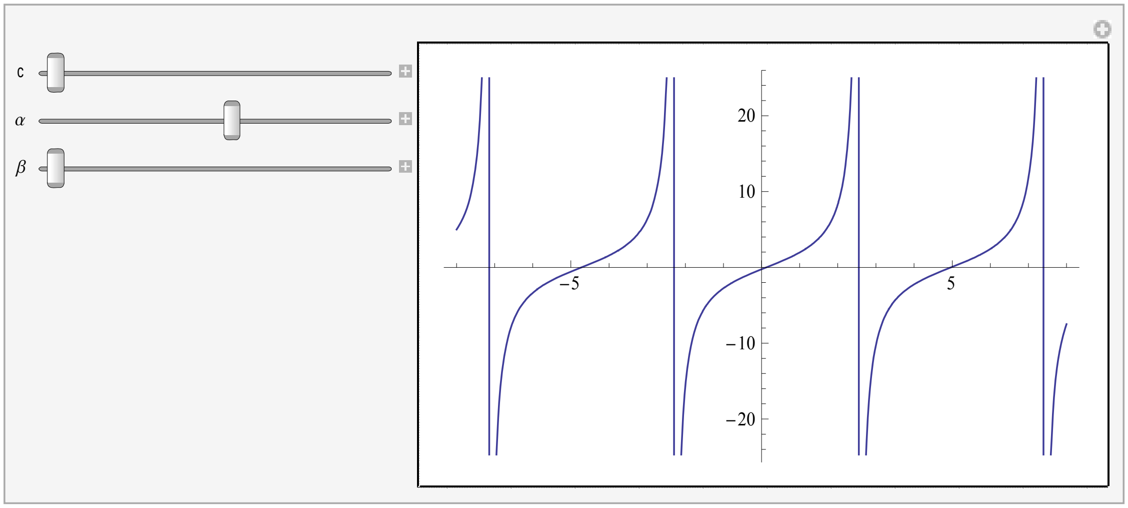

Figure 1. Graphical plot of solution of Riccati expression of Boltzmann equation

We can proceed further to transform from Exponential function to Trigonometric function:

$$\text{TrigToExp}[-\frac{\sqrt{\beta}\text{Tanh}[\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}]}{\sqrt{\alpha}}]$$

The result is shown below:

$$-\frac{(-e^{-\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}}+e^{\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}})\sqrt{\beta}}{(e^{-\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}}+e^{\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}})\sqrt{\alpha}}$$

Then we can plot its function:

$$\text{Manipulate}[\text{Plot}[-\frac{(-e^{-\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}}+e^{\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}})\sqrt{\beta}}{(e^{-\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}}+e^{\frac{(1+cx)\sqrt{\alpha}\sqrt{\beta}}{c}})\sqrt{\alpha}}, \{x,-8,8\}], \{c,-8,8\}, \{\alpha,-2,2\}, \{\beta,-2,2\}]$$

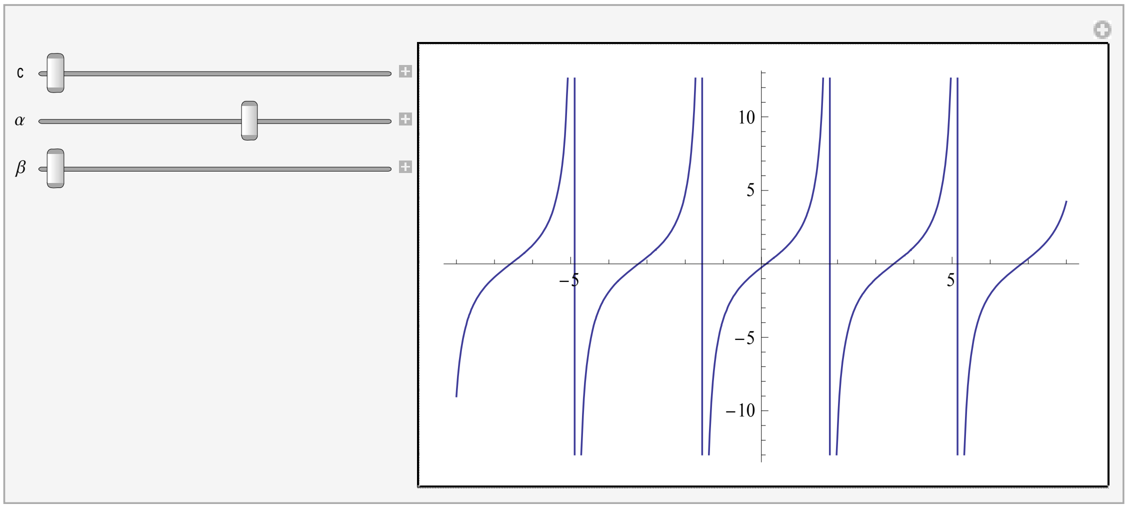

Figure 2. Graphical plot of solution of Riccati expression of Boltzmann equation after Trigonometric transformation

Concluding remarks

The Boltzmann equation is often used to calculate relic density of non-baryonic dark matter. The main idea is that particles were in thermal equilibrium with the early universe. There are many solutions for Boltzmann equation in the form of software packages, such as MadDM. This paper discusses an exact computational solution of Riccati form of Boltzmann equation with Mathematica. While in literature there is already another presentation of Mathematica routine to compute relic density, to our best knowledge, this solution has never been presented elsewhere before.

Acknowledgment
The author would like to express sincere gratitude to Lord Jesus Christ, who has helped him out from so many troubles. He is the true savior and the Good Shepherd.

Version 1.0: 31 October 2015, timestamp: 14.39

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