Abstract

The integral \( R(t) = \pi^{-1} \int \frac{d}{dt} \ln Z(t) dt = \pi^{-1} \left( \ln \left( \frac{1}{2} + it \right) + i \vartheta(t) \right) \) of the logarithmic derivative of the Hardy Z function \( Z(t) = e^{i \vartheta(t)} \left( \frac{1}{2} + it \right)^{1/2} \), where \( \vartheta(t) \) is the Riemann-Siegel vartheta function, and \( \zeta(t) \) is the Riemann zeta function, is used as a basis for the construction of a pair of transcendental entire functions \( \zeta(t) = -\frac{\zeta(1-t)}{t} = \frac{i}{\pi} \ln \left( \frac{1}{2} - it \right) \) where \( G = -(\Delta R(t))^{-1} \) is the derivative of the additive inverse of the reciprocal of the Laplacian of \( R(t) \) and \( \chi(t) = -\chi(1-t) = \frac{i}{\pi} H \left( \frac{1}{2} - it \right) \). When \( H(t) > 0 \) and \( H(t) = G(t) > 0 \), the point \( t \) marks a minimum of \( G(t) \) where it coincides with a Riemann zero, i.e., \( \zeta \left( \frac{1}{2} + it \right) = 0 \), otherwise when \( H(t) = 0 \) and \( H(t) = G(t) < 0 \), the point \( t \) marks a local maximum \( G(t) \), marking midway points between consecutive minima. Considered as a sequence of distributions or wave functions, \( \nu_n(t) = \nu(1 + 2n + 2t) \) converges to \( \nu_{\infty}(t) = \lim_{n \to \infty} \nu_n(t) = \sin^2(\pi t) \) and \( \chi_n(t) = \chi(1 + 2n + 2t) \) to \( \chi_{\infty}(t) = \lim_{n \to \infty} \chi_n(t) = -8 \cos(\pi t) \sin(\pi t) \).

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1 Derivations

1.1 Standard Definitions

Let \( \zeta(t) \) be the Riemann zeta function

\[
\zeta(t) = \sum_{n=1}^{\infty} n^{-s} \quad \forall \Re(s) > 1 \\
= (1 - 2^{1-s}) \sum_{n=1}^{\infty} n^{-s} (-1)^{n-1}
\]  (1)
and \( \vartheta(t) \) be Riemann-Siegel vartheta function \( \vartheta(t) \)

\[
\vartheta(t) = \frac{i}{2} \left( \ln \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) - \ln \Gamma \left( \frac{1}{4} - \frac{it}{2} \right) \right) - \frac{\ln(\pi) t}{2}
\]

(2)

where \( \arg(z) = \frac{\ln(z) - \ln(\bar{z})}{2i} \) and \( \Gamma(z) = \Gamma(\bar{z}) \). The Hardy \( Z \) function\[^{[13]} \] can then be written as

\[
Z(t) = e^{i \vartheta(t)} \zeta \left( \frac{1}{2} + it \right)
\]

(3)

which can be mapped isometrically back to the \( \zeta \) function

\[
\zeta(t) = e^{-i \vartheta \left( \frac{t}{2} - it \right)} Z \left( \frac{i}{2} - it \right)
\]

(4)

due to the isometry

\[
t = \frac{1}{2} + i \left( \frac{i}{2} - it \right)
\]

(5)

of the Mobius transforms\[^{1} \] \( f(t) = \frac{at + b}{ct + d} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) with

\[
\left( \begin{array}{cc} -i & \frac{i}{2} \\ 0 & 1 \end{array} \right)
\]

and its inverse \( \left( \begin{array}{cc} i & \frac{1}{2} \\ 0 & 1 \end{array} \right) \)

(6)

making possible the Riemann-Siegel-Hardy correspondence. Furthermore, let \( S(t) \) be argument of \( \zeta \) normalized by \( \pi \) defined by

\[
S(t) = \pi^{-1} \arg \left( \zeta \left( \frac{1}{2} + it \right) \right) = -\frac{i}{2\pi} \left( \ln \zeta \left( \frac{1}{2} + it \right) - \ln \zeta \left( \frac{1}{2} - it \right) \right)
\]

(7)

The Bäcklund counting formula gives the exact number of zeros on the critical strip up to level \( t \), not just on the critical line,

\[
N(t) = \text{Im}(R(t)) = \frac{\vartheta(t)}{\pi} + 1 + S(t)
\]

