The first evidence for M theory: Fractal nearly tri-bimaximal neutrino mixing and CP violation

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We propose an instructive possibility to generalize the tri-bimaximal neutrino mixing ansatz, such that leptonic CP violation and the fractal feature of the universe can naturally be incorporated into the resultant scenario of *fractal* nearly tri-bimaximal flavor mixing. The consequences of this new ansatze on the latest experimental data of neutrino oscillations are analyzed. Our theory is perfectly matched with the current experimental data, and we are surprised and excited to find that the existing neutrino oscillation experimental data is the first experimental evidence supporting one kind of higher dimensional unified theory, such as M theory. An interesting approach to construct lepton mass matrices in fractal universe under permutation symmetry is also discussed. Our theory opens an unexpected window on the physics beyond the Standard Model.

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I. INTRODUCTION

As is known to all, the atmospheric and solar neutrino oscillations observed in the Super-Kamiokande experiment [1–5] have provided robust evidence that neutrinos are massive and lepton flavors are mixed. All analyses of the atmospheric neutrino deficit favor $\nu_\mu \to \nu_\tau$ as the dominant oscillation mode with the mass-squared difference $\Delta m^2_{atm} \sim 10^{-3}\mathrm{eV}^2$ and the mixing factor $\sin^2 2\theta_{atm} = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy and $\sin^2 2\theta_{atm} = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy [6]. In addition, the solar neutrino anomaly is most likely attributed to the matter-enhanced $\nu_e \to \nu_\mu$ oscillation via the Mikheyev–Smirnov–Wolfenstein (MSW) mechanism [7, 8] with $\Delta m^2_{sum} \sim 10^{-5}\mathrm{eV}^2$ and $\sin^2 2\theta_{sun} = 0.846 \pm 0.021$ [6] (large-angle MSW solution). The strong hierarchy between Δm^2_{atm} and Δm^2_{sun}, together with the small ν_3-component in ν_e configuration restricted by the CHOOZ reactor neutrino oscillation experiment [9, 10], implies that atmospheric and solar neutrino oscillations decouple approximately from each other. Each of them is dominated by a single mass scale, which can be set as $\Delta m^2_{sum} \equiv \left|m_2^2 - m_1^2\right|$ or $\Delta m^2_{atm} \equiv \left|m_3^2 - m_2^2\right|$ The mixing factors of solar, atmospheric and CHOOZ neutrino oscillations are simply given by

$$\begin{aligned} \sin^2 2\theta_{sun} &= 4\left|V_{e1}\right|^2\left|V_{e2}\right|^2, \\ \sin^2 2\theta_{atm} &= 4\left|V_{\mu 3}\right|^2\left(1-\left|V_{\mu 3}\right|^2\right), \\ \sin^2 2\theta_{chz} &= 4\left|V_{e3}\right|^2\left(1-\left|V_{e3}\right|^2\right), \end{aligned} \tag{1}$$

where V is the 3×3 lepton flavor mixing matrix linking the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$. As current experimental data favor $\sin^2 2\theta_{chz} \ll \sin 2\theta_{sun} \sim \sin 2\theta_{atm} \sim o(1)$, two large flavor mixing angles can be drawn from Eq. (1) in a specific parametrization of V: one between the 2nd and 3rd lepton families and the other between the 1st and 2nd lepton families.

