GR as a Nonsingular Classical Field Theory*

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Abstract

Thirring and Feynman showed that the Einstein equation is simply a partial differential classical field equation, akin to Maxwell's equation, but it and its solutions are required to conform to the GR principles of general covariance and equivalence. It is noted, with examples, that solutions of such equations can contravene required physical principles when they exhibit unphysical boundary conditions. Using the crucially important tensor contraction theorem together with the equivalence principle, it is shown that metric tensors are physical only where all their components, and also those of their inverse matrix, are finite real numbers, and their signature is that of the Minkowski metric. Thus the "horizons" of the empty-space Schwarzschild solution metrics are unphysical, which is traced to the boundary condition that arises from the minimum energetically-allowed radius of a positive effective mass. It is also noted that "comoving" ostensible "coordinate systems" disrupt physical boundary conditions in time via their artificial "composite time" which can't be registered by the clock of any observer because it is "defined" by the clocks of an infinite number of observers. Spurious singularities ensue in such unphysical "coordinates", which fall away on transformation of metric solutions to non-"comoving coordinates".

Introduction

The Thirring-Feynman physical development of the Einstein equation within a purely Minkowskian framework [1, 2] is ample reason to regard that equation *as a straightforward partial differential classical field equation very akin to the Lorentz-covariant Maxwell electromagnetic field equation*, but one whose *solutions* are physical *only* when they are *consistent* with the *postulated* gravitational physical principles of equivalence and general covariance—the Einstein equation *itself* of course manifests *formal* general covariance.

Although this isn't usually *explicitly* pointed out, *solutions* of the partial differential equation of a classical field theory *sometimes violate* the physical *postulates* of that theory; such a seeming paradox *readily arises* when a solution *exhibits boundary conditions which happen to be inconsistent with those postulates.* For example, the four Maxwell equations of source-free electromagnetism,

$$\nabla\cdot\mathbf{E}=0,\quad \nabla\times\mathbf{E}=-(1/c)\dot{\mathbf{B}},\quad \nabla\cdot\mathbf{B}=0\quad\text{and}\quad \nabla\times\mathbf{B}=(1/c)\dot{\mathbf{E}},$$

are clearly satisfied by *all* static uniform $\mathbf{E}$ and $\mathbf{B}$ fields. However *unless* those particular solutions *completely vanish*, their corresponding *electromagnetic field energy*, namely $(1/2)\int d^3\mathbf{r}\,(|\mathbf{E}|^2+|\mathbf{B}|^2)$, *diverges*, revealing their *unphysical nature*; indeed nonzero static uniform electromagnetic fields, along with *all* field solutions of the above source-free Maxwell equations *which fail to be square-integrable*, must be *discarded* in order not to violate the physical postulate of finite energy. The *divergent energies* of these unphysical electromagnetic field solutions are reminiscent of the *divergent wave-function normalizations* which occur for a class of *unphysical wave-function solutions of Schrödinger equations.* (Complex-valued quantum Schrödinger equations are readily recast into the *form* of real-valued linear classical-field equation systems by separation of the real and imaginary parts of both their operators and wave functions.) The stationary-state Schrödinger equation for the simple harmonic oscillator,

$$(1/2)[-(\hbar^2/m)(d^2/dx^2)+m\omega^2x^2]\psi_{E_{\rm osc}}(x)=E_{\rm osc}\psi_{E_{\rm osc}}(x),$$

has for each nonnegative value of $E_{\rm osc}$ two linearly-independent parabolic cylinder function solutions. But it is *only* when $E_{\rm osc}$ takes on one of the *discrete* values $[n+(1/2)]\hbar\omega$, $n=0,1,2,\ldots$, that there exists a linear combination of those two solutions which *isn't* strongly unbounded as $|x|\to\infty$. All the remaining nonnegative values of $E_{\rm osc}$ are therefore associated to solutions *that are not normalizable and hence are*

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unphysical; such wave-function solutions *are all discarded.* The *discrete* negative energy spectrum of *the hydrogen atom* is *likewise* associated with the *discarding* of the non-normalizable, unphysical Schrödinger-equation solutions. Nor is non-normalizability the *only* unphysical wave-function attribute associated with a wave function's *exhibiting unphysical boundary conditions.* The simple rotator's discrete energy spectrum arises from the boundary-condition requirement that its *physical* wave-function solutions be *periodic* in the rotation angle θ *with period* 2π, and all *unphysical* wave-function solutions *are discarded.*

Turning now to the Einstein equation's compliance with the principle of general covariance, we note that because the Einstein tensor involves *contractions* of the Riemann tensor, the validity of the *tensor contraction theorem* is crucial to the Einstein equation's general covariance.