(8)

The relationship between the functions \( N(t) \), \( S(t) \), and \( Z(t) \) is demonstrated by

\[
\ln \zeta \left( \frac{1}{2} + it \right) = \ln |Z(t)| + i\pi S(t)
\]

(9)

These formulas are true independent of the Riemann hypothesis which posits that all complex zeros of \( \zeta(s + it) \) have real part \( s = \frac{1}{2} \). \[^{[13]} \] Corollary 1.8 p.13

1.2 The Logarithmic Derivative of \( Z(t) \) and its Integral

Let \( Q(t) \) be the logarithmic derivative of \( Z(t) \) given by

\[
Q(t) = \frac{\dot{Z}(t)}{Z(t)} = \frac{d}{dt} \ln Z(t)
\]

\[
= i \left( \frac{\zeta \left( \frac{1}{2} + it \right)}{\zeta \left( \frac{1}{2} + it \right)} + \Psi \left( \frac{1}{4} - \frac{i}{27} \right) + \Psi \left( \frac{1}{4} + \frac{i}{27} \right) - 2\ln(\pi) \right) - \frac{d}{dt} \ln |Z(t)|
\]

\[
= i \left( \frac{\dot{\zeta}(t)}{\zeta(t)} + g^{-}(t) + g^{+}(t) - 2\ln(\pi) \right) - \frac{d}{dt} \ln |Z(t)|
\]

(10)

1. Thanks to Matti Pitkänen for pointing out this is a Mobius transform pair, among other things
where
\[ \Psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{d}{dx} \frac{\Gamma(x)}{\Gamma(x)} \] (11)
is the digamma function, the logarithmic derivative of the \( \Gamma \) function and \( z(t) \) and \( g^\pm(t) \) have been introduced to simplify the expressions. The function \( Q(t) \) has singularities at \( \pm \frac{t}{\pi}(4n - 3) \) with residues
\[ \text{Res}(Q(t)) = -\frac{8\zeta(2 - 2n)n^2 + 4\zeta(2 - 2n)n}{16\zeta(2 - 2n)n^2 - 8\zeta(2 - 2n)n} = -\frac{1}{2} \] (12)
and
\[ \text{Res}(Q(t)) = \frac{8\zeta(2n - 1)n^2 - 4\zeta(2n - 1)n}{8\zeta(2n - 1)n - 4\zeta(2n - 1)} = \frac{1}{2} \] (13)

Now, the integral of the logarithmic derivative of \( Z \) is defined by
\[
R(t) = \pi^{-1} \int Q(t) dt \\
= \pi^{-1} \int \frac{Z(t)}{Z(t)} dt \\
= \pi^{-1} \int \frac{d}{dt} \ln Z(t) dt \\
= \pi^{-1} \int i \left( \frac{\dot{z}(t)}{z(t)} + \frac{g^-(t) + g^+(t) - 2\ln(\pi)}{4} \right) dt \\
= \pi^{-1} \int i \left( \frac{\dot{z}(t)}{z(t)} + \frac{\Psi\left( \frac{1}{4} - \frac{i}{2\pi} \right) + \Psi\left( \frac{1}{4} + \frac{i}{2\pi} \right) - 2\ln(\pi)}{4} \right) dt \\
= \pi^{-1} \left( \ln \zeta\left( \frac{1}{2} + it \right) + i\theta(t) \right) \\
= \pi^{-1} \left( \ln \zeta\left( \frac{1}{2} + it \right) + it \ln \Gamma\left( \frac{1}{4} - \frac{i}{2} \right) - it \ln \Gamma\left( \frac{1}{4} + \frac{i}{2} \right) - \ln(\pi) t \right) \] (14)

### 1.3 The Laplacian

The Laplacian \( \Delta = \nabla \cdot \nabla = \nabla^2 \) is a differential operator which corresponds to the divergence of the gradient of a function \( f(x) \) on a Euclidean space \( x \in X \subseteq \mathbb{R}^d \) and is denoted by
\[
\Delta f(x) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f(x_i) \forall x \in X \subseteq \mathbb{R}^d \] (15)
and is simply the second derivative of a function when \( d = 1 \)
\[
\Delta f(x) = \ddot{f}(x) = \frac{d}{dx^2} f(x) \forall x \in X \subseteq \mathbb{R} \] (16)