So far a number of phenomenological ansatzes of lepton flavor mixing with two large rotation angles, including the "democratic" ansatz [11–13] and the "bimaximal" ansatz [14], have been proposed and discussed [15]. In this paper we pay our particular attention to the ansatz of the form (up to a trivial sign or phase re-arrangement)

$$V_0 = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{2}$$

proposed by Harrison, Perkins and Scott [16]. This so-called "tri-bimaximal" flavor mixing pattern predicts $\sin^2 2\theta_{atm} = 1$ and $\sin^2 2\theta_{sun} = 8/9$, consistent with the atmospheric neutrino oscillation data and the large-angle MSW solution to the solar neutrino problem. However, it leads also to $\sin^2 2\theta_{chz} = 0$, implying the absence of both high-energy matter resonances and intrinsic CP violation in neutrino oscillations. Xing [17] discussed two possibilities to modify the tri-bimaximal neutrino mixing pattern in Eq. (2), such that CP violation can naturally be incorporated into the resultant scenarios of nearly tri-bimaximal flavor mixing. In Xing's article there is one scenario whose predictions of $\sin^2 2\theta_{sun}$ and $\sin^2 2\theta_{atm}$ are consistent well with the current neutrino oscillation data, but the prediction of $\sin^2 2\theta_{chz} \approx 0.01$ is not consistent with the current neutrino oscillation data [6]: $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$.

The main purpose of this work is to discuss one simple but instructive possibility to modify the tri-bimaximal neutrino mixing pattern in Eq. (2), such that CP violation and the fractal feature of the universe can naturally be incorporated into the resultant scenario of *fractal* nearly tri-bimaximal flavor mixing. One specific texture of the charged lepton mass matrix is taken into account, in order to obtain small but non-vanishing $|V_{e3}|$ or

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$\sin^2 2\theta_{chz}$. We find that when the current experimental data [6] $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$ is adopted to limit the value of parameter q, one has $10.46118470068419 \leq q \leq 12.931439345418308$; then substitute the q value into our theory, the predicted $\sin^2 2\theta_{atm}$ are consistent very well with the current data; and the current data limits the value of parameter φ, the range of which are much better than that in usual space-time. In addition, we also consider another way of thought. Given the q close to 11 and the closely relation $q = d_f$ between q and fractal dimension d_f when the Euclidean dimension is one [18], we assume $q = 11$, then our theory gives the predicted values of $\sin^2 2\theta_{chz}$ and $\sin^2 2\theta_{atm}$ which are consistent very well with the current data; and the range of parameter φ limited by current data is also much better than that in usual space-time. According to above two ways of thought the calculations suggest that the universe is fractal and it's dimension is high; the consistent fact between the calculation according to the second thought and the current data suggests that some high dimensional space-time theory, such as M theory, can be a theory in line with expectations. In addition, the predicted strength of CP or T violation in neutrino oscillations are given. We also discuss an interesting approach to construct lepton mass matrices in fractal universe under permutation symmetry, from which one may derive another *fractal* nearly tribimaximal neutrino mixing scenario with $|V_{e3}| \neq 0$ but with no intrinsic CP violation in neutrino oscillations.

II. FRACTAL NEARLY TRI-BIMAXIMAL NEUTRINO MIXING

The fact that masses of three active neutrinos are extremely small is presumably attributed to the Majorana nature of neutrino fields [19–21]. In this picture, the light (left-handed) neutrino mass matrix M_ν must be symmetric and can be diagonalized by a single unitary transformation:

$$U_\nu^\dagger M_\nu U_\nu^* = Diag\{m_1, m_2, m_3\}. \tag{3}$$

The charged lepton mass matrix M_l is in general non-Hermitian, hence the diagonalization of M_l needs a special bi-unitary transformation:

$$U_l^\dagger M_l \tilde{U}_l = Diag\{m_e, m_\mu, m_\tau\}. \tag{4}$$

The lepton flavor mixing matrix V, defined to link the neutrino mass eigenstates (v_1, ν_2, ν_3) to the neutrino flavor eigenstates (v_e, ν_μ, ν_τ), measures the mismatch between the diagonalization of M_l and that of M_ν: $V = U_l^\dagger U_\nu$. Note that (m_1, m_2, m_3) in Eq. (3) and (m_e, m_μ, m_τ) in Eq. (4) are physical (real and positive) masses of light neutrinos and charged leptons, respectively.