Space-time transformation constraints for valid tensor contraction

Demonstration of the tensor contraction theorem for the space-time transformation $\bar{x}^\alpha(x^\mu)$ at the space-time point x^μ requires that at x^μ the following relation must hold [3],

$$(\partial\bar{x}^\alpha/\partial x^\mu)(\partial x^\nu/\partial\bar{x}^\alpha) = \delta^\nu_\mu. \tag{1}$$

Eq. (1) of course follows from the chain rule of the calculus when every component of the Jacobian matrix $(\partial\bar{x}^\alpha/\partial x^\mu)$ is a well-defined finite real number and the same is true of every component of its inverse matrix. But if any component of that Jacobian matrix or of its inverse matrix is ill-defined as a finite real number, that will also be true of the left-hand side of Eq. (1), while its right-hand side remains well-defined as a finite real number, i.e., under those circumstances Eq. (1) *is self-inconsistent.* Therefore, in the context *of respecting the general covariance of the Einstein equation*, for which the tensor-contraction theorem is *crucial*, a space-time transformation is physical *only* at space-time points where *every component* of its Jacobian matrix and also of the *inverse* of that matrix is well-defined as a finite real number.

Since the principle of equivalence requires every metric tensor to locally be the *congruence* of the Minkowski metric tensor with a space-time transformation [4], the foregoing characterization of the *physical* space-time points of those transformations in the context of GR necessarily *also* impacts the characterization of the *physical* space-time points of metric tensors in the context of GR.

Metric constraints due to the constraints on physical transformations

The principle of equivalence as it affects metric tensors [4], taken together with the results of the previous section regarding the *physical* points of a space-time transformation in the context of GR, implies that a metric tensor can only be *physical* in the context of GR at space-time points where every component of both it and its inverse is a well-defined finite real number and where its signature is equal to the $(+,-,-,-)$ signature of the Minkowski metric tensor.

Therefore a metric tensor *solution* of the Einstein equation is *unphysical* in the context of GR at any space-time point where it or its inverse *has singular components.* As has been pointed out in the Introduction, such solution singularities would likely reflect *unphysical boundary conditions* exhibited by those solutions. In any case a *classical physics* theory ought not manifest *mathematical singularities that are regarded as physical.*

We now try to identify the unphysical boundary condition exhibited by the unphysical singular "horizon" points of spherically-symmetric static empty-space Schwarzschild metric tensor solutions of the Einstein equation. Those singular "horizon" points in empty space occur under the assumption that a static source of fixed *positive* effective mass $M > 0$ *can have arbitrarily small size*, just as is the case in *nonrelativistic* Newtonian gravity theory. But attempting to *assemble* an arbitrarily *small* source of fixed *positive* effective mass $M > 0$ unleashes a potentially *unlimited* source of *negative gravitational attractive energy* that enters into *the relativistic effective mass* and *blocks* the attempt.

Is the Schwarzschild "horizon" really located in empty space?

In the static picture, let's try to *assemble* a positive effective mass of arbitrarily small size by progressively *reducing the separation d* between *two* idealized point masses which each have positive mass $M_>/2$. The effective mass M of this system is given by,

$$Mc^2 = M_> c^2 - G(M_>/2)^2/d.$$

When $d \to \infty$, $M \to M_>$. But when $d \to 0$ for *fixed* $M_>$, $M \to -\infty$! To *avoid* this negative effective-mass catastrophe, we *optimally choose* $M_>$ *at each value of* d *so as to maximize* M. That *maximum* of M at d is attained when $M_>(d) = 2(c^2/G)d$, and it has the value,

$$M_{\max}(d) = (c^2/G)d. \tag{2}$$

The $M_{\max}(d)$ result of Eq. (2) shows conclusively that as $d \to 0$ a point object of *positive* effective mass *absolutely cannot ensue.* In *addition*, Eq. (2) draws our attention to an inherent self-gravitational *limit* on a system's effective mass that is proportional to *its largest linear dimension*, with a proportionality constant of order (c^2/G). Therefore a system of effective mass M *must have its largest linear dimension be of order* $(G/c^2)M$ *or greater.* But this is the *same* order as that of the radius of the unphysical Schwarzschild "horizon" resulting from effective mass M, making it plausible that that "horizon" *might always lie inside its source*, where the *empty-space* Schwarzschild solution *doesn't even apply.*