Let \( G(t) \) be the additive inverse of the reciprocal(also known as multiplicative inverse) of the Laplacian of \( R(t) \) interpreted as a partition function
\[
G(t) = -\frac{1}{\Delta R(t)} \\
= \frac{8 z(t)^2 \pi}{z(t)^2 (\dot{g}^-(t) - \dot{g}^+(t)) - 8 (z(t)^2 \ddot{z}(t) + \ddot{z}(t)^2)} \] (17)
Now, let $H(t)$ be the derivative of $G(t)$ given by

$$H(t) = \dot{G}(t) = \frac{d}{dt}\left(-\frac{1}{\Delta R(t)}\right)$$

$$= -4i\pi z(t)\zeta(t)^3 \Delta g(t) + 8(2z(t)^2 \bar{z}(t) - 6z(t)\bar{z}(t)\dot{z}(t) - 4\dot{z}(t)^3)$$

$$(z(t)^2 + 8(z(t)\bar{z}(t) - \dot{z}(t)^2))^2$$

When $H(t) = 0$ and $H'(t) = \dot{G}(t) = \Delta G(t) > 0$, the point $t$ marks a minimum of $G(t)$ where it coincides with a Riemann zero, i.e., $\zeta\left(\frac{1}{2} + it\right) = 0$, otherwise when $H(t) = 0$ and $H'(t) = \Delta G(t) < 0$, the point $t$ marks a local maximum $G(t)$, marking midway points between consecutive minima.

With the identity $t = \frac{1}{2} + i\left(\frac{i}{2} - it\right)$, define $\nu(t)$ as the Mobius transform of the partition function

$$\nu(t) = -\left(\Delta R\left(\frac{i}{2} - it\right)\right)^{-1} = -G\left(\frac{i}{2} - it\right)$$

then $\nu(t)$ has zeros at the positive odd integers, zero, and negative even integers. In the same way, let

$$\chi(t) = \hat{\nu}(t)$$

$$= -iH\left(\frac{i}{2} - it\right)$$

$$= -4i\pi\zeta(t)\zeta(t)^3(\hat{\Psi}\left(\frac{1}{2}\right) - \hat{\Psi}\left(\frac{1}{2} - t\right)) - 8(6\zeta(t)\tilde{\zeta}(t)\zeta(t) + 2\zeta(t)^3\tilde{\zeta}(t) + 4\zeta(t)^3)$$

$$\zeta(t)^2\left(\left(\hat{\Psi}\left(\frac{1}{2}\right) - \hat{\Psi}\left(\frac{1}{2} - t\right)\right) + 8(\zeta(t)\zeta(t) - \zeta(t)^2)^2\right)$$

Figure 1. The Real and Imaginary Parts of $R(t)$ compared with $G(t)$

Figure 2. $G(t)$, $H(t)$ and $\nu(t)$
which also has zeros on the real line at the positive odd integers, zero, and the negative even integers.

\[
\begin{align*}
\lim_{t \to 2n-1} \nu(t) &= \lim_{t \to 2n-1} \chi(t) = 0 \forall n > 0 \\
\lim_{t \to 0} \nu(t) &= \lim_{t \to 0} \chi(t) = 0 \\
\lim_{t \to -2n} \nu(t) &= \lim_{t \to -2n} \chi(t) = 0 \forall n < 0
\end{align*}
\]

Both \( \nu(t) \) and \( \chi(t) \) satisfy similar functional equations

\[
\nu(t) = \nu(1-t) \tag{22}
\]

and

\[
\chi(t) = -\chi(1-t) \tag{23}
\]

So the even \( \mu(t) \) and odd \( \psi(t) \) transcendental entire functions can be defined

\[
\mu(t) = \nu\left(t + \frac{1}{2}\right) \tag{24}
\]

\[
\psi(t) = \chi\left(t + \frac{1}{2}\right) \tag{25}
\]

which satisfy the functional symmetries

\[
\mu(t) = \mu(-t) \tag{26}
\]

and

\[
\psi(t) = -\psi(-t) \tag{27}
\]

The function \( \chi(t) \) has as a subset of its roots, the roots of the Riemann zeta function \( \zeta(t) \), the converse is not true, since \( \chi(t) \) is a function of \( \zeta(t) \) and its first, second, and third derivatives.

\[
\{t; \zeta(t) = 0\} \subset \{t; \chi(t) = 0\} \tag{28}
\]