In the flavor basis where M_l is diagonal (i.e., $U_l = \mathbf{1}$ being a unity matrix), the flavor mixing matrix is simplified to $V = U_\nu$. The tri-bimaximal neutrino mixing pattern $U_v = V_0$ can then be constructed from the product of two modified Euler rotation matrices:

$$\begin{aligned} R_{12}(\theta_x) &= \begin{pmatrix} c_x & s_x & 0 \\ -s_x & c_x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ R_{23}(\theta_y) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_y & s_y \\ 0 & -s_y & c_y \end{pmatrix}, \end{aligned} \tag{5}$$

where $s_x \equiv \sin_q \theta_x, c_y \equiv \cos_q \theta_y$, and so on. Functions $\sin_q u$ and $\cos_q u$ can be defined with $\exp_q(u)$ which is the one-dimensional q-exponential function that naturally emerges in nonextensive statistics [22]. For a pure imaginary iu, one defines $\exp_q(iu)$ as the principal value of

$$\begin{aligned} &\exp_q(iu) = [1 + (1-q)\, iu]^{1/(1-q)}, \\ &\exp_1(iu) \equiv \exp(iu). \end{aligned} \tag{6}$$

The above function satisfies [23]

$$\exp_q(\pm iu) = \cos_q(u) \pm i \sin_q(u), \tag{7}$$

$$\cos_q(u) = \rho_q(u) \cos\left\{\frac{1}{q-1} \arctan[(q-1)\, u]\right\}, \tag{8}$$

$$\sin_q(u) = \rho_q(u) \sin\left\{\frac{1}{q-1} \arctan[(q-1)\, u]\right\}, \tag{9}$$

$$\rho_q(u) = \left[1 + (1-q)^2 u^2\right]^{1/[2(1-q)]}, \tag{10}$$

$$\exp_q(iu)\exp_q(-iu) = \cos_q^2(u) + \sin_q^2(u) = \rho_q^2(u). \tag{11}$$

Notice that $\exp_q[i(u_1 + u_2)] \neq \exp_q(iu_1)\exp_q(iu_2)$ for $q \neq 1$[22]. Then we obtain

$$\begin{aligned} V_0 &= R_{23}(\theta_y) \otimes R_{12}(\theta_x) \\ &= \begin{pmatrix} c_x & s_x & 0 \\ -s_x c_y & c_x c_y & s_y \\ s_x s_y & -s_y c_x & c_y \end{pmatrix}. \end{aligned} \tag{12}$$

The vanishing of the (1, 3) element in V_0 assures an exact decoupling between solar $(\nu_e \to v_\mu)$ and atmospheric $(\nu_\mu \to v_\tau)$ neutrino oscillations. The generally form of the corresponding neutrino mass matrix M_ν is

$$M_\nu = V_0 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} V_0^T = \begin{pmatrix} c_x^2 m_1 + s_x^2 m_2 & -c_x c_y s_x (m_1 - m_2) & c_x s_x s_y (m_1 - m_2) \\ -c_x c_y s_x (m_1 - m_2) & c_y^2 s_x^2 m_1 + c_x^2 c_y^2 m_2 + s_y^2 m_3 & -c_y s_y (s_x^2 m_1 + c_x^2 m_2 - m_3) \\ c_x s_x s_y (m_1 - m_2) & -c_y s_y (s_x^2 m_1 + c_x^2 m_2 - m_3) & s_x^2 s_y^2 m_1 + c_x^2 s_y^2 m_2 + c_y^2 m_3 \end{pmatrix}. \tag{13}$$

Taking $q = 1$, $\theta_x = \arctan\left(1/\sqrt{2}\right) \approx 35.3^o$ and $\theta_y = 45^o$, the results in usually space-time are reproduced:

$$V_0 = R_{23}(\theta_y) \otimes R_{12}(\theta_x) = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{14}$$