That this is *indeed* the case *is strongly supported* by the fact that in spherically-symmetric "standard" coordinates the self-gravitationally shrinking dust ball of effective mass M treated by Oppenheimer and Snyder *never* (quite) shrinks to the radius $2(G/c^2)M$ [5], which is *precisely* the radius of the Schwarzschild-solution "horizon" that *also* has a source of effective mass M and is expressed in those *same* spherically-symmetric "standard" coordinates.

Recapitulating, the unphysical singular "horizon" of the *empty-space* Schwarzschild solution *is boundary-condition disallowed* because potentially unlimited negative gravitational attractive energy makes it *energetically impossible* for a spherically-symmetric source of *relativistic* positive effective mass $M > 0$ *to be as small as its Schwarzschild radius*, let alone arbitrarily small.

Unphysical boundary conditions in time from "comoving coordinates"

An artful insinuation into GR of *unphysical boundary conditions in time* (and with those, of *apparent* metric singularities that in fact *cannot transpire at physically realizable times*) occurs via "comoving" ostensible "coordinate systems" whose definition of "time" *cannot be registered by a clock possessed by any observer whatsoever* because it involves the clocks *of an infinite number of observers* [6]. The *purpose* of that *observationally unrealizable* definition of "time" is to *compel* g_{00} to equal unity *whether a gravitational field is present or not* [7], but this goal *clashes* with the GR *theorems* that in the weak-field static limit $\frac{1}{2}(g_{00} - 1)$ becomes the Newtonian gravitational potential ϕ [8], and that in the static limit $(g_{00})^{-\frac{1}{2}}$ is the gravitational time dilation factor [9].

Thus it is far from surprising that metric and energy-density *singularities* which occur at a definite "time" in unphysical "comoving coordinates" [10] *don't occur at any finite time in "standard" coordinates* [5], in which self-gravitationally contracting balls of uniform dust *never become as small as their Schwarzschild radii.* In like manner, in "standard" coordinates *expanding* balls of uniform dust *were at no time in the past* as small as their Schwarzschild radii [11]. Both of these *singularity-free results* in physically viable "standard" coordinates are *unavoidable* consequences of relativistic energy conservation in the face of potentially unlimited negative gravitational attractive energy. However, some researchers' failure *to transform out of the grossly unphysical "comoving coordinates"* has left them with the impression that expanding dust balls have a finite "age" *since the "occurrence" of their nonexistent "comoving coordinate" singularities* [12].

In singularity-free "standard" coordinates an expanding dust ball *also* has a finite "age", namely the time since the *peak* of its *expansion rate* [11], an "inflationary peak" *which necessarily exists* because at sufficiently early times gravitational time dilation immensely slows a dust ball's expansion, while at sufficiently late times a dust ball's expansion rate gravitationally decelerates in the familiar nonrelativistic way.

References

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[4] S. Weinberg, op. cit., Section 3.6, pp. 85–86.

[5] S. Weinberg, op. cit., The text around Eq. (11.9.40), p. 347.

[6] S. Weinberg, op. cit., Section 11.8, Second paragraph, p. 338.

[7] S. Weinberg, op. cit., Eq. (11.8.1), p. 339.

[8] S. Weinberg, op. cit., Eq. (3.4.5), p. 78.

[9] S. Weinberg, op. cit., Eq. (3.5.2), p. 79.

[10] S. Weinberg, op. cit., Sentence below Eq. (11.9.26), p. 345.

[11] S. K. Kauffmann, "Inflation sans Singularity in 'Standard' Transformed FLRW", viXra:1507.0153, vixra.org/abs/1507.0153 (2015).

[12] S. Weinberg, op. cit., Sentence below Eq. (15.1.24), p. 473.