Let

\[
\chi_n(t) = \chi(1+2n+2t) \tag{29}
\]

and

\[
\nu_n(t) = \nu(1+2n+2t) \tag{30}
\]

which both satisfy the symmetries

\[
\chi_n\left(-\frac{1}{2}\right) = \chi(t) \tag{31}
\]

\[
\nu_n\left(-\frac{1}{2}\right) = \nu(t) \tag{32}
\]

as well as

\[
\chi_n\left(-\frac{1}{2}\right) = \chi(2n) \tag{33}
\]

\[
\nu_n\left(-\frac{1}{2}\right) = \nu(2n) \tag{34}
\]

The sequence of wave functions \( \nu_n(t) \) converges, thanks to quantum ergodicity, to

\[
\nu_\infty(t) = \lim_{n \to \infty} \nu_n(t) = \sin^2(\pi t) \tag{35}
\]

which has a limiting maximum

\[
\lim_{n \to \infty} \max_{0 < t < 1} \nu_n(t) = \max_{0 < t < 1} \nu_\infty(t) = \lim_{n \to \infty} \nu(2n) \frac{8}{\pi} \approx 2.546479089470... \tag{36}
\]
and the associated differential is

\[ \chi_\infty(t) = \lim_{n \to \infty} \chi_n(t) = \frac{4}{\pi} \frac{d}{dt} \nu_\infty(t) = \frac{4}{\pi} \frac{d}{dt} \sin^2(\pi t) = -8 \sin(\pi t) \cos(\pi t) \]  

(37)

It is worth mentioning that the pair-correlation function for the zeros of \( \zeta(s) \) is

\[ r_2(x) = 1 - \frac{\sin^2(\pi x)^2}{\pi^2 x^2} \]

if the Riemann hypothesis is true. [11] It is also worth mentioning that in Dirac’s theory of the time-dependent Schroedinger there is a Hamiltonian of the form

\[ H = H_0 + \lambda H_1(t) \]  

(38)

which has a transition probability per unit time of

\[ w(p \to -p) = \frac{V_0^2 \sin^2\left(2t \sqrt{\frac{2mc}{\hbar}}\right)}{2\sqrt{8m\epsilon^2L}} \]

where \( V_0, \epsilon, \hbar, \) and \( L \) are suitable constants. [Gol61, 15.5] The point \( n = 1 \) is where \( \chi_n(-\frac{1}{2}) \) and \( \nu_n(-\frac{1}{2}) \) attain their greatest values among the integers.

<table>
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<th>( n )</th>
<th>( \nu_n(-\frac{1}{2}) )</th>
<th>( \chi_n(-\frac{1}{2}) )</th>
</tr>
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<td>9447.7593604712860</td>
</tr>
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<td>0000.2816402783351</td>
</tr>
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<tr>
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</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \frac{8}{\pi} )</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1. Even values of \( \nu(n) \) and \( \chi(n) \)

This phenomena of having the first wave function having a much larger size than all of the remainders is mentioned in [Kna99, Theorem 22] The convergence of \( \lim_{n \to \infty} \chi_n(t) \to \chi_\infty(t) \) is a manifestation of quantum ergodicity [11, C p.16]. There is an essential singularity [CB90, 54 p.169] at \( s_0 \equiv 1.98757823 \)

\[ \lim_{s \to s_0} \chi(s) = \infty \]  

(39)

The integral over this sine-squared kernel is

\[ \int_0^1 \sin^2(\pi t) \, dt = \frac{1}{2} \]  

(40)

whereas

\[ \int_0^1 \nu(t) \, dt \cong 0.3915066234114391640 \ldots \]  

(41)

or

\[ \int_0^{\nu(\frac{1}{2})} \frac{\nu(t)}{\frac{1}{2}} \, dt \cong 0.4669375515359653755 \ldots \]  

(42)
1.4 Figures

Figure 3. Left: The function $\nu(s)$ normalized by its maximum value on $[0,1]$ Right: The function $\nu(s)$ normalized by its maximum value and subtracted from $\sin^2(\pi x)$

Figure 4. Left: Convergence of $\chi_n(t) \to -8 \cos(\pi t) \sin(\pi t)$ Right: Convergence of $\nu_n(t) \to \sin^2(\pi t)\frac{a}{b}$

2 Appendix

2.1 Wave Mechanics

2.2 The One-Dimensional Wave Equation

The one-dimensional wave equation is

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} u(x, t) + V(x, t) u(t, x) = -\frac{\hbar}{i} \frac{\partial}{\partial t} u(x, t)$$