The corresponding neutrino mass matrix M_ν takes the form

$$M_\nu = V_0 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} V_0^T = \begin{pmatrix} A_\nu - B_\nu - C_\nu & C_\nu & -C_\nu \\ C_\nu & A_\nu & B_\nu \\ -C_\nu & B_\nu & A_\nu \end{pmatrix} \tag{15}$$

where

$$A_\nu = \frac{m_3}{2} + \frac{m_1 + 2m_2}{6}, \quad B_\nu = \frac{m_3}{2} - \frac{m_1 + 2m_2}{6}, \quad C_\nu = \frac{m_2 - m_1}{3}. \tag{16}$$

M_ν might have a meaningful interpretation in an underlying theory of neutrino masses with specific flavor symmetries [17].

The tri-bimaximal neutrino mixing pattern will be modified, if U_l deviates somehow from the unity matrix. This can certainly happen, provided that the charged lepton mass matrix M_l is not diagonal in the flavor basis where the neutrino mass matrix M_ν takes the form given in Eq. (15). As $U_\nu = V_0$ describes a product of two special Euler rotations in the real (2, 3) and (1, 2) planes, the simplest form of U_l which allows $V = U_l^\dagger U_v$ to cover the whole 3×3 space should be $U_l = R_{12}(\theta_x, q = 1)$ or $U_l = R_{31}(\theta_z, q = 1)$ (see Ref. [24] for a detailed discussion). Because when $U_l = R_{31}(\theta_z, q = 1)$ is adopted, the calculated result [17] $0.873 \le sin^2 2\theta_{sun}^{(z)} \le 0.903$ is not consistent with the experimental data [6] $\sin^2 2\theta_{12} = 0.846 \pm 0.021$, here we focus on calculation of case $U_l = R_{12}(\theta_x)$. For convenience θ_x will be replaced by θ below.

To make CP violation and the fractal feature of the universe can naturally be incorporated into V, we adopt the following complex rotation matrices:

$$R_{12}(\theta, \varphi) = \begin{pmatrix} c & s e_q^{i\varphi} & 0 \\ -s e_q^{-i\varphi} & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

where $c \equiv \cos_q \theta$, $s \equiv \sin_q \theta$, and $e_q^{i\varphi} = \exp_q(i\varphi)$. In this case, we arrive at lepton flavor mixing of the pattern

$$V = R_{12}^\dagger(\theta, \varphi) \otimes V_0 = \begin{pmatrix} \frac{1}{\sqrt{6}}\left(2c + s e_q^{i\varphi}\right) & \frac{1}{\sqrt{3}}\left(c - s e_q^{i\varphi}\right) & -\frac{1}{\sqrt{2}} s e_q^{i\varphi} \\ -\frac{1}{\sqrt{6}}\left(c - 2s e_q^{-i\varphi}\right) & \frac{1}{\sqrt{3}}\left(c + s e_q^{-i\varphi}\right) & \frac{1}{\sqrt{2}} c \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{18}$$

It is obvious that V represent one fractal nearly tri-bimaximal flavor mixing scenarios, if the rotation angle θ is small in magnitude. The parameters q and φ in V are the source of leptonic CP violation in neutrino oscillations.

III. CONSTRAINTS ON MIXING FACTORS AND CP VIOLATION

As the mixing angle θ arises from the diagonalization of M_l, it is expected to be a simple function of the ratios of charged lepton masses. Then the strong mass hierarchy of charged leptons naturally assures the smallness of θ as one can see later on.

Indeed a proper texture of M_l which may lead to the flavor mixing pattern V is

$$M_l = \begin{pmatrix} 0 & C_l & 0 \\ C_l^* & B_l & 0 \\ 0 & 0 & A_l \end{pmatrix}, \tag{19}$$

where $A_l = m_\tau$, $B_l = m_\mu - m_e$, and $C_l = \sqrt{m_e m_\mu} e_q^{i\varphi}$. Then the mixing angle θ in V is given by

$$\tan_q(\theta) = \frac{\sin_q \theta}{\cos_q \theta} = \sqrt{\frac{m_e}{m_\mu}}. \tag{20}$$

It is easy to prove that when $q \to 1$, the results in usual space-time are recovered, namely, [17]

$$C_l = \sqrt{m_e m_\mu} e^{i\varphi}, \quad \tan(2\theta) = 2\frac{\sqrt{m_e m_\mu}}{m_\mu - m_e}. \tag{21}$$

In view of the hierarchy of three charged lepton masses (i.e., $m_e \ll m_\mu \ll m_\tau$) and $q \sim o(1)$, we have $\tan_q \theta \approx \tan\theta \approx \sin\theta \approx \sqrt{m_e/m_\mu}$ to a good degree of accuracy. Numerically, we find $\theta \approx 3.978^o$ with the inputs $m_e = 0.511$MeV and $m_\mu = 105.658$MeV [6].

Now let us calculate the mixing factors of solar, atmospheric and reactor neutrino oscillations. With the help of Eqs. (1) and (18), we arrive straightforwardly at

$$\sin^2 2\theta_{sun} = \frac{8}{9}\left(1 - \frac{3}{4}s^2 - sc\cos_q\varphi + \frac{3}{2}s^3 c\cos_q\varphi - 2s^2 c\right.$$
$$\sin^2 2\theta_{atm} = 1 - s^4,$$
$$\sin^2 2\theta_{chz} = 1 - c^4. \quad (22)$$

Note that when $q \to 1$, the results in usual space-time are recovered [17]:

$$\sin^2 2\theta_{sun} = \frac{8}{9}\left(1 - \frac{3}{4}\sin^2\theta - \sin\theta\cos\theta\cos\varphi + \frac{3}{2}\sin^3\theta\right.$$
$$\sin^2 2\theta_{atm} = 1 - \sin^4\theta,$$
$$\sin^2 2\theta_{chz} = 1 - \cos^4\theta. \quad (23)$$

In this scenario, adopting experimental data [6] $\sin^2 2\theta_{chz} = (8.5 \pm 0.5) \times 10^{-2}$, one obtains $10.46118470068419 \leq q \leq 12.931439345418308$; thus there is $0.999987 \leq \sin^2 2\theta_{atm} \leq 0.99999$, which is consistent extremely well with the experimental data [6]: $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy and $\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy; in addition, to make $\sin^2 2\theta_{sun} \leq 0.867$ to accord with the experimental data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$, one needs only $-1537.79 \leq \varphi_{q=10.46} \leq 1537.79$ or $-6372.47 \leq \varphi_{q=12.93} \leq 6372.47$, which are much better than the usual space-time case $(0.485366 \leq \varphi_{q=1} \leq 1.27256)$.

In addition, given the q close to 11 and the closely relation $q = d_f$ between q and fractal dimension d_f when the Euclidean dimension is one [18], we assume $q = 11$, then this scenario gives the predicted values of $\sin^2 2\theta_{chz} = 0.082456$ and $\sin^2 2\theta_{atm} = 0.999987$ which are consistent amazingly well with the current data [6] $\sin^2(2\theta_{13}) = (8.5 \pm 0.5) \times 10^{-2}$ and $\sin^2(2\theta_{23}) = 0.999^{+0.001}_{-0.018}$ for normal mass hierarchy ($\sin^2(2\theta_{23}) = 1.000^{+0.000}_{-0.017}$ for inverted mass hierarchy), respectively; and the range of parameter $-2162.81 \leq \varphi_{q=11} \leq 2162.81$ limited by current data $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$ is also much better than that in usually space-time $(0.485366 \leq \varphi_{q=1} \leq 1.27256)$.

According to above calculations, we come to the following conclusions: i) the universe is fractal and the dimension is high; ii) some high dimensional space-time theory, such as M theory, can be a theory in line with expectations.