(43)

[Fli94, II.A.4 p.25][Gan06, 4.2.1 p.][Coo03, Ch. 8 p.147]

2.3 The Poisson Bracket and Lagrangian Mechanics

“The” Poisson bracket, expressed with Einstein summation convention (for the repeated index $i$)

$$\{A, B\} = \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial q_i}$$

(44)

has the antisymmetry property

$$\{A, B\} = -\{B, A\}$$

(45)

and the so-called Jacobi identity

$$\{A, \{B, C\}\} - \{B, \{A, C\}\} + \{C, \{A, B\}\} = 0$$

(46)
Two quantities $A$ and $B$ are said to commute if their Poisson bracket $\{A, B\}$ vanishes, that is, $\{A, B\} = 0$. Hamilton’s equations of motion for the system

$$\dot{p}_i = -\frac{H}{\dot{q}_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial \dot{q}_i}$$

where $H$ is a Legendre-transformed function of the Lagrangian called the Hamiltonian

$$H = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L(q_i, \dot{q}_i, t)$$

whose value for any given time $t$ gives the energy

$$A[q_i] = \int_{t_a}^{t_b} L(q_i, \dot{q}_i, t)dt$$

of the system where $L(q_i, \dot{q}_i, t)$ is the Lagrangian of the system and

$$q_i(t) = q_i^1(t) + \delta q_i(t)$$

is an arbitrary path where $q_i^1(t)$ is the classical orbit or classical path of the system and

$$\delta q_i(t) = q_i(t) - q_i^1(t)$$

[Kle04, 1.1]

2.3.1 The Euler-Lagrange equation

The Euler-Lagrange equation

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

indicates that the action $S$ given by

$$S = \int_{t_1}^{t_2} L(t)dt$$

where

$$L(t) = T - V(x(t))$$

is the Lagrangian and

$$T(t) = \frac{1}{2}m \dot{x}(t)^2$$

is the kinetic energy which is stationary for the physical solutions $q_i(t)$.[Gan06, 4.2.1]

2.3.2 Quantum Mechanics of General Lagrangian Systems

The coordinate transformation

$$x^i = x^i(q^\mu)$$

implies the relation

$$\partial_\mu = \frac{\partial}{\partial q^\mu} = e^{\mu}_i(q) \partial_i$$

between the derivatives $\partial_\mu$ and

$$\partial_i = \frac{\partial}{\partial x^i}$$

where

$$e^{\mu}_i(q) = \partial_\mu x^i(q)$$

is a transformation matrix called the basis $p$-ad where $p$ is the prefix corresponding to $n$, the dimension of $x$, monad when $n = 1$, dyad when $n = 2$, triad when $n = 3$, and so on. Let

$$e^{\mu}_i(q) = \frac{\partial q^\mu}{\partial x^i}$$
be the inverse matrix called the reciprocal $p$-ad. The basis $p – ad$ and its reciprocal satisfy the orthogonality and completeness relations

$$e_i^\nu e_\nu^\mu = \delta_i^\mu$$

and

$$e_i^\mu e_j^\mu = \delta_i^j$$

The inverse of $\partial_\mu$ is

$$\partial_\mu = e_\mu^\nu(q)\partial_\mu$$

which is related to the curvilinear transform of the Cartesian quantum-mechanical momentum operators by

$$\hat{\mathbf{p}}_i = -i\hbar \partial_\mu = -i\hbar e_\mu^\nu(q)\partial_\mu$$

The Hamiltonian operator for free particles is defined by

$$\hat{H}_0 = \hat{T} = \frac{1}{2M} \hbar^2 \Delta$$

where its metric tensor is given by

$$g^{\mu\nu}(q) = e_\mu^\nu(q) e_\nu^\nu(q)$$

and its inverse by

$$g_{\mu\nu}(q) = e_\mu^\nu(q) e_\nu^\mu(q)$$

The Laplacian of a metric tensor is then expressed

$$\Delta = \partial_\mu^2 = e_\mu^\nu\partial_\mu e_\nu^\nu\partial_\nu + (e_\mu^\nu\partial_\nu e_\nu^\nu)\partial_\nu$$

where

$$\Gamma_\mu^{\lambda\nu}(q) = -e_\nu^\lambda(q)\partial_\mu e_\nu^\nu(q) = e_\mu^\lambda(q)\partial_\nu e_\nu^\nu(q)$$

is the affine-connection [Kle04, 1.13][Pod28]

### 2.3.3 Noether’s Theorem and Lie Groups

From Noether’s theorem it is known that continuous symmetries have corresponding conservation laws.[Gan06, 4.2.1] Let

$$\alpha_s$$

be a continuous family of symmetries which is a 1-parameter subgroup $s \mapsto \alpha_s$ in the Lie group of symmetries. A Lie group is a group whose operations are compatible with the smooth structure. A smooth structure on a manifold allows for an unambiguous notion of smooth function.