A numerical illustration of $\sin^2 2\theta_{sun}$ as functions of q and φ is shown in Fig. 1, and the up and down two blue lines are respectively the upper and lower limits of experimental data. Based on Fig. 1 and the numerical calculations, one obtains the following table:

q	$\varphi_{x\,\min}$	$\varphi_{x\,\max}$
1.0	0.485366	1.27256
10.46118470068419	1537.79	1537.79
11	2162.81	2162.81
12.931439345418308	6372.47	6372.47

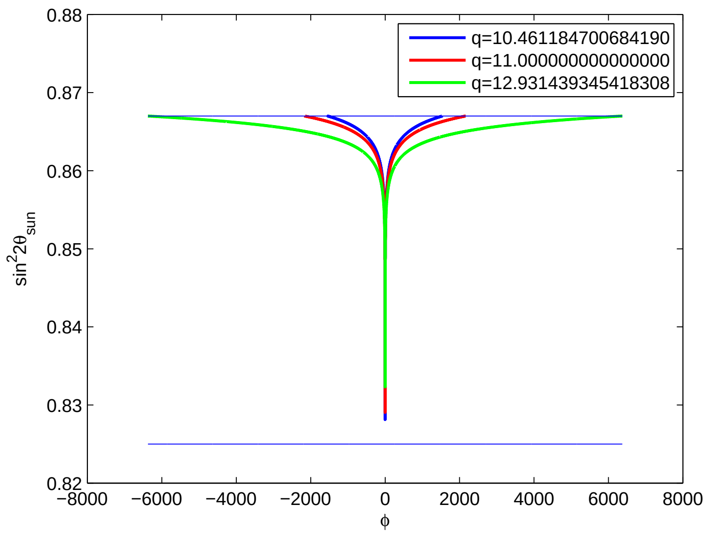

FIG. 1. The mixing factors $\sin^2 2\theta_{sun}$ against parameter φ under different values of q in fractal nearly tri-bimaximal neutrino mixing patterns.

The strength of CP or T violation in neutrino oscillations, no matter whether neutrinos are Dirac or Majorana particles, is measured by a universal and rephasing-invariant parameter J [25], defined through the following equation:

$$Im\left(V_{\alpha i}V_{\beta j}V^*_{\alpha j}V^*_{\beta i}\right) = J\sum_{\gamma,k}(\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk}), \quad (24)$$

in which the Greek subscripts run over (e, μ, τ), and the Latin subscripts run over (e, μ, τ). Considering the lepton mixing scenario proposed above, one obtains

$$J = \frac{1}{6}sc\sin_q\varphi\left(c^2 + s^2\rho_q^2(\varphi)\right). \quad (25)$$

Obviously, when $q \to 1$, the result in usual space-time is recovered [17]:

$$J = \frac{1}{6}sc\sin\varphi. \quad (26)$$

Based on Fig. 2 and the numerical calculations, one obtains the following table:

q	$J_{\min}$	$J_{\max}$
1.0	0.0054	0.0110
10.46118470068419	-0.0012	0.0012
11.0	-0.0011	0.0011
12.931439345418308	8.9595e-04	8.9595e-04

The strength of CP or T violation J in fractal nearly tri-bimaximal neutrino mixing patterns is predicted as:

$$-0.0011 \leq J_{q=11.00} \leq 0.0011. \quad (27)$$

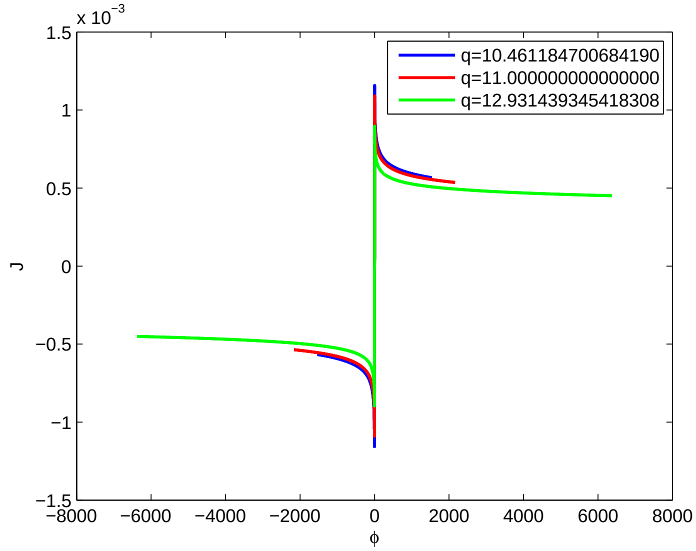

FIG. 2. The strength of CP or T violation J against parameter φ under different values of q in fractal nearly tri-bimaximal neutrino mixing patterns.

The experimental data of strength of CP or T violation may limit the range of parameter φ, but it is a pity that at present no experimental information on the Dirac and Majorana CP violation phases in the neutrino mixing matrix is available [6]. The former could be determined from the CP-violating asymmetry between $\nu_\mu \to \nu_e$ and $\bar{\nu}_\mu \to \bar{\nu}_e$ transitions or from the T-violating asymmetry between $\nu_\mu \to \nu_e$ and $\nu_\mu \to \nu_e$ transitions in a long-baseline neutrino oscillation experiment, if the terrestrial matter effects are insignificant or under control.

IV. FURTHER DISCUSSIONS AND REMARKS

We have discussed one simple possibility to construct the charged lepton and neutrino mass matrices, from which a fractal nearly tri-bimaximal neutrino mixing pattern can naturally emerge. This scenario is compatible with the large-angle MSW solution to the solar neutrino problem. The fact that the predictions of the fractal nearly tri-bimaximal neutrino mixing pattern are consistent perfect with the current experimental data strongly suggests that some high dimensional unified theory, such as M theory, could be the theory in line with expectations and neutrino oscillation experiment is the first robust evidence of M theory, which breaks the spell of the M theory no experimental evidence and opens an unexpected window on the physics beyond the Standard Model. The allowed range width of φ by the fractal nearly tri-bimaximal neutrino mixing pattern is 3 orders of magnitude larger than the usual theory. This theory also yields an prediction of CP- or T-violating asymmetry in long-baseline neutrino oscillation experiments.

The nearly tri-bimaximal neutrino mixing pattern, as Xing [17] expected serves as the leading-order approximation of a more complicated flavor mixing matrix, which is the $q \to 1$ limit case of the fractal nearly tri-bimaximal neutrino mixing pattern under discussion, although its prediction on $\sin^2 2\theta_{chz}$ is not consistent with the experimental data. There are certainly other possibilities to modify the tri-bimaximal neutrino mixing ansatz, such that nonvanishing $|V_{e3}|$ (and CP violation) can naturally be incorporated into the resultant scenario of fractal nearly tri-bimaximal mixing. Note that our scenario predicts that $-0.0011 \leq J_{q=11.00} \leq 0.0011$, and when $\varphi = 0$, $J_{q=11.00} = 0$, namely, there is no CP violation. Therefore our theory can adapt to the CP broken or not at the same time.

Finally let us remark that the fractal nearly tri-bimaximal mixing pattern and its possible extensions require some peculiar flavor symmetries to be imposed on the charged lepton and neutrino mass matrices. It is likely that the fractal nearly tri-bimaximal neutrino mixing pattern under discussion serves as the more complicated flavor mixing matrix that scientists are looking for [17] and one of the nearly tri-bimaximal neutrino mixing patterns is its leading-order approximation. We expect that more delicate neutrino oscillation experiments in the near future can verify the fractal nearly tri-bimaximal mixing pattern, from which one may get some insight into the underlying flavor symmetry and its breaking mechanism responsible for the origin of both lepton masses and leptonic CP violation.

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