### 2.4 Schroedinger’s Time-Dependent Equation and Nonstationary Wave Motion

The function $\cos^2(t) = 1 - \sin^2(t)$ arises in the Dirac phase averaging method of calculating transition probabilities of non-stationary states in the time-dependent Schroedinger equation. [Gol61, 15.1] 3

#### 2.4.1 Operators and Observables: Dirac’s Time-Dependent Theory

Let

$$y(x, t) = A \cos\left(\frac{2\pi px}{\hbar}\right) \cos\left(\frac{2\pi \epsilon t}{\hbar}\right)$$

where

$$\epsilon = \hbar v$$

(72)
which relates the energies of a “matter wave” system, whatever that is, maybe a fermionic system, to the frequencies of quanta it emits or absorbs. There is always some arbitrariness with choice of units. This wave function has the symmetry

$$\frac{\partial^2 y(x,t)}{x^2} = -\left(\frac{2\pi}{\hbar}\right)^2 p^2 y(x,t)$$

(73)

so that the kinetic energy of a particle is obtained from the wave function

$$\frac{(2\pi)^2}{2m} \frac{\partial^2 y(x,t)}{x^2} = \frac{p^2}{2m} y(x,t)$$

(74)

which means formally that

$$\frac{1}{2m}(p)^2 y(x,t) = -\frac{1}{2m} \left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right)^2$$

(75)

which suggests the identification of the Schrödinger momentum operator

$$\left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right) \leftrightarrow p$$

(76)

The equation for “nonfree” particles is augmented by

$$m \left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right)^2 y(x,t) = [\epsilon - V(x)] y(x,t)$$

(77)

where $\epsilon$ is the total energy of the particle and $V(x)$ is its potential energy. The first Schrödinger equation is then expressed

$$\epsilon y(x,t) = \left[ \frac{1}{2m} \left(\frac{2\pi}{i} \frac{\partial}{\partial x}\right)^2 + V(x) \right] y(x,t) = \left[ \frac{1}{2m} (p)^2 + V(x) \right] y(x,t)$$

(78)

or

$$H y(x,t) = \epsilon y(x,t)$$

where $H$ is the operator corresponding to the Hamiltonian of the (point) particle. [Gol61, 11.5] The time-dependent Schrödinger equation is written

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -\left(\frac{2\pi}{2}\right)^2 \epsilon^2 y(x,t) = -\left(\frac{H}{2\pi}\right)^2 y(x,t)$$

(79)

or equivalently as

$$0 = \left(\frac{2\pi}{2} \frac{\partial^2}{\partial t^2} + H \right) y(x,t) = \left( H + i2\pi \frac{\partial}{\partial t} \right) \left( H - i2\pi \frac{\partial}{\partial t} \right) y(x,t)$$

(80)

which suggest that

$$H y(x,t) = i2\pi \frac{\partial}{\partial t} y(x,t)$$

(81)

and can be interpreted in terms of the wave-function defined by (17) somehow.

2.4.2 The String Theoretic Partition Function

The string theoretic partition function is defined as

$$Z_{st}(\omega) = Z_{st,R}(\omega) = \text{Tr}[e^{2\pi i \omega_1 p - 2\pi \omega_2 H}] = (q\bar{q})^{-\frac{1}{12}} \text{Tr}(qL_0^+ \bar{q}L_0^-)$$

(82)

where $L_0^+$ and $L_0^-$ are the Virasoro generators defined by

$$L_0^\pm = \frac{(p^\pm)^2 + (p^-)^2}{2} + \sum_{n=1}^{\infty} \alpha_n^\pm \alpha_n^\pm$$

(83)

and $\alpha_{-n}^\pm \alpha_n^\pm$ are related to something called a Fubini-Veneziano field and $p^+$ is the left-momentum operator and $p^-$ is right-momentum operator. [Lap08, 2.2.1]
Bibliography


